Bright–dark solitons in the space-shifted nonlocal coupled nonlinear Schrödinger equation

Ping Ren · Jiguang Rao

October 25, 2021

Abstract Multiple bright-dark soliton solutions in terms of determinants for the space-shifted nonlocal coupled nonlinear Schrödinger (CNLS) equation are constructed by using the bilinear (Kadomtsev-Petviashvili) KP hierarchy reduction method. It is found that the bright-dark two-soliton only occur elastic collisions. Upon their amplitudes, the bright two solitons only admit one pattern whose amplitude are equal, and the dark two solitons have three different non-degenerated patterns and two different degenerated patterns. The bright-dark four-soliton is the superposition of the two-soliton pairs and can generated bound-state solitons. The multiple double-pole bright-dark soliton solutions are generated through the long wave limit of the obtained bright-dark soliton solutions, and their collision dynamics are also investigated.

Keywords Space-shifted nonlocal coupled nonlinear Schrödinger equation · Bright-dark solitons · Double-pole bright-dark solitons · Bilinear Kadomtsev-Petviashvili hierarchy reduction method.

PACS 02.30.Jr · 03.75.Lm · 04.20.Jb · 05.45.Yv

1 Introduction

The coupled nonlinear Schrödinger (CNLS) or CNLS-type equations have attracted considerable attentions because of their wide applications in a wide scope of physical fields, spanning from nonlinear optics to, water waves, atomic condensates, plasma physics, and others [1–5]. In general, these CNLS/CNLS-type equations are non-integrable. However, some variant forms of them with particular parametric choices can become integrable systems [6]. Since their mathematical and physical interests, the exact solutions of these integrable forms of

Ping Ren

¹ School of Mathematics and Statistics, Hubei University of Science and Technology, Xianning, Hubei, 437100, P. R. China.

Jiguang Rao (✉)

² School of Mathematics and Statistics, Hubei University of Science and Technology, Xianning, Hubei, 437100, P. R. China. E-mail: jgrao@szu.edu.cn, jgrao@mail.ustc.edu.cn

CNLS/CNLS-type equations were well investigated, including solitons, breathers, rogue wave, and coherent structures of them.

Nowadays, investigating nonlocal equations is a hot topic in the field of integrable systems. The first example is the nonlocal NLS equation, which was introduced by Ablowitz and Musslimani from the particular reductions of the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy [7]. Since their seminal works, hierarchies of nonlocal integrable equations and their exact solutions were proposed and studied [8–21]. Very recently, Ablowitz and Mussliman considered a new nonlocal reduction of the AKNS hierarchy called space-shifted nonlocal reduction [22], and proposed the following space-shifted nonlocal NLS equation:

$$iu_t + u_{xx} \pm u^2 u^*(x_0 - x, t) = 0, \quad (1)$$

where the space-shifted factor x_0 is an arbitrary real parameter. When $x = x_0$, the shifted nonlocal NLS equation (1) reduce back to the usual nonlocal NLS equation [7]. The space shifted nonlocal NLS equation (1) is solvable by the inverse scattering transform and admits an infinitely many conservation laws. The bright two-soliton solutions of the focusing case were studied in the framework of Riemann-Hilbert formulations [22]. The higher-order soliton solutions of Eq.(1) were investigated by using the bilinear method [23, 24].

In this paper, we study the space-shifted nonlocal CNLS equation

$$\begin{aligned} iu_t + u_{xx} + 2[\delta uu^*(x_0 - x, t) + \gamma vv^*(x_0 - x, t)]u &= 0, \\ iv_t + v_{xx} + 2[\delta uu^*(x_0 - x, t) + \gamma vv^*(x_0 - x, t)]v &= 0, \end{aligned} \quad (2)$$

where δ and γ are real coefficients. By scalings of u and v , the nonlinear coefficients δ and γ are normalized to be ± 1 without loss of generality. When the space-shifted factor $x_0 = 0$, Eq.(2) would reduce to the usual nonlocal CNLS equation [8]. The bright solitons and their collisions in the usual nonlocal CNLS equation have been in discussed in [25, 26]. Additionally, the double-pole solitons and their interaction behaviours were studied in details for the usual nonlocal CNLS equation with focusing (i.e., $\delta = -1, \gamma = -1$ in Eq.(2)) and the mixed focusing-defocusing (i.e. $\delta\gamma = -1$ in Eq.(2)) nonlinearities [26]. Here we have to emphasized that although the smooth bright single-pole soliton solutions exists in usual nonlocal CNLS equation with all types of nonlinearities, but the regular double-pole soliton solutions were not found in the usual nonlocal defocusing CNLS equation (i.e., $\delta = \gamma = -1, x_0 = 0$ in Eq.(2)).

The usual (single-pole) soliton solutions and higher-order pole soliton solutions have essential differences. The former one are expressed by superpositions of exponential functions, while the later one are given by combinations of exponential functions with rational functions. The higher-order pole soliton solutions can be derived from single pole-soliton solutions through limit procedures [29]. There are lot of researches were reported for the higher-order pole solitons in the local equations [27–46], which focus on the long time asymptotic behaviours and interaction properties of the higher-order pole solitons. However, related results on higher-order pole solitons are scarce for the nonlocal equations.

In this work, we will investigate the space-shifted nonlocal CNLS equation (2) in two aspects:

- The multiple bright-dark soliton solutions in terms of determinants for the space-shifted nonlocal CNLS equation (2) via the bilinear KP hierarchy reduction.
- The multiple double-pole bright-dark soliton solutions from the bright-dark solitons through the long wave limit procedure.

This paper is organized as follows. In Section 2, first we construct multiple bright-dark soliton solutions in terms of determinants for the space-shifted nonlocal CNLS equation (2) by using the bilinear KP hierarchy reduction, then we study the collision dynamics of these multiple bright-dark solitons. In Section 3, we derive the multiple double-pole bright-dark soliton solutions for the space-shifted nonlocal CNLS equation (2) by taking the long wave limit procedure from the obtained usual (single-pole) bright-dark solitons. The conclusions are given in the section 4.

2 Multiple bright-dark solitons in the space-shifted nonlocal CNLS equation

In this section, we investigate the multiple bright-dark soliton solutions for the space-shifted nonlocal CNLS equation (2) by using the bilinear Kadomtsev-Petviashvili (KP) hierarchy reduction method [47–51], in which the u and v components correspond to the bright and dark solitons, respectively.

