Parameter identification and the multi-switching sliding mode combination synchronization of fractional order non-identical chaotic system under stochastic disturbances

Weiqliu Pan, Tianzeng Li*, Yu Wang

*College of Mathematics and Statistics, Sichuan University of Science and Engineering,
Zigong 643000, China*

South Sichuan Center for Applied Mathematics, Yibin 644000, China.

Abstract This paper deals with the issue of the multi-switching sliding mode combination synchronization (MSSMCS) of fractional order (FO) chaotic systems with different structures and unknown parameters under double stochastic disturbances (S-D) utilizing the multi-switching synchronization method. The stochastic disturbances are considered as nonlinear uncertainties and external disturbances. Our theoretical part is divided into two cases, namely, the dimension of the drive-response system are different (or same). Firstly, a FO sliding surface was established in term of fractional calculus. Secondly, depended on the FO Lyapunov stability theory, the adaptive control technology and sliding mode control technique, the multi-switching adaptive controllers (MSAC) and some suitable multi-switching adaptive updating laws (MSAUL) are designed, so that the state variables of the drive systems are synchronized with the different state variables of the response systems. Simultaneously, the unknown parameters are assessed and the upper bound of stochastic disturbances are examined. Selecting the suitable scale matrices, the multi-switching projection synchronization, multi-switching complete synchronization, and multi-switching anti-synchronization will become special cases of MSSMCS. Finally, examples are displayed to certify the usefulness and validity of the demonstrated scheme via MATLAB.

Key words. Fractional-order chaotic system; Multi-switching combination synchronization (MSCS); Adaptive sliding mode control; Stochastic disturbance; Unknown parameters

1 Introduction

Chaos is an inherent characteristic of nonlinear dynamic systems and a common phenomenon in real life. The chaotic phenomenon exhibited by the chaotic system is

*Corresponding author. E-mail addresses: litianzeng27@163.com (T.Li), pwq237609@163.com (W.Pan), wangyu.813@163.com (Y.Wang)

uncertain, unrepeatable, and unpredictable. Therefore, experts in the fields of mathematics and control have carried out a series of researches on the control and synchronization of chaotic systems. So far, some effective synchronization control methods have been proposed, such as drive-response synchronization [1], projection synchronization [2, 3], adaptive fuzzy control [4–6], neural network synchronization [7, 8], feedback synchronization [9], pulse synchronization [10, 11], sliding mode control [12, 13] etc. People apply chaotic synchronization to secure communication, signal processing and life sciences, etc which has attracted great attention. Thus, chaos synchronization has gradually become a core research topic in the field of control science. Because chaotic systems are extremely sensitive to initial values, they are often subject to some SD. Whether it is the uncertainties within the system, such as parameter uncertainties, nonlinear uncertainties, or external disturbances, they cause an effect on the stability of the system. In the beginning, these synchronization methods are used by people to research the synchronization of single-drive system and single-response system [14–19]. Gradually, some scholars have considered the influence of SD on this basis [20–27]. Since Runzi et al [28] revealed the combination synchronization scheme, the synchronization of multi-drive and multi-response systems, multi-drive and single-response systems, single-drive and multi-response systems are suggested. Later, some new synchronization schemes appeared and have been developed by leaps and bounds, such as the combination-combination [29–32], compound synchronization [33–35] and double compound synchronization [36, 37], etc. It can be said to be a major breakthrough in the field of chaos synchronization. A major advantage of these new synchronization schemes is that they can ensure the security of information. However, as the transmission of signals become more and more complex, how to strengthen the security of information is a thought-provoking question.

