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Research Article

Keywords: Thermodynamic Uncertainty Relations (TUR), integrated currents, irreversible dissipative processes, small scale machines

Posted Date: November 11th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-1024620/v1>

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Absence of Thermodynamic Uncertainty Relations with Asymmetric Dynamic Protocols

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Abstract

Many versions of Thermodynamic Uncertainty Relations (TUR) have recently been discovered, which impose lower bounds on relative fluctuations of integrated currents in irreversible dissipative processes, and suggest that there may be fundamental limitations on the precision of small scale machines and heat engines. In this work we rigorously demonstrate that TUR can be evaded by using dynamic protocols that are asymmetric under time-reversal. We illustrate our results using a model heat engine using two-level systems, and also discuss heuristically the fundamental connections between TUR and time-reversal symmetry.

PACS numbers:

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I. INTRODUCTION

In recent years, several classes of inequalities have been discovered, which indicate trade-offs between various aspects of non-equilibrium processes. For example, many versions of thermodynamic uncertainty relations (TUR) [1–22][23] impose lower bounds of integrated current fluctuations in terms of entropy production, which signify a trade-off between dissipation and precision for small-scale machines. Another example is the so-called speed-limit inequalities [24–31], which indicate a trade-off between efficiency and power output for general Markov processes. Most recently, Dechant and Sasa [32] derived an upper bound for non-equilibrium response function in terms of fluctuations and relative entropy between the perturbed and reference states, which generated immediate interests [21, 33, 34]. Given the large number of works published in recent years, it is highly desirable and urgent to understand whether these inequalities are consequences of more fundamental aspects of non-equilibrium physics, such as Fluctuation Theorems and Markovian property, or rather depend on specific system details.

The current situation of TUR is particularly vibrant and complex [5]. The first version of TUR, which was proposed [1] and proved [2] for continuous time Markov jump processes with local detailed balance, has the form: $\langle \delta Q^2 \rangle / \langle Q \rangle^2 \leq 2 / \langle \Sigma \rangle$, where Q is certain integrated current, and $\langle \Sigma \rangle$ the average entropy production. It was very quickly generalized to other types of irreversible processes, including finite-time processes [11, 14, 15, 22] discrete time processes [17, 18], and most recently quantum processes [13, 21]. Its relation to the linear response theory has also been clarified [19]. More recently, Timpanaro *et. al.* [6] and Hasegawa *et. al.* [7] proved tighter TUR bounds in the form of $\langle \delta Q^2 \rangle / \langle Q \rangle^2 \leq f(\langle \Sigma \rangle)$ for general processes with symmetric dynamic protocols. For small $\langle \Sigma \rangle$, $f(\langle \Sigma \rangle) \sim 2 / \langle \Sigma \rangle$, so that the original TUR is restored. For large $\langle \Sigma \rangle$, $f(\langle \Sigma \rangle)$ vanishes exponentially. (Note that in the limit of large entropy production $\Sigma \rightarrow \infty$, both the original TUR bound in Refs. [1, 2] and the generalized bounds in Refs. [6, 7] vanish. This is completely expected, since large $\langle \Sigma \rangle$ means either large system size or long time. Either way, the law of large number come to play, so relative fluctuation of entropy production always reduces to zero. In another word, various TUR bounds discovered for symmetric protocols are effective only for small size systems in short time.) These generalized TUR has their origin in Fluctuation Theorem [35–37], and hence are universal and independent of model details. The deep connection

between fluctuation of entropy production and Fluctuation Theorem was previously studied by Merhav and Kafri [38].

While almost all previous works concern processes with dynamic protocols that are symmetric under time-reversal, employment of asymmetric dynamic protocol provides another dimension for the issue of TUR. Results about asymmetric dynamic protocols are scarce, but did indicate a very different scenario. Barrato and Seifert [39] showed that Brownian clock driven periodically can achieve arbitrary high precision with arbitrary low dissipation. Chun *et. al.* [40] showed that TUR may not work for system coupled to a magnetic field, which explicitly breaks the time-reversal symmetry. More recently, Cangemi *et. al.* [41, 42] discovered violation of TUR in quantum energy converters with asymmetric protocols. In light of these results, it is highly desirable to know whether in principle TUR-bound can always be evaded via clever design of asymmetric dynamic protocols. This question, which is independent of model details, is of fundamental importance for study of small scale machines [43–49].

