Maximum $\log_q$ Likelihood Estimation for Parameters of Weibull Distribution and Properties: Monte Carlo Simulation

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Abstract

The maximum $\log_q$ likelihood estimation method is a generalization of the known maximum log likelihood method to overcome the problem for modeling non-identical observations (inliers and outliers). The parameter q is a tuning constant to manage the modeling capability. Weibull is a flexible and popular distribution for problems in engineering. In this study, this method is used to estimate the parameters of Weibull distribution when non-identical observations exist. Since the main idea is based on modeling capability of objective function $\rho(x; \theta) = \log_q [f(x; \theta)]$, we observe that the finiteness of score functions cannot play a role in the robust estimation for inliers. The properties of Weibull distribution are examined. In the numerical experiment, the parameters of Weibull distribution are estimated by $\log_q$ and its special form, log, likelihood methods if the different designs of contamination into underlying Weibull distribution are applied. The optimization is performed via genetic algorithm. The modeling competence of $\rho(x; \theta)$ and insensitiveness to non-identical observations are observed by Monte Carlo simulation. The value of q can be chosen by use of the mean squared error in simulation and the p -value of Kolmogorov-Smirnov test statistic used for evaluation of fitting competence. Thus, we can overcome the problem about determining of the value of q for real data sets.

Keywords. Weibull distribution; inference; q -deformed logarithm; robustness.

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1 Introduction

After the study of a real-world phenomenon or the realization of an experiment, it may be desirable to model the experimental data by means of a proposed parametric model $f(x; \theta)$. In other words, the experimental data set is assumed to be a member of a parametric model. However, it cannot be a realistic assumption for the real world which will be modeled only with certain values of parameters in a model. The observations can be mixed with a different parameter of the same distribution or a different distribution. That is, a contamination exists into majority of the distribution, which leads to have non-identical observations. The type of contamination is defined as deviant observations. In other words, one or more observations are made to behave differently from what it is present when creating a deviant observation. Deviant observations can be divided into inlier and outlier deviations. The data set may have both inward/inlier and outward/outlier deviations at the same time. Inward deviations are generated by short-tailed distributions and outward deviations by thick-tailed distributions. For inward deviations, it can also be realized by generating random numbers from uniform distribution in the closed range $[a, b]$. In fact, deviant observations in the data set indicate that an assumption trusting on identically distributed random variables will not be realistic to model a phenomena (Tiku 1975; Lehmann and Casella 2006). If the assumption showing that the data set includes identically distributed random observations is violated, robust estimation method for the parameters of model $f(x; \theta)$ have been applied by use of the different objective functions. Robust methods trust the used objective function (Hampel et al. 2011). Deformed algebras such as Tsallis and Kaniakadis statistics are important to derive a neighborhood of a parametric model in order to overcome the problem which will occur when the assumption for identity is violated (Wada and Suyari 2007; Bercher 2011, 2012).

The origin of robust estimation method was started by biologist at 18th century (Hampel et al. 2011). The main working principle is based on the estimating equations (EEs) (Godambe 1960; Godambe and Thompson 1978). EEs can be derived by use of maximum composite likelihood estimation method. Tsallis q -entropy is creator of deformed logarithm ($\log_q$). In the direction of estimation method, $\log_q$ from Tsallis q -entropy has been studied recently. For this aim, maximum $\log_q$ likelihood estimation (MLqE) method are studied for outward observations. MLqE which is a generalization of the maximum likelihood (MLE) method Lehmann and Casella 2006 is used to obtain robust and also efficient estimators (Tsallis 2009; Ferrari, Yang, et al. 2010). Different deformed logarithms can be obtained from entropy functions (Wada and Suyari 2007). However, the deformed or the generalized logarithms should map the probability density function $f(x_i; \theta)$ as one-to-one and overlay (Lindsay et al. 1994; Çankaya and Korbel 2018). Kaniakadis' deformed logarithm ($\log_\kappa$) can have same property with $\log_q$ due to fact that (α, β) -difference operator in fractional calculus (FC) is used to generate entropies. The other generalized entropies can have same role in the estimation procedure (Wada and Suyari 2007; Hanel and Thurner 2011). Note that the performance of efficiency can be managed by using the generalized entropies and generalized logarithms from FC. MLqE is simple and its

computational implementation is not heavy. No extra condition is required to apply MLqE for estimation when it is compared with divergences (Cichocki and Amari 2010). The parameters of Gamma distribution are estimated by MLqE method (Xing, (2015)). As an another approach for the robust estimation, divergences have been used. Minimization of entropies and divergences are equivalent to maximization of the generalized maximum likelihood estimation method (Lindsay et al. 1994; Pardo 2018).

FC has been started to play an important role in the estimations of parameters of a probability density (p.d.) function $f(x; \theta)$. Tsallis distributions (q -distributions) have encountered a large success because of their remarkable agreement with experimental data. The parameter q behaves as a microscope and generates neighborhoods of p.d. function $f(x; \theta)$. Thus, the different behaviour of p.d. function $f(x; \theta)$ can be explored by use of $\log_q$ (see Bercher 2011, 2012 and references therein). After we use $\log_q$ from Tsallis q -entropy to estimate robustly the parameters θ of underlying distribution $f(x; \theta)$, that is, we will have estimators which are not affected by inliers and outliers in a data set, the optimization of $\sum_{i=1}^n \log_q [f(x_i; \theta)]$ according to the parameters θ is an another challenging problem for the estimation process if we have a nonlinear function. The optimization is performed by means of genetic algorithm (GA) which is not attached to local points. Thus, we will have estimators. GA mimicing evolutionary biology as a stochasticity is a derivative-independent method. We use hybrid method in ‘ga’ module in MATLAB2013a to decrease the computational error as well. In hybrid method, GA and derivative-based methods work simultenaously. Further, GA has an effective attitude to reach the global point in the fuction $\rho(x; \theta) = \Lambda[f(x; \theta)]$ (Mitchell 1998).

The aim is to estimate robustly shape and scale parameters of Weibull distribution when the different types of contamination into artificial data set are added. The properties of Weibull distribution are examined extensively. The roles of Λ from the deformed algebras and DPD, etc. as an objective function ρ , are observed easily by using the illustrative representations (see Fig. 1), which is a key to analyze their roles on p.d. function $f(x; \theta)$. Outliers and inliers are added into the artificial data sets at the simulation and the different types of contaminations to real data sets are applied to observe the robustness and the modeling capability of $\rho(x; \theta) = \log_q [f(x; \theta)]$. The value of parameter q is determined according to the p -value of KS test statistic of estimates of parameters for Weibull distribution. Fisher information based on $\log_q$ is used to evaluate the variance-covariance matrix of estimators derived by MLqE.

The organization of study is as follows. Section 2 includes Weibull distribution and its properties which indicate that Weibull can be used for the modeling fruitfully. We provide the essential tools to pass the modeling sketch of estimation procedure such as convexity, (concavity), entropies, etc Haberman, (1989) in the M-estimation (Huber 2004). We also provide main tools to get the elements of Fisher information matrix. Section 3 introduces estimation methods Varin, Reid, and Firth, (2011) and estimating equations (Godambe 1960; Godambe and Thompson 1978). Section 4 introduces the tools and robustness to outliers and also we propose a tool based on score function for robustness to inliers. Section 5 provides the tool used for optimization and numerical experiments. The last section

6 is given for conclusions.

2 Weibull Distribution

Weibull distribution is chosen because it has many applications in the field of the applied science. Further, it has many features such as existence of cumulative distribution (c.d.) function, moments and entropies, etc (Malik and Arora 1992; Çankaya 2018). We say that a non-negative random variable X has a Weibull distribution with vector parameter $\boldsymbol{\theta} = (\alpha, \beta)$, denoted by $X \sim \text{Weibull}(\boldsymbol{\theta})$, if its probability density function is given by

$$f(x; \boldsymbol{\theta}) = \frac{\alpha}{\beta} \left(\frac{x}{\beta} \right)^{\alpha-1} \exp \left[- \left(\frac{x}{\beta} \right)^{\alpha} \right], \quad x \geq 0; \alpha, \beta > 0, \quad (1)$$

where α is the shape parameter and β is the scale parameter (Weibull 1951).

2.1 Properties

If $X \sim \text{Weibull}(\boldsymbol{\theta})$ then the following properties are satisfied (see Almalki and Nadarajah, (2014) for review of modified Weibull distributions).

- 1) **Asymptotic behavior of $f(x; \boldsymbol{\theta})$.** The behavior of $f(x; \boldsymbol{\theta})$ with $x \rightarrow 0$ or $x \rightarrow \infty$ is as follows:

$$\lim_{x \rightarrow 0} f(x; \boldsymbol{\theta}) = \begin{cases} \infty & \text{for } 0 < \alpha < 1, \\ \frac{1}{\beta} & \text{for } \alpha = 1, \\ 0 & \text{for } \alpha > 1, \end{cases}$$

$$\lim_{x \rightarrow \infty} f(x; \boldsymbol{\theta}) = 0 \quad \forall \alpha > 0.$$

- 2) **Monotonicity, unimodality, concavity and convexity of $f(x; \boldsymbol{\theta})$.** The point x is a mode of the Weibull density, if and only if it is the solution of the following equation

$$\frac{df(x; \boldsymbol{\theta})}{dx} = \frac{f(x; \boldsymbol{\theta})}{x} \left[(\alpha - 1) - \alpha \left(\frac{x}{\beta} \right)^{\alpha} \right] = 0.$$

Solving this equation, we get the following critical point

$$x_0 = \begin{cases} \beta \left(\frac{\alpha - 1}{\alpha} \right)^{1/\alpha} & \text{for } \alpha > 1, \\ 0 & \text{for } \alpha \leq 1. \end{cases}$$

A simple calculation shows that

$$\frac{d^2 f(x; \boldsymbol{\theta})}{dx^2} = \frac{f(x; \boldsymbol{\theta})}{x^2} \left[\alpha^2 \left(\frac{x}{\beta} \right)^{2\alpha} - 3\alpha(\alpha - 1) \left(\frac{x}{\beta} \right)^\alpha + (\alpha - 1)(\alpha - 2) \right].$$

Note that

$$\frac{d^2 f(x_0; \boldsymbol{\theta})}{dx^2} = -\alpha(\alpha - 1) < 0, \quad \alpha > 1,$$

and

$$\frac{d^2 f(x; \boldsymbol{\theta})}{dx^2} = 0 \quad \Longleftrightarrow \quad x = x_{\pm} = \beta \left[\frac{3(\alpha - 1) \pm \sqrt{(\alpha - 1)(5\alpha - 1)}}{2\alpha} \right]^{1/\alpha}.$$

Therefore, the following properties follow immediately:

For $\alpha > 1$,

- $f(x; \boldsymbol{\theta})$ increases as $x \rightarrow x_0$ and decreases thereafter.
- The point x_0 is the unique mode for the Weibull density.
- For $\alpha \neq 2$, the inflection points $x_{\pm}$ satisfy inequality $x_- < x_+$. Furthermore, $f(x_0; \boldsymbol{\theta})$ is convex on $(0, x_-) \cup (x_+, \infty)$ and is concave on (x_-, x_+) .
- For $\alpha = 2$, $x_- = 0$ and $x_+ = 6\beta/4$. Then $f(x_0; \boldsymbol{\theta})$ is concave on $(0, x_+)$ and is convex on the interval (x_+, ∞) .

For $\alpha \leq 1$,

- $f(x_0; \boldsymbol{\theta})$ decreases monotonically and is convex.
- The mode is non-existent.

3) **Reliability.** If $F(t; \boldsymbol{\theta})$ denotes the cumulative distribution function of X , then the reliability function is written

as

$$R(t; \boldsymbol{\theta}) = 1 - F(t; \boldsymbol{\theta}) = \exp \left[- \left(\frac{t}{\beta} \right)^\alpha \right].$$

4) **Hazard rate.** The hazard rate is given by

$$H(t; \boldsymbol{\theta}) = \frac{f(t; \boldsymbol{\theta})}{R(t; \boldsymbol{\theta})} = \frac{\alpha}{\beta} \left(\frac{t}{\beta} \right)^{\alpha-1}.$$

The function $H(t; \boldsymbol{\theta})$ is increasing when $\alpha > 1$, decreasing when $\alpha < 1$ and constant when $\alpha = 1$.

5) **Truncated moment.** Integration by parts gives

$$\mathbb{E}[1_{\{X \geq t\}} X^s] = t^s \exp \left[- \left(\frac{t}{\beta} \right)^\alpha \right] + \frac{s\beta^s}{\alpha} \Gamma \left(\frac{s}{\alpha}, \frac{t^\alpha}{\beta^\alpha} \right), \quad s \geq 0,$$

where $\Gamma(s, x)$, $s > 0$, is the upper incomplete gamma function, and where we adopt the notation $\Gamma(0, x) = 0$.

6) **Moment of the residual life.** For each $t \geq 0$, we have

$$\begin{aligned} \phi_n(t; \boldsymbol{\theta}) &= \mathbb{E}[(X - t)^n | X \geq t] \\ &= \sum_{k=1}^n \binom{n}{k} (-t)^{n-k} \left\{ t^k + \beta^k \Gamma \left(1 + \frac{k}{\alpha}, \frac{t^\alpha}{\beta^\alpha} \right) \exp \left[\left(\frac{t}{\beta} \right)^\alpha \right] \right\} + (-t)^n. \end{aligned}$$

The proof of this identity is immediate since

$$\phi_n(t; \boldsymbol{\theta}) = \frac{1}{R(t; \boldsymbol{\theta})} \sum_{k=1}^n \binom{n}{k} (-t)^{s-k} \mathbb{E}[1_{\{X \geq t\}} X^n] + (-t)^n,$$

where $R(t; \boldsymbol{\theta})$ and $\mathbb{E}[1_{\{X \geq t\}} X^n]$ are given in Items 3) and 5), respectively.