In recent years, a new multi-switching combination synchronization (MSCS) scheme was analyzed by Vincent U et al [38] which means the state variables of the drive systems are synchronized with the different state variables of the response systems, breaking the conventional synchronization rules. Compared with the traditional synchronization or some extension of it, the MSCS scheme is very promising because it can provide greater security for the secure communication according to chaotic synchronization. The topic of dual combination-combination multi-switching synchronization in terms of eight chaotic systems was solved in [39]. The global exponential MSCS was introduced in terms of three different chaotic systems [40]. Reference [41] solved the problem of MSCS between three different integer-order chaotic systems. Adopting adaptive control technology, reference [42] investigated the multi-switching combination-combination synchronization of four integer-order chaotic systems which parameters are fully unknown. An further work of [42] has been developed by [43], which is indicated as integer-order time-delay chaotic system. Shafiq M et al [44] proposed a robust adaptive multi-switching technology to solve the issue of anti-synchronization for unknown hyper-chaotic systems under SD. The authors of references [45] and [46] considered the multi-switching synchronization of the single-drive and single-response system which the parameters are unknown. Their innovations lie in the order of the drive-response system is different in [45] and the dimension is different in [46]. On the contrary, Chen et al [47] considered the synchronization of multiple chaotic systems with unknown parameters and disturbances. It's a pity that the case of multi-switching is not considered. Although reference [48] take multi-switching scheme into account in terms of

multiple chaotic systems, it does not consider the influence of unknown parameters and SD.

Hence, to address this limitation, we plan to solve the multi-switching sliding mode combination synchronization (MSSMCS) of fractional order (FO) chaotic systems with different (or same) dimension under double stochastic disturbances (SD). Meanwhile, the parameters of two drive systems and one response system are fully unknown. The double SD are considered as nonlinear uncertainties and external disturbance. In the light of Lyapunov stability theory, adaptive control technology and sliding mode control technique, we introduce two multi-switching adaptive controllers (MSAC) and multi-switching adaptive updating laws (MSAUL) to realise the multi-switching synchronization of D-R systems and assess the unknown parameters. There are two innovations points in this article. 1. Considering the combination D-R systems with unknown parameters and double SD, the designed MSAC can make the state variables of drive systems are synchronized with the different state variables of the response system. Simultaneously, the designed MSAUL can identify the unknown parameters accurately and bounded estimates are made for uncertainties and external disturbances. 2. When the dimension of D-R systems are different (or same), scale matrices are chose as non-diagonal (or diagonal) matrices. If we adopt suitable scale matrices, the multi-switching projection synchronization, multi-switching complete synchronization, and multi-switching anti-synchronization will become the special cases of MSSMCS. Finally, numerical simulations via MATLAB demonstrate the multi-switching controllers we conducted have good robustness and anti-interference performance and this method we investigated can make the multi-switching error systems quickly converges to the equilibrium point. The rest of the paper is described as follows. In Section 2, some definitions, lemmas that need to be used are introduced. In Section 3, the problem statement are given. In Section 4, the MSAC and MSAUL are designed about D-R systems with the same dimension . In Section 5, the MSAC and MSAUL are designed about D-R systems with the different dimension. In Section 6, the numerical simulations conducted that our method is effective and dependable. In Section 7, there is a conclusion.

2 Preliminaries

The fractional calculus is an ancient and "fresh" concept. As early as the establishment of integer calculus, some scholars began to consider its meaning. Up till the present moment, there are some commonly used types of fractional derivatives, such as the Riemann-Liouville (R-L), Caputo and Grunwald-Letnikov (G-L) derivative etc. Among these definitions, Caputo's derivative definition is the most generally utilized.

Definition 1. [49] The mathematical expression of the fractional integral of the function $f(t)$ is following

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(v)}{(t-v)^{1-\alpha}} dv, \quad (1)$$

where $\Gamma(\alpha)$ indicates the Gamma function.

Definition 2. [49] The mathematical expression of Caputo derivative with order α is given as

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(p-\alpha)} \int_a^t (t-v)^{p-\alpha-1} f^{(p)}(v) dv, \quad (2)$$

where $p-1 < \alpha < p, p \in \mathbb{Z}^+$.

