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Robust Stability of Discrete-time Singularly Perturbed Systems with Nonlinear Perturbation

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Abstract: This paper is concerned with the robust stability and stabilization problems of discrete-time singularly perturbed systems (DTSPSs) with nonlinear perturbations. A proper sufficient condition via the fixed-point principle is proposed to guarantee that the given system is in a standard form. Then, based on the singular perturbation approach, a linear matrix inequality (LMI) based sufficient condition is presented such that the original system is standard and input-to-state stable (ISS) simultaneously. Thus, it can be easily verified for it only depends on the solution of an LMI. After that, for the case where the nominal system is unstable, the problem of designing a control law to make the resulting closed-loop system ISS is addressed. To achieve this, a sufficient condition is proposed via LMI techniques for the purpose of implementation. The criteria presented in this paper are independent of the small parameter and the stability bound can be derived effectively by solving an optimal problem. Finally, the effectiveness of the obtained theoretical results is illustrated by two numerical examples.

Key words: Discrete-time singularly perturbed systems, two-time scale, input-to-state stability, linear matrix inequality, robust stability

1. Introduction

Singularly perturbation problems are usually described by state-space models with a small singular perturbation parameter ε , which naturally arise in practical control engineering field, such as aerospace systems, robot control systems, nuclear reactor control systems and power systems. Such a parameter often can lead to the increased order and stiffness of systems. A so-called reduction technique is usually adopted for dealing with this kind of system [1]-[3], which is a basic characterization of singular perturbation and a useful tool. Thus the numerical stiffness problem resulting from the existence of multi-time-scale phenomena can be alleviated. During the past decades, the study of SPSs has received much attention, the survey on the recent developments in SPSs can be found in [4]-[15] and the references therein.

Recently, many efforts have been made in DTSPSs for their extensive applications in control theory and various engineering. As shown in [16], DTSPSs can be described by slow sampling rate model and fast sampling rate model [17]. As with its continuous-time counterpart, the study of DTSPSs still faces the same problems, such as ingredients-order reduction, boundary layer phenomena etc. [18]-[19]. So far, some results have been obtained for robust stability and stabilization of DTSPSs [20]-[22]. Based on the reduction technique, the composite control for DTSPSs is considered in [19], in which a theoretical framework is given for systems with slow and fast modes. In [22], a unified state-feedback is designed, under which the considered continuous- and discrete time SPS is robust stable. Recently, some LMIs approaches have been developed to deal with the robust control problems for DTSPSs [23]-[31]. However, it should be pointed that the most of the aforementioned works were limited to linear cases, in which the system matrices are time-invariant or state-independent uncertainties. Meanwhile, a prescribed scalar ε^* is needed when evaluating the upper bound of singular perturbation, which would bring some inconvenience for operating. The reason is that how to choose a proper ε^* is difficult to ensure the solvability of the proposed LMI. Moreover, the two-time scale decomposition technique is not used in the literature, in which the small parameter $\varepsilon > 0$ is seen as a static scalar. Whether the obtained result for the limit system (i.e. the slow and fast subsystems of the original system) would still remain as $\varepsilon \rightarrow 0$? Unfortunately, there are no definite answers right now. It will be more perfect if these problems can be solved.

Based on the above analysis, we, in this paper, consider the robust stability and synthesis for DTSPSs with nonlinear perturbation. Up to now, little work on this topic has been made in the literature. As a prerequisite, by resorting to the fixed-point principle, a sufficient condition is given to ensure that the isolate root is possible and unique, thus the given system is standard. Generally speaking, it is a basic requirement for SPSs [1], [19]. Furthermore, by singular perturbation approach, the ISS results for the slow and fast subsystems were established, respectively. The notion of ISS was first introduced by Sontag in 1989 [32], which play an important role in characterizing the effects of external input to a control system. Recently, the relevant results have been generalized to nonlinear time-delay systems [33], discrete-time systems [34], and singularly perturbed systems [35], etc. Based on the ISS notion, a unified LMI sufficient condition is given, under which the existence of the isolate root and ISS property of the original system is guaranteed. Then, a state feedback control is designed to render the resulting closed-loop system ISS. It is important to point out that the proposed method here is substantially different from the existing literature since the nonlinear term is considered here and the coordinate transformation of the original system is not used although the reduction technique is adopted.

Compared with the existing results, the advantages of this paper can be summarized as follow: 1) A more general system is considered, where the nonlinear term only knows their norm upper bounds; 2) A unified LMI-based sufficient condition is presented such that the isolate root and ISS are guaranteed. Thus, it is very easy to verify as it just depends on the solution of the LMI; 3) The method presented in this paper marries the reduced technique and LMI, thus the numerically stiff problem can be alleviated; 4) In [23], it is difficult to solve the controller gain, the reason is that much more complex equation are involved. However, our method overcomes this constraint, which can be solved easily; 5) Although the two-time scale decomposition technique is adopted in this paper, any coordinate transformations for the original system is not involved, while this point is necessary by the traditional method.

The rest of the paper is organized as follows. Section 2 gives the problem formulation. The main results are given in Section 3. Section 4 gives two examples to show the effectiveness of the proposed methods. Finally, the conclusion is drawn in Section 5.

Notation: Throughout the paper, the notations are fairly standard. $P > 0$ means that matrix P is positive definite; # denotes that the item will not be used in the

following.

2. Problem Formulation

Consider the following fast sampling DTSPSs with nonlinear disturbances described by

$$x_1(k+1) = (I + \varepsilon A_{11})x_1(k) + \varepsilon A_{12}x_2(k) + \varepsilon B_{u1}u(k) + \varepsilon H_1 f_1(x_1, x_2) + \varepsilon B_{w1}w(k), \quad (1)$$

$$x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k) + B_{u2}u(k) + H_2 f_2(x_1, x_2) + B_{w2}w(k), \quad (2)$$

where $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ ($n_1 + n_2 = n$) are the slow state and the fast state, respectively; $u \in R^q$ is the control input; $w \in R^m$ is the bounded disturbance input; $\varepsilon > 0$ is a positive singular perturbation parameter; $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$ are initial conditions. All matrices describing the above system are assumed to be known constant matrices with appropriate dimensions; $f_i(x_1, x_2)$ ($i=1,2$) are vector-value nonlinear functions with $f_i(0,0)=0$, which satisfy the following Lipschitz conditions for all $(x_1, x_2), (\tilde{x}_1, \tilde{x}_2) \in R^{n_1} \times R^{n_2}$:

$$\|f_i(x_1, x_2) - f_i(\tilde{x}_1, \tilde{x}_2)\| \leq \alpha \|F_{i1}(x_1 - \tilde{x}_1) + F_{i2}(x_2 - \tilde{x}_2)\|, \quad i=1,2, \quad (3)$$

where $\alpha > 0$ is a known constant; F_{ij} ($i, j=1,2$) are constant matrices with appropriate dimensions.