2.1 Multiple bright-dark soliton solutions in forms of determinants

Through the dependent variable transformations $u = e^{2i\gamma t} \frac{g}{f}$, $v = e^{2i\gamma t} \frac{h}{f}$, the space-shifted nonlocal CNLS equation (2) is transformed into the bilinear form:

$$\begin{aligned} (D_x^2 + iD_t)g \cdot f &= 0, (D_x^2 + iD_t)h \cdot f = 0, \\ (D_x^2 + 2\gamma)f \cdot f &= 2(\gamma h h^*(x_0 - x, t) + \delta g g^*(x_0 - x, t)), \end{aligned} \quad (3)$$

where the function f has to be subject to the phase-shifted symmetry and complex conjugate condition $f^*(x, t) = f(x_0 - x, t)$, and D is the Hirota's bilinear differential operator[49]. Based on this bilinear form, we present multiple bright-dark soliton solutions to the space-shifted nonlocal CNLS equation (2) in the following theorem.

Theorem 1 *The bright-dark soliton solutions for the space-shifted nonlocal CNLS equation (2) are*

$$u = e^{2i\gamma t} \frac{g}{f}, v = e^{2i\gamma t} \frac{h}{f} \quad (4)$$

where

$$\begin{aligned} f &= \begin{vmatrix} m_{s,j}^{(0)} & m_{s,N+j}^{(0)} \\ m_{N+s,j}^{(0)} & m_{N+s,N+j}^{(0)} \end{vmatrix}, g = \begin{vmatrix} m_{s,j}^{(0)} & m_{s,N+j}^{(0)} & e^{p_s x + i p_s^2 t + \xi_s^0} \\ m_{N+s,j}^{(0)} & m_{N+s,N+j}^{(0)} & e^{-p_s(x-x_0) + i p_s^2 t + \xi_s^{0*}} \\ -\alpha_j & -\alpha_j^* & 0 \end{vmatrix}, \\ h &= \begin{vmatrix} m_{s,j}^{(1)} & m_{s,N+j}^{(1)} \\ m_{N+s,j}^{(1)} & m_{N+s,N+j}^{(1)} \end{vmatrix}, \end{aligned} \quad (5)$$

and the matrix elements are given by

$$m_{s,j}^{(n)} = \left(-\frac{p_s}{p_j^*}\right)^n \frac{e^{(p_s+p_j^*)x+i(p_s^2-p_j^{*2})t+\xi_s^0+\xi_j^{0*}}}{p_s+p_j^*} + \frac{\alpha_s\alpha_j}{q_s+q_j^*}, \quad (6)$$

for $1 \leq s, j \leq N$. Here the parameters have to satisfy the following restrictions:

$$q_s = -\frac{p_s^2+\gamma}{\delta p_s}, p_{M+s} = -p_s, \xi_{N+s}^0 = \xi_s^{0*} + p_s x_0, \alpha_{N+j} = \alpha_j^*, \quad (7)$$

where p_s, ξ_s^0, α_j are freely complex parameters.

Proof. These bright-dark soliton solutions can be reduced from the following tau functions of three-component KP hierarchy:

$$\tau_0^{(n)} = |M|, \tau_1^{(n)} = \begin{vmatrix} M & \Phi^T \\ -\bar{\Psi} & 0 \end{vmatrix}, \tau_{-1}^{(n)} = \begin{vmatrix} M & \Psi^T \\ -\bar{\Phi} & 0 \end{vmatrix}, \quad (8)$$

where the elements of matrix M are

$$m_{s,j}^{(n)} = \left(-\frac{p_s}{\bar{p}_j}\right)^n \frac{e^{\xi_s+\bar{\xi}_j}}{p_s+\bar{p}_j} + \frac{e^{\eta_s+\bar{\eta}_j}}{q_s+\bar{q}_j}, \quad (9)$$

with

$$\begin{aligned} \xi_s &= \frac{1}{p_s} x_{-1} + p_s x_1 + p_s^2 x_2 + \xi_s^0, \eta_s = q_s y + \eta_s^0, \\ \bar{\xi}_j &= \frac{1}{\bar{p}_j} x_{-1} + \bar{p}_j x_1 - \bar{p}_j^2 x_2 + \bar{\xi}_j^0, \bar{\eta}_j = \bar{q}_j y + \bar{\eta}_j^0, \end{aligned} \quad (10)$$

for $1 \leq s, j \leq K$, and the superscript T represent the transpose, $\Phi, \bar{\Phi}, \Psi, \bar{\Psi}$ are row vectors defined by

$$\begin{aligned} \Phi &= (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_M}), \Psi = (e^{\eta_1}, e^{\eta_2}, \dots, e^{\eta_M}), \\ \bar{\Phi} &= (e^{\bar{\xi}_1}, e^{\bar{\xi}_2}, \dots, e^{\bar{\xi}_M}), \bar{\Psi} = (e^{\bar{\eta}_1}, e^{\bar{\eta}_2}, \dots, e^{\bar{\eta}_M}), \end{aligned} \quad (11)$$

which satisfy the following bilinear equations in KP hierarchy

$$\begin{aligned} (D_{x_1}^2 - D_{x_2})\tau_0^{(n+1)} \cdot \tau_0^{(n)} &= 0, (D_{x_1}^2 - D_{x_2})\tau_1^{(0)} \cdot \tau_{-1}^{(0)} = 0, \\ D_{x_1}D_y\tau_0^{(0)} \cdot \tau_0^{(0)} &= -2\tau_1^{(0)}\tau_{-1}^{(0)}, (D_{x_1}D_{x_{-1}} - 2)\tau_0^{(0)} \cdot \tau_0^{(0)} = -2\tau_0^{(n+1)}\tau_0^{(n-1)}. \end{aligned} \quad (12)$$

