Higher-Order Kinematics Modeling of 3-RRS Parallel Mechanism Based on CGA

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Title page

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ORIGINAL ARTICLE

Higher-order kinematics modeling of 3-RRS parallel mechanism

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Abstract: Higher-order kinematics of mechanisms has been applied in servo motor control, human-robot interaction and machinery life design fields, etc. The representations of acceleration and jerk by screws have been fully developed by researchers with the methods of the differential of the matrix representation of $SE(3)$ group. Clifford algebra, which is tighter and with higher computational efficiency than the matrix method, is another representation of the motions of rigid bodies. It has been used in position kinematics, grub task motion planning, and robot vision for its convenience of geometric representations and calculations. As far as we know, the work of higher-order kinematics of mechanisms based on Clifford algebra is rare. First, after recalling the based theory of motion representation in conformal geometric algebra (CGA), the mathematical relationships between flag and motor are built. Second, a method for the higher-order kinematics modeling of serial chain mechanisms is proposed. Finally, the higher-order kinematics of the 3-RRS parallel mechanism is built to prove the correctness of the algorithm. This work further enriches the application of CGA for the higher-order kinematics modeling of parallel mechanisms.

Keywords: CGA • Flag • Rigid body framing • Higher-order kinematics • 3-RRS parallel mechanism

1 Introduction

Pose (position and orientation), velocity, and acceleration

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calculations are the main topics of kinematics of mechanisms, and they play an essential role in motion planning, control, evaluation of robot [1, 2], etc. The main task of velocity analysis, which is also called the first-order kinematics, is to obtain Jacobian matrix that represents the mapping of velocities between actuators and output body. Acceleration analysis, which has relationships with the Hessian matrix, is mainly used to obtain the mapping of accelerations between actuators and output body. Second-order kinematics is the base for dynamic modeling of mechanisms. In recent years, third-order kinematics of mechanism, which is the mapping between jerks of actuators and the output, has attracted researchers. Compared with first- and second-order kinematics, jerk analysis seems less important. However, in some applications, such as servo control of motors[3], motion planning[4, 5], human-robot interaction[6], elevator control[7], and design of rollercoasters[8], third-order kinematics is also needed to be considered, for it has impacts on electron device, vibration of mechanisms, and body feelings[9]. For instance, if the jerk of a robot is large during fast motions, the acceleration changes frequently and will lead to premature wear of joints or actuators. In another case, if the jerk of a human body is large, the muscles of the body may fail to take actions to the change of the acceleration, and it could cause discomfort or even damage the human tissues. Higher-order kinematics modeling is also used to analyze the instantaneous mobility and the cusp of some mechanisms [10-14].

Screw theory, Lie group, and Lie algebra have been applied a lot in kinematics and dynamics modeling of mechanisms, especially the spatial multi-loop mechanisms. Screw theory is built in the 19th century. Plücker first used six coordinates to represent a line in 3D space. It was Ball

who first systematically built screw theory, and give guidance to the application of screw theory in the field of rigid body kinematics and dynamics[15]. From then on, a lot of researchers, such as Hunt[16], Waldron[17], Sugimoto[18], Duffy[19], Phillips[20], Selig[21], Dai[22] and Huang[23], have devoted themselves to the application and development of this tool. Until now, screw theory has been used in synthesis, kinematics, dynamics, and control of all kinds of mechanisms. Screws were initially used to represent the instantaneous motion or the force with the rigid bodies. Then Mises[24] defined the cross product and pointed out the necessity to take the derivative of screws with time. Brand [25] clearly defined acceleration motor as the derivation of velocity screw. Lie group and Lie algebra were developed paralleled with Ball's work, and it was Karger who developed the relationships between screw and Lie algebra and revealed that velocity screw is the element of $se(3)$ and wrench is the element of $se^*(3)$ [26]. Based on these developments, a more general Newton-Euler equation was discovered for the modeling of rigid body dynamics [27-29] and was used for kinematics and dynamics modeling of mechanisms. Higher-order kinematics of mechanisms based on these theories have also been significantly developed[11, 30].

The matrix representation of $SE(3)$, which is also called transformation matrix, is usually used to represent motions of rigid bodies in three-dimensional space. Based on the differential of transformation matrix, the screw velocity represented by Lie algebra can then be obtained. However, using a 4×4 matrix to describe the six-dimension motions of a rigid body has redundancy and will generate limitations in the derivation or calculation process. Compared with the matrix-vector represent of motions proposed by Gibbs, the geometric algebra methods proposed by Clifford have some other superiorities, such as the tightness of the representation and the higher calculation efficiency. For instance, some commercial software use the quaternions to represent the rotation to avoid the orthogonality error correction caused by numerical errors. The conformal group of space in Clifford algebra has been taken attention by Angles in the 1980s[31, 32], whose recently work about the conformal spin structure can be obtained in [33, 34]. Hestenes made some promotion of Clifford algebra (or geometric algebra) in terms of physical computing[35]. The conformal geometric algebra (CGA) introduced by Li and Hestenes [36] have been used to solve problems in vision, kinematics, and dynamics of robots due to its convenience and efficiency when representing points, lines, circles, spheres and planes. Ten years ago, Hestenes proposed the method to solve the

kinematics and dynamics of open chain kinematics and give the definition of screws and wrenches represented with Clifford algebra and CGA[37]. At the same time, Selig also proposed a method using an eight-dimensional algebra to build rigid body dynamics, and he claimed that inertias, velocities, and momenta all can be represented as the elements of that algebra and all the relationships between physical quantities could be given by the standard operations[38]. After that, attentions start to be paid by researchers in the robotic fields and amount of work has been done based on Clifford algebra[39, 40], such as the forward and inverse position kinematics[41-44], singularity analysis[45, 46], robot POC representation[47], type synthesis of parallel mechanisms[48], dynamics[49], freedom or constraint analysis[50-52], first-order kinematics and grub jobs[53]. As far as we know, the research of higher-order kinematics of parallel mechanisms based on CGA is relatively rare.

This research extends the application of CGA to mechanistic kinematics so that the positional solution and higher-order kinematics of mechanisms can be uniformly described in the same algebra system. Section 2 briefly gives a recall of Clifford algebra and CGA. Section 3 introduces the rotation, translation, and screw motion in the algebra, and introduces the framing method of rigid body based on flag. The analytic relationships between flag and motor is first revealed. Section 4 deduces the first-, second-, and third-order kinematics of serial chain, and the iteration formulas are given. Section 5 provides the modeling method of the position calculation and the higher-order kinematics of the 3-RRS parallel mechanism. Conclusions are drawn in Section 6.

We also refers to [35, 54-56] for the based theory and calculation of Clifford algebra, and the programs in this paper are built based on the project of Clifford algebra[57].

2 CGA foundations

The five-dimensional CGA, which uses three Euclidean basis vectors e_1, e_2, e_3 and two additional basis vectors e_4, e_5 with positive and negative signatures, respectively, is represented by $\mathbb{G}^{4,1}$. The operation rules of the basis vectors are listed as follows:

$$e_i \cdot e_j = \begin{cases} +1, 1 \leq i = j \leq 4 \\ -1, i = j = 5 \\ 0, i \neq j \end{cases} \quad \text{and} \quad e_i \wedge e_j = e_{i,j} = -e_j \wedge e_i, \quad (1)$$

where $\cdot$ is inner product and $\wedge$ is outer product. Another two bases are e_∞ and e_0 , with definitions by

$e_\infty = e_4 + e_5$ and $e_0 = \frac{1}{2} e_5 - e_4$. There are 32 special blades of $\mathbb{G}_7^{4,1}$ and their linear combinations are called multi-vectors. A k -blade is denoted by $A_{\langle k \rangle}$ that can be represented by the outer products of k linear independent 1-vectors. Besides outer product and inner product, geometric product of blades is also frequently used, and it is defined as

$$A_{\langle k \rangle} A_{\langle l \rangle} = \langle A_{\langle k \rangle} A_{\langle l \rangle} \rangle_{|k-l|} + \langle A_{\langle k \rangle} A_{\langle l \rangle} \rangle_{|k-l|+2} + \cdots + \langle A_{\langle k \rangle} A_{\langle l \rangle} \rangle_{|k+l|}, \quad (2)$$

where $\langle \cdot \rangle_m$ denotes the element of m grade, $\langle A_{\langle k \rangle} A_{\langle l \rangle} \rangle_{|k-l|}$ is the inner product $A_{\langle k \rangle} \cdot A_{\langle l \rangle}$, and $\langle A_{\langle k \rangle} A_{\langle l \rangle} \rangle_{|k+l|}$ is the outer product $A_{\langle k \rangle} \wedge A_{\langle l \rangle}$. More detail can be found in Charter 3 of [55]. The geometric product of two multi-vectors can also be expressed by the combinations of the products of blades according to the distributive property of the geometric product.