7) **First main tool.** By using the formula of Item (6) in Çankaya 2018, we have

$$\mathbb{E}[X^s f^{r-1}(X; \boldsymbol{\theta})] = \frac{\alpha^{r-1} \beta^{s-r+1}}{r^{r+(s-r+1)/\alpha}} \beta^s \Gamma \left(r + \frac{s-r+1}{\alpha} \right), \quad s-r+1 > -\alpha r.$$

• **Real moments.** By taking $r = 1$ in the first main tool, we reach

$$\mathbb{E}(X^s) = \beta^s \Gamma \left(1 + \frac{s}{\alpha} \right), \quad s > -\alpha,$$

where $\Gamma(z)$ is the complete gamma function. Taking $t \rightarrow 0$ in the moment of the residual life $\phi_n(t; \boldsymbol{\theta})$ we see that the entire moments are justified by the above identity.

- **Tsallis entropy.** As an immediate application of the first main tool, we have that if $q(\alpha - 1) > -1$ then the Tsallis entropy Tsallis 1988 is given by

$$\begin{aligned} S_q(X) &= \frac{1}{q-1} \left[1 - \int_0^\infty f^q(x; \boldsymbol{\theta}) dx \right] \\ &= \frac{1}{q-1} \left[1 - \left(\frac{\alpha}{\beta} \right)^{q-1} \frac{1}{q^{q+(1-q)/\alpha}} \Gamma\left(q + \frac{1-q}{\alpha}\right) \right], \quad q \in \mathbb{R} \setminus \{1\}. \end{aligned}$$

- * **Quadratic entropy.** It is followed directly by Tsallis entropy that, if $\alpha > 1/2$ then the quadratic entropy can be written as

$$\begin{aligned} H_2(X) &= -\log \left[\int_0^\infty f^2(x; \boldsymbol{\theta}) dx \right] \\ &= \log(\beta) - \log(\alpha) + \left(2 - \frac{1}{\alpha} \right) \log(2) - \log \left[\Gamma\left(2 - \frac{1}{\alpha} \right) \right]. \end{aligned}$$

- * **Shannon entropy.** By combining the Tsallis entropy with the well known relation $H_1(X) = \lim_{q \rightarrow 1} S_q(X)$, we have that the Shannon entropy can be written as

$$H_1(X) = - \int_0^\infty f(x; \boldsymbol{\theta}) \log [f(x; \boldsymbol{\theta})] dx = 1 + \log \left(\frac{\beta}{\alpha} \right) + \left(1 - \frac{1}{\alpha} \right) \gamma,$$

where $\gamma = -\frac{d\Gamma(x)}{dx} \Big|_{x=1} \approx 0.57721$ is the Euler-Mascheroni constant.

- 8) **Moment generating function.** By applying Fubini's Theorem we have that the moment generating function, $M_X(t) = \mathbb{E}[\exp(tX)]$, can be expressed as follows

$$M_X(t) = \begin{cases} \frac{1}{1 - \beta t} & \text{for } \alpha = 1 \text{ and } |t| < 1/\beta, \\ \sum_{n=0}^{\infty} \frac{(\beta t)^n}{n!} \Gamma\left(1 + \frac{n}{\alpha}\right) & \text{for } \alpha > 1 \text{ and } t \in \mathbb{R}. \end{cases}$$

- 9) **Light-tailed distribution.** From Item 8) it follows that, if $\alpha \geq 1$ then there exists $t_0 > 0$ such that $\mathbb{P}(X > x) \leq \exp(-t_0 x)$ for x large enough.

10) **Second main tool.** A simple change of variable shows that

$$\begin{aligned} \mathbb{E}[X^s \log(X) f^{r-1}(X; \boldsymbol{\theta})] \\ = \frac{\beta^{s-r+1} \alpha^{r-1}}{r^{1+\zeta/\alpha}} \Gamma\left(1 + \frac{\zeta}{\alpha}\right) \left[\log\left(\frac{\beta}{r^{1/\alpha}}\right) + \frac{1}{\alpha} \Psi^{(0)}\left(1 + \frac{\zeta}{\alpha}\right) \right], \quad \zeta > -\alpha, \end{aligned}$$

where $\zeta = s + (r-1)(\alpha-1)$, $r \in \mathbb{R}$, and $\Psi^{(m)}(z) = \frac{d^{m+1} \log[\Gamma(z)]}{dz^{m+1}}$ is the polygamma function of order m .

Indeed, by taking the change of variable $w = r\left(\frac{x}{\beta}\right)^\alpha$ we have, for each $\zeta > -\alpha$,

$$\begin{aligned} \mathbb{E}[X^s \log(X) f^{r-1}(X; \boldsymbol{\theta})] &= \left(\frac{\alpha}{\beta}\right)^r \int_0^\infty x^s \log(y) \left(\frac{x}{\beta}\right)^{r(\alpha-1)} \exp\left[-r\left(\frac{x}{\beta}\right)^\alpha\right] dx \\ &= \frac{\beta^{s-r+1} \alpha^{r-1}}{r^{1+\zeta/\alpha}} \int_0^\infty w^{\frac{\zeta}{\alpha}} \left[\log\left(\frac{\beta}{r^{1/\alpha}}\right) + \frac{1}{\alpha} \log(w) \right] \exp(-w) dw. \end{aligned} \quad (2)$$

By combining the following known formulas

$$\int_0^\infty w^{\frac{\zeta}{\alpha}} \exp(-w) dw = \Gamma\left(1 + \frac{\zeta}{\alpha}\right), \quad (3)$$

$$\int_0^\infty w^{\frac{\zeta}{\alpha}} \log(w) \exp(-w) dw = \Gamma\left(1 + \frac{\zeta}{\alpha}\right) \Psi^{(0)}\left(1 + \frac{\zeta}{\alpha}\right), \quad (4)$$

with the identity (2), the formula for the expectation $\mathbb{E}[X^s \log(X) f^{r-1}(X; \boldsymbol{\theta})]$ follows.

- In the particular case when $r = 1$ we have

$$\mathbb{E}[X^s \log(X)] = \beta^s \Gamma\left(1 + \frac{s}{\alpha}\right) \left[\log(\beta) + \frac{1}{\alpha} \Psi^{(0)}\left(1 + \frac{s}{\alpha}\right) \right], \quad s > -\alpha.$$

11) **Third main tool.** Analogously to the proof of Item 10), a simple change of variable shows that, for each

$s > -\alpha$,

$$\begin{aligned} \mathbb{E}[X^s \log^2(X) f^{r-1}(X; \boldsymbol{\theta})] \\ = \frac{2\beta^{s-r+1} \alpha^{r-2}}{r^{1+\zeta/\alpha}} \Gamma\left(1 + \frac{\zeta}{\alpha}\right) \log\left(\frac{\beta}{r^{1/\alpha}}\right) \Psi^{(0)}\left(1 + \frac{\zeta}{\alpha}\right) \\ + \frac{\beta^{s-r+1} \alpha^{r-1}}{r^{1+\zeta/\alpha}} \Gamma\left(1 + \frac{\zeta}{\alpha}\right) \left\{ \log^2\left(\frac{\beta}{r^{1/\alpha}}\right) + \frac{1}{\alpha^2} \left[\Psi^{(0)}\left(1 + \frac{\zeta}{\alpha}\right)^2 + \Psi^{(1)}\left(1 + \frac{\zeta}{\alpha}\right) \right] \right\}, \end{aligned}$$

where $\zeta = s + (r - 1)(\alpha - 1)$ and $r \in \mathbb{R}$ and $\zeta > -\alpha$.

The proof of this formula follows by taking the change of variable $w = r\left(\frac{x}{\beta}\right)^\alpha$ and by combining the formulas (3) and (4) with the following formula, for each $\zeta > -\alpha$,

$$\int_0^\infty w^{\frac{\zeta}{\alpha}} \log^2(w) \exp(-w) dw = \Gamma\left(1 + \frac{\zeta}{\alpha}\right) \left[\Psi^{(0)}\left(1 + \frac{\zeta}{\alpha}\right)^2 + \Psi^{(1)}\left(1 + \frac{\zeta}{\alpha}\right) \right].$$

- In the particular case when $r = 1$ we have, for $s > -\alpha$,

$$\mathbb{E}[X^s \log^2(X)] = \beta^s \Gamma\left(1 + \frac{s}{\alpha}\right) \left\{ \log^2(\beta) + \frac{1}{\alpha^2} \left[\Psi^{(0)}\left(1 + \frac{s}{\alpha}\right)^2 + \Psi^{(1)}\left(1 + \frac{s}{\alpha}\right) \right] + \frac{2 \log(\beta)}{\alpha} \Psi^{(0)}\left(1 + \frac{s}{\alpha}\right) \right\}.$$

3 Inference: Estimation Methods and Fisher information

3.1 Maximum likelihood estimation method

Maximum likelihood is the standard approach in parametric estimation, mainly due to the desirable asymptotic properties of consistency, efficiency and asymptotic normality under some regularity conditions (Lehmann and Casella 2006).

- The log-likelihood function for $\boldsymbol{\theta}$ is given by

$$l(\boldsymbol{\theta}; \mathbf{x}) = n \log\left(\frac{\alpha}{\beta}\right) + \sum_{i=1}^n \left\{ (\alpha - 1) \log\left(\frac{x_i}{\beta}\right) - \left(\frac{x_i}{\beta}\right)^\alpha \right\}.$$

A standard calculation shows that the first-order partial derivatives of $l(\boldsymbol{\theta}; \mathbf{x})$ are

$$\frac{\partial l(\boldsymbol{\theta}; \mathbf{x})}{\partial \alpha} = \frac{n}{\alpha} - n \log(\beta) + \sum_{i=1}^n \log(x_i) - \frac{1}{\beta^\alpha} \sum_{i=1}^n x_i^\alpha \log(x_i), \quad (5)$$

$$\frac{\partial l(\boldsymbol{\theta}; \mathbf{x})}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{\alpha}{\beta^{\alpha+1}} \sum_{i=1}^n x_i^\alpha. \quad (6)$$

The second-order partial derivatives of $l(\boldsymbol{\theta}; \mathbf{x})$ can be written as

$$\begin{aligned}\frac{\partial^2 l(\boldsymbol{\theta}; \mathbf{x})}{\partial \alpha^2} &= -\frac{n}{\alpha^2} + \frac{\log(\beta)}{\beta^\alpha} \sum_{i=1}^n x_i^\alpha \log(x_i) - \frac{1}{\beta^\alpha} \sum_{i=1}^n x_i^\alpha \log^2(x_i), \\ \frac{\partial^2 l(\boldsymbol{\theta}; \mathbf{x})}{\partial \beta^2} &= \frac{n\alpha}{\beta^2} - \frac{\alpha(\alpha+1)}{\beta^{\alpha+2}} \sum_{i=1}^n x_i^\alpha, \\ \frac{\partial^2 l(\boldsymbol{\theta}; \mathbf{x})}{\partial \alpha \partial \beta} &= \frac{\partial^2 l(\boldsymbol{\theta}; \mathbf{x})}{\partial \beta \partial \alpha} = -\frac{n}{\beta} + \frac{\alpha}{\beta^{\alpha+1}} \sum_{i=1}^n x_i^\alpha \log(x_i).\end{aligned}$$

- The $\log_q$ -likelihood function for $\boldsymbol{\theta}$ is given by

$$l_q(\boldsymbol{\theta}; \mathbf{x}) = \sum_{i=1}^n \frac{f^{1-q}(x_i; \boldsymbol{\theta}) - 1}{1 - q}.$$

Note that the first-order partial derivatives of $l_q(\boldsymbol{\theta}; \mathbf{x})$ are

$$\begin{aligned}\frac{\partial l_q(\boldsymbol{\theta}; \mathbf{x})}{\partial \alpha} &= \sum_{i=1}^n f^{-q}(x_i; \boldsymbol{\theta}) \frac{\partial f(x_i; \boldsymbol{\theta})}{\partial \alpha} = \sum_{i=1}^n f^{1-q}(x_i; \boldsymbol{\theta}) \left\{ \frac{1}{\alpha} + \left[1 - \left(\frac{x_i}{\beta} \right)^\alpha \right] \log \left(\frac{x_i}{\beta} \right) \right\}, \\ \frac{\partial l_q(\boldsymbol{\theta}; \mathbf{x})}{\partial \beta} &= \sum_{i=1}^n f^{-q}(x_i; \boldsymbol{\theta}) \frac{\partial f(x_i; \boldsymbol{\theta})}{\partial \beta} = \frac{\alpha}{\beta} \sum_{i=1}^n f^{1-q}(x_i; \boldsymbol{\theta}) \left[\left(\frac{x_i}{\beta} \right)^\alpha - 1 \right].\end{aligned}$$

3.2 Fisher information

The Fisher information matrix is defined by

$$F = n \begin{bmatrix} E_{\alpha\alpha} & E_{\alpha\beta} \\ E_{\beta\alpha} & E_{\beta\beta} \end{bmatrix}, \quad (7)$$

where n is sample size. E is integral for partial derivatives of $\log(L)$ according to parameters and it is taken by probability density function $f(x; \boldsymbol{\theta})$. The subscript in E represents second-order partial derivatives of $\log(L)$ according to parameters α and β . In other words, if $X \sim \text{Weibull}(\boldsymbol{\theta})$, by using Items 7), 10) and 11) of Subsection 2.1, we have

$$\begin{aligned}
E_{\alpha\alpha} &= -\frac{1}{\alpha^2} + \frac{\log(\beta)}{\beta^\alpha} \mathbb{E}[X^\alpha \log(X)] - \frac{1}{\beta^\alpha} \mathbb{E}[X^\alpha \log^2(X)] \\
&= -\frac{1}{\alpha^2} [1 + \Psi^{(1)}(2) + \Psi^{(0)}(2)^2 + \alpha \log(\beta) \Psi^{(0)}(2)], \\
E_{\beta\beta} &= \frac{\alpha}{\beta^2} - \frac{\alpha(\alpha+1)}{\beta^{\alpha+2}} \mathbb{E}(X^\alpha) = -\frac{\alpha^2}{\beta^2}, \\
E_{\alpha\beta} &= E_{\beta\alpha} = -\frac{1}{\beta} + \frac{\alpha}{\beta^{\alpha+1}} \mathbb{E}[X^\alpha \log(X)] = -\frac{1}{\beta} + \frac{\alpha}{\beta} \left[\log(\beta) + \frac{1}{\alpha} \Psi^{(0)}(2) \right].
\end{aligned}$$

Since, $\Psi^{(0)}(2) = \frac{d \log[\Gamma(x)]}{dx} \Big|_{x=2} = 1 - \gamma$ and $\Psi^{(1)}(2) = \frac{d^2 \log[\Gamma(x)]}{dx^2} \Big|_{x=2} = \frac{\pi^2}{6} - 1$, where $\gamma = -\frac{d\Gamma(x)}{dx} \Big|_{x=1} \approx 0.57721$ is the Euler-Mascheroni constant. Then the matrix (7) is given by

$$F = n \begin{bmatrix} -\frac{1}{\alpha^2} \left\{ \frac{\pi^2}{6} + (1-\gamma)[\alpha \log(\beta) + (1-\gamma)] \right\} & \frac{\alpha}{\beta} \left\{ \log(\beta) - \frac{\gamma}{\alpha} \right\} \\ \frac{\alpha}{\beta} \left\{ \log(\beta) - \frac{\gamma}{\alpha} \right\} & -\frac{\alpha^2}{\beta^2} \end{bmatrix}.$$

Theorem 3.1. Let $\Theta = \{\boldsymbol{\theta} = (\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \alpha \text{ is known and } \beta > 1\}$ be the parameter space. Then, with probability approaching 1, as $n \rightarrow \infty$, the likelihood equation $\frac{dl(\boldsymbol{\theta}; \mathbf{x})}{d\boldsymbol{\beta}} = 0$ has a consistent solution, denoted by $\hat{\boldsymbol{\beta}}$.

