Expansion of the Universe and cosmological redshift

Vaclav Vavrycuk (✉ vv@ig.cas.cz)

Czech Academy of Sciences <https://orcid.org/0000-0003-3962-2678>

Research Article

Keywords: dark energy, early universe, redshift, time dilation

Posted Date: November 2nd, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-1040995/v1>

License: This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Expansion of the Universe and cosmological redshift

V. Vavryčuk

The Czech Academy of Sciences

Boční II 1401, 141 00 Praha 4

vv@ig.cas.cz

ABSTRACT

The re-examination of light propagation in space described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric reveals surprisingly that this metric does not predict the cosmological redshift as so far incorrectly supposed. It is shown that the change in the frequency of light is always connected with time dilation, similarly as for the gravitational redshift. Therefore, the conformal time must be considered as the cosmic time at the high redshift universe and the original FLRW metric must be substituted by its conformal version. The correctness of the proposed conformal metric is convincingly confirmed by Type Ia supernovae (SNe Ia) observations. The standard FLRW metric produces essential discrepancy with the SNe Ia observations called the 'supernova dimming', and consequently dark energy has to be introduced to comply theoretical predictions with data. By contrast, the conformal FLRW metric fits data well with no need to introduce any new free parameter. Hence, the discovery of the supernova dimming actually revealed the failure of the FLRW metric and introducing dark energy was just an unsuccessful attempt to cope with the problem within this false metric. Obviously, adopting the conformal FLRW metric for describing the evolution of the Universe has fundamental cosmological consequences.

Subject headings: early universe – redshift – dark energy

1. Introduction

Friedmann (1922) applied the Einstein equations of General Relativity (GR) for describing the Universe and firstly showed that the space filled by uniformly distributed matter might evolve in time. The possibility that the Universe is really dynamic but not static was later supported by Lemaître (1927) and Hubble (1929), who observed a systematic redshift of nearby galaxies, which was roughly proportional to their distance. This observation (called the Hubble-Lemaître law) was interpreted as the Doppler effect produced by galaxies moving away from the Earth due to the universe expansion.

However, the intuitive idea of the redshift as the Doppler effect was later abandoned. At present, the Universe is described by the so-called Friedmann-Lemaître-Robertson-Walker (FLRW) metric (Friedmann 1999; Robertson 1935, 1936;

Walker 1935, 1937), which introduces the scale factor $a(t)$ for describing the space expansion. The redshift is not related to the speed of the expansion as for the Doppler effect but to the ratio between sizes of the space, in which the photons were emitted and received (Peebles 1993; Peacock 1999)

$$1 + z = \frac{a(r)}{a(e)}, \quad (1)$$

where z is the redshift, and $a(e)$ and $a(r)$ are the scale factors for the emitter and receiver, respectively. Hence the redshift of distant galaxies would be observed even in the case, when the Universe is not expanding anymore at the present epoch.

In contrast to the space coordinates, the time coordinate is assumed to be invariable during the universe history. This is somewhat strange and surprising, because other solutions in GR such as the well-known Schwarzschild solution (Misner

et al. 1973; Weinberg 1972; Carroll 2004) involve distortions in space and time together. Therefore, some authors pointed out to other alternative theories admissible in GR and introduced more general metrics for describing isotropic homogeneous universe evolving in time. In this case, another function is considered in the metric tensor $g_{\alpha\beta}$, which describes the evolution of the time component g_{00} . Among many possibilities how to define this function, the simplest way is to assume that the time and scale factors are defined by the same function $a(t)$. This metric is called the conformal FLRW metric (Endean 1994, 1997; Grøn & Johannesen 2011) and it is particularly interesting, because it leaves the Maxwell's equations unchanged from their form in the Minkowski spacetime (Infeld & Schild 1945, 1946; Ibison 2007). In addition, introducing the conformal time, the cosmological redshift can be defined by the same formula as the gravitational redshift

$$1 + z = \sqrt{\frac{g_{00}(r)}{g_{00}(e)}}, \quad (2)$$

where $g_{00}(e)$ and $g_{00}(r)$ are the time components of the metric tensor $g_{\alpha\beta}$ for the emitter and receiver, respectively.