Conformal geometric entities can be represented by both inner product null space (IPNS) and outer product null space (OPNS) in CGA[36, 55]. P , Π , S , C , L and Pp denote the IPNS representations of point, plane, sphere, circle, line and point pair, respectively. OPNS representation is denoted by adding the superscript *, e.g., Π^* is a plane, as shown in Table 1. Moreover, F , M , R , T and V represent the flag, motor, rotor, translator and screw, respectively. A point (or a vector) in three-dimensional space is $p = p_1 e_1 + p_2 e_2 + p_3 e_3$, and there exists a mapping from p to the point in conformal space, i.e.

$$P = \mathcal{P}(p) = p + \frac{1}{2} p^2 e_\infty + e_o. \quad (3)$$

Its inverse mapping is $p = \mathcal{P}^{-1}(P)$.

On the other hand, the vectors and matrices are represented by standard bold letters, i.e., $\mathbf{J}$ is Jacobian matrix and $\boldsymbol{\theta}$ is a column vector of joint angles.

Table 1 Representations of conformal geometric entities

Entity	IPNS representation	OPNS representation
Point	$P = \mathcal{P}(p) = p + \frac{1}{2} p^2 e_\infty + e_o$	
Plane	$\Pi = n + d e_\infty$, where n is the direction and d is the distance from the origin of the plane; $\Pi = P_1 - P_2$, where Π is	$\Pi^* = P_1 \wedge P_2 \wedge P_3 \wedge e_\infty$, where P_1, P_2, P_3 are three arbitrary points (except e_∞) on the plane.

	the middle plane of P_1, P_2 .	
Sphere	$S = P - \frac{1}{2} r^2 e_\infty$, where r is the diameter of the sphere, P is its center.	$S^* = P_1 \wedge P_2 \wedge P_3 \wedge P_4$, where P_1, P_2, P_3, P_4 are four arbitrary points on the sphere.
Circle	$C = S_1 \wedge S_2$, where C is the intersection circle of the two spheres S_1 and S_2 .	$C^* = P_1 \wedge P_2 \wedge P_3$, where P_1, P_2, P_3 are three arbitrary points on the circle.
Line	$L = \Pi_1 \wedge \Pi_2$, where L is the intersection line of the two planes Π_1 and Π_2 .	$L^* = P_1 \wedge P_2 \wedge e_\infty$, where P_1 and P_2 are any two points on the line; or, $L^* = C \wedge e_\infty$, where L^* is the line passing through the center and being norm to the face of a circle C .
Point pair	$Pp = S_1 \wedge S_2 \wedge S_3 = C \wedge S_3$, point pair determined by three spheres, or determined by a circle and a sphere.	$Pp^* = P_1 \wedge P_2$

3 Transformation and differential motion of conformal geometric entities and rigid bodies

3.1 Transformation of geometric entities

3.1.1 Rotation about the axis passing through origin

The IPNS representation of two planes that passing through the origin are $\Pi_1 = \bar{m}$, $\Pi_2 = \bar{n}$, where $\bar{m} = m_1 e_1 + m_2 e_2 + m_3 e_3$ and $\bar{n} = n_1 e_1 + n_2 e_2 + n_3 e_3$ are both unit vector, and the angle between them is $\theta/2$, as shown in Figure 1. If $A_{\langle k \rangle}$ reflects with Π_2 and Π_1 sequentially, then it rotates about the intersecting line of these two planes with the angle θ . The rotation result is

$$A'_{\langle k \rangle} = R_{L_0, \theta} A_{\langle k \rangle} \tilde{R}_{L_0, \theta}, \quad (4)$$

with the definitions

$$\begin{cases} R_{L_0, \theta} = \bar{m} \bar{n} = \cos\left(\frac{\theta}{2}\right) - L_0 \sin\left(\frac{\theta}{2}\right) \\ L_0 = \frac{\bar{n} \wedge \bar{m}}{\|\bar{n} \wedge \bar{m}\|} \end{cases} \quad (5)$$

where $R_{L_0, \theta}$ is a rotor, $\tilde{R}_{L_0, \theta}$ is the reverse of $R_{L_0, \theta}$, L_0 is the intersecting line of these two planes. $R_{L_0, \theta}$ is usually abbreviated as R_0 . The unit vector of the direction of L_0 is $\bar{\omega}$ which can be obtained by

$$\bar{\omega} = -L_0 e_{1,2,3}, \quad (6)$$

where $\bar{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3$. On the other hand, L_0 can be represented by $\bar{\omega}$ as follows:

$$L_0 = \bar{\omega} \mathbf{e}_{1,2,3} = \omega_3 \mathbf{e}_{1,2} - \omega_2 \mathbf{e}_{1,3} + \omega_1 \mathbf{e}_{2,3}. \quad (7)$$

Three special rotors are listed as follows:

- 1) The rotor about $\mathbf{e}_3$ is $R_3 \alpha = \cos\left(\frac{\alpha}{2}\right) - \mathbf{e}_{1,2} \sin\left(\frac{\alpha}{2}\right)$, where α is the rotation angle.
- 2) The rotor about $\mathbf{e}_2$ is $R_2 \beta = \cos\left(\frac{\beta}{2}\right) + \mathbf{e}_{1,3} \sin\left(\frac{\beta}{2}\right)$, where β is the rotation angle.
- 3) The rotor about $\mathbf{e}_1$ is $R_1 \gamma = \cos\left(\frac{\gamma}{2}\right) - \mathbf{e}_{2,3} \sin\left(\frac{\gamma}{2}\right)$, where γ is the rotation angle.

Figure 1 Rotation

Figure 2 Translation

3.1.2 Translation

The IPNS representation of two parallel planes are $\Pi_1 = \bar{\mathbf{d}} + d_1 \mathbf{e}_\infty$ and $\Pi_2 = \bar{\mathbf{d}} + d_2 \mathbf{e}_\infty$, where $\bar{\mathbf{d}}$ is the normal unit vector of them. If $A_{(k)}$ reflect with Π_2 and Π_1 sequentially, then it translates along $\bar{\mathbf{d}}$ with the length $d = 2(d_2 - d_1)$, the translator of this motion is

$$T \bar{\mathbf{d}}, d = \Pi_2 \Pi_1 = 1 - (d_2 - d_1) \bar{\mathbf{d}} \mathbf{e}_\infty = 1 - \frac{d}{2} \bar{\mathbf{d}} \mathbf{e}_\infty. \quad (8)$$

The translation result of $A_{(k)}$ is

$$A'_{(k)} = T \bar{\mathbf{d}}, d A_{(k)} \tilde{T} \bar{\mathbf{d}}, d. \quad (9)$$

3.1.3 Screw motion

Screw can usually represents the motions of joints of spatial mechanisms, and it can make geometric entities rotate about an axis with an angle θ and translate along the same axis with a distance $h\theta$, where h is pitch.

Figure 3 Screw motion

As shown in Figure 3, the intersecting line of two planes $\Pi_1 = \bar{\mathbf{m}} + d_1 \mathbf{e}_\infty$ and $\Pi_2 = \bar{\mathbf{n}} + d_2 \mathbf{e}_\infty$ is $L = \Pi_2 \wedge \Pi_1$. The angle of these two planes is $\theta/2$. The rotor part of the screw motion is

$$R L, \theta = \Pi_2 \Pi_1 = \bar{\mathbf{n}} + d_2 \mathbf{e}_\infty \bar{\mathbf{m}} + d_1 \mathbf{e}_\infty = \bar{\mathbf{n}} \bar{\mathbf{m}} + d_2 \mathbf{e}_\infty \bar{\mathbf{m}} + d_1 \bar{\mathbf{n}} \mathbf{e}_\infty. \quad (10)$$

There must exist a common point $\mathbf{r}$ lying on the intersecting line. The distance from the planes to the origin can be obtained with $d_2 = \mathbf{r} \cdot \bar{\mathbf{n}}$ and $d_1 = \mathbf{r} \cdot \bar{\mathbf{m}}$. Substituting them into Eq.(10) yields

$$R L, \theta = R_0 + \mathbf{e}_\infty \mathbf{r} \cdot \bar{\mathbf{n}} \bar{\mathbf{m}} - \mathbf{e}_\infty \mathbf{r} \cdot \bar{\mathbf{m}} \bar{\mathbf{n}} = R_0 + \mathbf{e}_\infty \mathbf{r} \cdot \bar{\mathbf{n}} \wedge \bar{\mathbf{m}} = R_0 + \mathbf{e}_\infty \mathbf{r} \times R_0, \quad (11)$$

where $\times$ is the commutator product with the definition as $A \times B = \frac{1}{2} AB - BA$, more detail can be obtained in Charter 3 of [55]. R_0 is with the definition in Eq.(5).

The translator along $\bar{\omega}$ with $h\theta$ is

$$T \bar{\omega}, h\theta = 1 - \frac{h\theta}{2} \bar{\omega} \mathbf{e}_\infty. \quad (12)$$

Thus, the combination of the rotor and the translator can be represented by a motor with the result as

$$M = T \bar{\omega}, h\theta R L, \theta. \quad (13)$$

when $h = \infty$, the screw motion becomes a pure translator

$$M = T \bar{\omega}, \theta. \quad (14)$$

Based on Chasles' theorem, any transformation in the

space can be represented by a motor as Eq.(13) or Eq.(14).