The bilinear equations (12) are (3+1) dimensional, while the bilinear equations (3) of the phase-shifted nonlocal CNLS equation are (1+1) dimensional, thus we have to construct a dimensional reduction for the bilinear equations (12). To this end, we take the parameter constraint $q_s = -\frac{p_s^2+\gamma}{\delta p_s}, \bar{q}_s = -\frac{\bar{p}_s^2+\gamma}{\delta \bar{p}_s}$, which can further yield the following relation:

$$(\partial_{x_1} + \delta \partial_y - \gamma \partial_{x_{-1}})m_{s,j}^{(0)} = 0, \text{ i.e., } (\partial_{x_1} + \delta \partial_y - \gamma \partial_{x_{-1}})\tau_0 = 0, \quad (13)$$

Under this dimension reduction, the last two bilinear equations in Eq. (12) generate the following bilinear equation:

$$(D_{x_1}^2 + 2\gamma)\tau_0^{(0)} \cdot \tau_0^{(0)} = -2(\delta \tau_1^{(0)}\tau_{-1}^{(0)} - \gamma \tau_0^{(n+1)}\tau_0^{(n-1)}), \quad (14)$$

Then this bilinear equation and the first two bilinear equations in Eq.(12) become to the bilinear equations (3) of the space-shifted nonlocal CNLS equation for $f = \tau_0^{(0)}, g = \tau_1^{(0)}, g^*(x_0 - x, t) = -\tau_{-1}^{(0)}, h = \tau_0^{(1)}, h^*(x_0 - x, t) = \tau_0^{(-1)}$ under the variable transformations $x_1 = x, x_2 = it$.

We further consider $2N \times 2N$ (i.e., $K = 2N$) matrices for tau functions $\tau_0^{(n)}, \tau_1^{(0)}$ and τ_{-1}^0 with the following parameter constraints:

$$\begin{aligned} \bar{\xi}_j^0 &= \xi_j^{0*}, \bar{\eta}_j^0 = \eta_j^0, p_{N+s} = -p_s, \\ \bar{p}_{N+s} &= -\bar{p}_s, \bar{p}_s = p_s^*, \eta_{N+s}^0 = \eta_j^{0*}, \xi_{N+s}^0 = \xi_s^{0*} + p_s x_0, \end{aligned} \quad (15)$$

for $j = 1, 2, \dots, 2N$ and $s = 1, 2, \dots, N$, then we can get the following relations:

$$\begin{aligned} (\xi_s + \bar{\xi}_j)^*(x_0 - x, t) &= (\xi_{N+s} + \bar{\xi}_{N+j})(x, t), \\ \bar{\xi}_s^*(x_0 - x, t) &= \xi_{N+s}(x, t), \\ (\xi_s + \bar{\xi}_{N+j})^*(x_0 - x, t) &= (\xi_j + \bar{\xi}_{N+s})(x, t), \\ (\xi_{K+s} + \bar{\xi}_j)^*(x_0 - x, t) &= (\xi_{N+j} + \bar{\xi}_s)(x, t), \\ (\xi_{N+j} + \bar{\xi}_{K+s})^*(x_0 - x, t) &= (\xi_s + \bar{\xi}_j)(x, t), \\ \bar{\xi}_{N+s}(x, t) &= \xi_s^*(-x, t), \end{aligned} \quad (16)$$

which implies

$$\begin{aligned} m_{N+s, N+j}^{(n)*}(x_0 - x, t) &= -m_{j, s}^{(-n)}(x, t), m_{N+s, j}^{(n)*}(x_0 - x, t) = -m_{N+j, s}^{(-n)}(x, t), \\ m_{s, N+j}^{(n)*}(x_0 - x, t) &= -m_{j, N+s}^{(-n)}(x, t), m_{s, j}^{(n)*}(x_0 - x, t) = -m_{N+j, N+s}^{(-n)}(x, t). \end{aligned} \quad (17)$$

Thus, one can derive

$$\tau_0^{(n)*}(x_0 - x, t) = \tau_0^{(-n)}(x, t), \tau_1^{(0)*}(x_0 - x, t) = -\tau_{-1}^{(0)}(x, t), \quad (18)$$

namely, the phase-shifted symmetry and complex conjugate condition is realized. By taking $f = \tau_0^{(0)}, g = \tau_1^{(0)}, g^*(x_0 - x, t) = \tau_{-1}^{(0)}, h = \tau_0^{(1)}, h^*(x_0 - x, t) = \tau_0^{(-1)}$, Theorem 1 is then proved.

2.2 Dynamics of the bright-dark soliton interactions.

By taking $N = 1$ in Theorem 1, the bright-dark two-soliton solutions can be derived, and the functions f, g and h are expressed as:

$$f = \begin{vmatrix} m_{1,1}^{(0)} & m_{1,2}^{(0)} \\ m_{2,1}^{(0)} & m_{2,2}^{(0)} \end{vmatrix}, g = \begin{vmatrix} m_{1,1}^{(0)} & m_{1,2}^{(0)} & e^{p_1 x + i p_1^2 t + \xi_1^0} \\ m_{2,1}^{(0)} & m_{2,2}^{(0)} & e^{-p_1(x-x_0) + i p_1^2 t + \xi_1^{0*}} \\ -\alpha_1 & -\alpha_1^* & 0 \end{vmatrix}, h = \begin{vmatrix} m_{1,1}^{(1)} & m_{1,2}^{(1)} \\ m_{2,1}^{(1)} & m_{2,2}^{(1)} \end{vmatrix}, \quad (19)$$

where $m_{1,1}^{(n)} = (-\frac{p_1}{p_1^*})^n \frac{e^{\xi_1 + \xi_1^*}}{p_1 + p_1^*} + \frac{\alpha_1^2}{q_1 + q_1^*}, m_{1,2}^{(n)} = (\frac{p_1}{p_1^*})^n \frac{e^{\xi_1 + \xi_2^*}}{p_1 - p_1^*} + \frac{\alpha_1 \alpha_1^*}{q_1 - q_1^*}, m_{2,1}^{(n)} = -m_{1,2}^{(n)*}, m_{2,2}^{(n)} = -(-\frac{p_1}{p_1^*})^n \frac{e^{\xi_2 + \xi_2^*}}{p_1 + p_1^*} - \frac{\alpha_1^2}{q_1 + q_1^*}$ and $\xi_1 = p_1 x + i p_1^2 t + \xi_1^0, \xi_2 = -p_1(x - x_0) + i p_1^2 t + \xi_1^{0*}$, and $q_1 = -\frac{p_1^2 + \gamma}{\delta p_1}$.