Nevertheless, introducing the conformal time into the FLRW metric is commonly viewed as a mathematical concept different from the physical cosmic time (Grøn & Johannesen 2011). Otherwise, we have to admit a variable speed of light dependent on the scale factor $a(t)$. Although, theories of variable speed of light (VSL) exist (Magueijo 2003; Ellis 2007), they are not paid much attention, because they are against a deeply rooted concept of the speed of light as a nature constant. Nevertheless, Dicke (1957) argues in his pioneering work on gravity that VSL is physically admissible. Also Dirac (1968) states that 'The laws may be changing, and in particular quantities which are considered to be constants of nature may be varying with cosmological time'.

In this paper, the problem of the cosmological redshift in the FLRW metric is revisited, It is shown that the redshift is not, actually, predicted by the original FLRW metric but only by the conformal FLRW metric. Cosmological consequences of this correction are discussed.

2. Theory

2.1. FLRW metric

The space filled by a homogenous and isotropic matter is described by the following general metric:

$$-c^2 d\tau^2 = -A^2(t) c^2 dt^2 + B^2(t) d\Sigma^2, \quad (3)$$

where c is the speed of light, τ is the proper time, t is the coordinate time, Σ is the 3-dimensional coordinate in space of uniform curvature, and $A(t)$ and $B(t)$ are arbitrary functions describing time evolution of time dilation and space expansion, respectively.

The FLRW metric is based on the assumption of the space expansion described by the scale factor $a(t) = B(t)$ and with no time dilation $A(t) = 1$. Hence, the metric reads in the spherical coordinate system as (Peebles 1993; Peacock 1999; Ryden 2016)

$$\begin{aligned} -c^2 d\tau^2 &= -c^2 dt^2 + a^2(t) \left(\frac{r^2}{1 - kr^2} + r^2 d\Omega^2 \right), \\ d\Omega^2 &= d\Theta^2 + \sin^2 \Theta d\phi^2, \end{aligned} \quad (4)$$

k is the Gaussian curvature of the space, r is the comoving distance, and Θ and ϕ are the spherical angles.

An alternative to Eq.(4) is the so-called conformal form of the FLRW metric (Grøn & Johannesen 2011), which assumes the same factor $a(t)$ for time dilation and space expansion, $A(t) = B(t) = a(t)$,

$$-c^2 d\tau^2 = a^2(t) \left(-c^2 dt^2 + \frac{r^2}{1 - kr^2} + r^2 d\Omega^2 \right). \quad (5)$$

Obviously, the Einstein's equations do not constrain functions $A(t)$ and $B(t)$ in Eq.(3) and they do not give us any preference between Eq.(4) for the original FLRW metric and Eq.(5) for the conformal FLRW metric. The correct form of the metric must be found from observations. Primarily, the correct metric should satisfactorily explain observations of the cosmological redshift.

2.2. Cosmological redshift

The cosmological redshift in the original FLRW metric is commonly explained as the change of

the photon wavelength due to the space expansion (Peebles 1993; Peacock 1999; Mukhanov 2005; Weinberg 2008; Ryden 2016). The standard derivation in textbooks is as follows. Light travels along the null geodesic, $cd\tau = 0$, hence

$$c^2 dt^2 = a^2(t) dr^2, \quad (6)$$

and consequently

$$c \frac{dt}{a(t)} = dr. \quad (7)$$

Suppose the distant galaxy emits photons at constant rate Δt_e and with wavelength λ_e . The photons are observed at rate Δt_r and with wavelength λ_r . The first photon is emitted at time t_e and received at time t_r . Taking into account that the comoving distance between the galaxy and the observer is the same for the two successive photons

$$\int_{t_e}^{t_r} \frac{cdt}{a(t)} = \int_{t_e+\Delta t_e}^{t_r+\Delta t_r} \frac{cdt}{a(t)}. \quad (8)$$

and subtracting the integral

$$c \int_{t_e+\Delta t_e}^{t_r} \frac{dt}{a(t)}, \quad (9)$$

we get

$$\int_{t_e}^{t_e+\Delta t_e} \frac{cdt}{a(t)} = \int_{t_r}^{t_r+\Delta t_r} \frac{cdt}{a(t)}. \quad (10)$$

Since the scale factor $a(t)$ varies slowly and does not change much during emission and observation of the two successive photons, we write

$$\frac{1}{a(t_e)} \int_{t_e}^{t_e+\Delta t_e} cdt = \frac{1}{a(t_r)} \int_{t_r}^{t_r+\Delta t_r} cdt, \quad (11)$$

hence

$$\frac{d_e}{a(t_e)} = \frac{d_r}{a(t_r)}, \quad (12)$$

where $d_e = c\Delta t_e$ and $d_r = c\Delta t_r$ are the distances between two successive photons at the emitter and the receiver, respectively. Subsequently, we can conclude that the wavelengths of photons λ_e and λ_r obey the same relation