Define

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad E_0 = \begin{pmatrix} I & O \\ O & O \end{pmatrix}, \quad E_\varepsilon = \begin{pmatrix} \varepsilon I & O \\ O & I \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B_u = \begin{pmatrix} B_{u1} \\ B_{u2} \end{pmatrix},$$

$$B_w = \begin{pmatrix} B_{w1} \\ B_{w2} \end{pmatrix}, \quad H = \begin{pmatrix} H_1 & O \\ O & H_2 \end{pmatrix}, \quad f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}.$$

Then system (1)-(2) can be rewritten as

$$x(k+1) = A_\varepsilon x(k) + B_{u\varepsilon}u(k) + H_\varepsilon f(x(k)) + B_{w\varepsilon}w(k), \quad x_2 = \varphi(x_1, w), \quad (4)$$

where $A_\varepsilon = E_0 + E_\varepsilon A$, $B_{u\varepsilon} = E_\varepsilon B_u$, $H_\varepsilon = E_\varepsilon H$, $B_{w\varepsilon} = E_\varepsilon B_w$.

It's easy to verify that the nonlinear term $f(x)$ is bounded by

$$f^T(x)f(x) \leq \alpha^2 x^T F^T F x. \quad (5)$$

Remark 2.1: Condition (3) is used to show that the system considered here is a standard SPSs. The perturbation in (5) has been widely studied; see [8], [36], [37] and the references therein. It is worth noting that the matched condition can be seen as a

special case of (5). However, for the discrete case, the robust stability for system (1)-(2) has not been considered.

A common idea for SPS is that the robust stability of the system (1)-(2) is considered by analyzing the corresponding slow and fast subsystems. Before we move on, some definitions and lemmas are given.

Definition 2.1: For any given (x_1, w) , if the algebraic equation

$$0 = A_{21}x_1 + (A_{22} - I)x_2 + H_2f_2(x_1, x_2) + B_{w2}w \quad (6)$$

has a unique isolate root $x_2 = \varphi(x_1, w)$, then system (1)-(2) with $u = 0$ is said to be standard.

It can be seen from the Definition 2.1 that the existence of the isolate root guarantees that the reduced-order model is well defined.

Definition 2.2: [34] Consider the discrete-time nonlinear system:

$$x(k+1) = f(x(k), w(k)), \quad (7)$$

where state $x(\cdot)$ is in R^n , and control input $w(\cdot)$ in R^m . $f: R^n \times R^m \rightarrow R^n$ is continuous and locally Lipschitz in x and w . The input w is a bounded function for all $k \geq 0$. Then the system (7) is said to be input-to-state stable (ISS) if there exist a class KL function β and a class K function γ such that for any initial state $x(0)$, the solution $x(k)$ exists for all $k \geq 0$ and satisfies:

$$\|x(k)\| \leq \beta(\|x(0)\|, k) + \gamma(\sup_{0 \leq \tau \leq k} \|w(\tau)\|). \quad (8)$$

Remark 2.2: The last inequality (8) guarantees that for any bounded input $w(\cdot)$, the state $x(\cdot)$ will be bounded, and as k increases, the state $x(k)$ will be ultimately bounded by a class K function of $\|w\|$. Furthermore, the inequality also shows that if $w(k)$ converges to zero as $k \rightarrow \infty$, so does $x(k)$.

Lemma 2.1: [34] Let $V: R^n \rightarrow R$ be a continuous function such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$

$$V(x(k+1)) - V(x(k)) \leq -W(x(k)), \quad \|x\| \geq \rho(\|w\|),$$

where α_1, α_2 are class K_∞ functions, ρ is a class K function, and $W(x)$ is a continuous and positive definite function on R^n . Then, the system (7) is input-to-state stable.

3. Main Results

A. The Existence of an Isolate Root Analysis

In the next two sections, the zero control input will be considered for system (1)-(2). We, in this subsection, first establish a sufficient condition to guarantee system (1)-(2) standard, which can be stated in the following Lemma.

Lemma 3.1: If there exist a scalar $\mu_1 > 0$, matrices P_{11} , P_{21} and P_{22} satisfying

$$\Phi_0 = \begin{pmatrix} A_1^T P_1 + P_1^T A_1 + \mu_1 \alpha^2 F^T F & P_1^T H \\ * & -\mu_1 I \end{pmatrix} < 0, \quad (9)$$

where $A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - I \end{pmatrix}$, $P_1 = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix}$. Then the system (1)-(2) is standard.

Proof: Consider the difference algebraic equation

$$x_1(k+1) = (I + \varepsilon A_{11})x_1(k) + \varepsilon A_{12}x_2(k) + \varepsilon H_{11}f_1(x_1, x_2) + \varepsilon B_{w1}w(k), \quad (10)$$

$$0 = A_{21}x_1(k) + (A_{22} - I)x_2(k) + B_{u2}u(k) + H_{21}f_2(x_1, x_2) + B_{w2}w(k). \quad (11)$$

From (9), one has

$$A_1^T P_1 + P_1^T A_1 + \mu_1 \alpha^2 F^T F < 0.$$

Noticing that $\mu_1 \alpha^2 F^T F$ is non-negative, thus

$$A_1^T P_1 + P_1^T A_1 < 0. \quad (12)$$

Now, we make a partition for (12)

$$(A_{22} - I)^T P_{22} + P_{22}^T (A_{22} - I) < 0.$$

So, $(A_{22} - I)^T P_{22}$ is nonsingular. It follow from $P_{22} > 0$ that $(A_{22} - I)$ is nonsingular too. Let

$$M = \begin{pmatrix} I & O \\ O & (A_{22} - I)^{-1} \end{pmatrix} \square \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \quad N = \begin{pmatrix} I & O \\ -(A_{22} - I)^{-1} A_{21} & I \end{pmatrix} \square \begin{pmatrix} N_1 & N_2 \end{pmatrix}.$$