These two solitons move along the lines $\zeta_1 = \xi_1 + \xi_1^* = x - 2p_1 t \approx 0$ and $\zeta_2 = \xi_2 + \xi_2^* = x + 2p_1 t \approx 0$, and for convenience they are denoted as soliton 1

and soliton 2, respectively. To study their behaviours, we obtain the asymptotic forms of these two solitons. For this purpose, we assume $p_{1R} > 0, p_{1I} > 0$ without loss of generality. After some simple algebraic calculations, then the asymptotic expressions of the two solitons are given by:

(i) Before collision ($t \rightarrow -\infty$):

Soliton 1 ($\eta_1 \approx 0, \eta_2 \approx +\infty$)

$$\begin{aligned} u_1^{(-)} &\approx \frac{e^{i(2\gamma t + \xi_{1I}) - \lambda_1}}{4} X_1 \operatorname{sech}(\xi_{1R} + \lambda_1), \\ v_1^{(-)} &\approx \frac{e^{2i\gamma t}}{2} (1 + y_1 + (y_1 - 1) \tanh \frac{\xi_1 + \xi_1 + \Phi_1}{2}), \end{aligned} \quad (20)$$

Soliton 2 ($\eta_2 \approx 0, \eta_1 \approx +\infty$)

$$\begin{aligned} u_2^{(-)} &\approx \frac{e^{i(2\gamma t + \xi_{2I}) - \lambda_2}}{4} X_2 \operatorname{sech}(\xi_{2R} + \lambda_2), \\ v_2^{(-)} &\approx \frac{e^{2i\gamma t}}{2} (1 + y_1 + (y_1 - 1) \tanh \frac{\xi_2 + \xi_2 + \Phi_2}{2}), \end{aligned} \quad (21)$$

(ii) After collision ($t \rightarrow +\infty$):

Soliton 1 ($\eta_1 \approx 0, \eta_2 \approx -\infty$)

$$\begin{aligned} u_1^{(+)} &\approx \frac{e^{i(2\gamma t + \xi_{1I}) - \hat{\lambda}_1}}{4} \hat{X}_1 \operatorname{sech}(\xi_{1R} + \hat{\lambda}_1), \\ v_1^{(+)} &\approx \frac{e^{2i\gamma t}}{2} (1 + y_1 + (y_1 - 1) \tanh \frac{\xi_1 + \xi_1 + \hat{\Phi}_1}{2}), \end{aligned} \quad (22)$$

Soliton 2 ($\eta_2 \approx 0, \eta_1 \approx -\infty$)

$$\begin{aligned} u_2^{(+)} &\approx \frac{e^{i(2\gamma t + \xi_{2I}) - \hat{\lambda}_2}}{4} \hat{X}_2 \operatorname{sech}(\xi_{2R} + \hat{\lambda}_2), \\ v_2^{(+)} &\approx \frac{e^{2i\gamma t}}{2} (1 + y_1 + (y_1 - 1) \tanh \frac{\xi_2 + \xi_2 + \hat{\Phi}_2}{2}), \end{aligned} \quad (23)$$

where

$$\begin{aligned} X_1 &= \frac{2p_1(q_1 + q_1^*)}{\alpha_1(p_1 - p_1^*)}, e^{2\lambda_1} = -\frac{4p_1p_1^*(q_1 + q_1^*)}{\alpha_1^2(p_1 + p_1^*)(p_1 - p_1^*)^2}, \\ \hat{X}_1 &= \frac{(q_1 + q_1^*)(q_1 - q_1^*)}{2\alpha q_1}, e^{2\hat{\lambda}_1} = -\frac{(q_1 + q_1^*)(q_1 - q_1^*)^2}{4\alpha_1^2 q_1 q_1^* (p_1 + p_1^*)}, \\ X_2 &= -\frac{2p_1(q_1 + q_1^*)}{\alpha_1^*(p_1 - p_1^*)}, e^{2\lambda_2} = -\frac{4p_1p_1^*(q_1 + q_1^*)}{\alpha_1^{*2}(p_1 + p_1^*)(p_1 - p_1^*)^2}, \\ \hat{X}_2 &= -\frac{(q_1 + q_1^*)(q_1 - q_1^*)}{\alpha^*(q_1 - q_1)}, e^{2\hat{\lambda}_2} = -\frac{(q_1 + q_1^*)(q_1 - q_1^*)^2}{4\alpha_1^{*2} q_1 q_1^* (p_1 + p_1^*)}, \\ e^{\Phi_1} &= -\frac{4p_1p_1^*(q_1 + q_1^*)}{\alpha_1^2(p_1 + p_1^*)(p_1 - p_1^*)^2}, e^{\hat{\Phi}_1} = -\frac{(q_1 + q_1^*)(q_1 - q_1^*)^2}{4\alpha_1^2 q_1 q_1^* (p_1 + p_1^*)}, \\ e^{\Phi_2} &= -\frac{4p_1p_1^*(q_1 + q_1^*)}{\alpha_1^{*2}(p_1 + p_1^*)(p_1 - p_1^*)^2}, e^{\hat{\Phi}_2} = -\frac{(q_1 + q_1^*)(q_1 - q_1^*)^2}{4\alpha_1^{*2} q_1 q_1^* (p_1 + p_1^*)}, \\ y_1 &= -\frac{p_1^*}{p_1}. \end{aligned} \quad (24)$$

The above asymptotic analysis indicate that $|u_j^{(+)}(\xi_{jR})| = |u_j^{(-)}(\xi_{jR} + \lambda_j - \hat{\lambda}_j)|$ and $|v_j^{(+)}(\xi_{jR})| = |v_j^{(-)}(\xi_{jR} + \frac{1}{2}(\Phi_j - \hat{\Phi}_j))|$, hence the collisions of these two solitons are elastic in both u and v components, namely, the natures of the bright-dark two solitons remain unaltered after collision except for finite shifts, including the amplitudes, speeds and shapes.

In u component, $|u_1^{(\pm)}| = |u_2^{(\pm)}|$ means that the amplitudes of the two bright solitons are equal, and the value is $|u_B| = \sqrt{|\frac{|q_1|^2(p_1+p_1^*)(q_1+q_1^*)}{(\alpha_1-\alpha_1^*)^2}|}$. Fig.1 displays the bright two-soliton solution in u component with same soliton parameters and different signs of nonlinearities δ and γ of Eq.(2). It is seen that the space-shifted nonlocal CNLS equation with three different types of nonlinearities admit smooth bright soliton solutions in u component. The different nonlinearities would generate diverse waveforms in the interaction region.