Proof. If $X \sim \text{Weibull}(\boldsymbol{\theta})$, a simple calculation shows that

1. $\mathbb{E}\left[\frac{d \log f(X; \boldsymbol{\theta})}{d\beta}\right] = -\frac{\alpha}{\beta} + \frac{\alpha}{\beta^{\alpha+1}} \mathbb{E}(X^\alpha) = 0$ for all $\boldsymbol{\theta} \in \Theta$;
2. $-\infty < \mathbb{E}\left[\frac{d^2 \log f(X; \boldsymbol{\theta})}{d\beta^2}\right] = -\frac{\alpha^2}{\beta^2} < 0$ for all $\beta \in \Theta$;
3. There exists a function $H(x)$ such that for all $\boldsymbol{\theta} \in \Theta$,

$$\begin{aligned}
\left| \frac{d^3 \log f(x; \boldsymbol{\theta})}{d\beta^3} \right| &= \left| -\frac{2\alpha}{\beta^3} - \frac{\alpha(\alpha+1)(\alpha+2)}{\beta^{2\alpha+4}} \beta^{\alpha+1} x^\alpha \right| \\
&< 2\alpha + \alpha(\alpha+1)(\alpha+2)x^\alpha = H(x),
\end{aligned}$$

because $\beta > 1$, and

$$\begin{aligned}
\mathbb{E}[H(X)] &= 2\alpha + \alpha(\alpha+1)(\alpha+2)\mathbb{E}(X^\alpha) \\
&= 2\alpha + \alpha(\alpha+1)(\alpha+2)\beta^\alpha = M(\beta) < \infty.
\end{aligned}$$

Hence, by Cramér, (1946) the proof follows. □

3.3 Estimating equations derived by objective functions in the composite likelihood

The maximum composite likelihood estimation (MCLE) is a generalization of maximum likelihood estimation (MLE), MCLE is given by

$$L_C[f(\mathbf{x}; \boldsymbol{\theta})] = \prod_{i=1}^n f(x_i; \boldsymbol{\theta})^{w_i} \quad (8)$$

$w_i \in \mathbb{R}$ is a weight function, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $f(x; \boldsymbol{\theta})$ is a p.d. function. MCLE can cover density power divergence and its generalized forms (Varin, Reid, and Firth, (2011); Cichocki and Amari 2010). The M-estimators are obtained by optimizing the objective function:

$$\sum_{i=1}^n \rho(x_i; \boldsymbol{\theta}) := \sum_{i=1}^n w_i \log[f(x_i; \boldsymbol{\theta})] = \sum_{i=1}^n \Lambda[f(x_i; \boldsymbol{\theta})]. \quad (9)$$

When $w_i = 1$ and $\log$ is replaced by $\log_q$, we have MLQE. $\log_q(f) = \frac{f^{1-q}-1}{1-q}$, $\lim_{q \rightarrow 1} \log_q(f) = \log(f)$. $\log_\kappa(f) = \frac{f^\kappa - f^{-\kappa}}{2\kappa}$, $\lim_{\kappa \rightarrow 0} \log_\kappa(f) = \log(f)$ and $\log_q$ and $\log_\kappa$ are deformed logarithm of $\log$. $q \in \mathbb{R} \setminus \{1\}$, $\kappa \in \mathbb{R}$ and $\gamma \geq 0$ are tuning constants used to adjust robustness and also efficiency (Ferrari, Yang, et al. 2010; Wada and Suyari 2007; Vajda 1986). The concavity property of Λ is examined by Çankaya and Korbel 2018; Korbel, Hanel, and Thurner 2020 and references therein to use Λ for the estimation process accurately.

The density power divergence (DPD) as an objective function between $f(x; \boldsymbol{\theta})$ and $g(x)$ which is free from parameter was proposed and the reorganized form of DPD is given by Basu et al., (1998):

$$\sum_{i=1}^n \rho(x_i; \boldsymbol{\theta}) := \sum_{i=1}^n \Lambda[f(x_i; \boldsymbol{\theta})] = \int f(x; \boldsymbol{\theta})^{1+\gamma} dx - \left(1 + \frac{1}{\gamma}\right) \frac{1}{n} \sum_{i=1}^n f(x_i; \boldsymbol{\theta})^\gamma. \quad (10)$$

Let us try to get $\log_q$ from DPD after algebraic rearrangement and $\gamma = 1 - q > 0$. Since $q < 1$, DPD puts a restriction for the values of tuning constant when we compare with $\log_q$. If equation (10) is rewritten for $\gamma = 1 - q$, then we have the following expression:

$$\frac{2-q}{n} \sum_{i=1}^n \log_q[f(x_i; \boldsymbol{\theta})] + \frac{n}{1-q} + \int f(x; \boldsymbol{\theta})^{2-q} dx = \sigma \sum_{i=1}^n \log_q[f(x_i; \boldsymbol{\theta})] + \mu, \quad (11)$$

where σ and μ represent scale and location for $\sum_{i=1}^n \log_q[f(x_i; \boldsymbol{\theta})]$. When we consider to apply the optimization for DPD and maximum $\log_q$ -likelihood, σ and μ will change where the optimized region is. In other words, there is an equivalence between DPD and $\log_q$ if they are optimized according to parameters $\boldsymbol{\theta}$. Further, the integral value of $I = \int f(x; \boldsymbol{\theta}) dx$ in DPD depends on Gamma function if f is Weibull (see Çankaya 2018). The arguman of Gamma function, i.e. $\Gamma, r > 0$, has to be a positive and requires that the values of parameters remain within certain values, which is disadvantageous for a case in estimation. In addition, the computation time Al Mohamad,

(2018) for numerical integration can be high according to the used p.d. function which is not tractable to calculate and get an expression for the result of integral. MCLE method is useful to get robust estimators for parameters. As a generalized form of MCLE, we can use MLqE which is for robust estimation, because MLqE is a simple method and does not have extra conditions. If I does not exist, it is mandatory to show that I is finite for the values of parameters in a p.d. function by use of tools in (Malik and Arora 1992). Otherwise, f in I does not have a finite value, which shows that we will not have $\hat{\theta}$. DPD for estimation of parameters of Weibull distribution does not work properly, as introduced by (Al Mohamad, (2018)).

Let us derive the estimating equations (EEs) to examine the role of objective functions. EEs are obtained after taking derivatives of objective functions, $\rho = \Lambda(f)$, according to parameters θ . For all of parameters, we have a system of EEs. The rearrangement forms of EEs are as follows:

$$\sum_{i=1}^n \frac{\partial \log_q [f(x_i; \theta)]}{\partial \theta} = \sum_{i=1}^n f(x_i; \theta)^{1-q} \mathcal{Z}(x_i; \theta) = \sum_{i=1}^n w_q(x_i; \theta) \mathcal{Z}(x_i; \theta) = \mathbf{0}, \quad (12)$$

$$\sum_{i=1}^n \frac{\partial \log_{\kappa} [f(x_i; \theta)]}{\partial \theta} = \sum_{i=1}^n [f(x_i; \theta)^{\kappa} + f(x_i; \theta)^{-\kappa}] \frac{\mathcal{Z}(x_i; \theta)}{2} = \sum_{i=1}^n w_{\kappa}(x_i; \theta) \mathcal{Z}(x_i; \theta) = \mathbf{0}, \quad (13)$$

$$\sum_{i=1}^n \frac{\partial \log [\infty + f(x_i; \theta)]}{\partial \theta} = \sum_{i=1}^n \frac{f(x_i; \theta)}{\infty + f(x_i; \theta)} \mathcal{Z}(x_i; \theta) = \sum_{i=1}^n w_{\infty}(x_i; \theta) \mathcal{Z}(x_i; \theta) = \mathbf{0}, \quad (14)$$

$$\frac{1}{n} \sum_{i=1}^n \mathcal{Z}(x_i; \theta) f(x_i; \theta)^{\gamma} - I = \sum_{i=1}^n w_{\gamma}(x_i; \theta) \mathcal{Z}(x_i; \theta) - I = \mathbf{0}, \quad (15)$$

where $I = \int \mathcal{Z}(x_i; \theta) f(x_i; \theta)^{1+\gamma} dx$. Eqs. (12)-(15) are the weighted score function with $w_q, w_{\kappa}, w_{\infty}$ and w_{γ} . $\mathcal{Z}(x_i; \theta) = \frac{\partial f(x_i; \theta)}{\partial \theta} / f(x_i; \theta)$ is a score function. Note that DPD has a weighting $0 \leq w_{\gamma} \leq 1$ which is disadvantageous because of the poorness in modeling capability (Basu et al., (1998)). Let us rewrite the system of estimating equations (EE) given by the following form:

$$\sum_{i=1}^n w(x_i; \theta) \mathcal{Z}(x_i; \theta) = \mathbf{0}. \quad (16)$$

Note that w_q is bigger than 1 if $q > 1$, which shows an advantage for us when we compare with $0 \leq w_{\infty}, w_{\gamma} \leq 1$ for $\infty, \gamma \geq 0$. If $0 \leq w \leq 1$, then a partial form of $\mathcal{Z}$ is produced. If $w > 1$, then $(1 + v)\mathcal{Z}$ as an extended form of $\mathcal{Z}$ is produced, which shows us that the information gained from the joint role of w and $\mathcal{Z}$ at same time is managed by not only $\mathcal{Z}$ from f'/f but also the different values of w from f^{1-q} . For this reason, $\log_q(f)$ as an objective function is used to perform an efficient fitting on a data set. Thus, it is seen that w can manage to produce the efficient estimators $\hat{\theta}$ from equation (16). Note that the chosen w affects the estimations of parameters which are shape and scale. After solving the systems of the estimating equations according to parameters, the estimators $\hat{\theta}$ are also

obtained instead of optimizing the objective function (Calin and Udrişte 2014; Amari 2016).

3.4 Investigation for behaviour of objective functions in the composite likelihood

Estimation is performed when we assume that the empirical data sets are a member of an objective function ρ . For this reason, we can remove integral or summation from relative entropy or divergence to have function part of analytical expression. Since f is on the closed interval $[0, 1]$, we have advantage to display the behaviour of objective functions ρ derived by $\Lambda(f)$ and the values of tuning parameters are chosen as the interval $[0, 1]$. Λ can be chosen as $\log_q$, $\log_\kappa$, $\log(\alpha + f)$, etc (Lindsay et al. 1994). Let us display their behaviours via Fig. 1: Figs. 1(a) and 1(b)

Figure 1: $f \in [0, 1]$ and $q, \kappa, \alpha, \gamma \in [0, 1]$

have big range for the values of functions when compared with the Figs. 1(c) and 1(d). As it seen from plots in Fig. 1, the concavity property of Figs. 1(a) and 1(c) can be better than Figs. 1(b) and 1(d). When we look at plots in Fig. 1, we can observe that $\log_q$ is better than other functions from Λ (Broniatowski and Vajda 2009; Vajda 1986). Thus, we test the role of Λ on p.d. function f .

3.5 q -Fisher information

Let us remind the definition of Fisher information based on $\log_q$ (Çankaya and Korbel 2018).

$$\begin{aligned} & {}_q\mathbb{E} \left\{ \frac{\partial \log f(X; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \cdot \left[\frac{\partial \log f(X; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right]^\top f^{1-q}(X; \boldsymbol{\theta}) \right\} \\ &= \int_0^\infty \frac{\partial \log f(x; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \cdot \left[\frac{\partial \log f(x; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right]^\top f^{2-q}(x; \boldsymbol{\theta}) dx. \end{aligned} \quad (17)$$

Let us rewrite the Equation (17) for the parameters α and β as follows

$${}_qF = n \begin{bmatrix} {}_qE_{\alpha\alpha} & {}_qE_{\alpha\beta} \\ {}_qE_{\beta\alpha} & {}_qE_{\beta\beta} \end{bmatrix}.$$

The elements of Fisher Information (FI) matrix based on $\log_q$ (${}_qF$) can be written as the following form.