It can be seen that the matrices M and N are non-singular, and the following decompositions can be given

$$ME_0 N = \begin{pmatrix} I_{n_1} & O \\ O & O \end{pmatrix}, \quad M\bar{A}_\varepsilon N = \begin{pmatrix} I + \varepsilon A_0 & \varepsilon A_{12} \\ O & I_{n_2} \end{pmatrix}, \quad MH_\varepsilon = \begin{pmatrix} \varepsilon H_1 & O \\ O & \bar{H}_2 \end{pmatrix}, \quad MB_{w\varepsilon} = \begin{pmatrix} \varepsilon B_{w1} \\ \bar{B}_{w2} \end{pmatrix},$$

where

$$\bar{A}_\varepsilon = \begin{pmatrix} I + \varepsilon A_{11} & \varepsilon A_{12} \\ A_{21} & A_{22} - I \end{pmatrix}, \quad A_0 = A_{11} - A_{12}(A_{22} - I)^{-1}A_{21},$$

$$\bar{H}_2 = (A_{22} - I)^{-1}H_2, \quad \bar{B}_{w2} = (A_{22} - I)^{-1}B_{w2}.$$

Applying the Schur's complement, one has from LMI (9) that

$$A_1^T P_1 + P_1^T A_1 + \mu_1 \alpha^2 F^T F + \mu_1^{-1} P_1^T H H^T P_1 < 0. \quad (13)$$

Premultiplying and postmultiplying (13) with N^T and N , respectively, it can be seen that

$$(MA_1 N)^T M^{-T} P_1 N + (M^{-T} P_1 N)^T M A_1 N + \mu_1 \alpha^2 N^T F^T F N$$

$$+ \mu_1^{-1} (M^{-T} P_1 N)^T M H H^T M^T M^{-T} P_1 N < 0. \quad (14)$$

Let

$$M^{-T} P_1 N = \begin{pmatrix} \bar{P}_1 & O \\ \bar{P}_3 & \bar{P}_4 \end{pmatrix},$$

By further calculation, it is shown in (14) that the block matrix at the second block row and the second block column is negative definite, that is

$$\bar{P}_4^T + \bar{P}_4 + \mu_1 \alpha^2 N_2^T F^T F N_2 + \mu_1^{-1} \bar{P}_4^T \bar{H}_2 \bar{H}_2^T \bar{P}_4 < 0.$$

The above inequality shows that there exists a scalar $\delta > 0$ satisfying

$$\bar{P}_4^T + \bar{P}_4 + \mu_1 \alpha^2 N_2^T F^T F N_2 + \mu_1^{-1} \bar{P}_4^T (\bar{H}_2 \bar{H}_2^T + \delta I) \bar{P}_4 < 0.$$

Denote $Q_\delta = (M_2 H H^T M_2^T + \delta I)^{-\frac{1}{2}}$. Then last inequality is equivalent to

$$\mu_1^{-1} (\bar{P}_4^T Q_\delta^2 + \mu_1 I) Q_\delta^{-2} (Q_\delta^2 \bar{P}_4 + \mu_1 I) - \mu_1 Q_\delta^{-2} + \mu_1 \alpha^2 N_2^T F^T F N_2 < 0,$$

which implies that

$$\alpha^2 N_2^T F^T F N_2 < Q_\delta^{-2}.$$

Thus there exists a scalar $\eta > 0$ satisfying

$$\|F_{22} Q_\delta\| < \frac{1}{\alpha \sqrt{1+\eta}}.$$

Furthermore, for $Q_\delta^{-1} \bar{H}_2$ we have

$$Q_\delta^{-1} \bar{H}_2 \bar{H}_2^T Q_\delta^{-T} = Q_\delta^{-1} (\bar{H}_2 \bar{H}_2^T + \delta I) Q_\delta^{-T} = I,$$

which implies that $\|Q_\delta^{-1} \bar{H}_2\| < 1$.

Next, the following transformation of coordinates is introduced by

$$(N_1, N_2 Q_\delta)^{-1} x = (x_{11}^T \quad x_{12}^T)^T, \quad (15)$$

where $x_{11} \in R^{n_1}$, $x_{12} \in R^{n_2}$. Then the equation (10)-(11) can be described as:

$$\begin{aligned} x_{11}(k+1) &= (I + \varepsilon A_0)x_{12}(k) + \varepsilon A_{12}Q_\delta x_2(k) + \varepsilon B_{w1}w(k) \\ &\quad + \varepsilon H_1 f_1(x_{11}(k), -(A_{22} - I)^{-1}A_{21}x_{11}(k) + Q_\delta x_{12}(k)), \\ 0 &= x_{12}(k) + Q_\delta^{-1}\bar{H}_2 f_2(x_{11}(k), -(A_{22} - I)^{-1}A_{21}x_{11}(k) + Q_\delta x_{12}(k)) + Q_\delta^{-1}\bar{B}_{w2}w(k). \end{aligned}$$

For any given x_{12} , $\bar{x}_{12} \in R^{n_2}$, we have

$$\begin{aligned} &\|Q_\delta^{-1}\bar{H}_2 f_2(x_{11}, -(A_{22} - I)^{-1}A_{21}x_{11} + Q_\delta x_{12}) - Q_\delta^{-1}\bar{H}_2 f(x_{11}, -(A_{22} - I)^{-1}A_{21}x_{11} + Q_\delta \bar{x}_{12})\| \\ &\leq \|Q_\delta^{-1}\bar{H}_2\| \|f(x_{11}, -(A_{22} - I)^{-1}A_{21}x_{11} + Q_\delta x_{12}) - f(x_{11}, -(A_{22} - I)^{-1}A_{21}x_{11} + Q_\delta \bar{x}_{12})\| \\ &\leq \alpha \|F_{22}Q_\delta^{-1}(x_{12} - \bar{x}_{12})\| \leq \alpha \|F_{22}Q_\delta^{-1}\| \|x_{12} - \bar{x}_{12}\| \\ &\leq (\sqrt{1+\eta})^{-1} \|x_{12} - \bar{x}_{12}\|. \end{aligned} \quad (16)$$

It can be seen from (16) that for any given (x_{11}, w) , there exists a unique solution $x_{12} = \phi(x_{11}, w)$ by resorting to the fixed-point principle. Therefore, the isolate root $x_2 = \phi(x_1, w)$ is guaranteed by (15), that is, the system (1) is standard.■

Remark 3.1: it is shown from Lemma 3.1 that the isolate root is property of the system itself, which is a basic requirement for the reduced technique. Moreover, it can be seen that $x_2 = \phi(x_1, w)$ is also Lipschitz. Specially, there exist two scalars $\alpha_1 > 0$ and $\alpha_2 > 0$, such that

$$\|\phi(x_1, w)\| \leq \alpha_1 \|x_1\| + \alpha_2 \|w\|. \quad (17)$$

A similar procedure to that of (16) will be adopted in the proof process, thus the detail is ignored here.

