The New Phi Function and some of it's Applications

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The New Phi Function and some of it's Applications.

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1. Abstract

In Mathematics, we see a large number of functions, each having their own properties. Some of these are very interesting and contribute greatly to the intensive research in the field of Mathematics. This paper deals with one such function (which we have termed as the phi function) which emerges from a chain of inequalities, established from the basic concepts of differential calculus. This paper establishes several inequalities which relate functions and their integrals. There is another important expression (from the point of view of notations), which links a class of divergent infinite series to the phi function. Finally we will dive into a brief overview of the *phi - form* of plane trigonometric functions and derive the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$, thus marking their importance. Throughout the paper, we will be analyzing functions in $\mathbb{R}^+$ such that the functions are always greater than 0. We will also consider that the functions are continuous and differentiable in the intervals under consideration.

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2. Background

Proposition 2.1. $\frac{d}{dx}(\prod_{k=1}^n f_k(x)) = (\prod_{k=1}^n f_k(x))(\sum_{i=1}^n (\frac{f'_i(x)}{f_i(x)}))$

Proof. By the product rule, $\frac{d}{dx}(\prod_{k=1}^n f_k(x)) = f'_1(\frac{\prod_{k=1}^n f_k(x)}{f_1}) + f'_2(\frac{\prod_{k=1}^n f_k(x)}{f_2}) + \dots + f'_n(\frac{\prod_{k=1}^n f_k(x)}{f_n})$. ■

Proposition 2.2. $\frac{d}{dx}(\sum_{i=1}^n \frac{1}{f_i(x)}) = (-1) \sum_{i=1}^n \frac{f'_i(x)}{(f_i(x))^2}$

Proof. $\frac{d}{dx}(\sum_{i=1}^n \frac{1}{f_i(x)}) = \sum_{i=1}^n \frac{d}{dx}(\frac{1}{f_i(x)}) = \sum_{i=1}^n \frac{-f'_i(x)}{(f_i(x))^2} \implies \frac{d}{dx}(\sum_{i=1}^n \frac{1}{f_i(x)}) = (-1) \sum_{i=1}^n \frac{f'_i(x)}{(f_i(x))^2}$ ■

Proposition 2.3. $\frac{d}{dx}(\frac{1}{\prod_{k=1}^n f_k(x)}) = -\frac{\sum_{i=1}^n \frac{f'_i(x)}{(f_i(x))}}{\prod_{m=1}^n f_m(x)}$

Proof. By the quotient rule, $\frac{d}{dx}(\frac{1}{\prod_{k=1}^n f_k(x)}) = \frac{0 - \frac{d}{dx}(\prod_{k=1}^n f_k(x))}{(\prod_{k=1}^n f_k(x))^2} = -\frac{\sum_{i=1}^n \frac{f'_i(x)}{(f_i(x))}}{\prod_{m=1}^n f_m(x)}$.
The last step was a change in the index from k to m. ■

3. Introduction to the Phi function

3.1. An Experiment

3.1.1. Let us examine the function $\frac{d}{dx}(\frac{f_i(x)}{\prod_{k=1}^n f_k(x)})$, where $i \in [1, n]$:

$$\begin{aligned} \frac{d}{dx}(\frac{f_i(x)}{\prod_{k=1}^n f_k(x)}) &= \frac{(\prod_{k=1}^n f_k(x))(f'_i(x)) - f_i(x)(\frac{d}{dx} \prod_{k=1}^n f_k(x))}{(\prod_{k=1}^n f_k(x))^2} \\ \implies \frac{d}{dx}(\frac{f_i(x)}{\prod_{k=1}^n f_k(x)}) &= \frac{f_i^0(x)(f'_i(x) - f_i(x)(\sum_{k=1}^n \frac{1}{f_k(x)} \frac{d}{dx} f_k(x)))}{\prod_{k=1}^n f_k(x)} = \frac{f'_i(x)}{\prod_{k=1}^n f_k(x)} - \frac{(f_i(x)(\sum_{k=1}^n \frac{f'_k(x)}{f_k(x)})}{\prod_{k=1}^n f_k(x)} \end{aligned}$$

From the above expression that we got, let us define a new function $\phi_1(x)$ whose value is $\frac{f_i^0(x)f'_i(x)}{\prod_{k=1}^n f_k(x)}$. Therefore, for $i \in [1, n]$,

$$\phi_1(x) = \frac{f_i^0(x)f'_i(x)}{\prod_{k=1}^n f_k(x)} \quad (0.1)$$

3.1.2. Let us examine the function $\frac{d}{dx}(\frac{f_i^2(x)}{\prod_{k=1}^n f_k(x)})$, where $i \in [1, n]$:

$$\begin{aligned} \frac{d}{dx}(\frac{f_i^2(x)}{\prod_{k=1}^n f_k(x)}) &= \frac{(\prod_{k=1}^n f_k(x))(2f_i(x)f'_i(x)) - f_i^2(x)(\frac{d}{dx} \prod_{k=1}^n f_k(x))}{(\prod_{k=1}^n f_k(x))^2} \\ \implies \frac{d}{dx}(\frac{f_i^2(x)}{\prod_{k=1}^n f_k(x)}) &= \frac{2f_i(x)f'_i(x)}{\prod_{k=1}^n f_k(x)} - \frac{(f_i^2(x)(\sum_{k=1}^n \frac{f'_k(x)}{f_k(x)})}{\prod_{k=1}^n f_k(x)} \quad [\text{applying Proposition 2.1}]. \end{aligned}$$

Similar to 3.1.1, let's define another function $\phi_2(x)$ whose value is equal to $\frac{2f_i(x)f'_i(x)}{\prod_{k=1}^n f_k(x)}$. Therefore, for $i \in [1, n]$,

$$\phi_2(x) = \frac{2f_i(x)f'_i(x)}{\prod_{k=1}^n f_k(x)} \quad (0.2)$$

Observation : We can now formulate a general form of the ϕ function that we were giving instances of, in (0.1) and (0.2). That is: $\phi_\Omega(x) = \frac{\Omega f_i^{\Omega-1}(x)f'_i(x)}{\prod_{k=1}^n f_k(x)}$.

