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Beam propagation factor of Hollow higher-order Cosh-Gaussian beams

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Abstract

Based on the second-order moments definition, we investigate in this paper the beam propagation factor of new mathematical model of Hollow higher-order Cosh-Gaussian (HhCG) beams. Two analytical formulae of the M^2 -factor of HhCG beams are derived. Moreover, numerical simulations are developed to illustrate the effects of the beams orders n and l , the parameter δ and the beam waist ω_0 on the M^2 -factor. The result shows a more general characteristic of higher-order Cosh-Gaussian, Cosh-Gaussian and the fundamental Gaussian beams can be obtained as specials cases of HhCG beams.

Keywords: Hollow higher-order Cosh-Gaussian; Second-order moment; M^2 factor.

1. Introduction

In recent decades, the parametric characterization of the laser beam and beam quality have received considerable interest based on the second-order-moment. Early, the beam-propagation factor (M^2 -factor) proposed by Siegman [1-2] as a better parameter to characterize different laser beams and their mode structure. It plays an important role for practical applications [3]. On the basis of higher-order irradiance moments, the symmetry and degree of flatness of laser beams can be obtained [4, 5]. On the other hands, Hermite–Sinusoidal–Gaussian (HSG) beam have been proposed as exact solutions of the paraxial wave equation [6,7]. Gaussian beams, cosh-Gaussian beams and higher order cosh-Gaussian beams can be got as special cases of HSG beams. The propagation properties of these beams family have been the subject of many papers in several optical systems [8-17]. The beam propagation factor of hard-edge

diffracted and their mode-coherence coefficients of cosh-Gaussian beams have been examined [8,9]. The propagation characteristic of Hermite-cosh-Gaussian beams and elegant Hermite-cosh-Gaussian beams have been studied, and the corresponding beam propagation factor has been given and expressed in a concise form [9-15]. Recently, a new method to generate flattened Gaussian beam by incoherent combination of cosh Gaussian beams is established [16]. More recently, Saad and Belafhal have investigated a novel light beam of Hollow higher-order cosh-Gaussian as a generalized beam of cosh Gaussian [17]. Similarly, the M^2 -factor of various dark hollow beams in free space have been studied [18-25]. Recently, the beam propagation factor and the kurtosis parameter of a higher-order cosh-Gaussian beam have been reported [22]. In addition, generalized M^2 -factor of hard-edged diffracted Hypergeometric-Gaussian type-II beams has been carried out [23]. The M^2 -factor of controllable dark-hollow beams through a multi-apertured ABCD optical system have also been presented [24]. More recently, Zhou et al. have investigated the propagation factor and kurtosis parameter of hollow vortex Gaussian beams [25]. Up to now, to our knowledge, the M^2 -factor of HhCG beams hasn't been reported. In this paper, our aim is to investigate the M^2 -factor of HhCG beams. In Section 2, two analytical formulae for M^2 -factor of HhCG beams on propagation are derived. Some numerical examples are given and discussed in Section 3. The main results are outlined in Section 4.

2. Theoretical models

In this Section, the M^2 -factor of HhCG beams at the source plane ($z=0$) in the Cartesian coordinate system is evaluated as [17]

$$E_{n,l}(x,y,0) = E_{n,l}(x,0)E_{n,l}(y,0), \quad (1)$$

where $E_{n,l}(x,0)$ and $E_{n,l}(y,0)$ are given by

$$E_{n,l}(j,0) = A_0 \left(\frac{j}{\omega_0} \right)^l \exp(-j^2/\omega_0^2) \cosh^n(\Omega j), \quad (2)$$

with $j = x$ or y , ω_0 is the Gaussian waist width, Ω is the parameter associated with the cosh part, n and l are the beams orders and A_0 is the amplitude. $E_{n,l}(j,0)$ can also be written in the form as follows

$$E_{n,l}(j,0) = \frac{A_0}{2^n} \left(\frac{j}{\omega_0} \right)^l \sum_{m=0}^n \frac{n!}{m!(n-m)!} \exp(a_{mm}^2 \delta) \exp \left[- \left(\frac{j}{\omega_0} - a_{mm} \sqrt{\delta} \right)^2 \right], \quad (3)$$

where

$$a_{mm} = \left(m - \frac{n}{2} \right) \text{ and } \delta = \omega_0^2 \Omega^2.$$