Now, using Lemma 3.1 and the reduced technique, the original system will be decomposed into the two subsystems.

A slow subsystem can be defined by setting $x_2(k+1) = x_2(k)$

$$x_s(k+1) = (I + \varepsilon A_{11})x_s(k) + \varepsilon A_{12}\bar{x}_2(k) + \varepsilon H_1 f_1(x_s, \bar{x}_2) + \varepsilon B_{w1}w_s(k), \quad x_s(0) = x_{10}, \quad (18)$$

$$0 = A_{21}x_s(k) + (A_{22} - I)\bar{x}_2(k) + H_2 f_2(x_s, \bar{x}_2) + B_{w2}w_s(k), \quad (19)$$

where $\bar{x}_2 = \phi(x_s, w_s)$ here is regarded as an intermediate variable, its introduction will avoid the coordinate transformation of the original system and contribute to the H_∞ performance analysis.

Equation (18)-(19) characterizes the evolution of the slow variables in the fast time scale k . Let $t = \varepsilon k$. Dividing both side by ε and taking the limit $\varepsilon \rightarrow 0$ yields

the following continuous-time slow subsystems:

$$\frac{dx_s}{dt} = A_{11}x_s + A_{12}\bar{x}_2 + H_1f_1(x_s, \bar{x}_2) + B_{w1}w_s, \quad x_s(0) = x_{10}, \quad (20)$$

$$0 = A_{21}x_s + (A_{22} - I)\bar{x}_2 + H_2f_2(x_s, \bar{x}_2) + B_{w2}w_s. \quad (21)$$

Define $\bar{x} = (x_s^T \quad \bar{x}_2^T)^T$, then slow subsystem (20)-(21) can be rewritten as

$$E_0\dot{\bar{x}} = A_1\bar{x} + Hf(\bar{x}) + B_w w_s. \quad (22)$$

In order to obtain the fast subsystems, we replace $\bar{x}(k+1) = \bar{x}(k)$, subtracting (19) from the equation (2) and setting $\varepsilon = 0$ yields the following fast subsystem

$$x_f(k+1) = A_{22}x_f(k) + H_2\Delta f_2 + B_{w2}w_f(k), \quad x_f(0) = x_{20} - \varphi, \quad (23)$$

where $x_f = x_2 - \varphi$, $w_f = w - w_s$, $\Delta f_2 = f_2(x_1, x_f + \varphi) - f_2(x_1, \varphi)$.

Remark 3.2: This hybrid form has been discussed in depth discussed in [19]. Meanwhile, the evolution of the slow subsystems has also been proven to be justified in the theory.

It is known that the ISS property of reduced order system does not imply the one of the original system. Therefore, we will focus on how the ISS property of the original system (1)-(2) can be derived from the subsystems.

B. Input-to-State Stability Analysis

Using the two-time scale decomposition technique, a sufficient condition will be proposed such that the full-order system (1)-(2) is ISS. First, the following result for the slow subsystem (20)-(21) is given.

Theorem 3.1: For Lemma 3.1, if the matrix $P_{11} > 0$ such that (9) holds, then the slow subsystem (20)-(21) is ISS with w_s .

Proof: Consider the Lyapunov function:

$$S_1(x_s) = x_s^T P_{11} x_s. \quad (24)$$

Clearly, $S_1(x_s) > 0$ for any $x_s \neq 0$. Noticing that $S_1(x_s) = x_s^T P_{11} x_s = \bar{x}^T E_0^T P_1 \bar{x}$, thus

$$\dot{S}_1(x_s) = (A_1\bar{x} + Hf + B_w w_s)^T P_1 \bar{x} + \bar{x}^T P_1^T (A_1\bar{x} + Hf + B_w w_s).$$

Furthermore, for any scalar $\mu_1 > 0$, using the constraint (5), we have

$$\begin{aligned} \dot{S}_1(x_s) &\leq (A_1\bar{x} + Hf + B_w w_s)^T P_1 \bar{x} + \bar{x}^T P_1^T (A_1\bar{x} + Hf + B_w w_s) \\ &\quad + \mu_1(\alpha^2 \bar{x}^T F^T F \bar{x} - f^T f) \end{aligned}$$

$$= (\bar{x}^T \quad f^T) \Phi_0 (\bar{x}^T \quad f^T)^T + 2\bar{x}^T P_1^T B_w w_s.$$

Let $a = \lambda_{\min}(-\Phi_0)$, it can be seen from (11) that $a > 0$. Thus, we have

$$\dot{S}_1(x_s) \leq -a \|x_s\|^2 + 2\bar{x}^T P_1^T B_w w_s.$$

The following inequality can be further obtained by (19)

$$\begin{aligned} \dot{S}_1(x_s) &\leq -a \|x_s\|^2 + 2(\|x_s\| + \|\varphi\|) \|P_1^T B_w\| \|w_s\| \\ &\leq -a \|x_s\|^2 + b \|x_s\| \|w_s\| + c \|w_s\|^2 \\ &\leq -a(1-\theta) \|x_s\|^2, \quad \forall \|x_s\| \geq \frac{b + \sqrt{b^2 + 4ca\theta}}{2a\theta} \|w_s\|, \end{aligned}$$

where $0 < \theta < 1$, $b = 2(1 + \alpha_1) \|P_1^T B_w\|$ and $c = 2\alpha_2 \|P_1^T B_w\|$. Hence, according to [30], there exist a class KL function β and a class K function γ such that for any initial state x_{10} , and satisfies

$$\|x_s(t)\| \leq \beta(\|x_{10}\|, t) + \gamma(\sup_{0 \leq \tau \leq t} \|w_s(\tau)\|). \quad (25)$$

That is, the slow subsystem (20)-(21) is ISS with w_s . ■

Theorem 3.2: For Lemma 3.1, if the matrix $P_{22} > 0$ such that (9) holds and satisfies

$$\Phi = \Phi_0 + \Phi_1 < 0, \quad (26)$$

where

$$\Phi_1 = \begin{pmatrix} A_1^T P_2 A_1 & A_1^T P_2 H \\ * & H^T P_2 H \end{pmatrix}, \quad P_2 = \begin{pmatrix} O & O \\ O & P_{22} \end{pmatrix}.$$

Then the fast subsystem (23) is ISS with w_f .