Proof. Let us examine the differential equation $\frac{d}{dx}(\frac{f_i^\Omega(x)}{\prod_{k=1}^n f_k(x)})$, where $i \in [1, n]$ and $\Omega \in \mathbb{Z}^+$.

$$\frac{d}{dx}(\frac{f_i^\Omega(x)}{\prod_{k=1}^n f_k(x)}) = \frac{(\prod_{k=1}^n f_k(x))(\Omega f_i^{\Omega-1}(x)f'_i(x)) - (f_i^\Omega(x))(\frac{d}{dx} \prod_{k=1}^n f_k(x))}{(\prod_{k=1}^n f_k(x))^2} = \frac{\Omega f_i^{\Omega-1}(x)f'_i(x)}{\prod_{k=1}^n f_k(x)} - \frac{f_i^\Omega(x)(\sum_{k=1}^n \frac{f'_k(x)}{f_k(x)})}{\prod_{k=1}^n f_k(x)}$$

Therefore, for $i \in [1, n]$ & $\Omega \in \mathbb{Z}^+$

$$\phi_\Omega(x) = \frac{\Omega f_i^{\Omega-1}(x)f'_i(x)}{\prod_{k=1}^n f_k(x)} \quad (0.3)$$

■

3.2 More Generalization

3.2.1. It does apparently seem that the function, $\phi_\Omega(x)$, which we defined in 3.1, is generalized.

But we can establish a more generalized expression for $\phi_\Omega(x)$:

Let us examine the differential equation $\frac{d}{dx} \frac{\sum_{k=1}^n f_k^2(x)}{\prod_{k=1}^n f_k(x)}$:

$$\begin{aligned} \frac{d}{dx} \frac{\sum_{k=1}^n f_k^2(x)}{\prod_{k=1}^n f_k(x)} &= \frac{(\prod_{k=1}^n f_k(x))(\frac{d}{dx} \sum_{k=1}^n f_k^2(x)) - (\sum_{k=1}^n f_k^2(x))(\frac{d}{dx} \prod_{k=1}^n f_k(x))}{(\prod_{k=1}^n f_k(x))^2} \\ \implies \frac{d}{dx} \frac{\sum_{k=1}^n f_k^2(x)}{\prod_{k=1}^n f_k(x)} &= \frac{2 \sum_{k=1}^n (f_k(x)f'_k(x))}{\prod_{k=1}^n f_k(x)} - \frac{(\sum_{k=1}^n f_k^2(x))(\sum_{k=1}^n \frac{f'_k(x)}{f_k(x)})}{\prod_{k=1}^n f_k(x)} \quad [\text{applying Proposition 2.1}] \end{aligned}$$

Let us define another function, $\phi'_2(x)$ [similar to that we defined in 2.1], whose value is equal to $\frac{2 \sum_{k=1}^n (f_k(x) f'_k(x))}{\prod_{k=1}^n f_k(x)}$. Therefore,

$$\phi'_2(x) = \frac{2 \sum_{k=1}^n (f_k(x) f'_k(x))}{\prod_{k=1}^n f_k(x)} \quad (0.4)$$

Remark: Here, $\phi'_2(x)$ does not mean the derivative of the function $\phi_2(x)$. In this context, $\phi'_2(x)$ depicts a function that is different from $\phi_2(x)$, as defined in 0.1.

3.2.2. Without taking further instances into consideration, let us compute now the differential equation $\frac{d}{dx} \frac{\sum_{i=1}^n f_i^\Omega(x)}{\prod_{i=1}^n f_i(x)}$, where $\Omega \in \mathbb{Z}^+$:

$$\begin{aligned} \frac{d}{dx} \frac{\sum_{i=1}^n f_i^\Omega(x)}{\prod_{i=1}^n f_i(x)} &= \frac{(\prod_{i=1}^n f_i(x)) (\frac{d}{dx} \sum_{i=1}^n f_i^\Omega(x)) - (\sum_{i=1}^n f_i^\Omega(x)) (\frac{d}{dx} \prod_{i=1}^n f_i(x))}{(\prod_{i=1}^n f_i(x))^2} \\ \implies \frac{d}{dx} \frac{\sum_{i=1}^n f_i^\Omega(x)}{\prod_{i=1}^n f_i(x)} &= \frac{\Omega(\sum_{i=1}^n (f_i^{\Omega-1}(x) f'_i(x)))}{(\prod_{i=1}^n f_i(x))} - \frac{(\sum_{i=1}^n f_i^\Omega(x)) (\sum_{i=1}^n \frac{f'_i(x)}{f_i(x)})}{(\prod_{i=1}^n f_i(x))} \quad [\text{applying Proposition 1.1}] \end{aligned}$$

Thus, we computed the new $\phi_\Omega(x)$, which is equal to $\frac{\Omega(\sum_{i=1}^n (f_i^{\Omega-1}(x) f'_i(x)))}{(\prod_{i=1}^n f_i(x))}$. Thus, for $\Omega \in \mathbb{Z}^+$:

$$\boxed{\phi_\Omega(x) = \frac{\Omega(\sum_{i=1}^n (f_i^{\Omega-1}(x) f'_i(x)))}{(\prod_{i=1}^n f_i(x))}} \quad (0.5)$$

Remark: From the above expression, we can argue that $\phi_\Omega(x)$ function is actually **an operator** which acts on several other functions.