Besides the screw representation, a motor can also be expressed as

$$M = T \bar{d}, d R_0, \quad (15)$$

which represents a rotation about $\bar{\omega}$ followed with a translation along $\bar{d}$. This form of motor can also represent the arbitrary transformation of geometric entities.

3.2 Rigid body framing based on the flag in CGA

A flag is a triple combination of a plane, a line on the plane, and a point on the line, and it can be used to frame a rigid body[58-60]. Hestenes wrote it in CGA form[37] as

$$F_X = P_X + L^* + \Pi^* \\ = P_X + P_X \wedge P_{X1} \wedge e_\infty + P_X \wedge P_{X1} \wedge P_{X2} \wedge e_\infty. \quad (16)$$

It shows that a flag is the sum of 1-blade, 3-blade and 4-blade.

For a better illustration, a flag is represented by a “flag-like” geometry, as shown in Figure 4. The red triangle represents the plane Π^* , the green line represents L^* and it is along one of the legs, and the point P_X is the vertex of the angle. If a flag is attached on a rigid body, the body is framed, and its position and orientation can be represented by the flag. A special flag F_0 can be set at the origin, and its plane and line are $\Pi^* = e_0 \wedge P_{01} \wedge P_{02} \wedge e_\infty$ and $L^* = e_0 \wedge P_{01} \wedge e_\infty$, respectively, where $P_{01} = \mathcal{P}(e_1)$ and $P_{02} = \mathcal{P}(e_2)$. Then this flag is

$$F_0 = e_0 + e_0 \wedge e_1 \wedge e_\infty + e_0 \wedge e_1 \wedge e_2 \wedge e_\infty. \quad (17)$$

Moreover, the triangle of F_0 is right and its legs are with unit length.

Figure 4 Rigid body framing by flags

A flag and a motor both can reflect a rigid body motion in 3D space. As far as we know, the relationships between the flag and motor have not been

studied. It will also show in Section 5.2 that the relationships between them can help us to solve the position kinematics of the parallel mechanism. The relationships are discussed as follows.

3.2.1 Obtain a flag by a motor

The flag at origin F_0 can be moved to the flag F_X by a motor M , i.e.,

$$F_X = M F_0 \tilde{M}. \quad (18)$$

What is noteworthy is that F_X in Eq.(18) has some limitations besides the form in Eq.(16), for it transforms from F_0 . These limitations are

$$\begin{cases} \|\mathcal{P}^{-1}(P_{X1}) - \mathcal{P}^{-1}(P_X)\| = 1 \\ \|\mathcal{P}^{-1}(P_{X2}) - \mathcal{P}^{-1}(P_X)\| = 1 \\ \mathcal{P}^{-1}(P_{X1}) - \mathcal{P}^{-1}(P_X) \cdot \mathcal{P}^{-1}(P_{X2}) - \mathcal{P}^{-1}(P_X) = 0 \end{cases} \quad (19)$$

which means that these two legs of the flag in Figure 4 are with unit length and orthogonal to each other. The flag is called a standard flag if it satisfies Eq.(19), and called non-standard flag if not. One standard flag F_X can be transformed from F_0 by a motor M , as shown in Figure 4.

3.2.2 Obtain a motor by a flag

If a standard flag F_X can be built directly, the transformation motor that transforms F_0 to F_X satisfies the following equation

$$F_X = M F_0 \tilde{M} = \underbrace{M e_0 \tilde{M}}_{P_X} + P_X \wedge \underbrace{M P_{01} \tilde{M}}_{P_{X1}} \wedge e_\infty \\ + P_X \wedge P_{X1} \wedge \underbrace{M P_{02} \tilde{M}}_{P_{X2}} \wedge e_\infty, \quad (20)$$

where $M = T \bar{d}, d R_0$. The elements of 1-vector of F_X can be expanded as

$$\langle F_X \rangle_1 = M e_0 \tilde{M} = d \bar{d} + \frac{1}{2} d^2 e_\infty + e_0. \quad (21)$$

Based on Eq.(21), d and $\bar{d}$ can be obtained, and $T \bar{d}, d$ can be calculated. Then Eq.(20) can be rewritten by the inverse translation with $\tilde{T} \bar{d}, d$ as follows:

$$\begin{aligned} \tilde{T} \bar{d}, d F_X T \bar{d}, d &= e_0 + e_0 \wedge R_0 P_{01} \tilde{R}_0 \wedge e_\infty + \\ e_0 \wedge R_0 P_{01} \tilde{R}_0 \wedge R_0 P_{02} \tilde{R}_0 \wedge e_\infty & \end{aligned} \quad (22)$$

Based on Eq. (5) and Eq.(7), we can set

$$R_0 = s + ae_{1,2} + be_{1,3} + ce_{2,3}, \quad (23)$$

where $s = \cos\left(\frac{\theta}{2}\right)$, $a = -\omega_3 \sin\left(\frac{\theta}{2}\right)$, $b = \omega_2 \sin\left(\frac{\theta}{2}\right)$, and $c = -\omega_1 \sin\left(\frac{\theta}{2}\right)$. Moreover, the quadratic sum of a , b , c and s is 1, i.e., $a^2 + b^2 + c^2 + s^2 = 1$. The sum of the elements of 3-vector and 4-vector of $\tilde{T} \bar{d}, d F_X T \bar{d}, d$ is

$$\begin{aligned} \langle \tilde{T} \bar{d}, d F_X T \bar{d}, d \rangle_{3,4} &= e_0 \wedge R_0 P_{01} \tilde{R}_0 \wedge e_\infty + \\ e_0 \wedge R_0 P_{01} \tilde{R}_0 \wedge R_0 P_{02} \tilde{R}_0 \wedge e_\infty &= \\ -a^2 - b^2 + c^2 + s^2 e_{1,4,5} + -2as - 2bc e_{2,4,5} + & (24) \\ 2ac - 2bs e_{3,4,5} + -a^2 + b^2 + c^2 - s^2 e_{1,2,4,5} + & \\ -2ab + 2cs e_{1,3,4,5} + -2ac - 2bs e_{2,3,4,5} & \end{aligned}$$

It also has the form as follows:

$$\begin{aligned} \langle \tilde{T} \bar{d}, d F_X T \bar{d}, d \rangle_{3,4} &= \kappa_{1,4,5} e_{1,4,5} + \kappa_{2,4,5} e_{2,4,5} + \\ \kappa_{3,4,5} e_{3,4,5} + \kappa_{1,2,4,5} e_{1,2,4,5} + \kappa_{1,3,4,5} e_{1,3,4,5} + \kappa_{2,3,4,5} e_{2,3,4,5} & \end{aligned} \quad (25)$$

which can be calculated directly by the expression of F_X and $T \bar{d}, d$. Based on the equalities of the terms corresponding to $e_{3,4,5}$, $e_{2,3,4,5}$, $e_{1,4,5}$, and $e_{1,2,4,5}$ in Eq.(25) and Eq. (24), the following equations can be obtained

$$ac = \kappa_A, \quad c^2 - a^2 = \kappa_B, \quad (26)$$

where $\kappa_A = \frac{\kappa_{3,4,5} - \kappa_{2,3,4,5}}{4}$ and $\kappa_B = \frac{\kappa_{1,4,5} + \kappa_{1,2,4,5}}{2}$. The solutions of Eq.(26) are listed as follows:

1) when $\sqrt{\kappa_B^2 + 4\kappa_A^2} \neq 0$, solving Eq.(26) yields

$$c^2 = \frac{\sqrt{\kappa_B^2 + 4\kappa_A^2} + \kappa_B}{2}, \quad a^2 = \frac{\sqrt{\kappa_B^2 + 4\kappa_A^2} - \kappa_B}{2}. \quad (27)$$

If $\kappa_A \geq 0$, a and c have the same sign, otherwise if

$\kappa_A < 0$, a and c have different signs. b and s can be obtained based on the equality of the terms corresponding to $e_{2,4,5}$ and $e_{1,3,4,5}$ in Eq.(25) and Eq. (24)

$$b = \frac{-\kappa_{2,4,5}c - (\kappa_{1,3,4,5})a}{2\sqrt{\kappa_B^2 + 4\kappa_A^2}}, \quad s = \frac{-\kappa_{2,4,5}a + (\kappa_{1,3,4,5})c}{2\sqrt{\kappa_B^2 + 4\kappa_A^2}}. \quad (28)$$

It means that two results of the rotor R_0 can be obtained by Eq.(27), and Eq.(28), and any one of them can transform F_0 to F_X .