In v component, the amplitudes of the two dark solitons are $|v_j| = \left| \frac{(p_1 Y_j - p_1^* |Y_j|)}{p_1(Y_j + |Y_j|)} \right|$ for $j = 1, 2$, where $Y_1 = -\frac{4p_1 p_1^*(q_1+q_1^*)}{\alpha_1^2(p_1+p_1^*)(p_1-p_1^*)^2}$, $Y_2 = -\frac{4p_1 p_1^*(q_1+q_1^*)}{\alpha_1^{*2}(p_1+p_1^*)(p_1-p_1^*)^2}$. Thus, the solitons in v can be classified into four different types: two dark solitons for $|v_j| < 1$ ($j = 1, 2$), a mixture of a dark soliton and an antidark soliton for $|v_j| < 1, |v_{3-j}| > 1$, two antidark solitons for $|v_j| > 1$, and a degenerated dark two-soliton for $|v_j| = 1, |v_{3-j}| < 1$, and a degenerated antidark two-soliton for $|v_j| = 1, |v_{3-j}| > 1$. These three types of non-degenerated two-soliton and the two types of degenerated dark two-soliton are displayed in Fig.2 and Fig.3, respectively. It should be noted that expressions of the soliton amplitudes are independent of the phase shift factor x_0 , thus the phase shift factor x_0 does not affect the amplitudes of the solitons.

Fig. 1 The bright two-soliton solutions (19) in u component of Eq.(2) with parameters $p_1 = \frac{1}{2} + i$, $\xi_1^0 = 0$, $x_0 = 0$, $\alpha_1 = 1 + (2 + \sqrt{5})i$ and different types of nonlinearities in the space-shifted nonlocal CNLS equation (2): (a) $\delta = 1, \gamma = 1$; (b) $\delta = 1, \gamma = -1$; (c) $\delta = -1, \gamma = -1$.

By taking $N = 2$ in Theorem 2, the bright-dark four-soliton solutions of the space-shifted nonlocal CNLS equation (2) can be obtained. Since the four solitons move along the line $x \pm 2p_{jI}t \approx 0$, thus they can form two pairs of bound-state two-soliton when $p_{1I} = \pm p_{2I}$. Here the subscripts R and I represent real and imaginary parts of a given parameter or a function, respectively. Fig.4 display the bright four-soliton in u component and the dark four-soliton in v component for $p_{1I} \neq \pm p_{2I}$. Fig.5 demonstrates the four solitons of bright type in u and dark

Fig. 2 The two different types of non-degenerated dark two-soliton solutions (19) in v component of Eq.(2) with parameters $\delta = 1, \gamma = 1, p_1 = \frac{1}{2} + i, \xi_1^0 = 0, x_0 = 2$ and different values of parameter α_1 : (a)Two antdark solitons with $\alpha_1 = \frac{1}{10} + 10i$; (b)A mixture of a dark soliton and a antdark soliton with $\alpha_1 = 2 + i$; (c)Two dark solitons with $\alpha_1 = 3 - \frac{1}{2}$;

Fig. 3 The two different types of degenerated dark two-soliton solutions (19) in v component of Eq.(2) with parameters $\delta = 1, \gamma = 1, p_1 = \frac{1}{2} + i, \xi_1^0 = 0, x_0 = 2$ and different values of parameter α_1 : (a)The degenerated dark two-soliton with $\alpha_1 = 1 + (2 + \sqrt{5})i$; (b)The degenerated antdark two-soliton with $\alpha_1 = 1 - (2 + \sqrt{5})i$.

type in v with $p_{1I} = p_{2I}$, which feature two pairs of bound-state two-soliton in u component and v component, respectively.

3 Multiple double-pole bright-dark solitons in the space-shifted nonlocal CNLS equation

Following the works in [26,27], the multiple double-pole bright-dark soliton solutions can be derived from the general soliton solutions in Theorem 1 through the long wave limiting procedure. This is done by choosing the parameters in Eq.(5) as

$$e^{\xi_s^0} = \tilde{\xi}_s p_{sI}, \alpha_s = \bar{\alpha}_s p_{sI} e^{i\theta_s}, |\tilde{\xi}_s|^2 (p_{sR}^2 - \delta_2) e^{p_{1R}x_0} - \delta_1 |\bar{\alpha}_s|^2 p_{sR}^2 = 0, \quad (25)$$

Fig. 4 The bright-dark four-soliton solutions (4) of the space-shifted nonlocal CNLS equation (2) with parameters $N = 2, \delta = 1, \gamma = 1, p_1 = 1 + i, p_2 = 1 + 2i, \xi_1^0 = 0, \xi_2^0 = 0, x_0 = 0, \alpha_1 = i, \alpha_2 = i$.

Fig. 5 The bright-dark four-soliton solutions (4) of the space-shifted nonlocal CNLS equation with parameters (4) $N = 2, \delta = 1, \gamma = 1, p_1 = 1 + i, p_2 = 2 + i, \xi_1^0 = 0, \xi_2^0 = 0, x_0 = 0, \alpha_1 = 5i, \alpha_2 = 5i$, which feature two pairs of bound-state two-soliton.

and then taking the limit $p_{sI} \rightarrow 0$ for $s = 1, 2, \dots, N$. After implementing this limit procedure, we can present the multiple double-pole bright-dark soliton solutions for the space-shifted nonlocal CNLS equation (2) by the following theorem.