Our goal is to evaluate the M^2 -factor, which is defined by

$$M^2 = \sqrt{M_x^2 M_y^2}. \quad (4)$$

Because their symmetry, the two components M_x and M_y are equivalents and consequently $M_x = M_y$ and the M^2 -factor of the considered beams family is evaluated as

$$M^2 = M_x^2 (= M_y^2). \quad (5)$$

In the following, we will evaluate only the component M_x which is defined by

$$M_x^2 = 4\pi \sigma_{0x} \cdot \sigma_{\infty x}, \quad (6)$$

where σ_{0x} and $\sigma_{\infty x}$ are the second-order moments of the intensity distribution in the spatial domain and in the spatial frequency domain, respectively. In accordance to the second order moments definition of σ_{0x} in the waist plane, we have [1]

$$\sigma_{0x}^2 = \frac{I_{v=2}}{I_{v=0}}, \quad (7)$$

where

$$I_v = \int_{-\infty}^{+\infty} x^v |E_{n,l}(x,0)|^2 dx. \quad (8)$$

If we substitute Eq. (3) into Eq. (8), and recalling the following integral formula [26]

$$\int_{-\infty}^{+\infty} x^m e^{-px^2 + 2qx} dx = e^{q^2/p} \sqrt{\frac{\pi}{p}} \left(\frac{1}{2i\sqrt{p}} \right)^m H_m \left(i \frac{q}{\sqrt{p}} \right), \text{ with } p > 0, \quad (9)$$

after tedious integral calculations, then, the result of Eq. (8) can be obtained as

$$I_v = \frac{A_0^2}{4^n} \sqrt{\frac{\pi}{2}} \omega_0^{v+1} \left(\frac{1}{2i\sqrt{2}} \right)^{2l+v} Q_{n,l}^0(v), \quad (10)$$

where

$$Q_{n,l}^0(v) = \sum_{m=0}^n \sum_{m'=0}^n \frac{(n!)^2}{m!m'!(n-m)!(n-m')!} e^{\frac{\delta}{2}(a_{mm}+a_{m'n})^2} H_{v+2l} \left(i\sqrt{\frac{\delta}{2}}(a_{mm} + a_{m'n}) \right). \quad (11)$$

The final result of σ_{0x} in the waist plane can be obtained by substituting Eqs. (10) and (11) into Eq. (7) to yield

$$\sigma_{0x} = \sqrt{\frac{I_2}{I_0}} = \frac{i\omega_0}{2\sqrt{2}} \sqrt{\frac{Q_{n,l}^0(2)}{Q_{n,l}^0(0)}}. \quad (12)$$

where $Q_{n,l}^0(2)$ and $Q_{n,l}^0(0)$ are given by Eq. (11).

Similar calculations can be performed in the spatial-frequency domain. As well-known, this moment is given for the intensity distribution of the far-field and is defined as $\theta/2\pi$ with θ is the far-field during angle. In the spatial-frequency domain, the considered optical field reads as [27]

$$E_{nl}(f_x, 0) = \int_{-\infty}^{+\infty} E_{nl}(x, 0) \exp(-i2\pi f_x x) dx, \quad (13)$$

where f_x is the transverse spatial frequency. Inserting Eq. (3) into Eq. (13) and integrating in the use of Eq. (9), we obtain

$$E_{nl}(f_x, 0) = \frac{\sqrt{\pi}\omega_0 A_0}{i^l 2^{n+l}} \sum_{m=0}^n \frac{n!}{m!(n-m)!} e^{X_{mn}^2} H_l(iX_{mn}), \quad (14)$$

with

$$X_{mn} = a_{mn} \sqrt{\delta} - i\pi f_x \omega_0. \quad (15)$$

According to the second-order moment definition of $\sigma_{\infty x}$ in spatial frequency, one obtains [1]