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_r}{a(t_r)}. \quad (13)$$

This derivation is not, however, correct. Using Eq.(11) we can also obtain the following equation

$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_r}{a(t_r)}, \quad (14)$$

which indicates that the coordinate time depends on the scale factor $a(t)$. Obviously, Eq.(14) is inconsistent with the original FLRW metric described by Eq.(4), where the coordinate time is invariant. Alternatively, we can keep the coordinate time independent of the scale factor, but then we have to assume that the light speed c depends on the scale factor $a(t)$ and we have to distinguish between the light speed in the emitter, c_e , and in the receiver, c_r . This is again inconsistent with Eq.(4).

The basic difficulty with the above derivation of redshift-dependent wavelengths of photons lies in an incorrect definition of the wavelength as distance between two different spacetime events. Obviously, the distance must be measured at one coordinate system, but not as a distance between points in two different coordinate systems connected with two photons measured at different times. Once we consider two photons travelling along the same ray path with distance d between them at the same coordinate time, the effect of increasing the distance between photons during the space expansion disappears. After any time t , both photons travel the same distance along the same ray, and consequently the distance between them keeps time independent.

Mathematically, we modify Eq.(8), in which we do not assume the equality of the comoving distance but the equality of the traveltime distance of the photons propagating along the same raypath from the emitter to the receiver:

$$\int_{t_e}^{t_r} cdt = \int_{t_e+\Delta t}^{t_r+\Delta t} cdt. \quad (15)$$

Using the same logic as above, we obtain that if time and speed of light is not changing, the wavelength of photons does not change. Hence, two successive photons travelling along the same ray-path keep their mutual distance constant and independent of redshift. However, the distance between two photons travelling along two parallel rays at the same time depends on redshift and increases with the space expansion. This is because the comoving distance between two photons

moving along parallel raypaths is constant, hence the proper distance must increase with the space expansion. Only the distance between two successive photons travelling along the same ray does not change.

The above derivation demonstrates that the original FLRW metric cannot be applied to the Universe because it does not predict the cosmological redshift. A disputable character of the original FLRW metric is also indicated by comparing this metric with other solutions in GR, where the expansion/contraction of space is tightly connected with time dilation. We conclude that the cosmological redshift is not a consequence of the space expansion but of time dilation. If we insist on no time dilation, no redshift will be observed. Hence, the correct metric is the conformal form of the FLRW metric described by Eq.(5) and the cosmological redshift obeys the same formula as the gravitational redshift:

$$\frac{\nu_e}{\nu_r} = 1 + z = \sqrt{\frac{g_{00}(r)}{g_{00}(e)}}, \quad (16)$$

where z is the redshift, ν_e and ν_r are the frequencies of the photon at the emitter and receiver, and $g_{00}(e)$ and $g_{00}(r)$ are the time components of the metric tensor $g_{\alpha\beta}$ at the emitter and receiver, respectively.

2.3. Properties of the conformal FLRW metric

The conformal FLRW metric is essentially different from the original FLRW metric with fundamental physical consequences:

- Eq.(5) implies that the comoving speed of light is constant but the proper speed of light depends on redshift. Hence the volume of the Universe and distance between galaxies were smaller at high redshift, but photons emitted by a galaxy reach a neighbouring galaxy after the same time at high redshift as well as at the present epoch. In other words, this universe model is conformal with the static universe.
- The frequency ν_e of photons emitted at redshift z is higher than the frequency ν_r of photons received as:

$$\frac{\nu_e}{\nu_r} = 1 + z. \quad (17)$$

- Not only the frequency ν of photons but also the rate of photons increases with redshift as $(1 + z)$.
- The speed of light c increases with the expansion of the Universe as $(1 + z)$.
- The wavelength λ_e of photons emitted at redshift z is lower than the wavelength λ_r of photons received as:

$$\frac{\lambda_e}{\lambda_r} = (1 + z)^{-2}. \quad (18)$$

This includes the decrease of frequency ν and the increase of the speed of light c with cosmic time.