2) when $\sqrt{\kappa_B^2 + 4\kappa_A^2} = 0$, it will satisfy $a = 0$, $c = 0$. On the other hand, according to the terms of $e_{1,4,5}$ and $e_{3,4,5}$ in Eq.(25) and Eq. (24), two equations can be obtained: $s^2 + b^2 = \kappa_{1,4,5}$, $-2bs = \kappa_{3,4,5}$. b and s can then be calculated as follows:

$$s^2 = \frac{\sqrt{\kappa_{1,4,5}^2 + \kappa_{3,4,5}^2} + \kappa_{1,4,5}}{2}, \quad b^2 = \frac{\sqrt{\kappa_{1,4,5}^2 + \kappa_{3,4,5}^2} - \kappa_{1,4,5}}{2}. \quad (29)$$

If $\kappa_{3,4,5} \geq 0$, b and s have different signs, otherwise if $\kappa_{3,4,5} < 0$, b and s have the same sign. It means that two results can be obtained by Eq.(29), and any one of them can transform F_0 to F_X .

If a non-standard flag F'_X is built as

$F'_X = P_X + L^* + \Pi^*$, the standard one can be obtained by the following standardization process. The line part of a flag is $L^* = P_X \wedge P_{XA} \wedge e_\infty$, unlike P_{X1} , P_{XA} does not satisfy Eq.(19). Suppose $P_X = \mathcal{P} x_1 e_1 + x_2 e_2 + x_3 e_3$ and $P_{XA} = \mathcal{P} a_1 e_1 + a_2 e_2 + a_3 e_3$, then the line is

$$\begin{aligned} L^* &= -a_1 x_2 + a_2 x_1 e_{1,2} \wedge e_\infty + -a_1 x_3 + a_3 x_1 e_{1,3} \wedge e_\infty + \\ -a_2 x_3 + a_3 x_2 e_{2,3} \wedge e_\infty + a_1 - x_1 e_1 \wedge e_\infty \wedge e_0 + & (30) \\ a_2 - x_2 e_2 \wedge e_\infty \wedge e_0 + a_3 - x_3 e_3 \wedge e_\infty \wedge e_0 & \end{aligned}$$

The unit vector from P_X to P_{XA} is

$$\bar{m}_1 = \frac{a_1 - x_1}{ax} e_1 + \frac{a_2 - x_2}{ax} e_2 + \frac{a_3 - x_3}{ax} e_3, \quad (31)$$

where $ax = \sqrt{a_1 - x_1^2 + a_2 - x_2^2 + a_3 - x_3^2}$. The point P_{X1} that satisfies Eq.(19) can then be obtained by

$$P_{X1} = \mathcal{P} x_1 e_1 + x_2 e_2 + x_3 e_3 + \bar{m}_1. \quad (32)$$

The OPNS representation of the plane of the non-standard flag F'_X is Π^* , then its IPNS representation is $\Pi' = -\Pi'^* e_{1,2,3,4,5}$. The normal vector n of the plane can be obtained by Π . The unit vector from P_X to P_{XB} is

$$\bar{m}_2 = \frac{-\bar{m}_1 \wedge \bar{n} e_{1,2,3}}{\|\bar{m}_1 \wedge \bar{n} e_{1,2,3}\|}. \quad (33)$$

Thus, the point satisfying Eq. (19) can be obtained by

$$P_{X2} = \mathcal{P} \quad x_1 e_1 + x_2 e_2 + x_3 e_3 + \bar{m}_2. \quad (34)$$

Based on the result in Eqs.(32), (34) and (16), the standard flag can be obtained according to P_X , P_{X1} , and P_{X2} .

If a geometric entity represented in flag F_X is denoted as ${}^X G$, then it can be represented in the origin frame F_0 as

$${}^0 G = M {}^X G \tilde{M}. \quad (35)$$

3.3 Twist and screw velocity

Spatial velocity (or screw) can be obtained by differentiating transform matrix with time[29]. Similarly, by differentiating motor with time, the spatial velocity in CGA can also be obtained, more details are in [37].

The motor in Eq.(13) contains $R L, \theta$ and $T \bar{\omega}, h\theta$ which are abbreviated as R and T , respectively. The derivative of T with respect to time is

$$\begin{aligned} \dot{T} &= -\frac{h\dot{\theta}}{2} \bar{\omega} e_\infty = \frac{h\dot{\theta}}{2} e_\infty \bar{\omega} \left(T + \frac{h\theta}{2} \bar{\omega} e_\infty \right) \\ &= \frac{1}{2} \dot{\theta} h e_\infty \bar{\omega} T \end{aligned} \quad (36)$$

The spatial angular velocity can be defined as $\Omega = 2\dot{R}_0 \tilde{R}_0$, i.e., $\dot{R}_0 = \frac{1}{2} \Omega R_0$. Based on Eq.(5), the spatial angular velocity can be calculated

$$\Omega = \frac{\dot{\theta}}{2} \sin \theta \quad \bar{\omega} \bar{\omega} - 1 \quad -\dot{\theta} \bar{\omega} e_{1,2,3} = -\dot{\theta} \bar{\omega} e_{1,2,3}. \quad (37)$$

Then, the derivative of R in Eq.(11) with respect to time is

$$\begin{aligned} \dot{R} &= \dot{R}_0 + e_\infty r \times \dot{R} = \frac{1}{2} \Omega R_0 + e_\infty r \times \Omega R_0 \\ &= \frac{1}{2} \Omega + e_\infty r \times \Omega \quad R \end{aligned} \quad (38)$$

The derivative of M can be obtained

$$\begin{aligned} \dot{M} &= \dot{T} R + T \dot{R} = \frac{1}{2} \dot{\theta} h e_\infty \bar{\omega} T R + \frac{1}{2} T \Omega + e_\infty r \times \Omega \quad R \\ &= \frac{1}{2} \dot{\theta} h e_\infty \bar{\omega} + T \Omega \tilde{T} + T e_\infty r \times \Omega \quad \tilde{T} M \end{aligned} \quad (39)$$

Based on the equation $T = 1 - \frac{h\theta}{2} \bar{\omega} e_\infty$, it will give the property that $T \Omega \tilde{T} = \Omega$ and $T e_\infty r \times \Omega \tilde{T} = e_\infty r \times \Omega$. Substituting these two properties into Eq.(39) yields

$$\begin{aligned} \dot{M} &= \frac{\dot{\theta}}{2} e_\infty h \bar{\omega} - r \times (\bar{\omega} e_{1,2,3}) - \bar{\omega} e_{1,2,3} \quad M \\ &= \frac{\dot{\theta}}{2} e_\infty h \bar{\omega} - r \wedge \bar{\omega} e_{1,2,3} - \bar{\omega} e_{1,2,3} \quad M \end{aligned} \quad (40)$$

If we set $v = h\bar{\omega} - r \wedge \bar{\omega} e_{1,2,3}$, then the twist can be defined as

$$\bar{V} = e_\infty v - \bar{\omega} e_{1,2,3}. \quad (41)$$

It can also be expanded as

$$\begin{aligned} \bar{V} &= v_1 e_\infty \wedge e_1 + v_2 e_\infty \wedge e_2 + v_3 e_\infty \wedge e_3 - \\ &\quad \bar{\omega}_1 e_2 \wedge e_3 + \bar{\omega}_2 e_1 \wedge e_3 - \bar{\omega}_3 e_1 \wedge e_2 \end{aligned} \quad (42)$$

Thus, the spatial velocity with the twist $\bar{V}$ is

$$V = \dot{\theta} \bar{V}, \quad (43)$$

and has the relationships with M as

$$\dot{M} = \frac{1}{2} V M, \quad \dot{M} = -\frac{1}{2} \tilde{M} V. \quad (44)$$

According to Eq.(14), when $h = \infty$, the twist with pure translation along $\bar{\omega}$ is

$$\bar{V} = e_\infty \bar{\omega}. \quad (45)$$

There exists a map $\mathcal{S}$ from the twist $\bar{V}$ to the vector in $\mathbb{R}^6$, which is

$$\mathcal{S} = \mathcal{S} \bar{V} = \omega_1 \ \omega_2 \ \omega_3 \ v_1 \ v_2 \ v_2^T \in \mathbb{R}^6. (46)$$

3.4 Differentiate blade with time

A k -blade at time 0 is $A_{(k)}$ and it moves to $A'_{(k)}$ by a motor M at time t , which means $A'_{(k)} = MA_{(k)}\tilde{M}$. The differential of $A_{(k)}$ is

$$\dot{A}'_{(k)} = \dot{M}A_{(k)}\tilde{M} + MA_{(k)}\dot{\tilde{M}} = \frac{1}{2} VA'_{(k)} - A'_{(k)}V = V \times A'_{(k)} \quad (47)$$

Thus, the differential of a point P in CGA is

$$\dot{P} = V \times P = \dot{\theta} \ e_{\infty} v - \bar{\omega} e_{1,2,3} \times \left(p + \frac{1}{2} p^2 e_{\infty} + e_0 \right). (48)$$

It will also generate the result as

$$\dot{p} = \dot{\theta} \ -\bar{\omega} \wedge p e_{1,2,3} + v, \quad (49)$$

which is the same as the result in Section 4.2 in [29].