Proof: A partition for LMI (26) is adopted, we have

$$\begin{pmatrix} \# & \# & \# & \# \\ * & (2,2) & \# & (2,4) \\ * & * & \# & \# \\ * & * & * & (4,4) \end{pmatrix} < 0,$$

where

$$(2,2) = A_{22}^T P_{22} A_{22} - P_{22} + \mu_1 \alpha^2 (F_{12}^T F_{12} + F_{22}^T F_{22}),$$

$$(2,4) = A_{22}^T P_{22} H_2, \quad (4,4) = -\mu_1 I + H_2^T P_{22} H_2.$$

This indicates that

$$\Phi_2 = \begin{pmatrix} A_{22}^T P_{22} A_{22} - P_{22} + \mu_1 \alpha^2 F_{22}^T F_{22} & A_{22}^T P_{22}^T H_2 \\ * & -\mu_1 I + H_2^T P_{22} H_2 \end{pmatrix} < 0. \quad (27)$$

Let $S_2(x_f) = x_f^T P_{22} x_f$, then one has

$$\begin{aligned} \Delta S_2 &= S_2(x_f(k+1)) - S_2(x_f(k)) \\ &\leq x_f^T(k+1) P_{22} x_f(k+1) - x_f^T(k) P_{22} x_f(k) + \mu_1 (\alpha^2 x_f^T F_{22}^T F_{22} x_f - \Delta^T f_2 \Delta f_2) \\ &= (x_f^T \quad \Delta^T f_2) \Phi_2 (x_f^T \quad \Delta^T f_2)^T + 2x_f^T A_{22}^T P_{22} B_{w2} w_f \\ &\quad + 2\Delta^T f_2 H_2^T P_{22} B_{w2} w_f + w_f^T B_{w2}^T P_{22} B_{w2} w_f. \end{aligned}$$

Let $\bar{a} = \lambda_{\min}(-\Phi_2)$, it can be seen from (26) that $\bar{a} > 0$. Thus, one has

$$\begin{aligned} \Delta S_2 &\leq -\bar{a} \|x_f\|^2 + \bar{b} \|x_f\| \|w_f\| + \bar{c} \|w_f\|^2 \\ &\leq -\bar{a}(1-\bar{\theta}) \|x_f\|^2, \quad \|x_f\| \geq \frac{\bar{b} + \sqrt{\bar{b}^2 + 4\bar{c}\bar{a}\bar{\theta}}}{2\bar{a}\bar{\theta}} \|w_f\|, \end{aligned}$$

where $0 < \bar{\theta} < 1$, $\bar{b} = 2\|A_{22}^T P_{22} B_{w2}\| + 2\alpha\|H_2^T B_{w2}\|\|F_{22}\|$ and $\bar{c} = \|B_{w2}^T B_{w2}\|$.

Hence, the conditions of Lemma 2.1 are satisfied, and we can conclude that there exist a class KL function $\bar{\beta}$ and a class K function $\bar{\gamma}$ such that for any initial state $x_f(0)$, the solution $x_f(k)$ exists for all $n \geq 0$

$$\|x_f(k)\| \leq \bar{\beta}(\|x_f(0)\|, k) + \bar{\gamma}(\sup_{0 \leq \tau \leq k} \|w_f(\tau)\|). \quad (28)$$

That is, the fast subsystem (20)-(21) is ISS with respect to the disturbance w_f . This completes the Proof. ■

Based on Theorems 3.1 and 3.2, we are now in the position to show the main result of this subsection, which is stated in the following theorem.

Theorem 3.3: If the conditions of Theorems 3.1 and 3.2 hold, then there exists an $\varepsilon^* > 0$, such that the following results hold:

- 1) system (1)-(2) is a standard singularly perturbed systems;
- 2) system (1)-(2) is made ISS with respect to disturbance w for any $\varepsilon \in (0, \varepsilon^*]$.

Proof:

1) The proof of Lemma 3.1 has shown that system (1)-(2) is in the standard form, which completes the proof of part 1).

2) We now show the ISS property of system (1)-(2). Under the condition of Theorems 3.1 and 3.2, it is shown that both P_{11} and P_{22} are positive definite matrices,

then there exists a sufficiently small scalar $\varepsilon_1 > 0$ such that $P_{11} - \varepsilon P_{12}^T P_{22}^{-1} P_{21} > 0$ for all $\varepsilon \in (0, \varepsilon_1]$. Thus, it yields

$$P_\varepsilon = \begin{pmatrix} \varepsilon^{-1} P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0, \quad \varepsilon \in (0, \varepsilon_1].$$

Define a Lyapunov function candidate for system (1)-(2) as follows

$$V(x(k)) = x^T(k) P_\varepsilon x(k), \quad (29)$$

Then, for any a constant $\mu_1 > 0$, one has

$$\begin{aligned} \Delta V &= V(x(k+1)) - V(x(k)) \\ &\leq x^T(k+1) P_\varepsilon x(k+1) - x^T(k) P_\varepsilon x(k) + \mu_1 (\alpha^2 x^T F^T F x - f^T f) \\ &\leq (x^T \quad f^T)(\Phi + \varepsilon \bar{\Phi})(x^T \quad f^T)^T + 2x^T A_\varepsilon^T P_\varepsilon B_{w\varepsilon} w + 2f^T H_\varepsilon^T P_\varepsilon B_{w\varepsilon} w + w^T B_{w\varepsilon}^T P_\varepsilon B_{w\varepsilon} w^T, \end{aligned}$$

where

$$\bar{\Phi} = \begin{pmatrix} A^T P_3 A & A^T P_3 H \\ * & H^T P_3 H \end{pmatrix}, \quad P_3 = \begin{pmatrix} P_{11} & P_{21}^T \\ P_{21} & O \end{pmatrix}. \quad (30)$$