- Energy flux of photons redshifted along their raypath is conserved. Energy flux is defined as energy of emitted/received photons per unit time. The frequency and the energy of single photons decreases with cosmic time but the time interval for receiving photons increases with the cosmic time.

2.4. Friedmann equations revisited

If the expansion of the Universe is described by the conformal FLRW metric, the Friedmann equations must be modified. The standard Friedmann equations for the pressureless fluid read (Peacock 1999; Ryden 2016)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{1}{3}\Lambda c^2, \quad (19)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho + \frac{1}{3}\Lambda c^2, \quad (20)$$

where $a = (1 + z)^{-1}$ is the scale factor, G is the gravitational constant, ρ is the mass density, k/a^2 is the spatial curvature of the universe, Λ is the cosmological constant, and c is the speed of light.

In order to express the Friedmann equations for the conformal FLRW metric, we have to substitute time t by the conformal time $t_c = t/a$. and time derivative d/dt by $d/dt_c = ad/dt$. Hence, the conformal Friedmann equations read

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho a^2 - kc^2, \quad (21)$$

$$\frac{a''}{a} = -\frac{4\pi G}{3}\rho a^2, \quad (22)$$

where a' denotes the derivative with respect to the conformal time, and we omitted the cosmological constant, because it was inserted into the Friedmann equations (19-20) artificially in order to fit Type Ia supernova observations. Considering mass density ρ in the matter-dominated universe, we get

$$\frac{8\pi G}{3}\rho = H_0^2 \Omega_m a^{-3}, \quad (23)$$

and Eq.(21) reads

$$H^2(a) = H_0^2 [\Omega_m a^{-1} + \Omega_k], \quad (24)$$

with the condition

$$\Omega_m + \Omega_k = 1, \quad (25)$$

where $H(a) = a'/a$ is the Hubble parameter, H_0 is the Hubble constant, and Ω_m , and Ω_k are the normalized matter and curvature terms. Since this model is basically the Einstein-de Sitter (EdS) model but applied to the conformal FLRW metric, it will be called as the 'conformal EdS model' in contrast to the standard EdS model based on the original FLRW metric.

3. Supernovae observations

The correctness of Eq.(24) for the time evolution of the Universe can be checked by Type Ia supernova (SNe Ia) observations, which provide the most accurate measurements of cosmological distances and of the expansion history of the Universe. A discrepancy between the supernova observations and the predictions of the standard EdS model was called the 'supernovae dimming' (Riess et al. 1998; Perlmutter et al. 1999), and led to reintroducing the cosmological constant Λ into the Einstein and Friedmann equations. The observation of the unexpected SNe Ia dimming motivated large-scale systematic searches for SNe Ia and resulted in a rapid extension of supernovae compilations.

The current supernovae compilations Union2.1 (Sullivan et al. 2011; Suzuki et al. 2012; Campbell et al. 2013; Betoule et al. 2014; Rest et al. 2014; Riess et al. 2018), and Pantheon (Scolnic et al. 2018; Jones et al. 2018) comprise of hundreds of SNe Ia discovered and spectroscopically confirmed. The Pantheon dataset is the most accurate SNe Ia compilation at present. Every SN

Ia is described by its apparent rest-frame B-band magnitude m_B , the absolute B-band magnitude M_B , the stretch parameter x_1 , and the colour parameter c . These parameters are used in the Tripp formula (Tripp 1998; Guy et al. 2007) for calculating the redshift-dependent distance modulus $\mu(z)$, which serves for testing the cosmological models,

$$\mu = m_B - M_B + \alpha x_1 - \beta c, \quad (26)$$

where coefficients α and β are the global nuisance parameters to be determined when seeking an optimum cosmological model. The expansion history is calculated from μ using the following equations,

$$\mu = 25 + 5\log_{10}(d_L), \quad d_L = (1+z) \int_0^z \frac{cdz'}{H(z')}, \quad (27)$$

where d_L is the luminosity distance expressed for the flat universe. The Hubble function $H(z)$ is expressed for the flat universe described by the standard Λ CDM model as

$$H^2(z) = H_0^2 [\Omega_m(1+z)^3 + \Omega_\Lambda], \quad (28)$$

by the standard EdS model as

$$H^2(z) = H_0^2 [\Omega_m(1+z)^3 + \Omega_k(1+z)^2], \quad (29)$$

and by the conformal EdS model as

$$H^2(z) = H_0^2 [\Omega_m(1+z) + \Omega_k]. \quad (30)$$

While the Λ CDM model contains dark energy Ω_Λ as a free parameter, which must be adjusted by fitting with the SNe Ia observations, the conformal EdS model requires no free parameter for the flat universe, and the curvature parameter Ω_k is needed for a curved universe. Since the Universe is nearly flat, this parameter should be close to zero and can be determined from other independent observations.