Particularly, the velocity of the point traveling through the origin is

$$V \times e_0 = \dot{\theta} v. \quad (50)$$

The higher-order differentials of the point are

$$\ddot{P} = \dot{V} \times P + V \times \dot{P} = \dot{V} \times P + V \times V \times P \quad (51)$$

$$\ddot{P} = \dot{V} \times P + 2\dot{V} \times V \times P + V \times \dot{V} \times P + V \times V \times P \quad (52)$$

3.5 Screws in different frames

As shown in Figure 5, a screw jV and a point jP are represented in F_j . The differential of the point under the motion of the screw is ${}^j\dot{P} = {}^jV \times {}^jP$. The same screw and the same point can be also represented in F_i , and the point velocity represented in F_i can be transformed by

$$\begin{aligned} {}^i\dot{P} &= M_{i,j} {}^j\dot{P} \tilde{M}_{i,j} = M_{i,j} {}^jV \times {}^jP \tilde{M}_{i,j} \\ &= M_{i,j} {}^jV \tilde{M}_{i,j} \times M_{i,j} {}^jP \tilde{M}_{i,j} \end{aligned} \quad (53)$$

where $M_{i,j}$ is the motor transforming F_i to F_j . It can be seen that the screw can be transformed by the motor as follows:

$${}^iV = M_{i,j} {}^jV \tilde{M}_{i,j}. \quad (54)$$

Actually, If we set $M_{i,j} = T_{i,j} R_{i,j}$, ${}^jV = e_{\infty} {}^jv - {}^j\omega e_{1,2,3}$, and ${}^iV = e_{\infty} {}^iv - {}^i\omega e_{1,2,3}$, where the translate vector of $T_{i,j}$ is $d_{i,j}$, then Eq.(54) can be rewritten as

$$\begin{aligned} &e_{\infty} {}^iv - {}^i\omega e_{1,2,3} \\ &= e_{\infty} T_{i,j} {}^jv \tilde{T}_{i,j} - T_{i,j} {}^j\omega \tilde{T}_{i,j} d_{i,j} e_{1,2,3} \wedge e_{\infty} + e_{1,2,3} \quad (55) \\ &= e_{\infty} {}^iv - d_{i,j} \wedge {}^i\omega e_{1,2,3} - {}^i\omega e_{1,2,3} \end{aligned}$$

where ${}^iv = R_{i,j} {}^jv \tilde{R}_{i,j}$ and ${}^i\omega = R_{i,j} {}^j\omega \tilde{R}_{i,j}$. Eq.(55) indicates that the results is consistent with the adjoint transformation of screws in p.55, ref.[29].

Figure 5 A screw in different frames

4 Higher-order kinematics of serial manipulators

4.1 First-order kinematics

As shown in Figure 6, two rigid body framed by F_i and F_j are connected by a joint which can be represented by a twist. At time 0, these two flags have the following relationships:

$$F_j(0) = M_{i,j}^0 F_i \tilde{M}_{i,j}^0. \quad (56)$$

At time t , the rigid body F_j is transformed to $F_j(t)$ by the screw motion determined by the joint. The screw motion is represented by the motor $M_{i,j}$ according to Eqs.(13) or (14). Then $F_j(t)$ can be calculated by

$$F_j(t) = M_{i,j} M_{i,j}^0 F_i \tilde{M}_{i,j}^0 \tilde{M}_{i,j} = M_{i,j} F_j(0) \tilde{M}_{i,j}. \quad (57)$$

Figure 6 Single joint kinematics

When $n+1$ rigid bodies (contain the fixed one) are connected by the joints in Figure 7, the flag of the end body $F_n(t)$ finally becomes

$$F_n(t) = M_{0-1} \cdots M_{n-2-n-1} M_{n-1-n} F_n(0) \tilde{M}_{n-1-n} \tilde{M}_{n-2-n-1} \cdots \tilde{M}_{0-1} \quad (58)$$

The following definitions are given:

$$\begin{aligned} M'_{i-j} &= M'_{i-i+1} M'_{i+1-i+2} \cdots M'_{j-1-j}, \\ \dot{M}'_{i-j} &= \frac{1}{2} V_{i-j} M'_{i-j}, \quad 0 \leq i < n, \quad i < j < n, \end{aligned} \quad (59)$$

where V_{i-j} is the screw motion of F_j relative to the frame F_i and it is represented in F_i . The differential of M'_{0-n} can be expanded as

$$\begin{aligned} \dot{M}'_{0-n} &= \dot{M}'_{0-1} M'_{1-2} \cdots M'_{n-1-n} + \\ M'_{0-1} \dot{M}'_{1-2} \cdots M'_{n-1-n} &+ \cdots M'_{0-1} M'_{1-2} \cdots \dot{M}'_{n-1-n} \end{aligned} \quad (60)$$

According to Eq.(59), the velocity of the end rigid body of the serial manipulator can be obtained as

$$\begin{aligned} V_{0-n} &= V_{0-1} M'_{1-2} \cdots M'_{n-1-n} + M'_{0-1} V_{1-2} \cdots M'_{n-1-n} + \\ \cdots M'_{0-1} M'_{1-2} \cdots \dot{M}'_{n-1-n} &\tilde{M}'_{0-n} \\ = V_{0-1} + M'_{0-1} V_{1-2} \tilde{M}'_{0-1} &+ M'_{0-2} V_{2-3} \tilde{M}'_{0-2} + \cdots M'_{0-n-1} V_{n-1-n} \tilde{M}'_{0-n-1} \end{aligned} \quad (61)$$

If we set ${}^0V_{i-i+1} = M'_{0-i} V_{i-i+1} \tilde{M}'_{0-i}$, i.e., transform V_{i-i+1} to the frame F_0 , then Eq.(61) can be rewritten as

$${}^0V_{0-n} = {}^0V_{0-1} + {}^0V_{1-2} + {}^0V_{2-3} + \cdots + {}^0V_{n-1-n}. \quad (62)$$

Figure 7 Kinematics of serial manipulator

The above equation can also be written as the linear combination of unit screws, i.e.,

$$\begin{aligned} {}^0V_{0-n} &= \dot{\theta}_{0-1} {}^0\bar{V}_{0-1} + \dot{\theta}_{1-2} {}^0\bar{V}_{1-2} + \dot{\theta}_{2-3} {}^0\bar{V}_{2-3} + \cdots + \dot{\theta}_{n-1-n} {}^0\bar{V}_{n-1-n} \\ &= \sum_{i=0}^{n-1} \dot{\theta}_{i-i+1} {}^0\bar{V}_{i-i+1} \end{aligned} \quad (63)$$

By considering the mapping ${}^0\bar{\$}_{i-i+1} = \mathcal{S} {}^0\bar{V}_{i-i+1}$, the Jacobian matrix can be obtained

$${}^0\$_{0-n} = J_{0-n} \dot{\theta}_{0-n}, \quad (64)$$

where $\dot{\theta}_{0-n} = [\dot{\theta}_{0-1} \quad \dot{\theta}_{1-2} \quad \cdots \quad \dot{\theta}_{n-1-n}]^T$ and

$$J_{0-n} = [{}^0\bar{\$}_{0-1} \quad {}^0\bar{\$}_{1-2} \quad \cdots \quad {}^0\bar{\$}_{n-1-n}].$$

The inverse of Eq.(64) is

$$\dot{\theta}_{0-n} = J_{0-n}^{\pm} {}^0\$_{0-n}, \quad (65)$$

where $J_{0-n}^{\pm}$ is the inverse or the Moore-Penrose inverse of the matrix J_{0-n} .

4.2 Higher-order kinematics

Derivate Eq. (63) with respect to time, it follows

$$\dot{{}^0V}_{0-n} = \sum_{i=1}^{n-1} \ddot{\theta}_{i-i+1} {}^0\bar{V}_{i-i+1} + \sum_{i=1}^{n-1} \dot{\theta}_{i-i+1} \dot{{}^0\bar{V}}_{i-i+1}. \quad (66)$$

As ${}^0\bar{V}_{i-i+1}$ has relationships with all the joints before it, the derivation of it can be written as

$$\dot{{}^0\bar{V}}_{i-i+1} = {}^0V_{0-i} \times {}^0\bar{V}_{i-i+1}, \quad (67)$$

where ${}^0V_{0-i} = \left(\sum_{j=0}^{i-1} \dot{\theta}_{j-j+1} {}^0\bar{V}_{j-j+1} \right)$. Thus, the acceleration of end body is

$$\dot{{}^0V}_{0-n} = \sum_{i=1}^{n-1} \ddot{\theta}_{i-i+1} {}^0\bar{V}_{i-i+1} + \sum_{i=1}^{n-1} \dot{\theta}_{i-i+1} {}^0V_{0-i} \times {}^0\bar{V}_{i-i+1}. \quad (68)$$

The above acceleration can also be written as the form of a vector in $\mathbb{R}^6$, that is

$${}^0\$_{0-n} = J_{0-n} \ddot{\theta}_{0-n} + \mathcal{S} \left(\sum_{i=1}^{n-1} \dot{\theta}_{i-i+1} {}^0V_{0-i} \times {}^0\bar{V}_{i-i+1} \right). \quad (69)$$

Derivating Eq. (68) with respect to time yields the jerk of end body as follows:

- Rotate F_0 by the angle γ about e_1 of F_0 to the first flag;

- 1) Rotate the first flag by the angle β about $\mathbf{e}_2$ of $\mathbf{F}_0$ to the second flag;
- 2) Rotate the second flag by the angle α about $\mathbf{e}_3$ of $\mathbf{F}_0$ to the third flag;
- 3) Translate the third flag by the vector $x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ to the final flag $\mathbf{F}_H$.