In this letter, we shall rigorously establish the absence of TUR-bound for processes with asymmetric dynamic protocols. We shall minimize the relative current fluctuations with the constraints of (1) fixed entropy production and (2) satisfaction of Detailed Fluctuation Theorem, which is valid for processes with asymmetric protocols, and show that the lower bound for the current fluctuations is strictly zero. Hence all TUR-bounds can be evaded via clever designs of asymmetric dynamic protocols. To illustrate our results, we also discuss a model of heat engine whose output fluctuation can be tuned arbitrarily small. Finally we present a heuristic discussion in terms of Detailed Fluctuation Theorem why TUR-bounds exist in processes with symmetric dynamic protocols but not in those with asymmetric dynamic protocols. Our results provide important insights for design of high precision microscopic machines and heat engines.

II. FLUCTUATION THEOREM

Consider an irreversible stochastic process with path probability distribution $P_U[\gamma]$, where $U = \{\lambda(t), t \in (0, T)\}$ denotes the dynamic protocol, and $\lambda(t)$ is the time-dependent external parameter, and $(0, T)$ is the time-range of the process. Associated with this process is the backward process, with a reversed protocol $\bar{U} = \{\bar{\lambda}(t) = \lambda^*(-t), t \in (-T, 0)\}$ and

a path probability distribution $P_U[\gamma]$. Note that λ^* is related to λ via reversal of odd parameters, such as magnetic field. Assuming that there is separation of time-scale, and that all slow variables included in the model, the entropy production $\Sigma[\gamma]$ of the forward process along the path γ satisfies [52, 53]

$$P_U[\gamma] = P_{\bar{U}}[\bar{\gamma}] e^{\Sigma[\gamma]}, \quad (2.1)$$

which automatically implies $\Sigma[\gamma] = -\bar{\Sigma}[\bar{\gamma}]$, i.e., entropy production changes sign under the reversal of both the path and the dynamic protocol. Here $\bar{\gamma}$ is the time-reversal of the path forward γ , which is obtained from γ via reversal of both time and odd variables. Equation (2.1) has been established for classical systems on very general ground [52, 53]. It is also known to hold for certain quantum systems [54]. We shall be interested in asymmetric protocols i.e., $U \neq \bar{U}$, which means that the forward and backward processes are macroscopically different.

Now consider a certain integrated current $Q[\gamma]$ as a functional of path, which is odd under time-reversal, i.e., $\bar{Q}[\bar{\gamma}] = -Q[\gamma]$. It may be the amount of matter transported or heat conducted along the path γ . The joint probability density functions (pdf) of $\Sigma[\gamma]$ and $Q[\gamma]$ for the forward and backward processes are

$$p(\sigma, q) = \sum_{\gamma} p_U[\gamma] \delta(\sigma - \Sigma[\gamma]) \delta(q - Q[\gamma]), \quad (2.2)$$

$$\bar{p}(\sigma, q) = \sum_{\gamma} p_{\bar{U}}[\gamma] \delta(\sigma - \bar{\Sigma}[\gamma]) \delta(q - \bar{Q}[\gamma]). \quad (2.3)$$

Generalizing a theorem due to van der Broeck and Cleuren [55], we can readily obtain a generalized version of Detailed Fluctuation Theorem (DFT):

$$p(\sigma, q) = \bar{p}(-\sigma, -q) e^{\sigma}. \quad (2.4)$$

All statistical properties of Σ and Q for the forward and backward processes can be obtained from $p(\sigma, q)$ and $\bar{p}(\sigma, q)$. If the dynamic protocol is symmetric, $\bar{U} = U$ and $\bar{p} = p$, and Eq. (2.4) reduces to:

$$p(\sigma, q) = p(-\sigma, -q) e^{\sigma}, \quad (2.5)$$

which was the starting point of studies in Ref. [6, 7]. Our aim is to minimize the variances $\langle \delta Q^2 \rangle$ and $\langle \delta \Sigma^2 \rangle$ under the constraint of fixed averages $\langle \Sigma \rangle$ and $\langle Q \rangle$, for all distributions that satisfying Eq. (2.4).