Lemma 1. [18] When $\mathbf{x}(t) \in \mathbb{R}^n$ has continuous first derivative, then

$${}_a D_t^\alpha \left(\frac{1}{2} \mathbf{x}^T(t) Q \mathbf{x}(t) \right) \leq \mathbf{x}^T Q {}_a D_t^\alpha \mathbf{x}(t), \quad (3)$$

where $\alpha \in (0, 1)$ and $Q \in \mathbb{R}^n \times \mathbb{R}^n$ indicates a positive definite matrix.

3 Problem description

The FO D-R systems with SD are demonstrated as

$${}_0 D_t^\alpha \mathbf{x}(t) = \mathbf{F}(\mathbf{x}(t)) \boldsymbol{\theta} + \mathbf{f}(\mathbf{x}(t)) + \Delta \mathbf{f}(\mathbf{x}(t)) + \mathbf{d}(\mathbf{x}, t), \quad (4)$$

$${}_0 D_t^\alpha \mathbf{y}(t) = \mathbf{G}(\mathbf{y}(t)) \boldsymbol{\beta} + \mathbf{g}(\mathbf{y}(t)) + \Delta \mathbf{g}(\mathbf{y}(t)) + \mathbf{D}(\mathbf{y}, t), \quad (5)$$

where the systems (4), (5) are regarded as drive systems. $\mathbf{x}(t) = (x_1, x_2, \dots, x_m)^T$ and $\mathbf{y}(t) = (y_1, y_2, \dots, y_m)^T$ are the state variables. $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_m)^T$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_m)^T$ are the parameter vectors; $\mathbf{F}_k(\mathbf{x}(t)), (k = 1, 2, \dots, n)$ is the k th row of $n \times m$ matrix $\mathbf{F}(\mathbf{x}(t))$ and $\mathbf{G}_j(\mathbf{y}(t)), (j = 1, 2, \dots, n)$ is the j th row of $n \times m$ matrix $\mathbf{G}(\mathbf{y}(t))$ whose elements are continuous nonlinear functions; $\mathbf{f}(\mathbf{x}(t)) = (f_1, f_2, \dots, f_m)^T, \mathbf{g}(\mathbf{y}(t)) = (g_1, g_2, \dots, g_m)^T$ are the nonlinear continuous functions; $\mathbf{d}(\mathbf{x}(t)) = (d_1, d_2, \dots, d_m)^T, \mathbf{D}(\mathbf{y}(t)) = (D_1, D_2, \dots, D_m)^T$ are the external disturbances; $\Delta \mathbf{f}(\mathbf{x}(t)) = (\Delta f_1, \Delta f_2, \dots, \Delta f_m)^T, \Delta \mathbf{g}(\mathbf{y}(t)) = (\Delta g_1, \Delta g_2, \dots, \Delta g_m)^T$ are the nonlinear uncertainties; $\alpha_i \in (0, 1)$ represents the fractional order.

$${}_0 D_t^\alpha \mathbf{z}(t) = \mathbf{H}(\mathbf{z}(t)) \boldsymbol{\vartheta} + \mathbf{h}(\mathbf{z}(t)) + \Delta \mathbf{h}(\mathbf{z}(t)) + \boldsymbol{\mu}(\mathbf{z}, t) + \mathbf{u}(t), \quad (6)$$

where the systems (6) is regarded as response system. $\mathbf{z}(t) = (z_1, z_2, \dots, z_n)^T$ and $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \dots, \vartheta_n)^T$ are the state variables and parameter vectors; $\mathbf{H}_i(\mathbf{z}(t)), (i = 1, 2, \dots, n)$ is the i th row of $n \times n$ matrix $\mathbf{H}(\mathbf{z}(t))$ whose elements are continuous nonlinear functions; $\mathbf{h}(\mathbf{z}(t)) = (h_1, h_2, \dots, h_n)^T$ are the nonlinear continuous functions; $\boldsymbol{\mu}(\mathbf{z}(t)) = (\mu_1, \mu_2, \dots, \mu_n)^T$ are the external disturbances; $\Delta \mathbf{h}(\mathbf{z}(t)) = (\Delta h_1, \Delta h_2, \dots, \Delta h_n)^T$ are the nonlinear uncertainties. $\mathbf{u}(t) = (u_1, u_2, \dots, u_n)^T$ represent the controller.