It follows from (26) that there exists a sufficiently small scalar $\varepsilon_2 > 0$ such that $\Phi + \varepsilon \bar{\Phi} < 0$ for any given $\varepsilon \in (0, \varepsilon_2]$. Let $\tilde{\alpha} = \lambda_{\min}(-(\Phi + \varepsilon \bar{\Phi}))$, then $\tilde{\alpha} > 0$ for $\varepsilon \in (0, \varepsilon_2]$ and the value is independent of the small parameter. Further, let $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2\}$, then we have $P_\varepsilon > 0$ and

$$\Delta V \leq -\tilde{\alpha} \|x\|^2 + 2x^T A_\varepsilon^T P_\varepsilon B_{w\varepsilon} w + 2f^T H_\varepsilon^T P_\varepsilon B_{w\varepsilon} w + w^T B_{w\varepsilon}^T P_\varepsilon B_{w\varepsilon} w^T$$

for any given $\varepsilon \in (0, \varepsilon^*]$. Thus, we have

$$\Delta V \leq -\tilde{\alpha} \|x\|^2 + \tilde{b} \|w\| \leq -\tilde{\alpha}(1 - \tilde{\theta}) \|x\|^2, \quad \|x\| \geq \frac{\tilde{b} + \sqrt{\tilde{b}^2 + 4\tilde{\alpha}\tilde{\theta}}}{2\tilde{\alpha}\tilde{\theta}} \|w\|, \quad (31)$$

where $\tilde{b} = 2 \max_{0 \leq \varepsilon \leq \varepsilon^*} (\|A_\varepsilon^T P_\varepsilon B_{w\varepsilon}\| + 2\alpha \|H_\varepsilon^T P_\varepsilon B_{w\varepsilon}\| \|F\|)$, $\tilde{c} = \max_{0 \leq \varepsilon \leq \varepsilon^*} \|B_{w\varepsilon}^T P_\varepsilon B_{w\varepsilon}\|$ and

$0 < \tilde{\theta} < 1$. According to Lemma 2.1, there exist KL function $\tilde{\beta}$ and K function $\tilde{\gamma}$ such that the solution $x(k)$ exists for all $n \geq 0$ and any initial state $x(0)$, and satisfies

$$\|x(k)\| \leq \tilde{\beta}(\|x(0)\|, k) + \tilde{\gamma}(\sup_{0 \leq \tau \leq k} \|w(\tau)\|), \quad \varepsilon \in (0, \varepsilon^*]. \quad (32)$$

That is, the system (1)-(2) is ISS with w . ■

Remark 3.3: Theorem 3.3 presents a unified sufficient condition for the standard

requirement and ISS of the DTSPS (1)-(2) by the reduced technique and LMI method, We find that this condition is very easy to verify as it just depends on the solution of the LMI. In addition, the proposed condition is independent of ε , thus, the numerically stiff problem can be alleviated when solving. In addition, it follows from (32) that the ISS property indicates that $w \rightarrow 0$ as $k \rightarrow \infty$, thus the asymptotical stability of the system (1)-(2) can be obtained for all $\varepsilon \in (0, \varepsilon^*]$.

By Theorem 3.3, we propose the following new sufficient condition to characterize and compute the stability bound.

Theorem 3.4: If there exist a constant $\lambda > 0$, positive definite matrices $\Pi > 0$, $P_{11} > 0$, $P_{22} > 0$ and matrix P_{21} , satisfying:

$$\Pi < \lambda P_{11}, \begin{pmatrix} \Pi & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0, \Phi < 0, \bar{\Phi} < -\lambda \Phi, \quad (33)$$

where Φ and $\bar{\Phi}$ are defined in (26) and (30), respectively. Then system (1)-(2) is in the standard and ISS with $w(t)$ for all $\varepsilon \in (0, \varepsilon^*]$ with $\varepsilon^* = \lambda^{-1}$.

Proof: From the condition (33), we have

$$\begin{pmatrix} \lambda P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0, \bar{\Phi} + \lambda \Phi < 0,$$

By $\varepsilon \in (0, \lambda^{-1}]$, it is easy to show that

$$\begin{pmatrix} \varepsilon^{-1} P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0, \Phi + \varepsilon \bar{\Phi} < 0$$

for all $\varepsilon \in (0, \lambda^{-1}]$. Thus, the ISS property for all $\varepsilon \in (0, \varepsilon^*]$ with $\varepsilon^* = \lambda^{-1}$ can be obtained by the proof of Theorem 3.3. ■

By Theorem 3.4, the upper bound ε^* can be solved by the following optimization problem ^[39]

$$\min \lambda \quad \text{subject to (33).}$$

A special case of system (1)-(2) is the simplified linear DTSPS in [16], [20].

$$\begin{cases} x_1(k+1) = (I + \varepsilon A_{11})x_1(k) + \varepsilon A_{12}x_2(k), \\ x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k). \end{cases} \quad (34)$$

Applying Theorem 3.3 to system (34)-(35), we have.

Corollary 3.1: If there exist positive definite matrices $P_{11} > 0$, $P_{22} > 0$ and matrix P_{21} , such that

$$\Xi = A_1^T P_1 + P_1^T A_1 + A_1^T P_2 A_1 < 0, \quad (35)$$

where P_1 and P_2 have same structures as those in Theorem 3.3. Then there exists an $\varepsilon^* > 0$ such that the system (34) is asymptotically stable for all $\varepsilon \in (0, \varepsilon^*]$.

In addition, the upper bound $\varepsilon^* = \lambda^{-1}$ for system (34) can be simultaneously obtained by solving the following GEVP:

$$\begin{aligned} & \min \lambda \quad \text{subject to} \\ & \Pi < \lambda P_{11}, \quad \begin{pmatrix} \Pi & P_{21}^T \\ P_{21} & P_{22} \end{pmatrix} > 0, \quad \Xi < 0, \quad A^T P_3 A + \lambda \Xi < 0, \end{aligned} \quad (33)$$

where Π and P_3 are given in Theorem 3.4.

Remark 3.4: Using the frequency domain approach, Li and Li in [20] considered the stability bound problem. The major drawback is that evaluating the exact value of stability bound by plotting the generalized Nyquist plot is extremely difficult unless the fast subsystem is a scalar. Further, based on the critical stability criteria, a new method was proposed for obtaining the stability bound in [16]. However, the method is still complex to operate. In contrast, Corollary 3.1 provides an LMI based sufficient condition for stability bound of system (34), which can be solved easily by LMI Toolbox. In [25]-[27], it is notice that a prescribed stability bound ε^* is required when using LMI technique. This leads to some inconvenience for implementation, because how to choose a proper ε^* is difficult in order to ensure the solvability of the resulting linear matrix inequality. However, from Corollary 3.1, we can see that this case has been avoided effectively by our method.

C. The Input-to-State Stability of Closed-loop Systems

It should be worth mentioning that Theorem 3.3 requires that the nominal system be stable. However, in practice, this condition may be unsatisfactory. In this case, a feedback control would be necessary to guarantee that the ISS property can be achieved. Instead of being stable, the system is assumed to be stabilizable. Therefore, a state feedback transformation is given by

$$u = K_1 x_1 + K_2 x_2, \quad (36)$$

where $K = (K_1 \ K_2)$ is a constant matrix, such that the resulting closed-loop system is ISS with w .