III. MINIMIZATION OF FLUCTUATIONS

In this section we will show that DFT (2.4) implies no TUR bound for fluctuations. Any continuous probability distribution can be approximated, to an arbitrary precision, by a discrete distribution. Hence we only need to study discrete distributions. We shall use the term *N-point distribution* to denote a probability distribution in the form of $p_N(\sigma, q) = \sum_i^N p_i \delta(\sigma - \sigma_i) \delta(q - q_i)$. Let $\{p_N(\sigma, q), \bar{p}_N(-\sigma, -q)\}$ be a pair of *N-point distributions* for the forward and backward processes satisfying Eq. (2.4). As shown in detail in Supplementary Informations (SI), for any $N \geq 3$, we can always construct a pair of $(N - 1)$ -point distributions $\{p_{N-1}(\sigma, q), \bar{p}_{N-1}(-\sigma, -q)\}$ which also satisfy Eq. (2.4) but has equal averages $\langle \Sigma \rangle$ and $\langle Q \rangle$ and smaller fluctuations $\langle \delta Q^2 \rangle$ and $\langle \delta \Sigma^2 \rangle$. Repeating this process, we eventually arrive at the following lemma:

Lemma 1 *For any pair of N-point distributions $p_N(\sigma, q), \bar{p}_N(\sigma, q)$ obeying DFT Eq. (2.4), we can always find a pair of 2-point distributions $p_2(\sigma, q), \bar{p}_2(\sigma, q)$ such that, comparing with $p_N(\sigma, q), \bar{p}_N(\sigma, q)$, $p_2(\sigma, q)$ has same averages $\langle \Sigma \rangle, \langle Q \rangle$ and smaller variances $\langle \delta \Sigma^2 \rangle$ and $\langle \delta Q^2 \rangle$.*

The detailed proof of this lemma is shown in SI.

Let us further try to find the pair $p_2(\sigma, q), \bar{p}_2(\sigma, q)$ that produces the minimal fluctuations $\langle \delta \Sigma^2 \rangle$ and $\langle \delta Q^2 \rangle$. We shall first treat the simple case $Q = \Sigma$, so that we only need to minimize $\langle \delta \Sigma^2 \rangle$ with fixed $\langle \Sigma \rangle$. Let us write:

$$p_2(\sigma) = p \delta(\sigma - \sigma_1) + (1 - p) \delta(\sigma - \sigma_2), \quad (3.1a)$$

$$\bar{p}_2(\sigma) = p e^{-\sigma_1} \delta(\sigma + \sigma_1) + (1 - p) e^{-\sigma_2} \delta(\sigma - \sigma_2), \quad (3.1b)$$

which satisfy Eq. (2.4). Whereas $p(\sigma)$ is already normalized, normalization of $\bar{p}(\sigma)$ fixes p in terms of σ_1, σ_2 :

$$p = \frac{1 - e^{-\sigma_2}}{e^{-\sigma_1} - e^{-\sigma_2}}. \quad (3.1c)$$

Using Eqs. (3.1) we find averages and fluctuations of Σ for the forward and backward pro-

cesses:

$$\langle \Sigma \rangle = \frac{\sigma_2(e^{-\sigma_1} - 1) + \sigma_1(1 - e^{-\sigma_2})}{e^{-\sigma_1} - e^{-\sigma_2}}, \quad (3.2a)$$

$$\langle \delta \Sigma^2 \rangle = \frac{(\sigma_1 - \sigma_2)^2 (e^{-\sigma_1} - 1)(1 - e^{-\sigma_2})}{(e^{-\sigma_1} - e^{-\sigma_2})^2}, \quad (3.2b)$$

$$\overline{\langle \Sigma \rangle} = \frac{-\sigma_1 e^{-\sigma_1} (1 - e^{-\sigma_2}) - \sigma_2 e^{-\sigma_2} (e^{-\sigma_1} - 1)}{e^{-\sigma_1} - e^{-\sigma_2}}, \quad (3.2c)$$

$$\overline{\langle \delta \Sigma^2 \rangle} = \frac{(\sigma_1 - \sigma_2)^2 (e^{\sigma_2} - 1)(1 - e^{\sigma_1})}{(e^{\sigma_1} - e^{\sigma_2})^2}. \quad (3.2d)$$