However, according to the DoF of 3-RRS, the above rotations and translation are not free. If the motion parameter z , β and γ are set to be free, x , y , and α can be determined by them. The points A_1 , A_2 , and A_3 represented in $\mathbf{F}_0$ are

$$\begin{aligned} \mathbf{P}_{A1} &= \mathcal{P} \mathbf{r} \mathbf{e}_2, \\ \mathbf{P}_{A2} &= \mathcal{P} \left(-\frac{\sqrt{3}}{2} \mathbf{r} \mathbf{e}_1 - \frac{r}{2} \mathbf{e}_2 \right), \\ \mathbf{P}_{A3} &= \mathcal{P} \left(\frac{\sqrt{3}}{2} \mathbf{r} \mathbf{e}_1 - \frac{r}{2} \mathbf{e}_2 \right), \end{aligned} \quad (75)$$

respectively. As the joints' distribution of the output platform is the same as the fixed platform determined by $A_1 A_2 A_3$, the points C_1 , C_2 , and C_3 represented in $\mathbf{F}_H$ is the same as the equations in Eq.(75), respectively. Then the points C_1 , C_2 , and C_3 represented in $\mathbf{F}_0$ can be obtained as follows

$$\begin{aligned} \mathbf{P}_{C1} &= \mathbf{M}_{0,H} \mathbf{P}_{A1} \tilde{\mathbf{M}}_{0,H}, \\ \mathbf{P}_{C2} &= \mathbf{M}_{0,H} \mathbf{P}_{A2} \tilde{\mathbf{M}}_{0,H}, \\ \mathbf{P}_{C3} &= \mathbf{M}_{0,H} \mathbf{P}_{A3} \tilde{\mathbf{M}}_{0,H}, \end{aligned} \quad (76)$$

respectively. Besides, the IPNS representations of the planes Π_1 , Π_2 , and Π_3 are

$$\begin{aligned} \Pi_1 &= \mathbf{e}_1, \\ \Pi_2 &= -\frac{1}{2} \mathbf{e}_1 + \frac{\sqrt{3}}{2} \mathbf{e}_2, \\ \Pi_3 &= -\frac{1}{2} \mathbf{e}_1 - \frac{\sqrt{3}}{2} \mathbf{e}_2, \end{aligned} \quad (77)$$

respectively. The constraint of the output platform is that the points C_1 , C_2 , and C_3 must lie on Π_1 , Π_2 , and Π_3 , respectively, i.e.

$$\mathbf{P}_{C1} \cdot \Pi_1 = 0, \quad \mathbf{P}_{C2} \cdot \Pi_2 = 0, \quad \text{and} \quad \mathbf{P}_{C3} \cdot \Pi_3 = 0. \quad (78)$$

By solving the above equations, x , y , and α can be expressed by z , β , and γ as follows:

- 1) When $\beta \neq 0$ or $\gamma \neq 0$, the results are

$$\alpha = \arctan \left(\frac{c\beta c\gamma - 1}{c\beta^2 c\gamma^2 - 1}, \frac{c\gamma + c\beta}{c\beta^2 c\gamma^2 - 1} \right), \quad (79)$$

$$x = -\frac{\frac{rs\beta s\gamma}{c\beta^2 c\gamma^2 - 1} \frac{c\beta c\gamma - 1}{c\beta^2 c\gamma^2 - 1}}{\sqrt{\frac{s\beta^2 s\gamma^2 + c\beta^2 + 2c\beta c\gamma + c\gamma^2}{c\beta^2 c\gamma^2 + 2c\beta c\gamma + 1}}}, \quad (80)$$

$$y = \frac{1}{2} \frac{c\beta^2 c\gamma^2 - 2c\beta^2 + 1}{c\beta^2 c\gamma^2 - 1} r \frac{c\beta c\gamma - 1}{c\beta^2 c\gamma^2 - 1}. \quad (81)$$

- 2) When $\beta = 0$ and $\gamma = 0$, the result are $\alpha = 0$, $x = 0$, $y = 0$.

If the motion laws of z , β and γ are given in Table 3, the six DoF motions of the output platform can be obtained and drawn in Figure 10 in the period of 10 s. The trace of the flag $\mathbf{F}_H$ is also given in Figure 11. It can be seen that the flag $\mathbf{F}_H$ can represent the motions of the output platform.

Table 3 The motion laws of the output platform

z (m)	β (rad)	γ (rad)
$1 + 0.2 \sin\left(\frac{2\pi}{10}t\right)$	$\frac{25}{180} \pi \sin\left(\frac{2\pi}{8}t\right)$	$\frac{20}{180} \pi \sin\left(\frac{2\pi}{5}t\right)$

Figure 10 Six DoF motions of the output platform

Figure 11 The trace of the flag attached on the output platform from 0 s to 10 s

Taking the differential of Eq.(74) yields

$$\begin{aligned} \dot{\mathbf{M}}_{0,H} &= \dot{\mathbf{T}}_{0,H} \mathbf{R}_z \alpha \mathbf{R}_y \beta \mathbf{R}_x \gamma + \mathbf{T}_{0,H} \dot{\mathbf{R}}_z \alpha \mathbf{R}_y \beta \mathbf{R}_x \gamma + \\ &\mathbf{T}_{0,H} \mathbf{R}_z \alpha \dot{\mathbf{R}}_y \beta \mathbf{R}_x \gamma + \mathbf{T}_{0,H} \mathbf{R}_z \alpha \mathbf{R}_y \beta \dot{\mathbf{R}}_x \gamma \\ &= \frac{1}{2} \mathbf{V}_{0,H} \mathbf{M}_{0,H} \end{aligned} \quad (82)$$

The velocity $\mathbf{V}_{0,H}$ of the output platform can be calculated by Eq.(82), and its results are given in Eq.(104) from **Appendix**.

As the motions of the three points C_1 , C_2 , and C_3 are restricted on the planes Π_1 , Π_2 , and Π_3 , respectively, the velocities of the points should always be perpendicular to the normal vector of these planes

$$\begin{aligned} \mathbf{V}_{0,H} \times \mathbf{P}_{C1} \cdot \Pi_1 &= 0, \\ \mathbf{V}_{0,H} \times \mathbf{P}_{C2} \cdot \Pi_2 &= 0, \\ \mathbf{V}_{0,H} \times \mathbf{P}_{C3} \cdot \Pi_3 &= 0. \end{aligned} \quad (83)$$

The solutions of the above equations are

$$\dot{x} = \frac{r A_x}{2 B_x}, \quad \dot{y} = \frac{r A_y}{4 B_x}, \quad \text{and} \quad \dot{\alpha} = -\frac{A_\alpha}{B_x}, \quad (84)$$

where the expressions of A_x , A_y , A_α , and B_x are listed in **Appendix**. Based on the motion laws in Table 3, the velocity of the output platform are calculated and illustrated in Figure 12.

Figure 12 Velocity of the output platform

By the multi-derivations of Eq.(104), the acceleration $\dot{\mathbf{V}}_{0,H}$ and the jerk $\ddot{\mathbf{V}}_{0,H}$ of the output platform can be obtained. By the multi derivations of Eq.(84), the constraint of acceleration and jerk can also be obtained.

5.2 Higher-order kinematics of 3-RRS

The kinematics of the 3-RRS mechanism is about the methods to obtain the positions, velocities, accelerations, and jerks of the actuators based on the motion of the output platform.