Theorem 2 *The double-pole bright-dark soliton solutions for the space-shifted nonlocal CNLS equation (2) are*

$$u = e^{2i\gamma t} \frac{g}{f}, v = e^{2i\gamma t} \frac{h}{f}, \quad (26)$$

where

$$f = \begin{vmatrix} \widehat{m}_{s,j}^{(0)} & \widehat{m}_{s,N+j}^{(0)} \\ \widehat{m}_{N+s,j}^{(0)} & \widehat{m}_{N+s,N+j}^{(0)} \end{vmatrix}, g = \begin{vmatrix} \widehat{m}_{s,j}^{(0)} & \widehat{m}_{s,N+j}^{(0)} & \widetilde{\xi}_s e^{p_{sR}x + ip_{sR}^2 t} \\ \widehat{m}_{N+s,j}^{(0)} & \widehat{m}_{N+s,N+j}^{(0)} & \widetilde{\xi}_s e^{-p_{sR}(x-x_0) + ip_{sR}^2 t} \\ -\bar{\alpha}_j e^{i\theta_j} & -\bar{\alpha}_j e^{-i\theta_j} & 0 \end{vmatrix}, \quad (27)$$

$$h = \begin{vmatrix} \widehat{m}_{s,j}^{(1)} & \widehat{m}_{s,N+j}^{(1)} \\ \widehat{m}_{N+s,j}^{(1)} & \widehat{m}_{N+s,N+j}^{(1)} \end{vmatrix}.$$

Here the matrix elements are defined as follows:

$$m_{s,j}^{(n)} = \frac{\widetilde{\xi}_s \widetilde{\xi}_j \left(-\frac{p_{sR}}{p_{jR}}\right)^n e^{(p_{sR}+p_{jR})(x+i(p_{sR}-p_{jR})t)}}{p_{sR} + p_{jR}} - \frac{\bar{\alpha}_1 \bar{\alpha}_2 \delta_1 p_{sR} p_{jR} e^{i(\theta_s + \theta_j)}}{(p_{sR} + p_{jR})(p_{sR} p_{jR} + \delta_2)},$$

$$m_{s,N+j}^{(n)} = -\frac{\widetilde{\xi}_s \widetilde{\xi}_j \left(\frac{p_{sR}}{p_{jR}}\right)^n e^{p_{sR}(x+2ip_{sR}t) - p_{jR}(x-x_0+ip_{jR}t)}}{p_{sR} - p_{jR}} - \frac{\alpha_s \alpha_j \delta_1 p_{sR} p_{jR} e^{i(\theta_s - \theta_j)}}{(p_{sR} - p_{jR})(p_{sR} p_{jR} - \delta_2)}, \quad (28)$$

for $1 \leq s \neq j \leq N$, and

$$m_{s,s}^{(n)} = \frac{c_s (-1)^n e^{2p_{sR}x}}{2p_{sR}} - \frac{\delta_1 p_{sR} \bar{\alpha}_s^2 e^{2i\theta_s}}{2(p_{sR}^2 + \delta_2)}, \quad (29)$$

$$m_{s,N+s}^{(n)} = \frac{\widetilde{\xi}_s e^{p_{sR}x_0}}{2p_{sR}} \left(x - \frac{x_0}{2} + 2ip_{sR}t + \frac{n}{p_{sR}}\right)$$

for $s = 1, 2, \dots, N$, and

$$m_{N+s,j}^{(n)} = -m_{j,N+s}^{(-n)*}(x_0 - x, t), m_{N+s,N+j}^{(n)} = -m_{N+s,N+j}^{(-n)*}(x_0 - x, t)$$

for $1 \leq s, j \leq N$. Here the real parameters $\widetilde{\xi}_s, \bar{\alpha}_s$ have to observe the restrictions

$$|\widetilde{\xi}_s|^2 (p_{sR}^2 - \delta_2) e^{p_{1R}x_0} - \delta_1 |\bar{\alpha}_s|^2 p_{sR}^2 = 0, \quad s = 1, 2, \dots, N. \quad (30)$$

By taking $N = 1$ in Theorem 2, the double-pole bright-dark two-soliton solutions can be derived, and the functions f, g and h are expressed as

$$f = \left(\frac{\widetilde{\xi}_1^2 e^{2p_{1R}x}}{2p_{1R}} - \frac{\delta_1 p_{1R} \bar{\alpha}_1^2 e^{2i\theta_1}}{2(p_{1R}^2 + \delta_2)} \right) \left(\frac{\widetilde{\xi}_1^2 e^{-2p_{1R}(x-x_0)}}{2p_{1R}} - \frac{\delta_1 p_{1R} \bar{\alpha}_1^2 e^{-2i\theta_1}}{2(p_{1R}^2 + \delta_2)} \right)$$

$$- \widetilde{\xi}_1^4 \left[\left(x - \frac{x_0}{2}\right)^2 + 4p_{1R}^2 t^2 \right] e^{2p_{1R}x_0},$$

$$g = -\bar{\alpha}_1 e^{i\theta_1} \left[\widetilde{\xi}_1^3 \left(x - \frac{x_0}{2} + 2ip_{1R}t\right) e^{-p_{1R}(x-2x_0) + ip_{1R}^2 t} + \left(\frac{\widetilde{\xi}_1^2 e^{-2p_{1R}(x-x_0)}}{2p_{1R}} \right. \right.$$

$$\left. - \frac{\delta_1 p_{1R} \bar{\alpha}_1^2 e^{-2i\theta_1}}{2(p_{1R}^2 + \delta_2)} \right) \widetilde{\xi}_1 e^{p_{1R}x + ip_{1R}^2 t} - \bar{\alpha}_1 e^{-i\theta_1} \left[\left(\frac{\widetilde{\xi}_1^2 e^{2p_{1R}x}}{2p_{1R}} \right. \right.$$

$$\left. - \frac{\delta_1 p_{1R} \bar{\alpha}_1^2 e^{2i\theta_1}}{2(p_{1R}^2 + \delta_2)} \right) \widetilde{\xi}_1 e^{-p_{1R}(x-x_0) + ip_{1R}^2 t} + \widetilde{\xi}_1^3 \left(x - \frac{x_0}{2} - 2ip_{1R}t\right) e^{p_{1R}(x+x_0) + ip_{1R}^2 t} \left. \right]$$

$$h = \left(\frac{\widetilde{\xi}_1^2 e^{2p_{1R}x}}{2p_{1R}} + \frac{\delta_1 p_{1R} \bar{\alpha}_1^2 e^{2i\theta_1}}{2(p_{1R}^2 + \delta_2)} \right) \left(\frac{\widetilde{\xi}_1^2 e^{-2p_{1R}(x-x_0)}}{2p_{1R}} + \frac{\delta_1 p_{1R} \bar{\alpha}_1^2 e^{-2i\theta_1}}{2(p_{1R}^2 + \delta_2)} \right)$$

$$- \widetilde{\xi}_1^4 \left[\left(x - \frac{x_0}{2}\right)^2 + (2ip_{1R}t - i)^2 \right] e^{2p_{1R}x_0}. \quad (31)$$

Fig.3 shows this double-pole two-soliton of bright-type in u component and dark-type in v component. As discussed previously, the usual (i.e., single pole) dark two solitons can possess unequal amplitudes. However, the amplitudes of the two solitons in the double-pole two-soliton solutions are always equal for all input parameters.