4. Inequalities Related to $\phi_\Omega(x)$ Function

4.1. Rewriting the Differential Equation

For establish the inequalities, we shall first write the differential equation in section 3.2.2 in a different form, using the Proposition 2.3 (mainly transforming the second term):

From, 3.2.2, we got the differential equation-

$$\frac{d}{dx} \frac{\sum_{i=1}^n f_i^\Omega(x)}{\prod_{i=1}^n f_i(x)} = \frac{\Omega(\sum_{i=1}^n (f_i^{\Omega-1}(x) f'_i(x)))}{(\prod_{i=1}^n f_i(x))} - \frac{(\sum_{i=1}^n f_i^\Omega(x)) (\sum_{i=1}^n \frac{f'_i(x)}{f_i(x)})}{(\prod_{i=1}^n f_i(x))}$$

Also, from Proposition 2.3, we know that:

$$\frac{d}{dx} \left(\frac{1}{\prod_{i=1}^n (f_i(x))} \right) = - \frac{\sum_{i=1}^n \frac{f'_i(x)}{f_i(x)}}{\prod_{i=1}^n f_i(x)}$$

Thus, applying Proposition 2.3 to the differential equation, we get:

$$\frac{d}{dx} \frac{\sum_{i=1}^n f_i^\Omega(x)}{\prod_{i=1}^n f_i(x)} = \frac{\Omega(\sum_{i=1}^n (f_i^{\Omega-1}(x) f'_i(x)))}{(\prod_{i=1}^n f_i(x))} + \left(\sum_{i=1}^n f_i^\Omega(x) \right) \left(\frac{d}{dx} \frac{1}{\prod_{i=1}^n f_i(x)} \right) \quad (0.6)$$

4.2. Establishing Inequalities

With reference to the differential equation 0.6, we can now establish our first inequality.

Theorem 0.1. For all $\Omega \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, if for all $i \in [1, n]$, $f_i(x)$ is an increasing and

positive function, then the following inequality holds:

$$\boxed{\frac{d}{dx} \frac{\sum_{i=1}^n f_i^\Omega(x)}{\prod_{i=1}^n f_i(x)} \geq \left(\sum_{i=1}^n f_i^\Omega(x) \right) \left(\frac{d}{dx} \left(\frac{1}{\prod_{i=1}^n f_i(x)} \right) \right)}$$

Equality occurs when the functions under consideration are all constant functions.

Before we move on to the next as well as important theorem, let us form an inequality bridge between integrals and the $\phi_\Omega(x)$ function:

Let there be a function $f(x)$ which is continuous on the interval (a,b) . Now let us compute the integral $\int_a^b [f(x)]\phi_\Omega(x)dx$, that is the integral of $[f(x)]\phi_\Omega(x)$ over the same interval. Thus,

$$\begin{aligned} \int_a^b [f(x)]\phi_\Omega(x)dx &= \int_a^b \Omega \frac{(f(x))^{\Omega-1} \frac{d}{dx} f(x)}{f(x)} dx = \int_a^b \Omega (f(x))^{\Omega-2} f'(x) dx \\ \implies \int_a^b [f(x)]\phi_\Omega(x)dx &= \frac{\Omega}{\Omega-1} \{ (f(b))^{\Omega-1} - (f(a))^{\Omega-1} \} \end{aligned}$$

As mentioned earlier, Ω can take up values like 2, 3, ... that is, $\Omega \geq 2$. Now we are fully equipped to analyze the following theorem. But before introducing the theorem, let us introduce another theorem with which we will relate Theorem 0.3. and subsequently Theorem 0.4.

Theorem 0.2. If f is integrable on $[a,b]$ with $m \leq f(x) \leq M$, then:

$$\boxed{m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)}$$

Consequently, the average value of f over $[a,b]$ lies between m and M . [2]

Similar to the above theorem, the next theorem gives a lower bound as well as an upper bound for linear functions under certain conditions.

Theorem 0.3. For a linear function $f(x) = cx + d$ which is continuous over the interval (a,b) as well as defined at $x=a$ and $x=b$ with $c, d \in \mathbb{Z}^+$, the following inequality holds:

$$\boxed{2\{f(b) - f(a)\} < \int_a^b f(x)dx < \frac{3}{2}\{f^2(b) - f^2(a)\}}$$

where a and b are positive integers with $b > a > 2$.

Proof. Applying the concept introduced before the theorem, we can say that $\frac{\Omega}{\Omega-1} \{ (f(b))^{\Omega-1} - (f(a))^{\Omega-1} \} |_{\Omega=2} = 2(f(b) - f(a))$ and $\frac{\Omega}{\Omega-1} \{ (f(b))^{\Omega-1} - (f(a))^{\Omega-1} \} |_{\Omega=3} = \frac{3}{2} \{ (f(b))^2 - (f(a))^2 \}$. We know that $f(x) = cx + d$ where c and d are positive integers. Therefore, $2(f(b) - f(a)) = 2(cb + d - ca - d) = 2c(b - a)$. When we compute the integral, we get:

$$\int_a^b f(x)dx = \int_a^b (cx + d)dx = c\left(\frac{b^2}{2} - \frac{a^2}{2}\right) + d(b - a) = \left(\frac{c}{2}(b + a) + d\right)(b - a).$$

Therefore, in order to prove the first part of the theorem (the case of lower bound), we have to prove $\frac{c}{2}(b + a) + d > 2c$:

Since $a > 2$ and $b > a$, therefore, $a + b > 4$. Also since c is a positive integer, we can write the equality as $(a + b)c > 4c$. Moreover, since d is also a positive integer, we can argue that: $(a + b)c + 2d > 4c$. Dividing 2 on both sides of the inequality, we get $\frac{c}{2}(a + b) + d > 2c$ as expected. Now, in order to prove the second half of the theorem, we first need to compute

$$\frac{3}{2}\{(f(b))^2 - (f(a))^2\} :$$

$$\begin{aligned} \text{Therefore, } \frac{3}{2}\{(f(b))^2 - (f(a))^2\} &= \frac{3}{2}\{(cb+d)^2 - (ca+d)^2\} = \frac{3}{2}\{c^2(b^2 - a^2 + 2cd(b-a))\} \\ \implies \frac{3}{2}\{(f(b))^2 - (f(a))^2\} &= 3c(b-a)\{\frac{c}{2}(b+a) + d\}. \end{aligned}$$

From our previous calculation, we know that $\int_a^b f(x)dx = (b-a)\{\frac{c}{2}(b+a) + d\}$. From these 2 expressions, we can observe that the terms $(b-a)$ and $\frac{c}{2}(b+a) + d$ are common in both expressions. Thus in order to prove that second half of the theorem, we need to show that $3c > 1$ is true because c is itself a positive integer and therefore, $\frac{3}{2}\{f^2(b) - f^2(a)\} > \int_a^b f(x)dx$ as stated in the theorem. $\blacksquare$

Now let us investigate more about Theorem 0.3. and check if this is valid even for rational c and d :

Let us define our function $f(x) = \frac{m}{n}x + \frac{w}{p}$ where m, n, w and p all are positive integers. Thus, let us verify whether our Theorem 4.3. is valid for this case or not.

$$\int_a^b f(x)dx = \frac{m}{2n}(b^2 - a^2) + \frac{w}{p}(b-a) = (b-a)\{\frac{m}{2n}(b+a) + \frac{w}{p}\}$$

$$2(f(b) - f(a)) = 2\{\frac{m}{n}(b-a)\}$$

$$\frac{3}{2}\{(f(b))^2 - (f(a))^2\} = \frac{3}{2}\{(\frac{m}{n})^2b^2 + 2b(\frac{mw}{np}) - (\frac{m}{n})^2a^2 - 2a(\frac{mw}{np})\} = \frac{3}{2}(b-a)\{(\frac{m}{n})^2(b+a) + 2\frac{mw}{np}\}.$$

Now let us verify whether $2\{\frac{m}{n}(b-a)\} < (b-a)\{\frac{m}{2n}(b+a) + \frac{w}{p}\}$ is true:

Since $(b-a)$ is common and in both the left hand side and right hand side of the inequality and positive as well, then it is sufficient to check if $\frac{2m}{n} < \frac{m}{2n}(b+a) + \frac{w}{p}$ is true.

We know that $a > 2$. Thus, $(a+b) > 4$. Dividing by 2 on both sides, we get $\frac{(a+b)}{2} > 2$. Multiplying by $\frac{m}{n}$ on both sides, we get $\frac{m}{2n}(b+a) > \frac{2m}{n}$. Since $\frac{w}{p}$ is also greater than 0, so we can also state that $\frac{m}{2n}(b+a) + \frac{w}{p} > \frac{2m}{n}$. Thus, the statement subject to verification is indeed true!

Since from our analysis it is clear that $\frac{m}{2n}(b+a) + \frac{w}{p} > \frac{2m}{n}$, let us also verify and check whether the inequality $\frac{3}{2}(b-a)\{(\frac{m}{n})^2(b+a) + 2\frac{mw}{np}\} > (b-a)\{\frac{m}{2n}(b+a) + \frac{w}{p}\}$ is also true or not:

Again, since the term $(b-a)$ is common in both the expressions, it will be sufficient to verify whether $\frac{3}{2}\{(\frac{m}{n})^2(b+a) + 2\frac{mw}{np}\} > \{\frac{m}{2n}(b+a) + \frac{w}{p}\}$ is true. For the sake of simplification, let us do some operations on the left hand side:

$\frac{3}{2}\{(\frac{m}{n})^2(b+a) + 2\frac{mw}{np}\} = 3(\frac{m}{n})\{\frac{m}{2n}(b+a) + \frac{w}{p}\}$. This simplification makes the fact clear that the inequality is true only when $3\frac{m}{n} > 1$, that is when $3m > n$. Thus we can now state our more generalized theorem 0.3. in the form of theorem 0.4.:

Theorem 0.4. For a linear function $f(x) = \frac{m}{n}x + \frac{w}{p}$, which is continuous and increasing on the interval (a,b) as well as defined at $x = a$ and $x = b$ and where $m, n, w, p \in \mathbb{Z}^+$, the following inequality holds true if $3m > n$:

$$2(f(b) - f(a)) < \int_a^b f(x)dx < \frac{3}{2}\{(f(b))^2 - (f(a))^2\}$$

where a and b are positive integers with $b > a > 2$.

Remark: It can be observed that the non-generalized theorem 4.3. is a special case of the generalized theorem 4.3 where n and p are both equal to 1 and that automatically satisfies the condition $3m > n$.

Comparing Theorems 0.2. and 0.4.: In this section, we will attempt to establish a comparison between the lower and upper bounds of the integral of the class linear functions established in both these theorems. We will first try to compare the lower bounds, $2(f(b) - f(a))$ and $m(b - a)$. For the purpose of this context, we can argue that $m = f(a)$. Therefore, the comparison is between $2(f(b) - f(a))$ and $f(a)(b - a)$.

Therefore, $2(f(b) - f(a)) = 2\{\frac{m}{n}b + \frac{w}{p} - \frac{m}{n}a - \frac{w}{p}\} = 2\frac{m}{n}(b - a)$. And,
 $f(a)(b - a) = (\frac{m}{n}a + \frac{w}{p})(b - a) = \frac{m}{n}a(b - a) + \frac{w}{p}(b - a)$.

From Theorem 0.4., we know that $b > a > 2$. Thus, $\frac{m}{n}a(b - a) > 2\frac{m}{n}(b - a)$. And since $\frac{w}{p}(b - a) > 0$, it becomes obvious that: $2(f(b) - f(a)) < f(a)(b - a)$ for the class of functions taken into account.