5.2.1 Calculate the length of the actuators

Based on $\mathbf{M}_{0,H}$ and Eq. (76), the points $\mathbf{P}_{C1}$, $\mathbf{P}_{C2}$, and $\mathbf{P}_{C3}$ represented in the flag $\mathbf{F}_0$ can be obtained. Then, the sphere, whose center is $\mathbf{P}_{Ci}$ ($i=1,2,3$) and radius is l , is expressed by IPNS as

$$\mathbf{S}_{Ci} = \mathbf{P}_{Ci} - \frac{1}{2} l^2 \mathbf{e}_\infty. \quad (85)$$

Besides, the sphere, whose center is $\mathbf{P}_{Ai}$ ($i=1,2,3$) and radius is l , is expressed by IPNS as

$$\mathbf{S}_{Ai} = \mathbf{P}_{Ai} - \frac{1}{2} l^2 \mathbf{e}_\infty. \quad (86)$$

The intersection of $\mathbf{S}_{Ai}$, $\mathbf{S}_{Ci}$, and Π_i is point pair, which can be expressed as

$$\mathbf{Pp}_i = \mathbf{S}_{Ai} \wedge \mathbf{S}_{Ci} \wedge \Pi_i \quad (87)$$

Based on $\mathbf{Pp}_i$, the point $\mathbf{B}_i$ can be obtained by

$$\mathbf{P}_{Bi} = \frac{\pm \sqrt{\mathbf{Pp}_i^* \cdot \mathbf{Pp}_i^*} + \mathbf{Pp}_i^*}{\mathbf{e}_\infty \cdot \mathbf{Pp}_i^*}, \quad (88)$$

where Π_i^* is the OPNS representation of Π_i in

Eq.(77). $\mathbf{P} \mathbf{p}_i^* = -\mathbf{P} \mathbf{p}_i \mathbf{e}_{1,2,3,4,5}$. Eq.(88) provides two results of $\mathbf{P}_{Bi}$, and the one which is closer to the origin is selected.

The flag $\mathbf{F}_{Di}$ attached on the link $B_i C_i$ is built at the midpoint D_i of $B_i C_i$. The point can be calculated by

$$\mathbf{P}_{Di} = \mathcal{P} \left(\frac{\mathcal{P}^{-1} \mathbf{P}_{Bi} + \mathcal{P}^{-1} \mathbf{P}_{Ci}}{2} \right). \quad (89)$$

According to Figure 9, the flag $\mathbf{F}_{Di}$ represented in $\mathbf{F}_0$ is

$${}^0 \mathbf{F}_{Di} = \mathbf{P}_{Di} + \mathbf{P}_{Di} \wedge \mathbf{P}_{Bi} \wedge \mathbf{e}_\infty + \mathbf{\Pi}_i^*, \quad (90)$$

where the standard form of ${}^0 \mathbf{F}_{Di}$ can be obtained based on Eqs. (30)~(34). Moreover, the motor $\mathbf{M}_{0_Di}$ from $\mathbf{F}_0$ to $\mathbf{F}_{Di}$ can be calculated by Eq (21), and Eqs. (27)~(29). The point F_i , which is on the head of the actuators, can be represented in $\mathbf{F}_{Di}$ as

$${}^{Di} \mathbf{P}_{Fi} = \mathcal{P} \ d_{F1} \mathbf{e}_1 + d_{F2} \mathbf{e}_2, \quad (91)$$

and it can be transformed into $\mathbf{F}_0$ by

$$\mathbf{P}_{Fi} = \mathbf{M}_{0_Di} {}^{Di} \mathbf{P}_{Fi} \tilde{\mathbf{M}}_{0_Di}. \quad (92)$$

Moreover, the fixed points E_i represented in $\mathbf{F}_0$ are

$$\begin{aligned} \mathbf{P}_{E1} &= \mathbf{P} \ d_{E1} \mathbf{e}_2 - d_{E2} \mathbf{e}_3, \\ \mathbf{P}_{E2} &= \mathbf{R}_z \left(\frac{2\pi}{3} \right) \mathbf{P}_{E1} \tilde{\mathbf{R}}_z \left(\frac{2\pi}{3} \right), \\ \mathbf{P}_{E3} &= \mathbf{R}_z \left(\frac{2\pi}{3} \right) \mathbf{P}_{E2} \tilde{\mathbf{R}}_z \left(\frac{2\pi}{3} \right) \end{aligned} \quad (93)$$

respectively. Then the distance between the point $\mathbf{P}_{Fi}$ and $\mathbf{P}_{Ei}$ can be obtained:

$$q_i = \sqrt{-2\mathbf{P}_{Fi} \cdot \mathbf{P}_{Ei}}. \quad (94)$$

The points and flags of the mechanism at different configurations are drawn in Figure 13, which can indicate the correctness of the calculation methods above.

a) $z=1$ m, $\beta=0$ deg, and $\gamma=0$ deg

b) $z=1$ m, $\beta=15$ deg, and $\gamma=20$ deg

Figure 13 Two configurations of the mechanism with points and flags

5.2.2 Calculate the velocity of the actuators

The twist of the k -th rigid body of the i -th limb represented in $\mathbf{F}_0$ is donated as ${}^0 \mathbf{V}_{j_k,i}$, and they can be constructed as follows:

$$\begin{aligned} {}^0 \bar{\mathbf{V}}_{0_1,i} &= \mathbf{e}_\infty \mathcal{P}^{-1} \mathbf{P}_{Ai} \wedge \mathbf{\Pi}_i \mathbf{e}_{1,2,3} - \mathbf{\Pi}_i \mathbf{e}_{1,2,3}, \\ {}^0 \bar{\mathbf{V}}_{1_2,i} &= \mathbf{e}_\infty \mathcal{P}^{-1} \mathbf{P}_{Bi} \wedge \mathbf{\Pi}_i \mathbf{e}_{1,2,3} - \mathbf{\Pi}_i \mathbf{e}_{1,2,3}, \\ {}^0 \bar{\mathbf{V}}_{2_3,i} &= \mathbf{e}_\infty \mathcal{P}^{-1} \mathbf{P}_{Ci} \wedge \mathbf{e}_1 \mathbf{e}_{1,2,3} - \mathbf{e}_1 \mathbf{e}_{1,2,3}, \\ {}^0 \bar{\mathbf{V}}_{3_4,i} &= \mathbf{e}_\infty \mathcal{P}^{-1} \mathbf{P}_{Ci} \wedge \mathbf{e}_2 \mathbf{e}_{1,2,3} - \mathbf{e}_2 \mathbf{e}_{1,2,3}, \\ {}^0 \bar{\mathbf{V}}_{4_5,i} &= \mathbf{e}_\infty \mathcal{P}^{-1} \mathbf{P}_{Ci} \wedge \mathbf{e}_3 \mathbf{e}_{1,2,3} - \mathbf{e}_3 \mathbf{e}_{1,2,3}, \end{aligned} \quad (95)$$

where ${}^0\bar{\mathbf{V}}_{0-1,i}$ and ${}^0\bar{\mathbf{V}}_{1-2,i}$ are corresponding to the first and second revolute joints, besides, ${}^0\bar{\mathbf{V}}_{2-3,i}$, ${}^0\bar{\mathbf{V}}_{3-4,i}$, and ${}^0\bar{\mathbf{V}}_{4-5,i}$ are corresponding to the three revolute axes of the spherical joint. As these axes of the spherical joint is arbitrary, their directions can be set as $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ for simplicity. The Jacobian matrix of the i -th limb can then be obtained as

$$\dot{\mathbf{0}}_{0-5,i} = \mathbf{J}_{0-5,i}^{\pm} \mathcal{S} {}^0\bar{\mathbf{V}}_{0-H} \quad (96)$$

According to Eq. (71), the velocity of each rigid body $\mathbf{V}_{0-k,i}$ can be obtained.

By calculating the differential of Eq. (94) and considering $\dot{\mathbf{P}}_{Fi} = \mathbf{V}_{0-2,i} \times \mathbf{P}_{Fi}$, the actuator's velocity of each limb can be obtained

$$\dot{q}_i = -\frac{\mathbf{V}_{0-2,i} \times \mathbf{P}_{Fi} \cdot \mathbf{P}_{Ei}}{q_i} \quad (97)$$

5.2.3 Calculate the acceleration and jerk of the actuators

According to Eq. (69), the angular accelerations of the joints on each limb can be obtained as follows:

$$\ddot{\mathbf{0}}_{0-5,i} = \mathbf{J}_{0-5,i}^{\pm} \mathcal{S} \left(\dot{{}^0\bar{\mathbf{V}}_{0-H}} - \sum_{i=1}^4 \dot{\theta}_{i-1} {}^0\mathbf{V}_{0-i} \times {}^0\bar{\mathbf{V}}_{i-1} \right) \quad (98)$$

The acceleration $\dot{\mathbf{V}}_{0-k,i}$ of each rigid body can then be calculated based on the angular velocities and accelerations. Derivate Eq. (97) with respect to time, it follows

$$\ddot{q}_i = \frac{-\dot{q}_i^2 - \ddot{\mathbf{P}}_{Fi} \cdot \mathbf{P}_{Ei}}{q_i} \quad (99)$$

where $\ddot{\mathbf{P}}_{Fi} = \dot{\mathbf{V}}_{0-2,i} \times \mathbf{P}_{Fi} + \mathbf{V}_{0-2,i} \times \mathbf{V}_{0-2,i} \times \mathbf{P}_{Fi}$ according to Eq.(51).