The double-pole bright-dark four-soliton solutions can be derived by taking $N = 2$ in Theorem 2, and their collision dynamics are shown in Fig.7. Similar to the bound-state two-soliton pairs displayed in Fig. 5, the double-pole bright-dark four-soliton can also generate temporal periodic line waves in their interaction region.

Fig. 6 The bright-dark double pole two-solitons solutions (4) of the space-shifted nonlocal CNLS equation with parameters $N = 2, \delta = 1, \gamma = -1, p_{1R} = \frac{1}{2}, \bar{\alpha}_1 = 1, x_0 = 2, \theta_1 = 0$.

Fig. 7 The double-pole bright-dark four-soliton solutions (4) of the space-shifted nonlocal CNLS equation with parameters $N = 2, \delta = 1, \gamma = -1, p_{1R} = \frac{1}{2}, p_{2R} = \frac{2}{3}, x_0 = 2, \alpha_1 = 1, \alpha_2 = 2, \theta_1 = 0, \theta_2 = 0$, which feature two pairs of bound-state double-pole two-soliton.

4 Conclusion

In this paper, the multiple bright-dark soliton solutions of the space-shifted nonlocal CNLS equation (2) were constructed via the bilinear KP hierarchy reduction method. According to asymptotic analysis of the two-soliton collisions, we found that bright solitons and dark solitons only occur elastic collision. The bright two solitons in component u possess equal amplitudes. The dark two-solitons in component v admits three different non-degenerated patterns (i.e., the dark-dark two-soliton, the dark-antidark two-soliton, the anti-dark-two soliton) (see Fig.2) and the two different degenerated patterns (i.e., degenerated dark/antidark two-soliton) (see Fig.3). The collisions of the bright-dark four-soliton illustrating the superposition of the paired solitons were also exhibited, which lead to two pairs of bound-state two-soliton. These bound-state two-soliton pairs can generate temporal periodic line waves in the interaction region (see Fig.5). By taking the long wave limit, multiple double-pole bright-dark solitons of the space-shifted nonlocal CNLS equation (2) were generated. The double-pole bright-dark two-soliton and four-soliton solutions were also investigated (see Figs. 6,7).

Acknowledgements This work is supported by the NSCF of China under Grant Nos. 12071304.

Conflict statement We declare we have no conflict of interests.

Ethical statement Authors declare that they comply with ethical standards.

Data availability statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. P. G. Kevrekidis, D. J. Frantzeskakis, Solitons in coupled nonlinear Schrödinger models: A survey of recent developments, *Rev. Phys.* **1**, 140(2016).
2. E. P. Bashkin, A. V. Vagov, Instability and stratification of a two-component Bose-Einstein condensate in a trapped ultracold gas, *Phys. Rev. B* **56**, 6207(1997).
3. M. J. Ablowitz, T. K. Horikis, Interacting nonlinear wave envelopes and rogue wave formation in deep water, *Phys. Fluids* **27**, 012107(2015).
4. H. Bailung, S. K. Sharma, Y. Nakamura, Observation of Peregrine Solitons in a Multi-component Plasma with Negative Ions, *Phys. Rev. Lett.* **107**, 255005(2011).
5. N. Akhmediev, A. Ankiewicz, Solitons: Nonlinear Pulses and Beams (Chapman and Hall, London, 1997).
6. V. G. Makhan'kov, O. K. Pashaev, Nonlinear Schrödinger equation with noncompact isogroup, *Theor. Math. Phys.* **53**, 979(1982).
7. M. J. Ablowitz, Z. H. Musslimani, Integrable nonlocal nonlinear Schrödinger equation, *Phys. Rev. Lett.* **110**, 064105(2013).
8. M. J. Ablowitz, Z. H. Musslimani, Integrable nonlocal nonlinear equations, *Stud. Appl. Math.* **139**, 7(2017).
9. A. S. Fokas, Integrable multidimensional versions of the nonlocal nonlinear Schrödinger equation, *Nonlinearity*, **29**, 319(2016).
10. K. Chen, X. Deng, S. Y. Lou, D. J. Zhang, Solutions of nonlocal equations reduced from the AKNS hierarchy, *Stud. Appl. Math.* **141**, 113 (2018).
11. B. Yang, J. Yang, Transformations between nonlocal and local integrable equations, *Stud. Appl. Math.* **140**, 178(2018).
12. B. F. Feng, X. D. Luo, A. J. Ablowitz, Z. H. Musslimani, General soliton solution to a nonlocal nonlinear Schrödinger equation with zero and nonzero boundary conditions, *Nonlinearity* **31**, 5385(2018).
13. M. Gürses, A. Pekcan, Nonlocal nonlinear Schrödinger equations and their soliton solutions, *J. Math. Phys.* **59**, 051501(2018).