Now let us move on to comparing the upper bounds. Interestingly, the scenario will be different from that of the lower bounds.

$$\frac{3}{2}\{(f(b))^2 - (f(a))^2\} = \frac{3}{2}\{(\frac{m}{n}b + \frac{w}{p})^2 - (\frac{m}{n}a + \frac{w}{p})^2\} = \frac{3}{2}(\frac{m}{n}(b - a))(\frac{m}{n}(b + a) + 2\frac{w}{p}).$$

Similar to above, since we are considering a class of continuous and increasing linear functions, we can also argue that $M = f(b)$. Therefore,

$$M(b - a) = f(b)(b - a) = (\frac{m}{n}b + \frac{w}{p})(b - a).$$

To compare $\frac{3}{2}(\frac{m}{n}(b - a))(\frac{m}{n}(b + a) + 2\frac{w}{p})$ and $(\frac{m}{n}b + \frac{w}{p})(b - a)$, let us check that whether the difference is positive and negative:

$$\frac{3}{2}(\frac{m}{n}(b - a))(\frac{m}{n}(b + a) + 2\frac{w}{p}) - (\frac{m}{n}b + \frac{w}{p})(b - a) = (\frac{3m^2}{2n^2}(b + a) + \frac{3mw}{np} - \frac{m}{n}b - \frac{w}{p})(b - a) = (\frac{3m^2}{2n^2}(b + a) - \frac{m}{n}b + (\frac{3m}{n} - 1)\frac{w}{p})(b - a).$$

Since we know that $3m > n$ from the conditions of the theorem, therefore $(\frac{3m}{n} - 1)$ is positive. But *no conclusion* can be drawn for $(\frac{3m^2}{2n^2}(b + a) - \frac{m}{n}b)$ because we cannot prove for sure that $3m > 2n$ [it depends completely on the choice of m and n].

Since we have successfully established our theorem for continuous and increasing linear functions, let us attempt to establish a similar theorem for quadratic polynomials:

Theorem 0.5. For a function $f(x) = cx^2 + dx + e$ where $f(x)$ is a continuous and increasing in the interval (a, b) as well as defined at $x = a, b$, then for $b > a > 2$

$$\boxed{\int_a^b f(x)dx < \frac{4}{3}\{(f(b))^3 - (f(a))^3\}} \quad (0.7)$$

Proof. We know $\int_a^b f(x)dx = \frac{c}{3}(b^3 - a^3) + \frac{d}{2}(b^2 - a^2) + e(b - a) = \frac{1}{6}(b - a)\{2c(b^2 + a^2 + ab) + 3d(b + a) + 6e\} = \frac{1}{6}(b - a)\{2b^2c + 2a^2c + 2abc + 3bd + 3ad + 6e\}$.

Also, R.H.S = $\frac{4}{3}(b - a)(cb + ca + d)\{(cb^2 + db + e)^2 + (ca^2 + da + e)^2 + (cb^2 + db + e)(ca^2 + ca + e)\}$
 $\implies \frac{4}{3}\{(f(b))^3 - (f(a))^3\} = \frac{4}{3}(b - a)(ca + cb + d)\{c^2b^4 + d^2b^2 + e^2 + 2b^3cd + 2bde + 2b^2ce + c^2a^4 + d^2a^2 + e^2 + 2a^3cd + 2ade + 2a^2ce + c^2a^2b^2 + ab^2cd + b^2ce + a^2bcd + abd^2 + bde + a^2ce + ade + e^2\} =$
 $\frac{4}{3}(b - a)(bc + ac + cd)\{c^2b^4 + c^2a^4 + d^2b^2 + d^2a^2 + 3e^2 + 2b^3cd + 2a^3cd + 3bde + 3ade + 3b^2ce + 3a^2ce + a^2b^2c^2 + ab^2cd + a^2bcd + abd^2\}$.

Comparing both the left and right hand sides, it becomes evident that $\frac{1}{6}(b - a) < \frac{4}{3}(b - a)$. Also it is worth noticing that for $b > a > 2$: $3a^2ce > 2a^2c$, $a^2b^2c^2 > 2abc$, $c^2b^4 > 2b^2$, $3bde > 3bd$, $3ade > 3ad$ and finally $3b^2ce > 6e$.

Thus, $\int_a^b f(x)dx < \frac{4}{3}\{(f(b))^3 - (f(a))^3\}$ ■

Now let us investigate Theorem 0.4. in more depth by relating it to Theorem 0.6.

Theorem 0.6. *If $f_1(x)$ and $f_2(x)$ are two functions of x , neither of which is a constant, such that if either one of them experiences an increase in any finite integral included in the interval of integration the other experiences a decrease, and conversely, then*

$$\boxed{\int_a^b f_1(x)dx \int_a^b f_2(x)dx > (b-a) \int_a^b f_1(x)f_2(x)dx}$$

If on the other hand the two functions experience increments in the same intervals and decrements in the same intervals, the inequality sign above is reversed.[3]

Now we will attempt to modify Theorem 0.6. and relate the concept with Theorems 0.4.and 0.5.

Corollary 0.1. If $f_1(x)$ is an increasing linear function in the interval from a to b and $f_2(x)$ is a function that is decreasing in the same interval of integration then, for $b > a > 2$,

$$\boxed{\{3bc^2 + 3ac^2 + 6cd - 2d\} \int_a^b f_2(x)dx > 2c \int_a^b x f_2(x)dx} \quad (0.8)$$

where $f_1(x) = cx + d$ for $c, d \in \mathbb{Z}^+$.