Substituting the feedback transformation (36) into (1)-(2), the resulting closed-loop

system is given as

$$x(k+1) = (A_\varepsilon + B_{u\varepsilon}K)x(k) + H_\varepsilon f(x(k)) + B_{w\varepsilon}w(k). \quad (37)$$

Applying Theorem 3.3 to (35), we have the following result.

Theorem 3.5: If there exist a constant $\mu_1 > 0$, positive definite matrices $X_{11} > 0$, $X_{22} > 0$ and matrices X_{21} , Y such that

$$\Omega_0 = \begin{pmatrix} \Lambda_0 & \mu_1^{-1}H & X^T A_2^T + Y^T B_{u2}^T & \alpha X^T F^T \\ * & -\mu_1^{-1}I & \mu_1^{-1}\tilde{H}_2^T & O \\ * & * & -X_{22} & O \\ * & * & * & -\mu_1^{-1}I \end{pmatrix} < 0, \quad (38)$$

where

$$\Lambda_0 = A_1 X + X^T A_1^T + B_u Y + Y^T B_u^T, \quad X = \begin{pmatrix} X_{11} & O \\ X_{21} & X_{22} \end{pmatrix}, \quad A_2 = (A_{21} \quad A_{22} - I),$$

$$\tilde{H}_2 = (O \quad H_2).$$

Then the resulting closed-loop system (37) is standard, and there exist an $\tilde{\varepsilon}^* > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}^*]$, the closed-loop system under the action of the feedback controller (36) with $K = YX^{-1}$ is ISS.

Proof: By substituting $K = YX^{-1}$ into (38), the inequality (38) is equivalent to

$$\begin{pmatrix} \Lambda_{11} & \mu_1^{-1}H & X^T (A_2 + B_{u2}K)^T \\ * & -\mu_1^{-1}I & \mu_1^{-1}\tilde{H}_2 \\ * & * & -X_{22} \end{pmatrix} < 0. \quad (39)$$

where $\Lambda_{11} = X^T (A_1 + B_u K)^T + (A_1 + B_u K)X + \mu_1 \alpha^2 X^T F^T F X$. Premultiplying and postmultiplying (39) by $\text{diag}(X^{-T}, \mu_1 I, I)$ and $\text{diag}(X^{-1}, \mu_1 I, I)$, respectively, let

$$X^{-1} = \tilde{P}_1 = \begin{pmatrix} \tilde{P}_{11} & O \\ \tilde{P}_{21} & \tilde{P}_{22} \end{pmatrix}, \text{ then } X_{22}^{-1} = \tilde{P}_{22} \text{ and}$$

$$\begin{pmatrix} \bar{\Lambda}_{11} & \tilde{P}_1^T H & (A_2 + B_{u2}K)^T \\ * & -\mu_1 I & \tilde{H}_2 \\ * & * & -\tilde{P}_{22}^{-1} \end{pmatrix} < 0, \quad (40)$$

where $\bar{\Lambda}_{11} = (A_1 + B_u K)^T \tilde{P}_1 + \tilde{P}_1^T (A_1 + B_u K) + \mu_1 \alpha^2 F^T F$. Applying the Schur Complement, we have

$$\begin{pmatrix} \bar{\Lambda}_{11} & \tilde{P}_1^T H \\ * & -\mu_1 I \end{pmatrix} + \begin{pmatrix} (A_2 + B_{u2}K)^T \\ \tilde{H}_2^T \end{pmatrix} \tilde{P}_{22} \begin{pmatrix} (A_2 + B_{u2}K)^T \\ \tilde{H}_2^T \end{pmatrix}^T < 0,$$

which is equivalent to

$$\Omega = \begin{pmatrix} \bar{\Lambda}_{11} & \tilde{P}_1^T H \\ * & -\mu_1 I \end{pmatrix} + \begin{pmatrix} (A_1 + B_u K)^T \tilde{P}_2 (A_1 + B_u K) & (A_1 + B_u K)^T \tilde{P}_2 H \\ * & H^T \tilde{P}_2 H \end{pmatrix} < 0,$$

where $\tilde{P}_2 = \begin{pmatrix} O & O \\ O & \tilde{P}_{22} \end{pmatrix}$. In light of Lemma 3.1 and Theorem 3.3, thus there exists an

$\tilde{\varepsilon}^* > 0$ such that the closed-loop system (37) is in the standard form and ISS with respect to w for any $\varepsilon \in (0, \tilde{\varepsilon}^*]$. This completes the proof. ■

According to Theorem 3.4, a direct result for solving the upper bound is derived as follows.

Theorem 3.6: Suppose that K has been solved from (38), and if there exist a scalar $\tilde{\lambda} > 0$, positive definite matrices $\tilde{\Pi} > 0$, $\tilde{P}_{11} > 0$, $\tilde{P}_{22} > 0$ and matrix $\tilde{P}_{21}$ such that

$$\tilde{\Pi} < \tilde{\lambda} \tilde{P}_{11}, \quad \begin{pmatrix} \tilde{\Pi} & \tilde{P}_{21}^T \\ \tilde{P}_{21} & \tilde{P}_{22} \end{pmatrix} > 0, \quad \Omega < 0, \quad \tilde{\Omega} < -\tilde{\lambda} \Omega, \quad (41)$$

where

$$\tilde{\Omega} = \begin{pmatrix} (A + B_u K)^T \tilde{P}_3 (A + B_u K) & (A + B_u K)^T \tilde{P}_3 H \\ * & H^T \tilde{P}_3 H \end{pmatrix}, \quad \tilde{P}_3 = \begin{pmatrix} \tilde{P}_{11} & \tilde{P}_{21}^T \\ \tilde{P}_{21} & O \end{pmatrix}.$$

Then the resulting closed-loop system (35) is standard and ISS with respect to w for all $\varepsilon \in (0, \tilde{\varepsilon}^*]$ with $\tilde{\varepsilon}^* = \tilde{\lambda}^{-1}$.

Remark 3.5: Theorem 3.5 presents an LMI based sufficient condition for the existence of isolate root and ISS of closed-loop systems (37), in which the control gain matrix can be obtained easily by solving the LMI (38). Compared with the existing result in [23], it is easy to show that our method is simpler although the system considered here is more complex. In [23], solving a nonlinear matrix inequality is required in order to get the stabilizing controller. Unfortunately, so far there is no effective algorithm for solving it. Thus, the method in [23] is rather conservative, and it will become more evident when dimension of the system is larger. Moreover, we found that no effective computation method for the upper bound is proposed in [23].