14. J. Rao, Y. Zhang, A. S. Fokas, J. He, Rogue waves of the nonlocal Davey-Stewartson I equation, *Nonlinearity* **31**, 4090–4107(2018).
15. B. Yang, Y. Chen, Several reverse-time integrable nonlocal nonlinear equations: Rogue-wave solutions. *Chaos* **28**, 053104(2018).
16. Z. Yan, Integrable PT -symmetric local and nonlocal vector nonlinear Schrödinger equations: A unified two-parameter model. *Appl. Math. Lett.* **47**, 61(2015).
17. Y. Shi, Y. S. Zhang, S. W. Xu, Families of nonsingular soliton solutions of a nonlocal Schrödinger–Boussinesq equation, *Nonlinear Dyn.* **94**, 2327–2334(2018).
18. Y. Shi, S. Shen, S. Zhao, Solutions and connections of nonlocal derivative nonlinear Schrödinger equations, *Nonlinear Dyn.* **95**, 1257–1267 (2019).
19. Y. Liu, B. Li, Dynamics of solitons and breathers on a periodic waves background in the nonlocal Mel’nikov equation, *Nonlinear Dyn.* **100**, 3717–3731(2020).
20. J. Chen, Q. Yan, H. Zhang, Multiple bright soliton solutions of a reverse-space nonlocal nonlinear Schrödinger equation, *Appl. Math. Lett.* **106**, 106375(2020).
21. J. Rao, J. He, D. Mihalache, Y. Cheng, PT -symmetric nonlocal Davey–Stewartson I equation: general lump–soliton solutions on a background of periodic line waves, *Appl. Math. Lett.* **106**, 106246(2020).
22. M. J. Ablowitz, Z.H. Musslimani, Integrable space-time shifted nonlocal nonlinear equations, *Phys. Lett. A*, **409**, 127516(2021).
23. M. Gürses, A. Pekcan, Soliton solutions of the shifted nonlocal NLS and MKdV equations, *arXiv:2106.14252v2 [nlin.SI]*.
24. S.M. Liu, J. Wang, D.J. Zhang, Solutions to integrable space-time shifted nonlocal equations, *arXiv:2107.04183v1 [nlin.SI]*.
25. S. Stalin, M. Senthilvelan, M. Lakshmanan, Energysharing collisions and the dynamics of degenerate solitons in the nonlocal Manakov system, *Nonlinear Dyn.* **95**, 343(2019).
26. J. Rao, J. He, T. Kanna, D. Mihalache, Nonlocal M-component nonlinear Schrödinger equations: Bright solitons, energy-sharing collisions, and positons, *Phys. Rev. E*, **102**, 032201(2021).
27. J. Rao, Multiple double–pole bright–bright and bright–dark solitons and energy–exchanging collision in the M-component nonlinear Schrödinger equations, *Phys. Rev. E*, **103**, 062214(2021).
28. Y. Zhang, X. Tao, T. Yao, J. He, The regularity of the multiple higher–order poles solitons of the NLS equation, *Stud. Appl. Math.* **145**, 812–827(2018).
29. Z. Zhang, B. Li, Q. Guo, Construction of higher–order smooth positons and breather positons via Hirota’s bilinear method, *Nonlinear Dyn.* **105**, 2611–2618(2021).
30. L. Gagon, N. Stiévenart, *N-soliton interaction in optical fibers: the multiple-pole case*, *Opt. Lett.*, **19**, 619 (1994).
31. T. Aktosun, F. Demontis, and C. van der Mee, *Exact solutions to the focusing nonlinear Schrödinger equation*, *Inverse Problems* **23**, 2171–2195 (2007).
32. T. Martinez, *Generalized inverse scattering transform for the nonlinear Schrödinger equation for bound states with higher multiplicities*, *Electron. J. Differ. Equ.* **179**, 1–15 (2017).
33. S. Tanaka, *Non-linear Schrödinger equation and modified Korteweg–de Vries equation; construction of solutions in terms of scattering data*, *Publ. RIMS, Kyoto Univ.* **10**, 329–357 (1975).
34. E. Olmedilla, *Multiple pole solutions of the non-linear Schrödinger equation*, *Physica D* **25**, 330–346 (1987).
35. C. Schiebold, *Asymptotics for the multiple pole solutions of the nonlinear Schrödinger equation*, *Nonlinearity*, **30**, 2930–2981 (2017).
36. D. Bilman and R. Buckingham, *Large-Order Asymptotics for Multiple-pole solitons of the focusing nonlinear Schrödinger Equation*, *J. Nonlinear Sci.* **29**, 2185–2229 (2019).
37. M. Wadati and K. Ohkuma, *Multiple-pole solutions of the modified Korteweg-de Vries equation*, *J. Phys. Soc. Jpn.* **51**, 2029–2035 (1982).
38. D. J. Zhang, S. L. Zhao, Y. Y. Sun, and J. Zhou, *Solutions to the modified Korteweg-de Vries equation*, *Rev. Math. Phys.*, **26**, 1430006 (2014).
39. T. Aktosun, F. Demontis, and C. van der Mee, *Exact solutions to the sine-Gordon equation*, *J. Math. Phys.* **51**, 123521 (2010).
40. C. Pöppe, *Construction of solutions of the sine-Gordon equation by means of Fredholm determinants*, *Physica D* **9**, 103–139 (1983).
41. H. Tsuru and M. Wadati, *The multiple pole solutions of the sine-Gordon equation*, *J. Phys. Soc. Jpn.* **53**, 2908–2921 (1984).

42. V. Shchesnovich and J. Yang, *Higher-order solitons in the N-wave system*, Stud. Appl. Math. **110**, 297–332 (2003).
43. Y. Kuang and J. Zhu, *The higher-order soliton solutions for the coupled Sasa-Satsuma system via the ∂ -dressing method*, Appl. Math. Lett. **66**, 47–53 (2017).
44. D.W.C. Lai and K.W. Chow, K. Nakkeeran, *Multiple-pole soliton interactions in optical fibres with higher-order effects*, J. Mod. Optic, **51**, 455–460 (2004).
45. M. Li, X. Zhang, T. Xu, and L. Li, *Asymptotic analysis and soliton interactions of the multi-pole solutions in the Hirota equation*, J. Phys. Soc. Jpn. **89**, 054004 (2020).
46. X. Zhang, L. Ling, *Asymptotic analysis of high-order solitons for the Hirota equation* Physica D, **426**, 132982(2021).
47. M. Jimbo, T. Miwa, *Solitons and infinite dimensional Lie algebras*. Publ. RIMS Kyoto Univ. **19**, 943–1001(1983).
48. E. Date, M. Kashiwara, M. Jimbo, T. Miwa, *Transformation groups for soliton equations*, in Nonlinear Integrable Systems—Classical Theory and Quantum Theory. eds. M. Jimbo and T. Miwa (World Scientific, Singapore, 1983).
49. R. Hirota, *The direct method in soliton theory*, (Cambridge University Press, Cambridge, 2004)
50. Y. Ohta, D. S. Wang, J. Yang, *General N-Dark-Dark Solitons in the Coupled Nonlinear Schrödinger Equations*. Stud. Appl. Math. **127**, 345–371(2011).
51. Y. Ohta, J. Yang, *General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation*. Proc. R. Soc. A **468**,1716(2012).