Proof. From our assumptions about $f_1(x)$ and $f_2(x)$, we can apply Theorem 0.6. to find that: $\int_a^b (cx + d)dx \int_a^b f_2(x)dx > (b-a) \int_a^b (cx + d)f_2(x)dx$. From Theorem 0.3., we can argue that $\int_a^b f_1(x)dx < \frac{3}{2}\{(f_1(b))^2 - (f_1(a))^2\}$.

$$\begin{aligned} \text{Thus, } & \frac{3}{2}((cb + d)^2 - (ca + d)^2) \int_a^b f_2(x)dx > (b-a) \int_a^b (cx + d)f_2(x)dx \\ \implies & \frac{3}{2}(c^2b^2 + d^2 + 2bcd - c^2a^2 - d^2 - 2acd) \int_a^b f_2(x)dx > (b-a) \int_a^b (cx + d)f_2(x)dx \\ \implies & \frac{3c}{2}(cb + ca + 2d) \int_a^b f_2(x)dx > d \int_a^b f_2(x)dx + c \int_a^b x f_2(x)dx \\ \implies & (3bc^2 + 3ac^2 + 6cd - 2d) \int_a^b f_2(x)dx > 2c \int_a^b x f_2(x)dx \end{aligned} \quad \blacksquare$$

After careful observation of Theorems 0.3. and 0.5., what comes to our minds is that there must be a general theorem that summarizes all of these. Let us introduce the general theorem, establishing a link between the integrals and the $\phi_\Omega(x)$ function:

Theorem 0.7. *For a polynomial function $f(x) = \sum_{i=0}^n a_i x^i$ where $f(x)$ is increasing on the interval (a, b) as well as defined at $x=a$ and $x=b$, the following inequality holds true:*

$$\boxed{\int_a^b f(x)dx < \frac{n+2}{n+1}\{(f(b))^{n+1} - (f(a))^{n+1}\}}$$

$\forall a_i \in \mathbb{Z}^+ \wedge b > a > 2$.

Proof. By Induction

Let $P(n)$ be the proposition that $\int_a^b (\sum_{i=0}^n a_i x^i)dx < \frac{n+2}{n+1}\{(f(b))^{n+1} - (f(a))^{n+1}\}$

Base Case: $P(1)$ is true since by Theorem 0.3.,

$$\int_a^b (a_1x + a_0)dx < \frac{3}{2}\{(a_1b + a_0)^2 - (a_1a + a_0)^2\}$$

Inductive Step: To show that for $n \geq 1$, $P(n) \implies P(n+1)$ is true.

Assume $P(n)$ to be true for the purpose of induction, i.e. $\int_a^b (\sum_{i=0}^n a_i x^i)dx < \frac{n+2}{n+1}\{(f(b))^{n+1} - (f(a))^{n+1}\}$

Need to show: $\int_a^b (\sum_{i=0}^{n+1} a_i x^i)dx < \frac{n+3}{n+2}\{(\sum_{i=0}^{n+1} a_i b^i)^{n+2} - (\sum_{i=0}^{n+1} a_i a^i)^{n+2}\}$

$$\int_a^b (\sum_{i=0}^{n+1} a_i x^i)dx = \int_a^b a_{n+1}x^{n+1}dx + \int_a^b (a_n x^n + a_{n-1}x^{n-1} + \dots + a_0)dx.$$

From our consideration of $P(n)$ to be true, we can argue that:

$$\int_a^b (\sum_{i=0}^{n+1} a_i x^i)dx < \int_a^b a_{n+1}x^{n+1}dx + \frac{n+2}{n+1}\{(a_n b^n + a_{n-1}b^{n-1} + \dots a_0)^{n+1} - (a_n a^n + a_{n-1}a^{n-1} + \dots + a_0)^{n+1}\}$$

$$\implies \int_a^b (\sum_{i=0}^{n+1} a_i x^i)dx < \frac{n+2}{n+1}a_{n+1}(b^{n+2} - a^{n+2}) + \frac{n+2}{n+1}\{(a_n b^n + a_{n-1}b^{n-1} + \dots a_0)^{n+1} - (a_n a^n + a_{n-1}a^{n-1} + \dots + a_0)^{n+1}\} \text{ [since } \int_a^b a_{n+1}x^{n+1}dx = \frac{1}{n+2}(b^{n+2} - a^{n+2}), \text{ and } \frac{1}{n+2} < \frac{n+2}{n+1}]$$

$$\implies \int_a^b (\sum_{i=0}^{n+1} a_i x^i)dx < \frac{n+2}{n+1}\{a_{n+1}(b^{n+2} - a^{n+2}) + (\sum_{i=0}^n a_i b^i)^{n+1} - (\sum_{i=0}^n a_i a^i)^{n+1}\}$$

$$\implies \int_a^b (\sum_{i=0}^{n+1} a_i x^i)dx < \frac{n+2}{n+1}\{(\sum_{i=0}^{n+1} a_i b^i)^{n+2} - (\sum_{i=0}^{n+1} a_i a^i)^{n+2}\}$$

$$\text{We know that } \int_a^b (\sum_{i=0}^{n+1} a_i x^i)dx = \frac{a_{n+1}}{n+2}(b^{n+2} - a^{n+2}) + \frac{a_n}{n+1}(b^{n+1} - a^{n+1}) + \dots + a_0(b - a).$$

From this we can argue that

$$\int_a^b (\sum_{i=0}^{n+1} a_i x^i)dx < a_{n+1}(b^{n+2} - a^{n+2}) + a_n(b^{n+1} - a^{n+1}) + \dots + a_0(b - a).$$

$$\text{And since, } a_{n+1}(b^{n+2} - a^{n+2}) + a_n(b^{n+1} - a^{n+1}) + \dots + a_0(b - a) < \{(\sum_{i=0}^{n+1} a_i b^i)^{n+2} - (\sum_{i=0}^{n+1} a_i a^i)^{n+2}\},$$

we finally arrive to the conclusion that:

$$\int_a^b (\sum_{i=0}^{n+1} a_i x^i)dx < \frac{n+3}{n+2}\{(\sum_{i=0}^{n+1} a_i b^i)^{n+2} - (\sum_{i=0}^{n+1} a_i a^i)^{n+2}\}$$

since $\frac{n+3}{n+2} > 0$.