- By taking $n = 1$ in (5), we get

$$\begin{aligned} {}_qE_{\alpha\alpha} &= {}_q\mathbb{E} \left\{ \left[\frac{\partial \log f(X; \boldsymbol{\theta})}{\partial \alpha} \right]^2 f^{1-q}(X; \boldsymbol{\theta}) \right\} \\ &= \left[\frac{1}{\alpha} - \log(\beta) \right]^2 \mathbb{E}[f^{1-q}(X; \boldsymbol{\theta})] \\ &\quad + \sum_{\ell \in \{0, \alpha\}} a_\ell \mathbb{E}[X^\ell \log(X) f^{1-q}(X; \boldsymbol{\theta})] + \sum_{\ell \in \{0, \alpha, 2\alpha\}} \tilde{a}_\ell \mathbb{E}[X^\ell \log^2(X) f^{1-q}(X; \boldsymbol{\theta})], \end{aligned}$$

where $a_0 = 2 \left[\frac{1}{\alpha} - \log(\beta) \right]$, $a_\alpha = -\frac{2}{\beta^\alpha} \left[\frac{1}{\alpha} - \log(\beta) \right]$, $\tilde{a}_0 = 1$, $\tilde{a}_\alpha = -\frac{2}{\beta^\alpha}$ and $\tilde{a}_{2\alpha} = \frac{1}{\beta^{2\alpha}}$. By using Items 7), 10) and 11) of Subsection 2.1, we have

$$\begin{aligned} {}_qE_{\alpha\alpha} &= \left[\frac{1}{\alpha} - \log(\beta) \right]^2 \left(\frac{\alpha}{\beta} \right)^{1-q} \frac{1}{q^{2-q+(q-1)/\alpha}} \Gamma \left(2 - q + \frac{q-1}{\alpha} \right) \\ &\quad + \sum_{\ell \in \{0, \alpha\}} a_\ell \frac{\beta^{\ell+q-1} \alpha^{1-q}}{(2-q)^{1+\zeta/\alpha}} \Gamma \left(1 + \frac{\zeta}{\alpha} \right) \left[\log \left(\frac{\beta}{\{2-q\}^{1/\alpha}} \right) + \frac{1}{\alpha} \Psi^{(0)} \left(1 + \frac{\zeta}{\alpha} \right) \right] \\ &\quad + \sum_{\ell \in \{0, \alpha, 2\alpha\}} \tilde{a}_\ell \frac{\beta^{\ell+q-1} \alpha^{1-q}}{(2-q)^{1+\zeta/\alpha}} \Gamma \left(1 + \frac{\zeta}{\alpha} \right) \left\{ \log^2 \left(\frac{\beta}{\{2-q\}^{1/\alpha}} \right) + \right. \\ &\quad \left. + \frac{1}{\alpha^2} \left[\Psi^{(0)} \left(1 + \frac{\zeta}{\alpha} \right)^2 + \Psi^{(1)} \left(1 + \frac{\zeta}{\alpha} \right) \right] + \frac{2}{\alpha} \log \left(\frac{\beta}{\{2-q\}^{1/\alpha}} \right) \Psi^{(0)} \left(1 + \frac{\zeta}{\alpha} \right) \right\}, \end{aligned}$$

where $\zeta = \ell + (1-q)(\alpha-1)$.

- Letting $n = 1$ in (6), we get

$${}_qE_{\beta\beta} = {}_q\mathbb{E} \left\{ \left[\frac{\partial \log f(X; \boldsymbol{\theta})}{\partial \beta} \right]^2 f^{1-q}(X; \boldsymbol{\theta}) \right\} = \sum_{\ell \in \{0, \alpha, 2\alpha\}} b_\ell \mathbb{E}[X^\ell f^{1-q}(X; \boldsymbol{\theta})],$$

where $b_0 = \left(\frac{\alpha}{\beta}\right)^2$, $b_\alpha = -\frac{2\alpha^2}{\beta^{\alpha+2}}$ and $b_{2\alpha} = \frac{\alpha^2}{\beta^{2(\alpha+1)}}$. By using Item 7) of Subsection 2.1, we reach

$${}_qE_{\beta\beta} = \sum_{\ell \in \{0, \alpha, 2\alpha\}} b_\ell \frac{\alpha^{1-q} \beta^{\ell+q-1}}{(2-q)^{r+(\ell+q-1)/\alpha}} \beta^\ell \Gamma\left(r + \frac{\ell+q-1}{\alpha}\right).$$

- Multiplying (5) and (6) with $n = 1$, we obtain

$$\begin{aligned} {}_qE_{\alpha\beta} &= {}_q\mathbb{E} \left[\frac{\partial \log f(X; \boldsymbol{\theta})}{\partial \alpha} \frac{\partial \log f(X; \boldsymbol{\theta})}{\partial \beta} f^{1-q}(X; \boldsymbol{\theta}) \right] \\ &= \sum_{\ell \in \{0, \alpha\}} c_\ell \mathbb{E}[X^\ell f^{1-q}(X; \boldsymbol{\theta})] + \sum_{\ell \in \{0, \alpha, 2\alpha\}} d_\ell \mathbb{E}[X^\ell \log(X) f^{1-q}(X; \boldsymbol{\theta})], \end{aligned}$$

where $c_0 = -\frac{\alpha}{\beta} \left[\frac{1}{\alpha} - \log(\beta) \right]$, $c_\alpha = \frac{\alpha}{\beta^{\alpha+1}} \left[\frac{1}{\alpha} - \log(\beta) \right]$, $d_0 = -\frac{\alpha}{\beta}$, $d_\alpha = \frac{\alpha}{\beta} \left(1 + \frac{1}{\beta^\alpha} \right)$ and $d_{2\alpha} = -\frac{\alpha}{\beta^{2\alpha+1}}$. By using Items 7) and 10) of Subsection 2.1, we have

$$\begin{aligned} {}_qE_{\alpha\beta} &= \sum_{\ell \in \{0, \alpha\}} c_\ell \frac{\alpha^{1-q} \beta^{\ell+q-1}}{(2-q)^{r+(\ell+q-1)/\alpha}} \beta^\ell \Gamma\left(r + \frac{\ell+q-1}{\alpha}\right) \\ &\quad + \sum_{\ell \in \{0, \alpha, 2\alpha\}} d_\ell \frac{\beta^{\ell+q-1} \alpha^{1-q}}{(2-q)^{1+\zeta/\alpha}} \Gamma\left(1 + \frac{\zeta}{\alpha}\right) \left[\log\left(\frac{\beta}{\{2-q\}^{1/\alpha}}\right) + \frac{1}{\alpha} \Psi^{(0)}\left(1 + \frac{\zeta}{\alpha}\right) \right], \end{aligned}$$

where $\zeta = \ell + (1-q)(\alpha-1)$.

Note that arguments in log and Γ functions should be positive.

4 Robustness

Influence function is a measure which is used to evaluate the robustness of M-estimators (Hampel et al. 2011). Let $\boldsymbol{\theta}$ be a vector for parameters α and β , i.e. $\boldsymbol{\theta} = (\alpha, \beta)$, and $f(x; \boldsymbol{\theta})$ represents p. d. function of Weibull distribution. In this case, the objective function is defined as

$$\rho(x; \boldsymbol{\theta}) = \Lambda[f(x; \boldsymbol{\theta})], \quad \psi_\xi(x) = \frac{\partial \rho(x; \boldsymbol{\theta})}{\partial \xi}, \quad \xi \in \{\alpha, \beta\},$$

where $\Psi = (\psi_\alpha, \psi_\beta)$ is a vector of score functions ψ_α and ψ_β and $\Lambda(z) = \log_q(z)$.

Robustness trusts on the finiteness of score functions from $\frac{\partial \rho(x; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ when x goes to infinity, i.e. $\lim_{x \rightarrow \infty} \frac{\partial \rho(x; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$. By using same way from robustness to outliers, the finiteness of score functions should be tested for the case in which we have $\lim_{x \rightarrow 0} \frac{\partial \rho(x; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$. Thus, we will imply the robustness to inlier observations in a data set.

4.1 Examination of score functions of parameters in objective function $\log_q(f)$

In order to get MLqE of parameters α and β , $\log_q$ is applied to $f(x; \boldsymbol{\theta})$. Thus, we have objective function $\rho(x; \boldsymbol{\theta}) = \log_q [f(x; \boldsymbol{\theta})]$. The score functions derived by $\rho(x; \boldsymbol{\theta})$ for the corresponding parameters are given in the following order:

$$\psi_\alpha(x) = \frac{\partial \rho(x; \boldsymbol{\theta})}{\partial \alpha} = f^{1-q}(x; \boldsymbol{\theta}) \left\{ \frac{1}{\alpha} + \left[1 - \left(\frac{x}{\beta} \right)^\alpha \right] \log \left(\frac{x}{\beta} \right) \right\}, \quad (18)$$

$$\psi_\beta(x) = \frac{\partial \rho(x; \boldsymbol{\theta})}{\partial \beta} = \frac{\alpha}{\beta} f^{1-q}(x; \boldsymbol{\theta}) \left[\left(\frac{x}{\beta} \right)^\alpha - 1 \right]. \quad (19)$$

A simple calculation shows that

$$\lim_{x \rightarrow 0} \psi_\alpha(x) = \begin{cases} +\infty & \text{for } \alpha \geq 1 \text{ and } q > 1, \\ 0 & \text{for } \alpha \geq 1 \text{ and } 0 < q < 1, \\ 0 & \text{for } 0 < \alpha < 1 \text{ and } q > 1, \\ +\infty & \text{for } 0 < \alpha < 1 \text{ and } 0 < q < 1, \end{cases} \quad \lim_{x \rightarrow 0} \psi_\beta(x) = \begin{cases} -\infty & \text{for } \alpha \geq 1 \text{ and } q > 1, \\ 0 & \text{for } \alpha \geq 1 \text{ and } 0 < q < 1, \\ 0 & \text{for } 0 < \alpha < 1 \text{ and } q > 1, \\ -\infty & \text{for } 0 < \alpha < 1 \text{ and } 0 < q < 1, \end{cases}$$

$$\lim_{x \rightarrow +\infty} \psi_\alpha(x) = \begin{cases} -\infty & \text{for } \alpha \geq 1 \text{ and } q > 1, \\ 0 & \text{for } \alpha \geq 1 \text{ and } 0 < q < 1, \\ -\infty & \text{for } 0 < \alpha < 1 \text{ and } q > 1, \\ 0 & \text{for } 0 < \alpha < 1 \text{ and } 0 < q < 1, \end{cases} \quad \lim_{x \rightarrow +\infty} \psi_\beta(x) = \begin{cases} +\infty & \text{for } \alpha \geq 1 \text{ and } q > 1, \\ 0 & \text{for } \alpha \geq 1 \text{ and } 0 < q < 1, \\ +\infty & \text{for } 0 < \alpha < 1 \text{ and } q > 1, \\ 0 & \text{for } 0 < \alpha < 1 \text{ and } 0 < q < 1, \end{cases}$$

Briefly, the limits above can be written using the following tables:

Table 1: Limit values of vector $\Psi(x) = (\psi_\alpha(x), \psi_\beta(x))$

$x \rightarrow 0$	$q > 1$	$0 < q < 1$
$\alpha \geq 1$	$(+\infty, -\infty)$	$(0, 0)$
$0 < \alpha < 1$	$(0, 0)$	$(+\infty, -\infty)$

Table 2: Limit values of a vector $\Psi(x) = (\psi_\alpha(x), \psi_\beta(x))$

$x \rightarrow +\infty$	$q > 1$	$0 < q < 1$
$\alpha \geq 1$	$(-\infty, \infty)$	$(0, 0)$
$0 < \alpha < 1$	$(-\infty, \infty)$	$(0, 0)$

$\Psi = (\psi_\alpha, \psi_\beta)$ is a vector of score functions ψ_α and ψ_β . Influence function is proportional to score function. If score functions ψ_α and ψ_β are finite, then we have finite influence function, which shows that the estimators $\hat{\alpha}$ and $\hat{\beta}$ from MLqE and MLE will be robust (Hampel et al. 2011). When MLqE is used, the influence function of the estimators $\hat{\alpha}$ and $\hat{\beta}$ is finite if $q \in (0, 1)$ (see Table 2). However, one can show that there are cases for $\log(f)$ in which the score functions of parameters of f are finite or infinite according to property of f (Çankaya et al., (2019)). In addition, for an arbitrary f , we cannot get FI and the definition of influence function also includes the inverse of Fisher. There can be cases in which an element of FI matrix is not defined for some values of parameters such as $\Gamma(r), r > 0, a/b, b \neq 0$, etc and the inverse of FI matrix cannot exist. However, we can get estimates of parameters for these cases in which FI and its inverse does not exist. Instead of calling robustness and trusting on the tools in robustness (M. Thompson, e-mail communication, March 2, 2016), the main approach should be based on the modelling competence of an arbitrary function. Further, Table 1 includes the infinity cases. The modeling is carried out for the finite sample size. The robustness is for the case in which we take limit at the values which are zero and infinity. However, it is expected that the robustness should be supported by simulation. There is an open question: Even if the score functions are infinite for limit values at zero and infinity, can we perform a modeling capability on a finite sample size? Yes and we can find a counter example against the robustness theory from simulation results (see Case 4 in Table 5).

5 Optimization and numerical experiments

5.1 Optimization via genetic algorithm for $\Lambda(f)$

The genetic algorithm (GA) can be applied to solve a variety of optimization problems that are unsuitable for standard optimization algorithms which include problems where the objective function is undifferentiated in Radon Nikodym

derivative, highly nonlinear, discontinuous, (absolutely) continuous, non-smooth, even stochastic and random subsets from the real line. GA method is preferred to ensure that such objective functions converge to a global point. When $\log_q$ is used as the objective function, GA method has been used by (Çankaya and Korbel 2018). The codes used to get estimates of MLqE are given by Appendix 6.