$$\sigma_{\infty x} = \frac{K_2}{K_0} \quad (16)$$

where

$$K_v = \int_{-\infty}^{+\infty} f_x^v |E_{nl}(f_x, 0)|^2 df_x. \quad (17)$$

On substituting from Eq. (14) into Eq. (17) and used the integral formula of Eq. (9), and after some calculations, the result of Eq. (17) can be arranged as

$$K_v = \frac{\pi A_0^2 \omega_0^2}{2^{2n+2l}} \sum_{m=0}^n \sum_{m'=0}^n \frac{(n!)^2}{m!m'!(n-m)!(n-m')!} J_{v,mn}, \quad (18)$$

where $J_{v,mn}$ is given by

$$J_{v,mn} = \int_{-\infty}^{+\infty} f_x^v e^{(X_{mn}^2 + X_{m'n}^2)} H_l(iX_{mn}) H_l(-iX_{m'n}) df_x. \quad (19)$$

By using the following formula [28,29]

$$H_n(x+y) = \frac{1}{2^{n/2}} \sum_{k=0}^n \binom{n}{k} H_k(\sqrt{2}x) H_{n-k}(\sqrt{2}y), \quad (20)$$

Eq. (19) becomes

$$\begin{aligned} J_{v,mn} = & \frac{1}{2^l} e^{(a_{mn}^2 + a_{m'n}^2)\delta} \sum_{r=0}^l \sum_{p=0}^l \binom{l}{r} \binom{l}{p} H_{l-r}(\varepsilon_{mn} \sqrt{2}) H_{l-p}(-\varepsilon_{m'n} \sqrt{2}) \\ & \times \sum_{k=0}^{[r/2]} \sum_{k'=0}^{[p/2]} \frac{(-1)^{k+k'} r! p!}{k!(r-2k)! k'!(p-2k')!} (2\pi\sqrt{2})^{r+p-2k-2k'} \\ & \times \int_{-\infty}^{+\infty} f_x^{v+r+p-2k-2k'} \exp(-af_x^2 + 2bf_x) df_x, \end{aligned} \quad (21)$$

with $a = 2\pi^2 \omega_0^2$, $b = i\pi\omega_0 \sqrt{\delta}(a_{m'n} - a_{mn})$, $\varepsilon_{mn} = i\sqrt{\delta}a_{mn}$ and $\varepsilon_{m'n} = i\sqrt{\delta}a_{m'n}$.

If we recall the integral formula of Eq. (9), and insert Eq. (21) into Eq. (18), we then obtain

$$K_v = A_{n,l}(v) Q_{n,l}^\infty(v), \quad (22)$$

where

$$\begin{aligned} Q_{n,l}^\infty(v) = & \sum_{m=0}^n \sum_{m'=0}^n \frac{(n!)^2}{m! m'! (n-m)! (n-m')!} e^{\frac{\delta}{2}(a_{mn} + a_{m'n})^2} \\ & \times \sum_{r=0}^l \sum_{p=0}^l \binom{l}{r} \binom{l}{p} H_{l-r}(i\sqrt{2\delta}a_{mn}) H_{l-p}(-i\sqrt{2\delta}a_{m'n}) \\ & \times \sum_{k=0}^{[r/2]} \sum_{k'=0}^{[p/2]} \frac{(-1)^{k+k'} (r)! p!}{k!(r-2k)! k'!(p-2k')!} \frac{1}{i^{r+p-2k-2k'}} H_{v+r+p-2k-2k'} \left(\sqrt{\frac{\delta}{2}} (a_{mn} - a_{m'n}) \right), \end{aligned} \quad (23)$$

and

$$A_{n,l}(v) = \frac{A_0^2}{\sqrt{2\pi}} \frac{\omega_0}{2^{2n+2l}} \frac{1}{(2i\sqrt{2\pi}\omega_0)^v}. \quad (24)$$

Finally, the second moment of the intensity distribution in the spatial frequency domain $\sigma_{\infty x}$ can be written as follows

$$\sigma_{\infty x} = \frac{K_2}{K_0} = \sqrt{\frac{A_{n,l}(2) Q_{n,l}^\infty(2)}{A_{n,l}(0) Q_{n,l}^\infty(0)}} = \frac{1}{2i\pi\sqrt{2\omega_0}} \sqrt{\frac{Q_{n,l}^\infty(2)}{Q_{n,l}^\infty(0)}}, \quad (25)$$

where $Q_{n,l}^\infty(2)$ and $Q_{n,l}^\infty(0)$ are given by Eq. (23).