Therefore, $\forall n \geq 1$, $P(n) \implies P(n+1)$. ■

5. More Applications of the $\phi_\Omega(x)$ Function

Before introducing the next applications of the function, let us first go through some preliminary steps mentioned in 5.1 and 5.2:

5.1. The Notation

In this section we shall be introducing a new notation for the convenience of identifying the functions being considered inside the $\phi_\Omega(x)$ function:

Let $[f_1(x), f_2(x), f_3(x), \dots, f_n(x)]\phi_\Omega(x)$ denote that the functions $f_i(x)$ for $i \in [1, n]$ are taken inside the $\phi_\Omega(x)$ function. Or, in other words, it depicts that we shall be calculating $\phi_\Omega(x)$ for all these functions. In the following section, we shall be taking some examples for better understanding of this notation.

Let us now look into some examples using our new notation and applying them on hyperbolic trigonometric functions. Let us first consider only one function-

$$\text{Let } f_1(x) = \sinh(x). \text{ Also, we know that } \sinh(x) = \frac{e^x - e^{-x}}{2}. \text{ Therefore,}$$

$$[\sinh(x)]\phi_\Omega(x) = \frac{\Omega(\sinh(x))^{\Omega-1}(\cosh(x))}{\sinh(x)} = \frac{\Omega(e^x - e^{-x})^{\Omega-1}(e^x + e^{-x})}{2\Omega(\frac{e^x - e^{-x}}{2})} = \frac{\Omega(e^x - e^{-x})^{\Omega-2}(e^x + e^{-x})}{2^{\Omega-1}}$$

Remark : In the above example, we considered only one function, that is $\sinh(x)$. And this is

why, for this example, $n=1$.

Now, let us consider another example with 2 hyperbolic trigonometric functions $\sinh(x)$ and $\tanh(x)$. Therefore,

$$\begin{aligned} [\sinh(x), \tanh(x)]\phi_\Omega(x) &= \frac{\Omega(\sinh(x))^{\Omega-1}(\cosh(x) + (\tanh(x))^{\Omega-1}\text{sech}^2(x))}{(\sinh(x)\tanh(x))} \\ \implies [\sinh(x), \tanh(x)]\phi_\Omega(x) &= \frac{\Omega\left(\frac{e^x - e^{-x}}{2}\right)^{\Omega-1}\left(\frac{e^x + e^{-x}}{2}\right) + \Omega\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^{\Omega-1}\left(\frac{4}{(e^x + e^{-x})^2}\right)}{\frac{(e^x - e^{-x})^2}{2(e^x + e^{-x})}} \end{aligned}$$

After few more steps of manipulation, we shall get:

$$[\sinh(x), \tanh(x)]\phi_\Omega(x) = \frac{\Omega((e^x + e^{-x})^{2\Omega} + 2^{\Omega+2}(e^x - e^{-x})^{\Omega-2})}{2^{\Omega-1}(e^x + e^{-x})^\Omega} \quad (0.9)$$

5.1.1. *Applying the new notation:* We shall now attempt to write the general form of a diverging series using the new notation introduced in 5.1-

Let us focus on logarithmic functions. We shall select a class of functions of the form $\ln(x^k)$ for $k \in \mathbb{Z}^+$ and examine their relation with the $\phi_\Omega(x)$ function.

$$[\ln(x^k)]\phi_\Omega(x) = \frac{k\Omega(\ln(x))^{\Omega-2}}{x} \quad (0.10)$$

Therefore, applying the previously defined function $\delta_\Omega(x)$, we get:

$$\begin{aligned} \delta_\Omega(x) &= \frac{\Omega(\ln(x))^{\Omega-2}}{x} + \frac{2\Omega(\ln(x))^{\Omega-2}}{x} + \dots \\ \implies \delta_\Omega(x) &= \frac{\Omega(\ln(x))^{\Omega-2}}{x}(1 + 2^{\Omega-1} + 3^{\Omega-1} + 4^{\Omega-1} + \dots) \\ \implies 1 + 2^{\Omega-1} + 3^{\Omega-1} + 4^{\Omega-1} + \dots &= \frac{x\delta_\Omega(x)}{\Omega(\ln(x))^{\Omega-2}} \end{aligned}$$

Therefore, we can write this whole expression as:

$$\lim_{n \rightarrow \infty} \sum_{m=1}^n m^{\Omega-1} = \frac{x}{\Omega(\ln(x))^{\Omega-2}} \left(\lim_{n \rightarrow \infty} \sum_{k=1}^n [\ln(x^k)]\phi_\Omega(x) \right)$$

Thus we can write the infinite series of the form $\lim_{n \rightarrow \infty} \sum_{m=1}^n m^{\Omega-1}$ using the new notation which gives a completely new form involving the logarithm function.