If $\sigma_1 = 0$ ($\sigma_2 = 0$), $p = 1$ ($p = 0$), the entropy production is always zero, which corresponds to a reversible process. This is not what we aim to study in this work, hence we shall assume that neither of σ_1, σ_2 vanish. From Eq. (3.1c) we see to make p or $1 - p$ both positive, σ_1, σ_2 must have different signs. Let us assume $\sigma_1 < 0 < \sigma_2$.

Let us now fix $\sigma_2 > 0$, and make $|\sigma_1| = -\sigma_1 \gg 1$, hence $e^{-\sigma_1} \gg 1$, and $p \sim e^{\sigma_1} = e^{-|\sigma_1|} \ll 1$, according to Eq. (3.1c). From Eqs. (3.2) we obtain the following asymptotics:

$$\langle \Sigma \rangle \sim \sigma_2, \quad (3.3a)$$

$$\langle \delta \Sigma^2 \rangle \sim \sigma_1^2 (1 - e^{-\sigma_2}) e^{\sigma_1} \rightarrow 0, \quad (3.3b)$$

$$\langle \delta \Sigma^2 \rangle / \langle \Sigma \rangle^2 \sim \sigma_2^{-2} \sigma_1^2 (1 - e^{-\sigma_2}) e^{\sigma_1} \rightarrow 0. \quad (3.3c)$$

Hence in the forward process, $\langle \Sigma \rangle$ converges to a finite limit, whereas $\langle \delta \Sigma^2 \rangle$ becomes exponentially small in σ_1 . Similarly for the backward process we obtain

$$\overline{\langle \Sigma \rangle} \sim -\sigma_1 + \sigma_1 e^{-\sigma_2} \rightarrow \infty, \quad (3.3d)$$

$$\overline{\langle \delta \Sigma^2 \rangle} \sim \sigma_1^2 e^{-2\sigma_2} (e^{\sigma_2} - 1) \rightarrow \infty, \quad (3.3e)$$

$$\overline{\langle \delta \Sigma^2 \rangle} / \overline{\langle \Sigma \rangle}^2 \sim 1 / (e^{\sigma_2} - 1). \quad (3.3f)$$

Hence in the backward process, both $\langle \Sigma \rangle$ and $\langle \delta \Sigma^2 \rangle$ diverge, whereas $\overline{\langle \delta \Sigma^2 \rangle} / \overline{\langle \Sigma \rangle}^2$ converges to a finite limit.

Let us now consider the general case where Q is an integrated current distinct from entropy production. The joint distributions of Σ and Q have the following forms:

$$\begin{aligned} p_2(\sigma, q) &= p \delta(\sigma - \sigma_1) \delta(q - q_1) \\ &+ (1 - p) \delta(\sigma - \sigma_2) \delta(q - q_2), \end{aligned} \quad (3.4a)$$

$$\begin{aligned} \bar{p}_2(\sigma, q) &= p e^{-\sigma_1} \delta(\sigma + \sigma_1) \delta(q + q_1) \\ &+ (1 - p) e^{-\sigma_2} \delta(\sigma + \sigma_2) \delta(q + q_2), \end{aligned} \quad (3.4b)$$

where p is still given by Eq. (3.1c). Whilst the average entropy productions are still given by Eqs. (3.2), the average and variance of Q can be calculated using Eqs. (3.4). Recall that we fix $\sigma_2 > 0$ and let $\sigma_1 \rightarrow -\infty$. In this limit, we have for the forward process:

$$\langle \delta Q^2 \rangle \sim (q_1 - q_2)^2 e^{\sigma_1} (1 - e^{-\sigma_2}), \quad (3.5a)$$

$$\langle Q \rangle \sim q_2. \quad (3.5b)$$

For all reasonable physical models, we expect that entropy production scales quadratically (or at least polynomially) with the current. Hence q_1 diverges with σ_1 , whereas q_2 remains finite. From Eqs. (3.5) we find that $\langle \delta Q^2 \rangle$ converges to zero, whereas $\langle Q \rangle$ remains fixed. The relative fluctuation of Q converges to zero:

$$\langle \delta Q^2 \rangle / \langle Q \rangle^2 \sim q_2^{-2} q_1^2 (1 - e^{-\sigma_2}) e^{\sigma_1} \rightarrow 0. \quad (3.5c)$$