Fig. 1 shows a comparison of the SNe Ia measurements with predictions of the Λ CDM model and of the standard and conformal EdS models. The standard EdS model is in a visible disagreement with the SNe Ia measurements and this disagreement led to developing the Λ CDM model by introducing dark energy Ω_Λ into Eq.(28) to get a satisfactory fit. Strikingly, the conformal EdS model (30) fits data equally well as the Λ CDM model with no assumption of dark energy (see

Fig. 1.— The Hubble diagram with Type Ia supernovae observations. Blue dots show measurements of the SNe Pantheon compilation (Scolnic et al. 2018; Jones et al. 2018). The red line in (a) shows the Λ CDM model described by Eq.(28) with $H_0 = 71.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. The red line in (b) shows the conformal EdS model described by Eq.(30) with $H_0 = 71.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_m = 1.1$ and $\Omega_k = -0.1$. The black line in (a-b) shows the standard EdS model described by Eq.(29) with $H_0 = 71.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_m = 1.0$ and $\Omega_k = 0$.

Fig. 2.— Residual Hubble plots for (a-b) the individual SNe Ia data and (c-d) the binned SNe Ia data. (a,c) The flat Λ CDM model, (b,d) the conformal EdS model. For parameters of the models, see caption of Fig. 1. The error bars in (c-d) show the 99% confidence intervals. Data are taken from the SNe Pantheon compilation (Scolnic et al. 2018; Jones et al. 2018).

Fig. 2). This confirms that the solution of the puzzle with the supernovae dimming does not lie in introducing dark energy but in correcting the metric used in the Friedmann equations.

4. Discussion

The Friedmann equations introduce the expansion of the Universe and form fundamentals of modern cosmology. Intuitively, the space expansion can explain the cosmological redshift, because the distant galaxies are moving away due to the expansion and we observe their light distorted by the Doppler effect. This was probably the motivation for describing the Universe by the FLRW metric. The problem is, however, more involved, and we know that the cosmological redshift is not due to the Doppler effect but due to distortion of the spacetime described by GR. The redshift of distant galaxies would be observed even for a non-expanding universe at the present epoch. From this point of view, there is no clear argument, why the FLRW metric introduces just the space expansion with no time dilation. In fact, it is surprising to assume distortion of space only, because other solutions in GR such as the well-known Schwarzschild solution involve distortions in space and time together.

The re-examination of light propagation in space described by the FLRW metric reveals that this metric does not predict the cosmological redshift, because the change in the frequency of light is always connected with time dilation and with a variation of the time metric g_{00} in GR, similarly as for the gravitational redshift. By contrast, the conformal FLRW metric predicts the cosmological redshift properly. Therefore, the cosmological redshift is a consequence of time dilation but not of the space expansion and the conformal FLRW metric should be applied.

Obviously, we can ask a question: why atoms radiate photons with the same (rest-frame) frequency at all redshifts and why this frequency is not affected by time dilation? The answer is straightforward: the frequency of emitted photons is independent of redshift, because it depends on quantized energy levels of electrons in atoms and these energy levels are redshift independent. Once the photon is emitted, its frequency decreases due to time dilation when photon propagates along the

ray path from the emitter to the receiver. Since the comoving speed of light is constant, the coordinate speed of light must be variable. In this way, the emitted photons with frequency ν have shorter proper wavelengths at high redshift than the photons with the same frequency ν but emitted at the present epoch.

The correctness of the conformal FLRW metric is convincingly confirmed by SNe Ia observations. In fact, observations of the SNe Ia were originally proposed by Riess et al. (1998) and Perlmutter et al. (1999) for testifying the existing cosmological model and the SNe Ia observations surprisingly revealed essential discrepancy between theoretical predictions and measurements. However, instead of questioning the validity of the FLRW metric and the Friedmann equations, Riess et al. (1998) and Perlmutter et al. (1999) introduced a free parameter into the Friedmann equations to comply with data. In this way, the model is capable to fit the SNe Ia observations, but at the cost of introducing a physically controversial concept of dark energy. By contrast, the EdS model based on the conformal FLRW metric fits the SNe Ia data with no need to introduce any new free parameter.