5.2 Structure of contamination: The distributions and their parameter values used at Monte Carlo simulation

Contamination makes a disorder in the identically distributed random variables represented by $X_1, X_2, \dots, X_n$. If these random variables are distributed non-identically, then we express the structure of non-identity from mixing of two p.d. functions f_0 and f_1 , as given by following form:

$$f_\varepsilon(x; \boldsymbol{\theta}; \boldsymbol{\tau}) = (1 - \varepsilon)f_0(x; \boldsymbol{\theta}) + \varepsilon f_1(x; \boldsymbol{\tau}) \quad (20)$$

is the contaminated distribution. The constant ε is the contamination rate. f_0 is the underlying and f_1 is contamination. f_1 can be same distribution with f_0 , but the parameter values of f_1 are different from f_0 , i.e. $f_0 = \text{Weibull}(\alpha_0, \beta_0)$ and $f_1 = \text{Weibull}(\alpha_1, \beta_1)$. We can select f_1 distribution with the given values of parameters. For example, $f_1 = \text{BurrIII}(\alpha_1, \beta_1)$. In the real world, after assuming that a data set is a member of f_0 distribution, a contamination to f_0 by means of f_1 can occur in an empirical distribution. We do not know how much rate ε of contamination into the data set exist. Further, there are two types of contamination. These are inliers and outliers. A data set can include both of contaminations. The aim is to estimate robustly parameters $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \dots, \boldsymbol{\theta}_p$ of f_0 under contamination.

The distributions and parameter values selected to generate different forms of contamination are those in which both inliers and outliers are at the same time. The plots of functions of the selected values of parameters are given in Figures 2-4. Thus, it can be observed the structure of underlying and contamination distributions. We provide three blocks for Figures 2-4. Each of plots in the blocked Figures 2-4 are given to depict the role of underlying and contamination distributions clearly. They are depicted separately and same length x -axis to avoid bad illustration which can occur the different values of parameters for values of p.d. function at y -axis. Thus, we observe the contamination structure and so that we can test the performance of MLqE for such contaminations in all cases given by Figures 2-4. Blue and red lines representing p. d. function abbreviated as PDF show the underlying f_0 and the contamination f_1 into underlying distribution respectively. Simulation in section 5.3 includes the robust estimations of parameters of blue lines. As it is proven theoretically by 2.1, Weibull distribution has one unimodal for $\alpha > 1$. Note that the mode of Weibull does not exist and makes asymptotic to y -axis for $\alpha \leq 1$. The uniform and BurrIII Çankaya et al., (2019) distributions have one mode as well.

For the estimation process, the modality of p.d. function $f(x; \theta)$ and the concavity of Λ are taken in account to get $\Lambda(f)$ and apply for the estimation. Thus, the cooperation between $f(x; \theta)$ and Λ for conducting an accurate modeling on a data set can be performed successfully. For example, the smoothness property of objective function $\rho(x; \theta) = \Lambda[f(x; \theta)]$ is important to apply for estimation if a data set does not have several jumpings on an interval on the real line, i.e. the smoothness of data set should be provided. The structure of inliers in Monte Carlo simulation will not be strictly existing. In other words, the frequencies of artificial data sets for a narrow interval on the real line are not extremely high degree. One can observe the schema of underlying and contamination distributions in Figures 2-4.

Figure 2: The underlying and contamination distributions are Weibull (color online)

Figure 3: The underlying is Weibull and contamination is Uniform distributions (color online)

Figure 4: The underlying is Weibull and contamination is BurrIII distributions (color online)

5.3 Monte Carlo simulation: The design of artificial random numbers and results of simulation

A comprehensive simulation study is performed to make a comparison between MLqE and MLE methods. There are three contamination structures in the simulation. Contamination makes a disorder in the identically distributed random observations represented by $x_1, x_2, \dots, x_n$. Contaminations are as follows:

1. $(1 - \varepsilon)f_0(x; \alpha_0, \beta_0) + \varepsilon f_1(x; \alpha_1, \beta_1) = (1 - \varepsilon)\text{Weibull}(x; \alpha_0, \beta_0) + \varepsilon \text{Weibull}(x; \alpha_1, \beta_1),$
2. $(1 - \varepsilon)f_0(x; \alpha_0, \beta_0) + \varepsilon f_1(x; \alpha_1, \beta_1) = (1 - \varepsilon)\text{Weibull}(x; \alpha_0, \beta_0) + \varepsilon \text{Uniform}(x; \alpha_1, \beta_1),$
3. $(1 - \varepsilon)f_0(x; \alpha_0, \beta_0) + \varepsilon f_1(x; \alpha_1, \beta_1) = (1 - \varepsilon)\text{Weibull}(x; \alpha_0, \beta_0) + \varepsilon \text{BurrIII}(x; \alpha_1, \beta_1).$

The contamination rate ε is chosen as 10% and 20%. The parameters α_0 and β_0 of underlying distribution $\text{Weibull}(\alpha_0, \beta_0)$ are estimated under contamination. f_1 can be chosen as $\text{Weibull}(\alpha_1, \beta_1)$, single-peak $\text{BurrIII}(\alpha_1, \beta_1)$ Çankaya et al., (2019) and $\text{Uniform}(\alpha_1, \beta_1)$ with the same mode for all values of $x \in [a, b]$. Thus, inliers and outliers can be observed through shape of function (see Figures 2-4). Four different sample sizes $n = 50, 100, 150$ and $n = 200$ are used in the designs of simulation. n_0 and n_1 represent the number of random numbers from f_0 and f_1 distributions respectively. The number of replication for n is 10^4 .

Simulation variance $\widehat{Var}(\hat{\theta})$ and simulation mean error squares $\widehat{MSE}(\hat{\theta})$ are computed by using the following forms:

$$\begin{aligned} \widehat{MSE}(\hat{\theta}) &= \widehat{Var}(\hat{\theta}) + \widehat{Bias}(\hat{\theta})^2 \\ \widehat{MSE}(\hat{\theta}) &= \hat{E}[(\hat{\theta} - \hat{E}(\hat{\theta}))^2] + (\hat{E}(\hat{\theta}) - \theta)^2 \end{aligned} \quad (21)$$

$\widehat{Bias}^2$ and $\hat{E}$ show the bias and the expected values obtained from the simulation, respectively. $\widehat{MSE}$ and $\widehat{Var}$ obtained through simulation are sampling forms of theoretical MSE and Var, respectively. If the estimators $\hat{\theta}$ obtained by MLqE are unbiased, then $\widehat{MSE}(\hat{\theta}) = \widehat{Var}(\hat{\theta})$. For comparison between MLqE and MLE methods, Tables

3-6 give the estimates of parameters α_0 and β_0 , $\widehat{Var}(\hat{\theta})$ and $\widehat{MSE}(\hat{\theta})$. Note that other objective functions from $\log_\kappa$ and DPD did not give results which are better than MLqE for different types of contaminations. For this reason, we do not give them for the sake of not increasing the page numbers (see Figures 1(a)-1(d) for objective functions).

Table 3: The underlying and contamination distributions are Weibull

$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$		
	Case 1: $(1-\varepsilon)W(4,2)+\varepsilon W(1,5), \varepsilon=0.1$								Case 1: $(1-\varepsilon)W(4,2)+\varepsilon W(1,5), \varepsilon=0.2$							
	$n=50, q=0.84$	$n=100, q=0.84$	$n=150, q=0.84$	$n=200, q=0.84$	$n=50, q=0.77$	$n=100, q=0.77$	$n=150, q=0.77$	$n=200, q=0.77$								
MLqE($\hat{\theta}$)	4.0117	1.9984	3.9449	1.9981	3.9294	1.9967	3.9238	1.9970	3.9537	1.9951	3.8851	1.9938	3.8627	1.9939	3.8454	1.9928
$\widehat{Var}(\hat{\theta})$	0.3883	0.0073	0.1775	0.0036	0.1161	0.0023	0.0883	0.0018	0.4907	0.0092	0.2389	0.0044	0.1516	0.0030	0.1108	0.0022
$\widehat{MSE}(\hat{\theta})$	0.3884	0.0073	0.1806	0.0036	0.1210	0.0024	0.0941	0.0018	0.4928	0.0092	0.2522	0.0045	0.1705	0.0030	0.1347	0.0023
MLE($\hat{\theta}$)	1.7602	2.3742	1.5993	2.3786	1.5511	2.3800	1.5226	2.3812	1.3897	2.6791	1.3278	2.6798	1.3082	2.6802	1.2964	2.6796
$\widehat{Var}(\hat{\theta})$	0.2919	0.0518	0.1015	0.0259	0.0576	0.0174	0.0372	0.0130	0.0771	0.0869	0.0289	0.0435	0.0181	0.0284	0.0124	0.0214
$\widehat{MSE}(\hat{\theta})$	5.3086	0.1919	5.8647	0.1692	6.0546	0.1618	6.1746	0.1583	6.8909	0.5480	7.1698	0.5056	7.2639	0.4911	7.3221	0.4832
	Case 2: $(1-\varepsilon)W(6,4)+\varepsilon W(1,5), \varepsilon=0.1$								Case 2: $(1-\varepsilon)W(6,4)+\varepsilon W(1,5), \varepsilon=0.2$							
	$n=50, q=0.82$	$n=100, q=0.82$	$n=150, q=0.82$	$n=200, q=0.82$	$n=50, q=0.76$	$n=100, q=0.76$	$n=150, q=0.76$	$n=200, q=0.76$								
MLqE($\hat{\theta}$)	6.0814	3.9807	5.9821	3.9823	5.9549	3.9824	5.9371	3.9828	5.8268	3.9667	5.7126	3.9646	5.6736	3.9658	5.6606	3.9650
$\widehat{Var}(\hat{\theta})$	0.8033	0.0126	0.3677	0.0065	0.2322	0.0043	0.1738	0.0032	1.0473	0.0159	0.4823	0.0079	0.3153	0.0054	0.2322	0.0040
$\widehat{MSE}(\hat{\theta})$	0.8099	0.0130	0.3680	0.0068	0.2342	0.0046	0.1777	0.0035	1.0773	0.0170	0.5648	0.0092	0.4219	0.0066	0.3474	0.0053
MLE($\hat{\theta}$)	2.9198	4.2649	2.5628	4.2746	2.4039	4.2797	2.3270	4.2847	2.1074	4.4416	1.9521	4.4407	1.8948	4.4488	1.8674	4.4494
$\widehat{Var}(\hat{\theta})$	1.0803	0.0952	0.5275	0.0509	0.2924	0.0334	0.1991	0.0254	0.3171	0.1437	0.1272	0.0721	0.0753	0.0490	0.0539	0.0364
$\widehat{MSE}(\hat{\theta})$	10.5682	0.1654	12.3418	0.1263	13.2245	0.1116	13.6903	0.1064	15.4692	0.3388	16.5125	0.2663	16.9276	0.2504	17.1320	0.2384
	Case 3: $(1-\varepsilon)W(5,5)+\varepsilon W(2,6), \varepsilon=0.1$								Case 3: $(1-\varepsilon)W(5,5)+\varepsilon W(2,6), \varepsilon=0.2$							
	$n=50, q=0.83$	$n=100, q=0.83$	$n=150, q=0.83$	$n=200, q=0.83$	$n=50, q=0.73$	$n=100, q=0.73$	$n=150, q=0.73$	$n=200, q=0.73$								
MLqE($\hat{\theta}$)	5.0053	5.0198	4.9333	5.0161	4.9117	5.0177	4.9002	5.0177	5.0820	5.0168	4.9997	5.0160	4.9665	5.0140	4.9475	5.0166
$\widehat{Var}(\hat{\theta})$	0.5207	0.0289	0.2328	0.0144	0.1590	0.0097	0.1163	0.0072	0.6637	0.0365	0.3162	0.0178	0.1996	0.0117	0.1496	0.0090
$\widehat{MSE}(\hat{\theta})$	0.5207	0.0293	0.2372	0.0146	0.1667	0.0100	0.1262	0.0075	0.6704	0.0368	0.3162	0.0181	0.2008	0.0119	0.1524	0.0093
MLE($\hat{\theta}$)	3.8745	5.1391	3.6735	5.1411	3.6032	5.1439	3.5537	5.1446	3.2677	5.2637	3.1351	5.2671	3.0810	5.2674	3.0652	5.2669
$\widehat{Var}(\hat{\theta})$	0.5877	0.0465	0.3243	0.0237	0.2239	0.0158	0.1695	0.0118	0.3543	0.0665	0.1678	0.0333	0.1046	0.0223	0.0796	0.0170
$\widehat{MSE}(\hat{\theta})$	1.8545	0.0659	2.0839	0.0436	2.1750	0.0365	2.2613	0.0327	3.3551	0.1360	3.6455	0.1046	3.7871	0.0938	3.8231	0.0883
	Case 4: $(1-\varepsilon)W(10,8)+\varepsilon W(4,10), \varepsilon=0.1$								Case 4: $(1-\varepsilon)W(10,8)+\varepsilon W(4,10), \varepsilon=0.2$							
	$n=50, q=0.82$	$n=100, q=0.82$	$n=150, q=0.82$	$n=200, q=0.82$	$n=50, q=0.74$	$n=100, q=0.73$	$n=150, q=0.73$	$n=200, q=0.73$								
MLqE($\hat{\theta}$)	10.0432	8.0523	9.9416	8.0521	9.9305	8.0521	9.8742	8.0522	10.0133	8.0936	10.0648	8.0798	10.0319	8.0819	10.0015	8.0829
$\widehat{Var}(\hat{\theta})$	2.4484	0.0201	1.1634	0.0096	0.8164	0.0065	0.6065	0.0051	3.1352	0.0264	1.6086	0.0126	1.1025	0.0085	0.8469	0.0067
$\widehat{MSE}(\hat{\theta})$	2.4502	0.0229	1.1668	0.0123	0.8212	0.0092	0.6223	0.0078	3.1354	0.0352	1.6128	0.0190	1.1035	0.0152	0.8469	0.0135
MLE($\hat{\theta}$)	6.3489	8.2799	5.9287	8.2881	5.8087	8.2890	5.7108	8.2928	5.2740	8.5174	5.0506	8.5238	4.9863	8.5280	4.9650	8.5302
$\widehat{Var}(\hat{\theta})$	2.2001	0.0424	0.9516	0.0206	0.5923	0.0145	0.4180	0.0109	0.9024	0.0577	0.3525	0.0292	0.2149	0.0190	0.1645	0.0143
$\widehat{MSE}(\hat{\theta})$	15.5306	0.1207	17.5273	0.1036	18.1596	0.0980	18.8150	0.0966	23.2372	0.3253	24.8492	0.3035	25.3524	0.2977	25.5160	0.2954
	Case 5: $(1-\varepsilon)W(3,5)+\varepsilon W(2,7), \varepsilon=0.1$								Case 5: $(1-\varepsilon)W(3,5)+\varepsilon W(2,7), \varepsilon=0.2$							
	$n=50, q=0.84$	$n=100, q=0.82$	$n=150, q=0.82$	$n=200, q=0.82$	$n=50, q=0.75$	$n=100, q=0.75$	$n=150, q=0.74$	$n=200, q=0.74$								
MLqE($\hat{\theta}$)	3.0231	5.0735	3.0327	5.0527	3.0191	5.0543	3.0108	5.0532	3.0520	5.1156	3.0003	5.1164	3.0117	5.1089	3.0024	5.1063
$\widehat{Var}(\hat{\theta})$	0.1642	0.0781	0.0782	0.0382	0.0519	0.0253	0.0381	0.0192	0.1995	0.0964	0.0931	0.0468	0.0624	0.0308	0.0456	0.0231
$\widehat{MSE}(\hat{\theta})$	0.1647	0.0835	0.0793	0.0410	0.0523	0.0283	0.0383	0.0221	0.2022	0.1098	0.0931	0.0604	0.0625	0.0426	0.0456	0.0344
MLE($\hat{\theta}$)	2.6699	5.2183	2.6007	5.2154	2.5756	5.2189	2.5600	5.2188	2.4435	5.4241	2.3859	5.4267	2.3642	5.4295	2.3518	5.4286
$\widehat{Var}(\hat{\theta})$	0.1476	0.0888	0.0733	0.0424	0.0513	0.0285	0.0371	0.0216	0.1148	0.1111	0.0552	0.0542	0.0348	0.0360	0.0261	0.0269
$\widehat{MSE}(\hat{\theta})$	0.2565	0.1364	0.2327	0.0888	0.2315	0.0764	0.2307	0.0695	0.4245	0.2910	0.4324	0.2363	0.4391	0.2204	0.4462	0.2106