Remark 1. It is worth noting that $\mathbf{x}(t), \mathbf{y}(t) \in \mathbb{R}^m, \mathbf{z}(t) \in \mathbb{R}^n, m = n$ or $m \neq n$; And the parameters of the D-R systems are unknown. We use $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\vartheta}}$ to represent the estimation of parameter $\boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\vartheta}$.

Definition 3. If there exist three constant matrices $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times n}$, $C \neq 0$ such that

$$\lim_{t \rightarrow \infty} \|\mathbf{e}\| = \lim_{t \rightarrow \infty} \|C\mathbf{z} - (B\mathbf{y} + A\mathbf{x})\| = 0, \quad (7)$$

then the drive systems (4), (5) and response system (6) can be reach combination synchronization.

Remark 2. We can redefined the error system (7) as

$$\lim_{t \rightarrow \infty} {}^{(l w v)}e_p = \lim_{t \rightarrow \infty} \left[\sum_{i=1}^n (c_{li} z_i) - \left\{ \sum_{j=1}^m (b_{wj} y_j) + \sum_{k=1}^m (x_{vk} x_k) \right\} \right], \quad (8)$$

where $i, j, k, p \in (1, 2, \dots, n)$ represent p th error component of $\mathbf{e}$, i th components of $\mathbf{z}$, j th components of $\mathbf{y}$, and k th components of $\mathbf{x}$ respectively, l, w, v represent the switching mode. Suppose that $l = w = v$, then the error variables are expressed in the form of definition 3; if $l \neq w \neq v$, definition 3 will no longer apply.

Definition 4. We redefine the error state in definition 3 as

$$\lim_{t \rightarrow \infty} {}^{(l w v)}e_p = \lim_{t \rightarrow \infty} \left[\sum_{i=1}^n (c_{li} z_i) - \left\{ \sum_{j=1}^m (b_{wj} y_j) + \sum_{k=1}^m (x_{vk} x_k) \right\} \right] = 0, \quad (9)$$

then the drive systems (4), (5) and the response systems (6) realize the MSCS, where $l = w \neq v$ or $l \neq w = v$ or $l = w = v$ or $l \neq w \neq v$.

Remark 3. The matrices $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times n}$, $C \neq 0$ are hailed as the scaling matrices. Furthermore, they can be either constant matrices or the functions with regard to state variables $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$.

Remark 4. If $A = B = C = I$, the MSCS becomes the multi-switching combination complete synchronization; If $A = 0, B = C = I$ or $B = 0, A = C = I$, the MSCS becomes the multi-switching complete synchronization.

Remark 5. If $A = B = -I, C = I$, the MSCS becomes the multi-switching combination anti-synchronization; If $A = 0, B = -I, C = I$ or $B = 0, A = -I, C = I$, the MSCS becomes the multi-switching anti-synchronization.

Remark 6. If $A \neq I, B \neq I, C = I$, the MSCS becomes the multi-switching combination projective synchronization; If $A = I, B \neq I, C = I$ or $B = I, A \neq I, C = I$, the MSCS becomes the multi-switching projective synchronization.

4 The synchronization of multi-switching FO chaotic system with same dimensions

In Section 4, the MSSMCS of FO chaotic systems with same dimension is formulated which means the dimension of D-R systems (4), (5) and (6) satisfies $m = n$. Thus, the scaling matrices A, B, C are given as diagonal matrices. Firstly, we know

that even if a chaotic system is slightly disturbed, its state orbit will change drastically over time. Therefore, it is crucial to suppose them as bounded. Then, we designed appropriate multi-switching adaptive controllers (MSAC) and some multi-switching adaptive updating laws (MSAUL) to realize the synchronization of the D-R system which are proved in Theorem 1.