4. Numerical Examples

In this section, two illustrative examples are given to verify the feasibility of the obtained results.

Example 4.1: Consider the following fast sampling DTSPS in [23]

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} I + \varepsilon A_{11} & \varepsilon A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} \varepsilon I & O \\ O & I \end{pmatrix} \begin{pmatrix} \Delta A_{11} x_1(k) \\ \Delta A_{22} x_2(k) \end{pmatrix} + \begin{pmatrix} \varepsilon B_{u1} \\ B_{u2} \end{pmatrix} u(k), \quad (42)$$

where

$$\begin{aligned} A_{11} &= \begin{pmatrix} 0.87 & 1 \\ 1.6953 & 0.87 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} -0.7 & -0.14 \\ -0.0758 & -0.1836 \end{pmatrix}, \\ A_{21} &= \begin{pmatrix} 0.3424 & 0.35 \\ 0.3713 & 0.34 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0.5807 & 0.1894 \\ -0.4546 & -0.1013 \end{pmatrix}, \\ B_{u1} &= \begin{pmatrix} 0.5 \\ 0.1351 \end{pmatrix}, \quad B_{u2} = \begin{pmatrix} -0.1747 \\ -0.1894 \end{pmatrix}, \quad \Delta A_{11} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \quad \Delta A_{22} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ &\quad -0.5 \leq \sigma \leq 0.5, \quad -0.4 \leq \rho \leq 0.4. \end{aligned}$$

Let $f(x(k)) = \begin{pmatrix} \Delta A_{11} x_1(k) \\ \Delta A_{22} x_2(k) \end{pmatrix}$, then it is not difficult to verify that f satisfies the condition (5) with

$$F_{11} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix}, \quad F_{22} = \begin{pmatrix} 0.4 & 0 \\ 0 & 0 \end{pmatrix}, \quad F_{12} = F_{21} = 0.$$

By utilizing Theorem 3.5, we can get the following solutions

$$X = \begin{pmatrix} 2.0029 & -1.0963 & 0 & 0 \\ -1.0963 & 1.1096 & 0 & 0 \\ 1.4796 & 0.6454 & 2.1227 & -0.6711 \\ 0.5828 & 0.0339 & -0.6711 & 2.3946 \end{pmatrix},$$

$$Y = (-3.4987 \quad -2.3188 \quad 1.0676 \quad -0.3233), \quad \mu_1 = 0.9031.$$

The controller gain is chosen as

$$K = YX^{-1} = (-7.4627 \quad -9.7571 \quad 0.5050 \quad 0.0065).$$

Moreover, an upper bound $\varepsilon^* = \lambda^{-1} = 0.2090$ can be found via solving GEVP (41), which means that system is standard and asymptotically stable for $\varepsilon \in (0, \varepsilon^*]$. From the numerical computation, it has been clearly shown that solving control gain matrix is simple. However, it should be pointed out that the derived sufficient condition in [23] for solving the control gain matrix involves much more complex nonlinear

matrix inequalities, which is numerically inefficient. In addition, with the best effort, the authors found that the method of [23] is also infeasible for solving the stability bound.

Example 4.2: Consider the following nuclear reactor model, which is borrowed from [21].

$$\dot{x}_1 = -\lambda x_1 + \lambda x_2, \quad (43)$$

$$\dot{x}_2 = \frac{\beta}{\nu} x_1 + \frac{\beta}{\nu} x_2 + \frac{\rho}{\nu}, \quad (44)$$

the parameters are $\lambda = 0.001$, $\beta = 0.0064$ and $\nu = 0.08$. Let $\rho = u + f_1(x_1, x_2)$, u and f_1 here are linear and nonlinear inputs, respectively. By discretizing the model with a sampling period $T = 0.05s$ and a zero-order holder, the DTSPS is given by

$$A = \begin{pmatrix} -0.3417 & 0.3417 \\ 0.2733 & 0.7267 \end{pmatrix}, \quad B_u = \begin{pmatrix} 9.0021 \\ 42.7983 \end{pmatrix}, \quad H = \begin{pmatrix} 9.0021 & 0 \\ 0 & 42.7983 \end{pmatrix}, \quad B_w = \begin{pmatrix} 0 \\ 0.2 \end{pmatrix}. \quad (45)$$

Assume that $f_1(x_1, x_2) = 10^{-2} \times \sin(10x_1 + 0.3x_2)$, which satisfy the condition (5) with

$$F_{11} = F_{21} = 0.1, \quad F_{12} = F_{22} = 0.003, \quad \alpha = 1.$$

Utilizing the LMI toolbox, the following solutions via (38) can be given by

$$X = 10^3 \times \begin{pmatrix} 0.1376 & 0 \\ -1.6472 & 2.3798 \end{pmatrix}, \quad Y = (-15.5524 \quad -26.8486), \quad \mu_1 = 1.0195.$$

The controller gain is given by

$$K = YX^{-1} = (-0.2481 \quad -0.0113).$$

Furthermore, an upper bound $\varepsilon^* = \lambda^{-1} = 0.1786$ can be derived by solving the GEVP (41). Therefore, the system is standard and ISS with w for $\varepsilon \in (0, \varepsilon^*]$. However, it is noticed that the method in [23] and [31] is infeasible for this system. To facilitate simulation, given the initial condition $x(0) = (-1.5 \quad 0.9)^T$, then the simulations with $w(k) = (1 + k^2)^{-1}$ and $w(k) = 5 \cos k$ are shown in Figs. 1 and 2, respectively. It can be seen from the simulations that the closed-loop system (45) is asymptotically stable when the disturbance input tends to zero, and ultimately bounded when the disturbance input is bounded, respectively.

Fig. 1. States of the closed-loop system (45) with $\varepsilon = 0.1$ and $w(k) = (1 + k^2)^{-1}$

Fig. 2. States of the closed-loop system (45) with $\varepsilon = 0.1$ and $w(k) = 5 \cos k$.

5. Conclusion

A unified sufficient condition for the existence of isolate root and ISS property has been presented by marrying the reduced technique and LMI method. Thus the numerically ill-conditioned problem is avoided. Based on the established results, a proper state feedback control law has been constructed to render the closed-loop systems ISS. Moreover, a workable way for evaluating the stability bound has been obtained via solving GEVP problem, in which a prescribed stability bound is not required. Thus, we has improved and generalized the results and technique in the literature. Finally, two illustrative examples have been given to verify the feasibility of the methods.

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