When the Tables 3-6 are examined;

- It is generally observed that the modeling capability of $\log_q(f_0=\text{Weibull}(\alpha_0, \beta_0))$ from MLqE is better than that of $\log(f_0=\text{Weibull}(\alpha_0, \beta_0))$ from MLE.
- In MLqE, it is observed especially that the estimated values of shape parameter α can be significantly small for some tried designs.
- As it is logical to expect, $\widehat{MSE}(\hat{\theta})$ of MLE cannot get the smaller values when the sample size n is increased for some designs of contamination; because $n = n_0 + n_1$ increases sample size n_1 of a contamination, which makes more contaminated data set when it is compared with small sample sizes, such as $n = 50, 100$, etc. In addition,

Table 4: The underlying and contamination distributions are Weibull (continuation of the Table 3)

	Case 6: $(1 - \varepsilon)W(1,5) + \varepsilon W(2,8)$, $\varepsilon=0.1$								Case 6: $(1 - \varepsilon)W(1,5) + \varepsilon W(2,8)$, $\varepsilon=0.2$							
	$n=50, q=0.97$	$n=100, q=0.97$	$n=150, q=0.97$	$n=200, q=0.97$	$n=50, q=0.87$	$n=100, q=0.87$	$n=150, q=0.87$	$n=200, q=0.87$	$n=50, q=0.87$	$n=100, q=0.87$	$n=150, q=0.87$	$n=200, q=0.87$	$n=50, q=0.87$	$n=100, q=0.87$	$n=150, q=0.87$	$n=200, q=0.87$
MLqE($\hat{\theta}$)	1.0720	5.1854	1.0565	5.1629	1.0519	5.1597	1.0513	5.1636	1.1229	5.0455	1.1103	5.0387	1.1053	5.0389	1.1020	5.0298
$\widehat{Var}(\hat{\theta})$	0.0157	0.5523	0.0074	0.2762	0.0049	0.1820	0.0037	0.1358	0.0217	0.5414	0.0106	0.2750	0.0070	0.1796	0.0051	0.1326
$\widehat{MSE}(\hat{\theta})$	0.0209	0.5866	0.0106	0.3027	0.0076	0.2075	0.0063	0.1626	0.0368	0.5435	0.0228	0.2765	0.0181	0.1811	0.0155	0.1335
MLE($\hat{\theta}$)	1.0727	5.2321	1.0568	5.3022	1.0521	5.2999	1.0514	5.3035	1.1238	5.6089	1.1099	5.6057	1.1044	5.6091	1.1008	5.6013
$\widehat{Var}(\hat{\theta})$	0.0151	0.5622	0.0071	0.2814	0.0047	0.1853	0.0035	0.1390	0.0173	0.5579	0.0084	0.2808	0.0055	0.1824	0.0041	0.1360
$\widehat{MSE}(\hat{\theta})$	0.0204	0.6666	0.0103	0.3727	0.0074	0.2752	0.0062	0.2311	0.0326	0.9287	0.0205	0.6477	0.0164	0.5534	0.0142	0.4976
	Case 7: $(1 - \varepsilon)W(5,1) + \varepsilon W(2,8)$, $\varepsilon=0.1$								Case 7: $(1 - \varepsilon)W(5,1) + \varepsilon W(2,8)$, $\varepsilon=0.2$							
	$n=50, q=0.87$	$n=100, q=0.87$	$n=150, q=0.87$	$n=200, q=0.87$	$n=50, q=0.8$	$n=100, q=0.8$	$n=150, q=0.8$	$n=200, q=0.8$	$n=50, q=0.8$	$n=100, q=0.8$	$n=150, q=0.8$	$n=200, q=0.8$	$n=50, q=0.8$	$n=100, q=0.8$	$n=150, q=0.8$	$n=200, q=0.8$
MLqE($\hat{\theta}$)	5.3941	1.0002	5.3295	1.0007	5.3101	1.0006	5.2996	1.0009	5.6228	1.0009	5.5442	1.0007	5.5227	1.0006	5.5089	1.0010
$\widehat{Var}(\hat{\theta})$	0.5300	0.0011	0.2401	0.0005	0.1615	0.0003	0.1201	0.0003	0.6812	0.0015	0.3046	0.0006	0.2071	0.0004	0.1522	0.0003
$\widehat{MSE}(\hat{\theta})$	0.6854	0.0011	0.3487	0.0005	0.2577	0.0003	0.2098	0.0003	1.0691	0.0015	0.6008	0.0006	0.4803	0.0004	0.4113	0.0003
MLE($\hat{\theta}$)	1.0943	1.5828	1.0772	1.5837	1.0698	1.5858	1.0685	1.5851	0.9719	2.0993	0.9644	2.1000	0.9620	2.1006	0.9612	2.1001
$\widehat{Var}(\hat{\theta})$	0.0156	0.0129	0.0062	0.0064	0.0037	0.0042	0.0027	0.0031	0.0048	0.0249	0.0021	0.0123	0.0014	0.0084	0.0011	0.0063
$\widehat{MSE}(\hat{\theta})$	15.2700	0.3525	15.3944	0.3472	15.4501	0.3474	15.4596	0.3455	16.2302	1.2334	16.2880	1.2223	16.3068	1.2197	16.3127	1.2166
	Case 8: $(1 - \varepsilon)W(5,3) + \varepsilon W(2,7)$, $\varepsilon=0.1$								Case 8: $(1 - \varepsilon)W(5,3) + \varepsilon W(2,7)$, $\varepsilon=0.2$							
	$n=50, q=0.84$	$n=100, q=0.84$	$n=150, q=0.84$	$n=200, q=0.84$	$n=50, q=0.77$	$n=100, q=0.77$	$n=150, q=0.77$	$n=200, q=0.77$	$n=50, q=0.77$	$n=100, q=0.77$	$n=150, q=0.77$	$n=200, q=0.77$	$n=50, q=0.77$	$n=100, q=0.77$	$n=150, q=0.77$	$n=200, q=0.77$
MLqE($\hat{\theta}$)	5.0450	3.0315	4.9924	3.0291	4.9679	3.0253	5.0395	3.0237	5.0236	3.0612	4.9861	3.0476	5.0784	3.0395	5.0649	3.0376
$\widehat{Var}(\hat{\theta})$	0.7555	0.0112	0.3561	0.0053	0.2309	0.0035	0.1613	0.0026	1.1168	0.0181	0.4973	0.0076	0.3107	0.0046	0.2178	0.0033
$\widehat{MSE}(\hat{\theta})$	0.7575	0.0121	0.3561	0.0062	0.2320	0.0042	0.1629	0.0032	1.1174	0.0218	0.4975	0.0099	0.3169	0.0061	0.2221	0.0047
MLE($\hat{\theta}$)	2.2789	3.4869	2.1464	3.4979	2.1086	3.4970	2.0953	3.4989	1.9144	3.9000	1.8628	3.9065	1.8508	3.9044	1.8441	3.9029
$\widehat{Var}(\hat{\theta})$	0.2579	0.0414	0.0829	0.0205	0.0486	0.0141	0.0333	0.0103	0.0700	0.0636	0.0286	0.0316	0.0175	0.0201	0.0129	0.0156
$\widehat{MSE}(\hat{\theta})$	7.6625	0.2785	8.2258	0.2684	8.4086	0.2611	8.4705	0.2592	9.5907	0.8736	9.8707	0.8533	9.9347	0.8381	9.9725	0.8308