When we choose $m = n$ and the diagonal matrices $A = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$, $B = \text{diag}(b_{11}, b_{22}, \dots, b_{nn})$, $C = \text{diag}(c_{11}, c_{22}, \dots, c_{nn})$, the error system (9) can be described as

$$\lim_{t \rightarrow \infty} {}^{(ijk)}e_p = \lim_{t \rightarrow \infty} [c_{ii}z_i - (b_{jj}y_j + a_{kk}x_k)] = 0, \quad (10)$$

where $i, j, k, p \in (1, 2, \dots, n)$ represent p th error component of $\mathbf{e}$, i th components of $\mathbf{z}$, j th components of $\mathbf{y}$, and k th components of $\mathbf{x}$.

Assumption 1. Assumed that the external disturbances d_k , D_j , μ_i , uncertain non-linear vectors Δf_k , Δg_j , Δh_i all have a bounded norm. Namely, there are suitable positive constants ${}^{(ijk)}r_p$, ${}^{(ijk)}q_p$ that satisfy

$$\begin{aligned} |c_{ii}\mu_i - (b_{jj}D_j + a_{kk}d_k)| &\leq {}^{(ijk)}r_p, \\ |c_{ii}\Delta f_k - (b_{jj}\Delta g_j + a_{kk}\Delta h_i)| &\leq {}^{(ijk)}q_p. \end{aligned} \quad (11)$$

where $p = 1, 2, \dots, n$.

Remark 7. The positive constants ${}^{(ijk)}r_p$, ${}^{(ijk)}q_p$ are unknown. ${}^{(ijk)}\hat{r}_p$, ${}^{(ijk)}\hat{q}_p$ are used to represent the estimation of parameter ${}^{(ijk)}r_p$, ${}^{(ijk)}q_p$.

According to the definition of the error vector (10), we get the FO error system as

$$\begin{aligned} {}_0D_t^\alpha \{ {}^{(ijk)}e_p \} &= c_{ii} \{ {}_0D_t^\alpha z_i \} - b_{jj} \{ {}_0D_t^\alpha y_j \} - a_{kk} \{ {}_0D_t^\alpha x_k \} \\ &= c_{ii} \mathbf{H}_i(z(t)) \boldsymbol{\vartheta} + c_{ii} h_i(z(t)) + c_{ii} \Delta h_i(z(t)) + c_{ii} \mu_i(z, t) + c_{ii} \{ {}^{(ijk)}u_p(t) \} \\ &\quad - b_{jj} \mathbf{G}_j(y(t)) \boldsymbol{\beta} - b_{jj} g_j(y(t)) - b_{jj} \Delta g_j(y(t)) - b_{jj} D_j(y, t) \\ &\quad - a_{kk} \mathbf{F}_k(x(t)) \boldsymbol{\theta} - a_{kk} f_k(x(t)) - a_{kk} \Delta f_k(x(t)) - a_{kk} d_k(x, t). \end{aligned} \quad (12)$$

For convenience, we define the errors of unknown parameters $\boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\vartheta}$, ${}^{(ijk)}r_p$, ${}^{(ijk)}q_p$ as $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$, $\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}} - \boldsymbol{\beta}$, $\tilde{\boldsymbol{\vartheta}} = \hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}$, ${}^{(ijk)}\tilde{r}_p = {}^{(ijk)}\hat{r}_p - {}^{(ijk)}r_p$, ${}^{(ijk)}\tilde{q}_p = {}^{(ijk)}\hat{q}_p - {}^{(ijk)}q_p$. When the sliding mode surface is designed as ${}^{(ijk)}s_p = {}^{(ijk)}\lambda_p ({}^{(ijk)}e_p)$ which can be rewritten as $\mathbf{s} = ({}^{(ijk)}s_1, {}^{(ijk)}s_2, \dots, {}^{(ijk)}s_n)$, we can get the following MSAC (13) and MSAUL (16) are designed:

$$\begin{aligned} {}^{(ijk)}u_p &= \frac{1}{c_{ii}} \{ -c_{ii}h_i + b_{jj}g_j + a_{kk}f_k + a_{kk} \mathbf{F}_k(x(t)) \hat{\boldsymbol{\theta}} + b_{jj} \mathbf{G}_j(y(t)) \hat{\boldsymbol{\beta}} \\ &\quad - c_{ii} \mathbf{H}_i(z(t)) \hat{\boldsymbol{\vartheta}} - ({}^{(ijk)}r_p + {}^{(ijk)}q_p) \text{sign}({}^{(ijk)}s_p) - k_1 \tanh({}^{(ijk)}s_p) \}, \end{aligned} \quad (13)$$

where ${}^{(ijk)}\lambda_p$ is a constant.

Substituting (13) into Eq. (12), we obtain

$$\begin{aligned} {}_0D_t^\alpha \{ {}^{(ijk)}e_p \} &= -c_{ii} \mathbf{H}_i(z(t)) \tilde{\boldsymbol{\vartheta}} + b_{jj} \mathbf{G}_j(y(t)) \tilde{\boldsymbol{\beta}} + a_{kk} \mathbf{F}_k(x(t)) \tilde{\boldsymbol{\theta}} \\ &\quad + \{ c_{ii} \Delta h_i(z(t)) - b_{jj} \Delta g_j(y(t)) - a_{kk} \Delta f_k(x(t)) \} + \{ c_{ii} \mu_i(z, t) \\ &\quad - b_{jj} D_j(y, t) - a_{kk} d_k(x, t) \} - ({}^{(ijk)}\hat{r}_p + {}^{(ijk)}\hat{q}_p) \text{sign}({}^{(ijk)}s_p(t)) \\ &\quad - k_1 \tanh({}^{(ijk)}s_p). \end{aligned} \quad (14)$$

Thus, it follows from $^{(ijk)}s_p = ^{(ijk)}\lambda_p(^{(ijk)}e_p)$ and Eq. (14) that we obtain the following summation result.

$$\begin{aligned}
& \sum_{p=1}^n \{^{(ijk)}s_p\}_0 D_t^\alpha \{^{(ijk)}s_p\} \\
&= \sum_{i=1}^n ^{(ijk)}s_i(^{(ijk)}\lambda_i) \{-c_{ii}\mathbf{H}_i(z(t))\tilde{\boldsymbol{\vartheta}} + c_{ii}\Delta h_i(z(t)) + c_{ii}\mu_i(z, t)\} \\
&+ \sum_{j=1}^n ^{(ijk)}s_j(^{(ijk)}\lambda_j) \{b_{jj}\mathbf{G}_j(y(t))\tilde{\boldsymbol{\beta}} - b_{jj}\Delta g_j(y(t)) - b_{jj}D_j(y, t)\} \\
&+ \sum_{k=1}^n ^{(ijk)}s_k(^{(ijk)}\lambda_p) \{a_{kk}\mathbf{F}_k(x(t))\tilde{\boldsymbol{\theta}} - a_{kk}\Delta f_k(x(t)) - a_{kk}d_k(x, t)\} \\
&- \sum_{p=1}^n (^{(ijk)}\lambda_p)(^{(ijk)}\hat{r}_p + ^{(ijk)}\hat{q}_p) \text{sign}(^{(ijk)}s_p(t)) - \sum_{p=1}^n (^{(ijk)}\lambda_p)k_1 \tanh(^{(ijk)}s_p).
\end{aligned} \tag{15}$$