5. Conclusions

In summary, we conclude that the conformal FLRW metric is the only correct metric for describing the evolution of the Universe, which can predict the cosmological redshift. The variable rate of time during the expansion of the Universe is inevitable and implies the following fundamental consequences: (1) The gravitational and cosmological redshifts are described by the same formula. (2) The Universe is conformal with the static model. (3) The frequency of photons increases with redshift as $(1+z)$. (4) Not only the frequency of photons but also the rate of photons increases with redshift as $(1+z)$. (5) The speed of light increases with the expansion of the Universe as $(1+z)$. (6) The wavelength of photons decreases with redshift as $(1+z)^{-2}$. (7) Energy flux of photons redshifted along their raypath is conserved. (8) The expansion of the Universe is decelerating at the present epoch. The conformal FLRW metric fits the SN Ia observations and no dark energy is needed in the Einstein and Friedmann equations.

REFERENCES

- Betoule, M., Kessler, R., Guy, J., et al. 2014, *A&A*, 568, A22
- Campbell, H., D’Andrea, C. B., Nichol, R. C., et al. 2013, *ApJ*, 763, 88
- Carroll, S. M. 2004, *Spacetime and geometry. An introduction to general relativity*
- Dicke, R. H. 1957, *Rev. Mod. Phys.*, 29, 363
- Dirac, P. 1968, *On Methods in Theoretical Physics*
- Ellis, G. F. R. 2007, *General Relativity and Gravitation*, 39, 511
- Endean, G. 1994, *ApJ*, 434, 397
- Endean, G. 1997, *ApJ*, 479, 40
- Friedmann, A. 1922, *Zeitschrift fur Physik*, 10, 377
- Friedmann, A. 1999, *General Relativity and Gravitation*, 31, 1991
- Grøn, Ø. & Johannessen, S. 2011, *European Physical Journal Plus*, 126, 28
- Guy, J., Astier, P., Baumont, S., et al. 2007, *A&A*, 466, 11
- Hubble, E. 1929, *Proceedings of the National Academy of Science*, 15, 168
- Ibison, M. 2007, *Journal of Mathematical Physics*, 48, 122501
- Infeld, L. & Schild, A. 1945, *Physical Review*, 68, 250
- Infeld, L. & Schild, A. E. 1946, *Physical Review*, 70, 410
- Jones, D. O., Scolnic, D. M., Riess, A. G., et al. 2018, *ApJ*, 857, 51
- Lemaître, G. 1927, *Annales de la Société Scientifique de Bruxelles*, 47, 49
- Magueijo, J. 2003, *Reports on Progress in Physics*, 66, 2025
- Misner, C. W., Thorne, K. S., & Wheeler, J. A. 1973, *Gravitation*
- Mukhanov, V. 2005, *Physical Foundations of Cosmology*
- Peacock, J. A. 1999, *Cosmological Physics*
- Peebles, P. J. E. 1993, *Principles of Physical Cosmology*
- Perlmutter, S., Aldering, G., Goldhaber, G., et al. 1999, *ApJ*, 517, 565
- Rest, A., Scolnic, D., Foley, R. J., et al. 2014, *ApJ*, 795, 44
- Riess, A. G., Casertano, S., Yuan, W., et al. 2018, *ApJ*, 861, 126
- Riess, A. G., Filippenko, A. V., Challis, P., et al. 1998, *AJ*, 116, 1009
- Robertson, H. P. 1935, *ApJ*, 82, 284
- Robertson, H. P. 1936, *ApJ*, 83, 257
- Ryden, B. 2016, *Introduction to Cosmology*
- Scolnic, D. M., Jones, D. O., Rest, A., et al. 2018, *ApJ*, 859, 101
- Sullivan, M., Guy, J., Conley, A., et al. 2011, *ApJ*, 737, 102
- Suzuki, N., Rubin, D., Lidman, C., et al. 2012, *ApJ*, 746, 85
- Tripp, R. 1998, *A&A*, 331, 815
- Walker, A. G. 1935, *MNRAS*, 95, 263
- Walker, A. G. 1937, *Proceedings of the London Mathematical Society*, 42, 90
- Weinberg, S. 1972, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*
- Weinberg, S. 2008, *Cosmology*