Thus, from Eq. (6) and by using Eqs. (12) and (25), the M^2 -factor of HhCG beams turns out to be

$$M_x^2 = 4\pi\sigma_{0x} \cdot \sigma_{\infty x} = \frac{1}{2} \sqrt{\frac{Q_{n,l}^0(2)Q_{n,l}^\infty(2)}{Q_{n,l}^0(0)Q_{n,l}^\infty(0)}}. \quad (26)$$

This last expression is the closed form propagation formula of the considered beam which describes the M^2 -factor of HhCG beams.

3. Numerical results and discussions

In this Section, the numerical calculations of the M^2 -factor of HhCG beams are performed based on our main analytical result given by Eq. (26). The numerical simulations are illustrated by studying the effects of the beam parameters (l, n, δ, ω_0 and Ω) on the beam propagation factor of HhCG beams at the source plane ($z=0$). In Fig. 1(a-d), the M^2 -factor is plotted as a function of parameter δ , for various values of beam orders n and l .

Figure 1: The M^2 -factor of HHCGs as a function of parameter δ and with different values of l : (a) $n=1$, (b) $n=4$, (c) $n=7$ and (d) $n=10$.

It can be seen from these plots that, the M^2 -factor of HhCG beams increases with the growth of δ ($\delta > 0$) and as $\delta = 0$, it still maintains the same minimum values, for three values of the hollowness parameter l and for four values of the beam order n (1, 4, 7 and 10), e.g., when $l = 0$, 1 and 5, the M^2 -factor always equals to 1, 3 and 4.819, respectively, this means that the central dark size increases. One can also note that the larger the beam order n is, the maximum value of M^2 -factor is. This characteristic may be considered to describe highly divergent HhCG beams. In addition, our result agrees well with Fig. 1 ($\alpha=\infty$) of Ref. [9], as $l=0$, $n=1$, and as $l=0$, $n=2$, it is consistent with Ref. [22].

Also, the M^2 -factor of HhCG beams as a function of beam order n is investigated for $l=1$ at different values of δ , which is shown in Fig. 2.

Figure 2: The M^2 -factor of HhCG beams as a function of the beam order n and with the hollowness order $l = 1$: (a) $\delta=0.1$, (b) $\delta=1$, (c) $\delta=3$ and (d) $\delta=10$.

One can easily see that, as n increases the maximum value of M^2 -factor augments, while, its minimum value keeps at the same value (here, it equals to 3). One can notice that, the large δ gives the large maximum value to M^2 -factor, e.g., as $\delta = 0.1, 1, 3$ and 10 , the maximum M^2 -factor corresponds to $4.67, 12.29, 20.95$ and 34.89 .

In Fig. 3, we investigate the effect of the beam waist ω_0 on the M^2 -factor of the considered beams.

Figure 3: The M^2 -factor as a function of ω_0 , with $l=1$, $\Omega=10 \text{ m}^{-1}$ and for n : (a) $n=1$ (b) $n=4$, (c) $n=7$ and (d) $n=10$.

The calculation parameters are $l=1$ and $\Omega=10 \text{ m}^{-1}$. The curves show that the M^2 -factor is fixed at about 3 as $\omega_0 \leq 6 \text{ cm}$ for all values of δ and then strongly augments with n and δ are increased. These results are consistent with the results of the previous figure, as expected.

Finally, from the above numerical results, it follows that the M^2 -factor of HhCG beams closely related to their initial beam parameters (l , n , ω_0 and δ).

6. Conclusion

In summary, the propagation factor of HhCG beams has been investigated. Two analytical expressions of M^2 factor for HhCG beams have been derived in detail. According to the derived formula, the properties of M^2 factor beams are illustrated numerically as a function of initial beam parameters. The change of the M^2 -factor of HhCG beams closely related to their initial beam parameters (l , n , ω_0 and δ). The findings of this propagation factor are helpful to the practical applications of a hollow higher-order cosh–Gaussian beam, such as in fiber communication and laser optics.

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