5.2. Relation of $\phi_\Omega(x)$ with the trigonometric functions - A Brief Overview

5.2.1. *Plane Trigonometric Functions:* To establish our relations, first let us establish the relations between the $\phi_\Omega(x)$ function and the plane trigonometric functions.

$$\begin{aligned} [\sin(x)]\phi_\Omega(x) &= \frac{\Omega(\sin(x))^{\Omega-1}\cos(x)}{\sin(x)} = (\sin(x))^{\Omega-2}\cos(x) \\ \implies [\sin(x)]\phi_\Omega^2(x) &= \Omega^2(\sin(x))^{2\Omega-4}\cos^2(x) = \Omega^2(\sin(x))^{2\Omega-4}(1 - \sin^2(x)) \end{aligned}$$

$$\text{Therefore, } \boxed{[\sin(x)]\phi_\Omega(x) = \Omega\sqrt{(\sin(x))^{2\Omega-4} - (\sin(x))^{2\Omega-2}}} \text{ ————— [Eq. 5.2.1(a)]}$$

Let us sketch the graph of $[\sin(x)]\phi_\Omega(x)$ for different values of Ω :

Fig. 5.2.(a)

For each of these curves in Fig. 5.2.(a), the function $f(x) = \Omega \sqrt{(\sin(x))^{2\Omega-4} - (\sin(x))^{2\Omega-2}}$, where for each curve, the value of Ω has been treated as a constant. For instance, the curve in red represents the graph of the function for $\Omega = 2$, the curve in blue represents the graph for value of $\Omega = 3$, while the curve in green represents the graph for values $\Omega = 4$. Excluding the case for $\Omega = 2$, one can observe that the height of the graphs increase with increase in the value of Ω .

Similarly, let us also analyze $[\cos(x)]\phi_\Omega(x)$ and examine it's behaviour:

$$\begin{aligned} [\cos(x)]\phi_\Omega(x) &= \Omega \frac{(\cos(x))^{\Omega-1} \cos'(x)}{\cos(x)} = -\Omega (\cos(x))^{\Omega-2} \sin(x) \\ \implies [\cos(x)]\phi_\Omega^2(x) &= \Omega^2 (\cos(x))^{2\Omega-4} (\sin(x))^2 = \Omega^2 (\cos(x))^{2\Omega-4} (1 - (\cos(x))^2) \\ \implies [\cos(x)]\phi_\Omega^2(x) &= \Omega^2 (\cos(x))^{2\Omega-4} - (\cos(x))^{2\Omega-2} \end{aligned}$$

Therefore, $\boxed{[\cos(x)]\phi_\Omega(x) = -\Omega \sqrt{(\cos(x))^{2\Omega-4} - (\cos(x))^{2\Omega-2}}}$ ————— [Eq 5.2.1(b)]

Similar to what we did in case of $[\sin(x)]\phi_\Omega(x)$, let us also analyse the graph for $[\cos(x)]\phi_\Omega(x)$

Fig. 5.2.(b)

Similar to above, graph 5.2.(b) represents the curve of the function $f(x) = -\Omega\sqrt{(\cos(x))^{2\Omega-4} - (\cos(x))^{2\Omega-2}}$, where for each curve, the value of Ω is treated as constant. Just like the graph of $[\sin(x)]\phi_\Omega(x)$ function, the graph in red represents the curve of the function for $\Omega = 2$, while the graphs in blue and green represents for $\Omega = 3$ and $\Omega = 4$ respectively.

Now let us attempt to derive the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$ using equations 5.2.1(a) and 5.2.1(b):

We know that, $[\cos(x)]\phi_\Omega^2(x) = \Omega^2(\cos(x))^{2\Omega-4}(\sin(x))^2$ and $[\sin(x)]\phi_\Omega^2(x) = \Omega^2(\sin(x))^{2\Omega-4}(\cos(x))^2$.

Let us focus on computing $[\cos(x)]\phi_\Omega^2(x) + [\sin(x)]\phi_\Omega^2(x)$. Therefore,

$$[\cos(x)]\phi_\Omega^2(x) + [\sin(x)]\phi_\Omega^2(x) = \Omega^2(\cos^{2\Omega-4}(x) - \cos^{2\Omega-2}(x) + \sin^{2\Omega-4}(x) - \sin^{2\Omega-2}(x))$$

To see in more detail, let us graph the addition of these 2 functions taking the value of Ω constant as done previously. Let us first plot the graph of $[\cos(x)]\phi_\Omega^2(x) + [\sin(x)]\phi_\Omega^2(x) = \Omega^2(\cos^{2\Omega-4}(x) - \cos^{2\Omega-2}(x) + \sin^{2\Omega-4}(x) - \sin^{2\Omega-2}(x))$ for $\Omega = 2$:

Fig. 5.2.(c)

It can be observed from Fig. 5.2.(c) that the graph of $[\cos(x)]\phi_{\Omega}^2(x) + [\sin(x)]\phi_{\Omega}^2(x)$ versus x , we get a line, that is we get a constant value for all values of x and for $\Omega = 2$, in this case 4. That is, $[\cos(x)]\phi_{\Omega}^2(x) + [\sin(x)]\phi_{\Omega}^2(x)$ has value 4 for values of x and for $\Omega = 2$. Thus,

$$\begin{aligned}
 & [\cos(x)]\phi_2^2(x) + [\sin(x)]\phi_2^2(x) = 4 \\
 \implies & 4(\cos(x))^{2.2-4}\sin^2(x) + (\sin(x))^{2.2-4}\cos^2(x) = 4 \\
 \implies & 4(\sin^2(x) + \cos^2(x)) = 4 \\
 \implies & \boxed{\sin^2(x) + \cos^2(x) = 1}
 \end{aligned}$$

Thus we have proven the fact that $\sin^2(x) + \cos^2(x) = 1$ with the help of phi-trigonometric functions and through graphical method. The derivation of the very basic yet powerful trigonometric identity will lead to the path for deriving other identities as well and in a similar fashion, we can also derive the identities of hyperbolic trigonometric functions which we will leave up to the readers to compute and derive.

Conclusion

From all the analysis we did, we can suggest that the new phi function leads the path to the derivation of various useful inequalities, lets us write expressions using the new notation and

the phi-trigonometric functions can be used in several theorems which can in-turn lead to many new and important theorems as well as occasionally ease our computation process. Besides these, there are several important results that are linked to inequalities and the $\phi_{\Omega}(x)$ function which will present in my next research paper.

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