For the backward process we have

$$\overline{\langle \delta Q^2 \rangle} \sim (q_1 - q_2)^2 e^{-\sigma_2} (1 - e^{-\sigma_2}), \quad (3.5d)$$

$$\overline{\langle Q \rangle} \sim -(1 - e^{-\sigma_2}) q_1 - e^{-\sigma_2} q_2. \quad (3.5e)$$

hence both $\overline{\langle \delta Q^2 \rangle}$ and $\overline{\langle Q \rangle}$ diverge with q_1 . The relative fluctuation of Q converges to the same limit as in Eq. (3.3f):

$$\overline{\langle \delta Q^2 \rangle} / \overline{\langle Q \rangle}^2 \sim 1 / (e^{\sigma_2} - 1), \quad (3.5f)$$

Naturally one may wonder whether it is possible to make current fluctuations small for both the forward and backward process. This is impossible. In Refs. [8, 9] the following inequality was derived from DFT (2.4):

$$\frac{\langle \delta Q^2 \rangle + \overline{\langle \delta Q^2 \rangle}}{(\langle Q \rangle + \overline{\langle Q \rangle})^2} \geq \exp - \frac{1}{2} [\langle \Sigma \rangle + \overline{\langle \Sigma \rangle}]. \quad (3.6)$$

If $\langle Q \rangle, \overline{\langle Q \rangle}, \langle \Sigma \rangle, \overline{\langle \Sigma \rangle}$ are all finite, and $\langle \delta Q^2 \rangle, \overline{\langle \delta Q^2 \rangle}$ are very small, Eq. (3.6) must be violated. On the other hand, for the two-point distributions we studied above, using Eqs. (3.5), we see that the LHS of Eq. (3.6) converges to $1 / (e^{\sigma_2} - 1) > 0$, whereas RHS of Eq. (3.6) converges to zero. Hence the inequality (3.6) is not tight.

These results show unambiguously that there is no TUR bound for entropy production as long as asymmetric dynamic protocols are allowed. In Ref. [56], Brandner et. al. studied an irreversible process with magnetic field and certain dynamic protocol, and discovered a

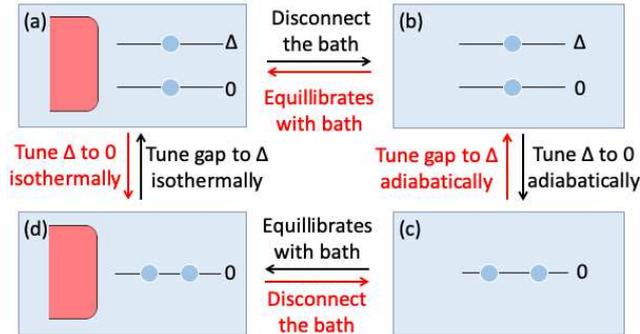


Figure 1: Schematics of the qubit model. The red regions denote heat baths. Black arrows and red arrows denote respectively forward and backward dynamic protocols. Two energy levels are degenerate in (c) and (d).

non-vanishing TUR bound for current fluctuations. This does not contradict our theory. Instead our theory means that the TUR bound discovered in Ref. [56] can be evaded if more sophisticated protocols are used.