According to Eq. (70), the angular jerk of the joints on each limb can be obtained by Eq.(105) in **Appendix**. Derivate Eq. (99) with respect to time, it follows

$$\ddot{\ddot{q}}_i = \frac{-3\dot{q}_i \ddot{q}_i - \ddot{\ddot{\mathbf{P}}}_{Fi} \cdot \mathbf{P}_{Ei}}{q_i} \quad (100)$$

where $\ddot{\ddot{\mathbf{P}}}_{Fi} = \ddot{\mathbf{V}}_{0-2,i} \times \mathbf{P}_{Fi} + 2\dot{\mathbf{V}}_{0-2,i} \times \mathbf{V}_{0-2,i} \times \mathbf{P}_{Fi} +$

$\mathbf{V}_{0-2,i} \times \dot{\mathbf{V}}_{0-2,i} \times \mathbf{P}_{Fi} + \mathbf{V}_{0-2,i} \times \mathbf{V}_{0-2,i} \times \mathbf{P}_{Fi}$ according to Eq. (52).

The lengths, velocities, accelerations, and jerks of the actuators under the motion laws of the output platform in

Table 3 are simulated and drawn in Figure 14, Figure 15, Figure 16 and Figure 17, respectively. Based on the derivation relationships between them, the algorithms above are verified.

Figure 14 Lengths of actuators

Figure 15 Velocities of actuators

Figure 16 Accelerations of actuators

Figure 17 Jerks of actuators

6 Conclusions

This paper mainly solves higher-order kinematics modeling of serial and parallel mechanisms based on CGA representations and calculations. In order to frame rigid body and transform geometric entities between different frames, the mathematical relationships between flag and motor are given. The higher-order kinematics of serial chain mechanisms based on the motors and screws in CGA is built. The calculation methods of the position, velocity, acceleration, and jerk of the 3-RRS parallel mechanism are given and their correctness is verified by the calculating examples. It shows that CGA is more concise in expression, formula derivation, and calculation when applied in the higher-order kinematics of the 3-RRS parallel mechanism. The methods have potential applications in real-time motion planning, control, and computer simulation of mechanisms.

7 Declaration

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Availability of data and materials

Authors' contributions

The author' contributions are as follows: Chang Wang wrote the manuscript; Tie-Shi Zhao was in charge of the

whole trial; Er-Wei Li assisted with the programming. Yan-Zhi Zhao, Hui Bian and Ming-Chao Geng assisted with the basis theory of CGA.

Competing interests

The authors declare no competing financial interests.

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Not applicable

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Appendix

The expressions of A_x , A_y , A_z , and B_x are listed as follows:

$$B_x = s \alpha + \gamma + \beta - s \alpha - \gamma + \beta - s \alpha + \gamma - \beta + s \alpha - \gamma - \beta - 2c \alpha - \gamma - 2c \alpha + \gamma - 2c \alpha - \beta - 2c \alpha + \beta . (101)$$

$$\begin{aligned} A_x = & s \alpha + \gamma + \beta \dot{\gamma} c \alpha - \beta + s \alpha + \gamma + \beta \dot{\gamma} c \alpha + \beta + 2s \alpha + \gamma + \beta \dot{\beta} c \alpha - \beta + \\ & s \alpha - \gamma + \beta \dot{\gamma} c \alpha - \beta + s \alpha - \gamma + \beta \dot{\gamma} c \alpha + \beta - 2s \alpha - \gamma + \beta \dot{\beta} c \alpha - \beta - \\ & s \alpha + \gamma - \beta \dot{\gamma} c \alpha - \beta - s \alpha + \gamma - \beta \dot{\gamma} c \alpha + \beta + 2s \alpha + \gamma - \beta \dot{\beta} c \alpha + \beta - \\ & s \alpha - \gamma - \beta \dot{\gamma} c \alpha - \beta - s \alpha - \gamma - \beta \dot{\gamma} c \alpha + \beta - 2s \alpha - \gamma - \beta \dot{\beta} c \alpha + \beta + \\ & 2c \alpha - \gamma \dot{\gamma} c \alpha - \beta + 2c \alpha - \gamma \dot{\gamma} c \alpha + \beta - 2c \alpha - \gamma \dot{\beta} c \alpha - \beta + \\ & 2c (\alpha - \gamma) \dot{\beta} c (\alpha + \beta) - 2c (\alpha + \gamma) \dot{\beta} c (\alpha - \beta) - 2c (\alpha + \gamma) \dot{\beta} c (\alpha + \beta) - \\ & 2c (\alpha + \gamma) \dot{\beta} c (\alpha - \beta) - 2c (\alpha + \gamma) \dot{\beta} c (\alpha + \beta) \end{aligned} . (102)$$

$$\begin{aligned} A_z = & s \alpha + \gamma + \beta \dot{\gamma} + s \alpha + \gamma + \beta \dot{\beta} + s \alpha - \gamma + \beta \dot{\gamma} - s \alpha - \gamma + \beta \dot{\beta} - s \alpha + \gamma - \beta \dot{\gamma} + \\ & s \alpha + \gamma - \beta \dot{\beta} - s \alpha - \gamma - \beta \dot{\gamma} - s \alpha - \gamma - \beta \dot{\beta} + 2c \alpha - \gamma \dot{\gamma} - 2c \alpha + \gamma \dot{\gamma} + \\ & 2\dot{\beta} c \alpha - \beta - 2\dot{\beta} c \alpha + \beta \end{aligned} . (103)$$

The expressions of the velocity of output platform V_{0-H} are listed as follows:

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$$\begin{aligned}
V_{0_H} &= e_\infty \dot{x}e_1 + \dot{y}e_2 + \dot{z}e_3 + T_{0,H} -\dot{\alpha}e_{1,2} \tilde{T}_{0,H} + T_{0,H} R_z \alpha \beta e_{1,3} \tilde{R}_z \alpha \tilde{T}_{0,H} \\
&+ T_{0,H} R_z \alpha R_y \beta -\dot{\gamma}e_{2,3} \tilde{R}_y \beta \tilde{R}_z \alpha \tilde{T}_{0,H} \\
&= -\dot{\alpha} + s\beta\dot{\gamma} e_{1,2} + \left(\frac{1}{2}\dot{\gamma}s \alpha - \beta + \frac{1}{2}\dot{\gamma}s \alpha + \beta + \dot{\beta}c\alpha \right) e_{1,3} + \left(-\frac{1}{2}c \alpha - \beta \dot{\gamma} - \frac{1}{2}c \alpha + \beta \dot{\gamma} + \dot{\beta}s\alpha \right) e_{2,3} + \\
&\left(\dot{\beta}zc\alpha - y\dot{\alpha} + y\dot{\gamma}s\beta + \frac{1}{2}z\dot{\gamma}s \alpha + \beta + \frac{1}{2}z\dot{\gamma}s (\alpha - \beta) \dot{x} \right) e_{1,\infty} + \left(\dot{\beta}zs\alpha + x\dot{\alpha} - x\dot{\gamma}s\beta - \frac{1}{2}z\dot{\gamma}c (\alpha - \beta) \dot{y} + \frac{1}{2}z\dot{\gamma}c (\alpha + \beta) \dot{y} \right) e_{2,\infty} + \\
&\left(-\dot{\beta}ys\alpha + \frac{1}{2}y\dot{\gamma}c (\alpha - \beta) \dot{x} + \frac{1}{2}y\dot{\gamma}c (\alpha + \beta) \dot{x} - \frac{1}{2}x\dot{\gamma}s (\alpha + \beta) \dot{y} - \frac{1}{2}x\dot{\gamma}s (\alpha - \beta) \dot{y} + \dot{\beta}xc\alpha - \dot{z} \right) e_{3,\infty}
\end{aligned} \tag{104}$$

The angular velocities of joints from limb i :

$$\ddot{\theta}_{0_5,i} = J_{0_5,i}^\pm S \left({}^0\dot{\bar{V}}_{0_H} - 2 \sum_{i=1}^{n-1} \ddot{\theta}_{i_i+1} {}^0V_{0_i} \times {}^0\bar{V}_{i_i+1} - \sum_{i=1}^{n-1} \dot{\theta}_{i_i+1} {}^0\dot{V}_{0_i} \times {}^0\bar{V}_{i_i+1} + {}^0V_{0_i} \times {}^0V_{0_i} \times {}^0\bar{V}_{i_i+1} \right). \tag{105}$$

Figures

Figure 1

Rotation

Figure 2

Translation

Figure 3

Screw motion

Figure 4

Rigid body framing by flags

Figure 5

A screw in different frames

Figure 6

Single joint kinematics

Figure 7

Kinematics of serial manipulator

Figure 8

The 3-RRS mechanism

Figure 9

The geometry parameters of the i-th limb and actuator

Figure 10

Six DoF motions of the output platform

Figure 11

The trace of the flag attached on the output platform from 0 s to 10 s

Figure 12

Velocity of the output platform

a) $z = 1$ m, $\beta = 0$ deg, and $\gamma = 0$ deg

b) $z = 1$ m, $\beta = 15$ deg, and $\gamma = 20$ deg

Figure 13

Two configurations of the mechanism with points and flags

Figure 14

Lengths of actuators

Figure 15

Velocities of actuators

Figure 16

Accelerations of actuators

Figure 17

Jerks of actuators