Table 5: The underlying is Weibull and contamination is Uniform distributions

$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
	Case 1: $(1 - \varepsilon)W(8,10) + \varepsilon U(4,10), \varepsilon=0.1$								Case 1: $(1 - \varepsilon)W(8,10) + \varepsilon U(4,10), \varepsilon=0.2$							
	$n=50, q=0.84$	$n=100, q=0.84$	$n=150, q=0.84$	$n=200, q=0.84$	$n=50, q=0.7$	$n=100, q=0.7$	$n=150, q=0.7$	$n=200, q=0.7$	$n=50, q=0.7$	$n=100, q=0.7$	$n=150, q=0.7$	$n=200, q=0.7$	$n=50, q=0.7$	$n=100, q=0.7$	$n=150, q=0.7$	$n=200, q=0.7$
MLqE($\hat{\theta}$)	8.0064	9.8590	7.9204	9.8638	8.0010	9.8689	7.9822	9.8692	8.1291	9.7391	8.0101	9.7455	7.9531	9.7452	7.9363	9.7461
$\widehat{Var}(\hat{\theta})$	0.9125	0.0385	0.4443	0.0204	0.3111	0.0138	0.2331	0.0100	1.3036	0.0470	0.6139	0.0238	0.4175	0.0161	0.3002	0.0122
$\widehat{MSE}(\hat{\theta})$	0.9125	0.0584	0.4507	0.0389	0.3111	0.0310	0.2334	0.0271	1.3202	0.1151	0.6140	0.0886	0.4197	0.0810	0.3043	0.0766
MLE($\hat{\theta}$)	7.1776	9.8106	7.1036	9.8159	7.0731	9.8169	7.0499	9.8152	6.4031	9.6178	6.3368	9.6245	6.3101	9.6246	6.3005	9.6251
$\widehat{Var}(\hat{\theta})$	0.6376	0.0365	0.3069	0.0191	0.2049	0.0127	0.1524	0.0093	0.4950	0.0386	0.2386	0.0197	0.1605	0.0133	0.1140	0.0098
$\widehat{MSE}(\hat{\theta})$	1.3140	0.0724	1.1105	0.0530	1.0639	0.0462	1.0552	0.0435	3.0452	0.1848	3.0048	0.1607	3.0162	0.1542	3.0022	0.1503
	Case 2: $(1 - \varepsilon)W(3,5) + \varepsilon U(5,15), \varepsilon=0.1$								Case 2: $(1 - \varepsilon)W(3,5) + \varepsilon U(5,15), \varepsilon=0.2$							
	$n=50, q=0.75$	$n=100, q=0.75$	$n=150, q=0.74$	$n=200, q=0.74$	$n=50, q=0.65$	$n=100, q=0.65$	$n=150, q=0.65$	$n=200, q=0.65$	$n=50, q=0.65$	$n=100, q=0.65$	$n=150, q=0.65$	$n=200, q=0.65$	$n=50, q=0.65$	$n=100, q=0.65$	$n=150, q=0.65$	$n=200, q=0.65$
MLqE($\hat{\theta}$)	3.1173	5.1332	3.0509	5.1345	3.0231	5.1392	3.0161	5.1385	3.2591	5.2930	3.1292	5.2903	3.0989	5.2890	3.0789	5.2872
$\widehat{Var}(\hat{\theta})$	0.2868	0.0827	0.1362	0.0412	0.0871	0.0277	0.0665	0.0209	0.5459	0.1214	0.2484	0.0618	0.1641	0.0415	0.1236	0.0311
$\widehat{MSE}(\hat{\theta})$	0.3006	0.1004	0.1388	0.0593	0.0876	0.0471	0.0668	0.0400	0.6130	0.2072	0.2650	0.1461	0.1738	0.1250	0.1299	0.1135
MLE($\hat{\theta}$)	2.2236	5.6668	2.1925	5.6698	2.1816	5.6751	2.1756	5.6775	2.0557	6.3144	2.0339	6.3136	2.0278	6.3161	2.0252	6.3139
$\widehat{Var}(\hat{\theta})$	0.0656	0.0843	0.0280	0.0415	0.0173	0.0274	0.0125	0.0207	0.0363	0.0990	0.0165	0.0492	0.0107	0.0336	0.0079	0.0249
$\widehat{MSE}(\hat{\theta})$	0.6684	0.5288	0.6800	0.4901	0.6870	0.4832	0.6921	0.4797	0.9280	1.8265	0.9499	1.7746	0.9560	1.7656	0.9580	1.7513
	Case 3: $(1 - \varepsilon)W(10,8) + \varepsilon U(4,10), \varepsilon=0.1$								Case 3: $(1 - \varepsilon)W(10,8) + \varepsilon U(4,10), \varepsilon=0.2$							
	$n=50, q=0.85$	$n=100, q=0.85$	$n=150, q=0.85$	$n=200, q=0.85$	$n=50, q=0.74$	$n=100, q=0.73$	$n=150, q=0.7$	$n=200, q=0.7$	$n=50, q=0.74$	$n=100, q=0.73$	$n=150, q=0.7$	$n=200, q=0.7$	$n=50, q=0.74$	$n=100, q=0.73$	$n=150, q=0.7$	$n=200, q=0.7$
MLqE($\hat{\theta}$)	9.9399	7.9939	9.8051	7.9961	9.7881	7.9996	9.7611	7.9982	9.7928	7.9929	9.7810	7.9984	10.0043	8.0030	9.9722	8.0032
$\widehat{Var}(\hat{\theta})$	1.3651	0.0185	0.6472	0.0092	0.4316	0.0062	0.3394	0.0049	1.6467	0.0226	0.8393	0.0119	0.6388	0.0080	0.4839	0.0062
$\widehat{MSE}(\hat{\theta})$	1.3687	0.0186	0.6852	0.0092	0.4765	0.0062	0.3964	0.0049	1.6896	0.0226	0.8873	0.0119	0.6388	0.0080	0.4847	0.0062
MLE($\hat{\theta}$)	9.0213	7.9756	8.8908	7.9779	8.8674	7.9813	8.8412	7.9802	8.0909	7.9542	8.0147	7.9587	7.9808	7.9593	7.9593	7.9605
$\widehat{Var}(\hat{\theta})$	1.0064	0.0186	0.4656	0.0091	0.3062	0.0062	0.2331	0.0048	0.7783	0.0221	0.3634	0.0114	0.2341	0.0074	0.1771	0.0058
$\widehat{MSE}(\hat{\theta})$	1.9643	0.0192	1.6960	0.0096	1.5889	0.0065	1.5759	0.0052	4.4230	0.0242	4.3048	0.0131	4.3113	0.0091	4.3414	0.0074
	Case 4: $(1 - \varepsilon)W(3,5) + \varepsilon U(3,5), \varepsilon=0.1$								Case 4: $(1 - \varepsilon)W(3,5) + \varepsilon U(3,5), \varepsilon=0.2$							
	$n=50, q=1.07$	$n=100, q=1.09$	$n=150, q=1.09$	$n=200, q=1.09$	$n=50, q=1.09$	$n=100, q=1.1$	$n=150, q=1.09$	$n=200, q=1.1$	$n=50, q=1.09$	$n=100, q=1.1$	$n=150, q=1.09$	$n=200, q=1.1$	$n=50, q=1.09$	$n=100, q=1.1$	$n=150, q=1.09$	$n=200, q=1.1$
MLqE($\hat{\theta}$)	3.0854	4.9569	3.0185	4.9681	2.9972	4.9665	2.9902	4.9669	3.1509	4.9048	3.0867	4.9119	3.0773	4.9113	3.0555	4.9142
$\widehat{Var}(\hat{\theta})$	0.1239	0.0569	0.0556	0.0282	0.0362	0.0191	0.0272	0.0143	0.1287	0.0536	0.0587	0.0261	0.0391	0.0179	0.0289	0.0136
$\widehat{MSE}(\hat{\theta})$	0.1312	0.0588	0.0559	0.0292	0.0362	0.0203	0.0272	0.0154	0.1514	0.0626	0.0662	0.0338	0.0451	0.0258	0.0319	0.0210
MLE($\hat{\theta}$)	3.1676	4.9365	3.1250	4.9384	3.1052	4.9384	3.0986	4.9386	3.2627	4.8741	3.2150	4.8761	3.1958	4.8784	3.1864	4.8775
$\widehat{Var}(\hat{\theta})$	0.1317	0.0567	0.0599	0.0279	0.0392	0.0189	0.0292	0.0141	0.1372	0.0521	0.0630	0.0251	0.0414	0.0174	0.0308	0.0131
$\widehat{MSE}(\hat{\theta})$	0.1598	0.0607	0.0755	0.0315	0.0503	0.0227	0.0390	0.0178	0.2062	0.0680	0.1092	0.0405	0.0797	0.0322	0.0655	0.0281

Table 6: The underlying is Weibull and contamination is BurrIII distributions

$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$				
Case 1: $(1-\varepsilon)W(1,5)+\varepsilon B(2,20)$, $\varepsilon=0.1$								Case 1: $(1-\varepsilon)W(1,5)+\varepsilon B(2,20)$, $\varepsilon=0.2$								
	$n=50, q=0.94$		$n=100, q=0.94$		$n=150, q=0.94$		$n=200, q=0.94$		$n=50, q=0.89$		$n=100, q=0.89$		$n=150, q=0.89$		$n=200, q=0.89$	
$MLqE(\hat{\theta})$	1.0576	5.0057	1.0422	5.0046	1.0374	4.9953	1.0347	4.9909	1.1055	5.0499	1.0895	5.0407	1.0855	5.0421	1.0832	5.0372
$\widehat{Var}(\hat{\theta})$	0.0179	0.5367	0.0083	0.2677	0.0054	0.1777	0.0041	0.1363	0.0221	0.5177	0.0103	0.2601	0.0069	0.1736	0.0050	0.1310
$\widehat{MSE}(\hat{\theta})$	0.0212	0.5367	0.0100	0.2677	0.0068	0.1777	0.0053	0.1363	0.0332	0.5202	0.0183	0.2617	0.0142	0.1753	0.0119	0.1324
$MLE(\hat{\theta})$	1.0530	5.3062	1.0344	5.3124	1.0274	5.3053	1.0236	5.3019	1.0795	5.6179	1.0560	5.6162	1.0484	5.6207	1.0424	5.6168
$\widehat{Var}(\hat{\theta})$	0.0174	0.5776	0.0089	0.2903	0.0061	0.1932	0.0050	0.1495	0.0218	0.6135	0.0116	0.3085	0.0088	0.2059	0.0070	0.1576
$\widehat{MSE}(\hat{\theta})$	0.0202	0.6714	0.0100	0.3879	0.0069	0.2864	0.0055	0.2407	0.0281	0.9953	0.0148	0.6882	0.0111	0.5911	0.0088	0.5381
$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
Case 2: $(1-\varepsilon)W(3,5)+\varepsilon B(2,20)$, $\varepsilon=0.1$								Case 2: $(1-\varepsilon)W(3,5)+\varepsilon B(2,20)$, $\varepsilon=0.2$								
	$n=50, q=0.89$		$n=100, q=0.87$		$n=150, q=0.87$		$n=200, q=0.87$		$n=50, q=0.81$		$n=100, q=0.81$		$n=150, q=0.81$		$n=200, q=0.81$	
$MLqE(\hat{\theta})$	3.0249	5.0598	3.0102	5.0515	2.9923	5.0422	2.9935	5.0425	3.0853	5.0898	3.0407	5.0752	3.0137	5.0756	3.0157	5.0760
$\widehat{Var}(\hat{\theta})$	0.1834	0.0740	0.0874	0.0351	0.0562	0.0227	0.0429	0.0183	0.2157	0.0858	0.0967	0.0417	0.0636	0.0272	0.0491	0.0209
$\widehat{MSE}(\hat{\theta})$	0.1840	0.0776	0.0875	0.0378	0.0562	0.0245	0.0430	0.0201	0.2230	0.0939	0.0984	0.0473	0.0638	0.0329	0.0494	0.0266
$MLE(\hat{\theta})$	2.4092	5.3579	2.1905	5.3762	2.0784	5.3782	2.0086	5.3820	2.0350	5.7129	1.8471	5.7146	1.7609	5.7196	1.7135	5.7233
$\widehat{Var}(\hat{\theta})$	0.4280	0.1964	0.2898	0.1031	0.2456	0.0725	0.2043	0.0569	0.3500	0.3050	0.2018	0.1572	0.1518	0.1118	0.1191	0.0862
$\widehat{MSE}(\hat{\theta})$	0.7771	0.3245	0.9450	0.2446	1.0950	0.2155	1.1872	0.2028	1.2812	0.8132	1.5309	0.6679	1.6872	0.6296	1.7742	0.6094
Case 3: $(1-\varepsilon)W(5,1)+\varepsilon B(2,20)$, $\varepsilon=0.1$								Case 3: $(1-\varepsilon)W(5,1)+\varepsilon B(2,20)$, $\varepsilon=0.2$								
	$n=50, q=0.86$		$n=100, q=0.88$		$n=150, q=0.88$		$n=200, q=0.88$		$n=50, q=0.8$		$n=100, q=0.8$		$n=150, q=0.8$		$n=200, q=0.8$	
$MLqE(\hat{\theta})$	5.4852	0.9989	5.3394	1.0000	5.3218	0.9995	5.3078	0.9992	5.6303	1.0037	5.5981	0.9994	5.5816	0.9987	5.5781	0.9993
$\widehat{Var}(\hat{\theta})$	0.5120	0.0010	0.2345	0.0005	0.1585	0.0004	0.1169	0.0003	0.9365	0.0028	0.3323	0.0009	0.1932	0.0004	0.1482	0.0003
$\widehat{MSE}(\hat{\theta})$	0.7474	0.0010	0.3497	0.0005	0.2621	0.0004	0.2117	0.0003	1.3338	0.0028	0.6900	0.0009	0.5315	0.0004	0.4824	0.0003
$MLE(\hat{\theta})$	1.1026	1.5800	1.0635	1.5789	1.0379	1.5827	1.0278	1.5812	0.9505	2.1014	0.9278	2.0925	0.9142	2.0963	0.9075	2.0951
$\widehat{Var}(\hat{\theta})$	0.0410	0.0223	0.0243	0.0121	0.0174	0.0082	0.0134	0.0060	0.0208	0.0498	0.0110	0.0220	0.0089	0.0160	0.0071	0.0114
$\widehat{MSE}(\hat{\theta})$	15.2304	0.3587	15.5201	0.3473	15.7159	0.3478	15.7914	0.3437	16.4190	1.2628	16.5937	1.2155	16.7024	1.2177	16.7557	1.2105
Case 4: $(1-\varepsilon)W(10,8)+\varepsilon B(2,20)$, $\varepsilon=0.1$								Case 4: $(1-\varepsilon)W(10,8)+\varepsilon B(2,20)$, $\varepsilon=0.2$								
	$n=50, q=0.82$		$n=100, q=0.82$		$n=150, q=0.82$		$n=200, q=0.8$		$n=50, q=0.7$		$n=100, q=0.7$		$n=150, q=0.69$		$n=200, q=0.69$	
$MLqE(\hat{\theta})$	9.8204	7.9524	9.6892	7.9526	9.6657	7.9558	9.8622	7.9581	10.0918	7.9249	9.8777	7.9264	9.9913	7.9367	9.9574	7.9362
$\widehat{Var}(\hat{\theta})$	1.7467	0.0190	0.8708	0.0094	0.5688	0.0064	0.4775	0.0048	2.8331	0.0239	1.3950	0.0122	0.9690	0.0082	0.7346	0.0067
$\widehat{MSE}(\hat{\theta})$	1.7789	0.0212	0.9673	0.0116	0.6806	0.0084	0.4965	0.0066	2.8415	0.0295	1.4100	0.0176	0.9691	0.0123	0.7364	0.0108
$MLE(\hat{\theta})$	5.5193	8.2448	4.4450	8.2947	3.8883	8.3273	3.5565	8.3300	3.5681	8.4284	2.9810	8.4351	2.6693	8.4701	2.5040	8.4876
$\widehat{Var}(\hat{\theta})$	5.4952	0.3408	4.0032	0.2034	3.0931	0.1523	2.2446	0.1183	2.5827	0.4989	1.4444	0.2649	0.9102	0.1950	0.6641	0.1617
$\widehat{MSE}(\hat{\theta})$	25.5720	0.4007	34.8611	0.2903	40.4455	0.2594	43.7628	0.2272	43.9517	0.6824	50.7104	0.4542	54.6498	0.4160	56.8545	0.3994

$\log(f)$ cannot model well. However, for some designs of contamination, $\widehat{MSE}(\hat{\theta})$ of $\log(f)$ can be smaller than that of $\log_q(f)$, because the used objective function is an important indicator for modeling capability. For example, $(1-\varepsilon)\text{Weibull}(\alpha_0=1, \beta_0=5) + \varepsilon \text{Weibull}(\alpha_1=2, \beta_1=8)$.

- $\widehat{MSE}(\hat{\theta})$ were used to compare the performance of MLqE and MLE. Depending on the structure of contamination, the unbiasedness of estimators $\hat{\theta}$ obtained by MLqE were examined by simulation. However, the contamination structure, i.e. the selected f_1 distribution and its parameter values and contamination rate ε can lead to observe the biasedness in Tables 3-6. In addition, in the some cases of MLqE, $\widehat{MSE}(\hat{\alpha}) \approx \widehat{Var}(\hat{\alpha})$ and $\widehat{MSE}(\hat{\beta}) \approx \widehat{Var}(\hat{\beta})$ for the shape α and the scale β parameters, respectively.
- When the sample size is increased, the values of $\widehat{MSE}(\hat{\alpha})$ and $\widehat{MSE}(\hat{\beta})$ for MLqE decreased, as expected. However, this is not observed by the results from MLE. The structure of inliers and outliers affected partially the estimation of shape parameter α .
- For some contamination and the values of parameters α_0 and β_0 for Weibull, the values of q can differ for different sample sizes. In addition, q differs for different contamination structure and $\text{Weibull}(\alpha_0, \beta_0)$, as expected. The value of q is chosen until the smallest values of $\widehat{MSE}(\hat{\theta})$ are obtained. If the constant q with

MLqE provides the smallest values for $\widehat{MSE}(\hat{\theta})$, then it is not possible to find another value for q which will give the smallest values for $\widehat{MSE}(\hat{\theta})$.