IV. A MODEL HEAT ENGINE

We shall now discuss a concrete irreversible process with finite $\langle \Sigma \rangle$ and vanishingly small current fluctuations. Consider a two-level system with a Hamiltonian

$$\hat{H}(\Delta) = \Delta \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Delta > 0, \quad (4.1)$$

where the energy of the excited state Δ that can be tuned externally. We shall construct an irreversible cycle that starts from a Gibbs state $\hat{\rho}(\Delta) = Z^{-1}e^{-\beta\hat{H}(\Delta)}$, illustrated by panel (a) of Fig. 1 (top left), which can be obtained by connecting the system to a heat bath with temperature $T = 1/\beta$. As illustrated by the black arrows in Fig. 1, the cycle consists of the following four steps (a) $\xrightarrow{1}$ (b) $\xrightarrow{2}$ (c) $\xrightarrow{3}$ (d) $\xrightarrow{4}$ (a):

1. *Disconnection* of the system from the heat bath.
2. *Adiabatic process*: We adiabatically tune the parameter Δ to 0, during which the system remains in the state $\hat{\rho} = Z^{-1}e^{-\beta\hat{H}(\Delta)}$.
3. *Equilibration process*: We reconnect the system to the bath, and the system re-equilibrates to a new Gibbs state $\hat{\rho} = Z^{-1}e^{-\beta\hat{H}(0)} = \hat{\mathbf{I}}/2$. This step is irreversible

process with an entropy increase.

4. *Isothermal process*: With the system connected to the bath, the gap is tuned isotatically back to Δ so that the density matrix returns to the initial state $\hat{\rho}(\Delta) = Z^{-1}e^{-\beta\hat{H}(\Delta)}$.

In stages (a) and (d), the system is in the relevant Gibbs-Boltzmann state. In stages (b) and (c), the system is either at the excited level (with probability $p = e^{-\beta\Delta}/(1 + e^{-\beta\Delta})$) or at the ground level (with probability $1 - p = 1/(1 + e^{-\beta\Delta})$). There are two paths for the adiabatic process, $\gamma_1 = \{\text{Gibbs} \rightarrow \text{excited} \rightarrow \text{excited} \rightarrow \text{Gibbs}\}$ with probabilities p , and $\gamma_2 = \{\text{Gibbs} \rightarrow \text{ground} \rightarrow \text{ground} \rightarrow \text{Gibbs}\}$, with probabilities $1 - p$. These two paths merge in stages (a) and (d).

The only integrated current in this problem is the entropy production. Since the process is cyclic, the entropy production is related to the work done by the external force via $\Sigma = \beta W$. Let us consider the adiabatic process. Along the path γ_1 , the system remains in the excited state, and the external force does work $-\Delta$. Along the path γ_2 , the work is identically zero. During the isothermal process, two path coincide, and the work done equals to the change of system free energy, which is $T \log 2/(1 + e^{-\beta\Delta})$. Hence the total works along two paths are respectively:

$$W[\gamma_1] = -\Delta + T \log \frac{2}{1 + e^{-\beta\Delta}} = T \log(2p), \quad (4.2a)$$

$$W[\gamma_2] = T \log \frac{2}{1 + e^{-\beta\Delta}} = T \log 2(1 - p). \quad (4.2b)$$

Hence the path probabilities and the entropy production of γ_1, γ_2 are respectively

$$p_1 = p = \frac{e^{-\beta\Delta}}{1 + e^{-\beta\Delta}}, \quad p_2 = 1 - p, \quad (4.3a)$$

$$\sigma_1 = \log(2p), \quad \sigma_2 = \log 2(1 - p). \quad (4.3b)$$

This is a special case of Eqs. (3.1), with only one independent parameter p (or equivalently $\beta\Delta$).

The backward process is obtained by reversing the dynamic protocols, shown as all red arrows in Fig. 1. In the backward process, the system starts from the equilibrium state $\hat{\rho} = Z^{-1}e^{-\beta\hat{H}(\Delta)}$ (a), and goes through consecutively isothermal, disconnection, and adiabatic, and connection steps. Note the reversed operations of disconnection/connection with the bath is connection/disconnection with the bath. For example, in the forward process,

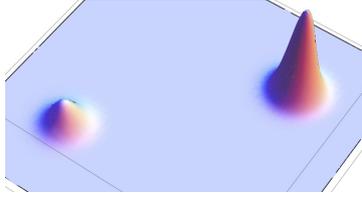


Figure 2: A cartoon of pdf $p(\sigma, q)$ for processes with symmetric dynamic protocol. The high peak in the right is centered at (q_0, σ_0) , whereas the mirror peak in the left, which is dictated by the Detailed Fluctuation Theorem, is located at $(-q_0, -\sigma_0)$. The mirror peak can be diminished by using asymmetric dynamic protocols, thereby evading TUR bounds.

the system gets disconnected from the bath with $\Delta > 0$, whereas in the backward process, disconnection happens when $\Delta = 0$. This shows explicitly that the protocol is indeed asymmetric under time-reversal. The two paths diverge from each other during the adiabatic process (c) \rightarrow (b). In the stage (c), two levels are degenerate, and the system is in each of them with probability $1/2$. The works done by the external force during each path in the backward process are the negative of those in the forward process. Hence we have for the backward process:

$$\bar{p}_1 = \frac{1}{2}, \quad \bar{p}_2 = \frac{1}{2}, \quad (4.3c)$$

$$\bar{\sigma}_1 = -\log(2p), \quad \bar{\sigma}_2 = -\log 2(1-p). \quad (4.3d)$$

We easily verify that Eqs. (4.3) satisfy the entropy production formula, Eq. (2.1).

As we make $\beta\Delta$ large, $p \rightarrow 0$, and $\sigma_1 \rightarrow 0$, $\sigma_2 \rightarrow \log 2$, and we recover all the asymptotics in Eqs. (3.3). More concretely, in the forward process, $\langle \Sigma \rangle \rightarrow \log 2$, and $\langle \delta\Sigma^2 \rangle \sim (\beta\Delta)^2 e^{-\beta\Delta} \rightarrow 0$, whereas in the backward process, we have $\overline{\langle \Sigma \rangle} \rightarrow \beta\Delta/2$, and $\overline{\langle \delta\Sigma^2 \rangle} \sim (\beta\Delta)^2/2$. The relative fluctuation of entropy production of the backward process converges to a finite limit $\overline{\langle \delta\Sigma^2 \rangle} / \overline{\langle \Sigma \rangle}^2 \rightarrow 2$.

V. HEURISTIC DISCUSSION

We conclude our work with a heuristic discussion about TUR bounds in general irreversible processes. We aim, via tuning of dynamic protocols, to reduce the fluctuations of Σ and Q , with their averages fixed and finite. The distribution $p(\sigma, q)$ then should have significant values only near a single point (σ_0, q_0) . Everywhere else $p(\sigma, q)$ must be negligibly small. However, if the dynamic protocol is symmetric under time-reversal, Eq. (2.5)

must be respected, according to which there is a peak at the mirror point $(-\sigma_0, -q_0)$, as illustrated in Fig. 2. Equation (2.5) also dictates that the probabilities of the peak and of the mirror peak are respectively $1/(1 + e^{-\sigma_0})$ and $1/(1 + e^{\sigma_0})$. Using these probabilities one easily obtain $\langle \Sigma \rangle = \sigma_0 \tanh(\sigma_0/2)$ and $\langle \delta Q^2 \rangle / \langle Q \rangle^2 = 2 \operatorname{csch}^2(\sigma_0/2)$. The latter is in fact exactly the sharp TUR bound obtained recently by Timpanaro *et. al.* [6]. For processes with asymmetric dynamic protocols, however, the relevant Detailed Fluctuation Theorem is Eq. (2.4) rather than Eq. (2.5). But Eq. (2.4) does not imposes any constraint on $p(\sigma, q)$. Instead, it determines $\bar{p}(\sigma, q)$ in terms of $p(\sigma, q)$. Hence by using asymmetric protocols, the mirror peak can be arbitrarily diminished, and the TUR bound can be evaded, with average entropy production finite. Note that this heuristic discussion is applicable to models with continuous distributions of dynamic paths as well.

VI. ACKNOWLEDGEMENT

X.X. acknowledges support from NSFC via grant #11674217, as well as additional support from a Shanghai Talent Program. This research is also supported by Shanghai Municipal Science and Technology Major Project (Grant No.2019SHZDZX01).

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