In overall assesment about robustness and modeling, the modeling is a procedure applied the finite sample size. The robustness of score function is based on the limit values at point zero and inifinity. Since the main idea is based on the modeling capability of $\log_q(f)$, using finiteness of score functions for implying robustness is not necessary (see Table 1 for $q > 1$ and $\alpha \geq 1$). When we look at the results of simulation, we have a result in which $q > 1$ and $\alpha \geq 1$. When we use p.d. $f(x; \hat{\theta})$ and $q > 1$, the values of $w_q = f(x; \hat{\theta})^{1-q}$ are bigger than 1. $\log_q(f)$ gives an advantageous for us to have $w \in [0, \infty]$ and so it is flexible to perform an efficient modeling.

5.4 Real data application

Real data sets are applied to test the performance of MLE and MLqE methods. The estimates of $\hat{\theta}$ and Kolmogorov-Smirnov (KS) test statistics are provided by Tables 7-10. We give the plots of c.d. and p.d. functions (CDF and PDF) in Figures 5-9 for illustrative purpose and make a comparison among the fitting competence with estimates from MLE and MLqE. The variance-covariance of M-estimators Huber 2004; Godambe 1960; Godambe and Thompson 1978 are mainly based on the Taylor expansion of score function around the true value of parameter. Even if M-estimation method provides a family for estimation methodology, a tool from information geometry was provided by Çankaya and Korbel 2018 as well. The Taylor expansion approach Huber 2004 is also used by Xing, (2015) to determine the value of q . Instead of using variance-covariance based on Taylor expansion as a rough approach for score function based on $\log_q$, the p -value of KS test statistic should be preferred. The different q -values are tried until the highest p -value of KS test statistic from c.d. function of Weibull is reached. This approach is also supported by Figures 5-9 which depict the fitting of c.d. and p.d. functions of Weibull distribution with $\hat{\alpha}$ and $\hat{\beta}$ (Çankaya 2020).

It is difficult to know the nature of reality and also knowing modality and bimodality in an empirical distribution or a real data set. We have to assume a parametric model and estimate the parameters of underlying distribution as much as we can do. We use two real data sets which are modeled by objective functions $\log_q[f(x; \alpha, \beta)]$ and $\log[f(x; \alpha, \beta)]$. Contaminations are applied into real data sets. Thus, we will test the performance of MLE and MLqE when the different types of contamination exist in real data sets. Three types of contamination to the real data set were performed. These are given by the following items:

1. Inliers: $\alpha_1 = 5; \beta_1 = 10; n_1 = 100$ and $\alpha_1 = \min(x) + 0.5; \beta_1 = \max(x) - 0.5; n_1 = 10$; for examples 1 and 2, respectively. $x_1 = \text{randuni}(n_1, \alpha_1, \beta_1); x = [x; x_1]$;

randuni is a function written in MATLAB2013a to generate uniform artificial data set from the interval $[a, b]$. $\min(x)$ and $\max(x)$ represent the minimum and maximum value of observations, respectively.

2. Outliers: $x = [x; 2 \max(x); 3 \max(x); 4 \max(x); 5 \max(x)]$.
3. Both of inliers and outliers: $x = [x; 2 \max(x); 3 \max(x); 4 \max(x); 5 \max(x); x_1]$;

5.4.1 Real data application: Example 1

This section consists of the numerical example for the application of real data set. R Version 4.0.2 and some packages such as `source("http://bioconductor.org/biocLite.R")`, `biocLite("GEOquery")` and `require(GEOquery)` are used to reach the real data set. We use the real data set from "test\$myMean". The sample size n for this data set is 6136. The value of tuning constant is chosen until the highest p -value of KS test statistic is obtained when q is 0.85, which means that the best values for estimates of parameters can be obtained.

Table 7: The estimates of parameters α and β via MLE and MLqE methods

	Without contamination		With inliers		With outliers		With both	
θ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
MLE(θ)	3.5427(.00045)	7.9935(.00038)	3.5623(.00045)	7.9937(.00038)	2.5596(.00065)	7.9907(.00044)	2.5869(.00032)	8.0862(.00053)
MLqE(θ)	3.9137(.00065)	7.9942(.00044)	3.9382(.00064)	7.9985(.00043)	3.9133(.00033)	7.9940(.00054)	3.9398(.00064)	7.9961(.00043)

Table 7 shows that MLqE can be robust to inliers and outliers at each case. The estimates of scale parameter β can be similar to each other at each case for MLE and MLqE methods. MLE and MLqE with inliers cannot be more different than that of without the contamination case. However, when MLE and MLqE methods are compared for the case of outliers, it is seen that the shape parameter α obtained by MLE is very sensitive outlier. For the real data set, the results show that MLqE is robust to outliers, because the score functions derived from $\log_q$ for parameters α and β are finite. For this real data set, the sensitivity of estimates of β from MLE cannot be more, however when the estimates of β is compared with that of MLqE, MLqE is insensitive to contamination in all of cases. The estimates of α from MLqE can be insensitive to contamination in all of cases. Especially, the estimates of β from MLqE for both contamination are insensitive when compared with that of MLE.

Table 8: The p -value of KS test statistics computed by the estimates of parameters α and β via MLE and MLqE

	Without contamination	With inliers	With outliers	With both
θ	p -value	p -value	p -value	p -value
MLE(θ)	4.3225e-07	2.3326e-07	6.8452e-76	2.6225e-67
MLqE(θ)	1.2516e-04	1.2895e-04	1.2767e-04	1.3000e-04

Table 8 shows the p -value of KS test statistics. According to p -values of KS test statistic, there does not exist enough evidence to accept the null hypothesis H_0 which shows that the real data set is a member of Weibull distribution. However, note that if the significance level of test statistic is chosen to be 10^{-5} , then Weibull with MLqE(θ) provides sufficient evidence not to reject the null hypothesis H_0 . Let us focus on the p -values instead of considering whether or not the data set does really come from Weibull distribution with parameters α and β which were estimated by MLqE and MLE methods. The p -values of KS test statistic obtained from two cases which are

Figure 5: CDF, PDF and histogram when real data set does not have inliers and outliers for bioconductor test data(color online)

Figure 6: CDF, PDF and histogram when real data set has inliers for bioconductor test data (color online)

Figure 7: CDF, PDF and histogram when real data set has outliers for bioconductor test data(color online)

Figure 8: CDF, PDF and histogram when real data set has inliers and outliers for bioconductor test data(color online)

outliers and both contaminations are very small when they are compared with that of MLqE, which shows that adding outliers to real data set makes more far from Weibull distribution with the estimates obtained from MLE. However, the p -values of KS test statistic with estimates of MLqE(θ) did not differ much for three cases which are with inliers, outliers and both contaminations. Let us focus on the comparison of MLE for without contamination and with inliers cases, p -values from $4.3225e-07$ to $2.3326e-07$ tend to be near to zero due to the fact that adding inliers affects the estimates of MLE. In MLqE, such tendency being zero is not observed because of robustness property of MLqE.

5.4.2 Real data application: Example 2

The data are the strengths of 1.5 cm glass fibres measured by the National Physical Laboratory, England. The sample size is $n = 63$ (Smith and Naylor 1987). The distributions and references therein Nassar et al. 2017 are used to model the data set. The estimates from Weibull with $\log_q(f)$ as an objective function give the p -value which is bigger than that of distributions in (Nassar et al. 2017). Note that the parametric model and objective functions should be tried for the case in which we can improve the modeling competence.

Table 9: The estimates of parameters α and β via MLE and MLqE methods

θ	Without contamination		With inliers		With outliers		With both	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
MLE(θ)	5.7762(.07149)	1.6275(.00471)	5.8626(.06262)	1.6164(.00398)	1.4587(.01697)	2.1153(.02278)	1.4961(.01515)	2.0564(.01879)
MLqE(θ)	7.5423(.10755)	1.6401(.00393)	7.6146(.09359)	1.6325(.00334)	7.5136(.10077)	1.6392(.00371)	7.5692(.08819)	1.6231(.00317)

Table 9 shows that MLqE can be robust to inliers and outliers at each case. For all of scenarios of contaminations which are inliers, outliers and both of them, the estimates $\hat{\alpha}$ and $\hat{\beta}$ from MLE are sensitive to contamination. However, such sensitivity has not been observed at MLqE. When outliers and both contaminations are examined, it is observed that the estimates from MLE are very sensitive to contamination schemas. However, the estimates from MLqE can have similar values at which there are not contaminations into data set, which shows that MLqE are robust.

Table 10: The p -value of KS test statistics computed by the estimates of parameters α and β via MLE and MLqE

θ	Without contamination	With inliers	With outliers	With both
	p -value	p -value	p -value	p -value
MLE(θ)	0.0936	0.0740	9.6151e-07	1.1024e-07
MLqE(θ)	0.7283	0.4501	0.4504	0.2700

Table 10 shows the p -value of KS test statistics. According to p -values of Kolmogorov-Smirnov (KS) as a goodness of fit test, there exists enough evidence to accept the null hypothesis H_0 which shows that the real data set is a member of Weibull distribution. MLqE and MLE depend on $\log_q$ and $\log$ functions respectively. So the importance of the used objective function has been observed when we make a comparison between the p -values which are 0.0936 and 0.7283 of MLE and MLqE respectively. The value of tuning constant is chosen until the highest p -value of KS

test statistics is obtained when q is 0.8. Even though there is no strict changing of the values of estimates from MLqE, the p -values of KS test statistics from MLqE go to lower values. This is due to the definition of KS test statistic which uses values of x (see codes for the computation of p -value in Appendix 6). Figures 9-12 illustrate that there exist a good fitting by MLqE, i.e. CDF_{MLqE} and PDF_{MLqE} , as supported by p -values of KS test statistics in Table 10.

Figure 9: CDF, PDF and histogram when real data set does not have inliers and outliers for 1.5 cm glass fibres(color online)

Figure 10: CDF, PDF and histogram when real data set has inliers for 1.5 cm glass fibres(color online)

Figure 11: CDF, PDF and histogram when real data set has outliers for 1.5 cm glass fibres(color online)

The insensitivities for $\hat{\alpha}$ and $\hat{\beta}$ in Tables 7-9 to contaminations show not only the robustness of MLqE but also we can conclude that the modeling competence of $\log_q(f)$ is better than $\log(f)$.

Figure 12: CDF, PDF and histogram when real data set has inliers and outliers for 1.5 cm glass fibres(color online)

6 Conclusions and discussions

If the random variables $X_1, X_2, \dots, X_n$ are not distributed identically, different values of parameters of Weibull distribution have been estimated robustly by using MLqE. Numerical experiments have been applied to assess the performance of MLqE. In the simulation, Weibull with different values of parameters from underlying Weibull, BurrIII and uniform distributions have been used for schema of contamination which occurs non-identically. The performance of MLqE was tested for the functions having one mode property. The contaminated distributions f_1 have one mode property as well. GA which is powerful tool to avoid the local points of the optimized function has been used to get the estimates of parameters. Researchers can use MLqE to estimate the parameters of Weibull distribution having contamination from an one mode function. The p -value of KS test statistic should be used to determine the value of constant q for the real data set. Thus, the evaluation of fitting performance of Weibull with the estimated parameters can be tested easily. The score functions derived by $\log(f)$ and $\log_q(f)$ are infinite and finite for $0 < q < 1$ and $\alpha \geq 1$, respectively. The results in real data show that if $0 < q < 1$, then we have estimates which can be insensitive to the added inliers and outliers. When we consider on the results of simulation, we can have for inlier case in which $q > 1$ which shows that the score functions of parameters are infinite for $q > 1$ and $\alpha \geq 1$. Consequently, the numerical experiments show that robustness is not enough to imply that the best modeling is accomplished. By using the MLqE, the values of the parameters representing the majority of the distribution were estimated with small $\widehat{MSE}(\hat{\theta})$ for different scenarios of contaminations. Many results from simulation and the application of real data sets show that MLqE is capable to model efficiently and gives an advantage to obtain the estimates for parameters of underlying distribution.

Information geometry will be used (Amari 2016; Calin and Udriște 2014); and the adopted goodness of fit test will be proposed for further advance to determine the value of q by means of tools in statistics. In our future works, we will try to find counter examples from deformation family and its generalization as theoretical results and numerical experiments for different types of contamination will be used to test their modeling competence even if their score function is infinite.

Appendix

Computation of parameters via GA

- **Run:**

```
opt=gaoptimset('CrossoverFcn',{@crossoversinglepoint},'display','off');
```

```
1. lb=[0 0];ub=[10^10 10^10];
```

```
MLqE(t,:)=ga(@(p)fMLqE(p,x),2,[],[],[],[],lb,ub,[],opt);
```

- **Apply:** 'fMLqE' is a function given by

```
1. function [sumL] = fMLqE(p,x)
```

```
2. a=p(1);b=p(2);f=(a/b)*(x/b).^(a-1).*exp(-(x/b).^a);
```

```
3. sumL=-sum((f.^(1-q)-1)./(1-q));
```

When $q = 1$, line 3 is replaced by $\log(f)$.

Computation of p -value of KS test statistic

Apply: WeiCDF is a function written in MATLAB 2013a to compute the CDF values.

```
1. function F=WeiCDF(a,s,x)
```

```
2. F=1-exp(-(x/s).^a);
```

The p -value is given by

```
1. F=WeiCDF(alpha,beta,x);
```

```
2. test_cdf=[x,F];
```

```
3. [p_value]=kstest(x,'CDF',test_cdf);
```

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