The MSAUL with regard to unknown parameters $\boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\vartheta}, ^{(ijk)}r_p, ^{(ijk)}q_p$ are selected to

$$\begin{aligned}
{}_0D_t^\alpha \tilde{\boldsymbol{\theta}} &= -\mathbf{F}^T(x(t))\boldsymbol{\gamma} - \varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}})|(\tilde{\boldsymbol{\theta}})|^w, \\
{}_0D_t^\alpha \tilde{\boldsymbol{\beta}} &= -\mathbf{G}^T(y(t))\boldsymbol{\xi} - \varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}})|(\tilde{\boldsymbol{\beta}})|^w, \\
{}_0D_t^\alpha \tilde{\boldsymbol{\vartheta}} &= \mathbf{H}^T(z(t))\boldsymbol{\varpi} - \varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}})|(\tilde{\boldsymbol{\vartheta}})|^w, \\
{}_0D_t^\alpha \{^{(ijk)}\tilde{r}_p\} &= m_1(^{(ijk)}\lambda_p)|^{(ijk)}s_p(t)|, \\
{}_0D_t^\alpha \{^{(ijk)}\tilde{q}_p\} &= m_2(^{(ijk)}\lambda_p)|^{(ijk)}s_p(t)|,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\boldsymbol{\gamma} &= \{(^{(ijk)}\lambda_1)a_{11}^*(^{(ijk)}s_1), (^{(ijk)}\lambda_2)a_{22}^*(^{(ijk)}s_2), \dots, (^{(ijk)}\lambda_n)a_{nn}^*(^{(ijk)}s_n)^T\}, \\
\boldsymbol{\xi} &= \{(^{(ijk)}\lambda_1)b_{11}^*(^{(ijk)}s_1), (^{(ijk)}\lambda_2)b_{22}^*(^{(ijk)}s_2), \dots, (^{(ijk)}\lambda_n)b_{nn}^*(^{(ijk)}s_n)^T\}, \\
\boldsymbol{\varpi} &= \{(^{(ijk)}\lambda_1)c_{11}^*(^{(ijk)}s_1), (^{(ijk)}\lambda_2)c_{22}^*(^{(ijk)}s_2), \dots, (^{(ijk)}\lambda_n)c_{nn}^*(^{(ijk)}s_n)^T\},
\end{aligned}$$

m_1, m_2 are positive constants and the meaning of matrices $A^* = \text{diag}(a_{11}^*, a_{22}^*, \dots, a_{nn}^*)$, $B^* = \text{diag}(b_{11}^*, b_{22}^*, \dots, b_{nn}^*)$, $C^* = \text{diag}(c_{11}^*, c_{22}^*, \dots, c_{nn}^*)$ is the rearrangement of the original matrices A, B, C .

Theorem 1. For any given initial conditions $\mathbf{x}(0), \mathbf{y}(0), \mathbf{z}(0)$, if the Assumption 1 is hold, the synchronization error system (10) will achieve the multi-switching sliding mode combination synchronization (MSSMCS) via the multi-switching adaptive controller (MSAC) (13) and multi-switching adaptive updating laws (MSAUL) (16).

Proof. Adopting the Lyapunov function as:

$$\begin{aligned}
V &= \frac{1}{2} \sum_{p=1}^n (^{ijk)}s_p^2 + \frac{1}{2m_1} \sum_{p=1}^n \{^{(ijk)}\tilde{r}_p^T\} \{^{(ijk)}\tilde{r}_p\} + \frac{1}{2m_2} \sum_{p=1}^n \{^{(ijk)}\tilde{q}_p^T\} \{^{(ijk)}\tilde{q}_p\} \\
&+ \frac{1}{2} \tilde{\boldsymbol{\theta}}^T \boldsymbol{\theta} + \frac{1}{2} \tilde{\boldsymbol{\beta}}^T \boldsymbol{\beta} + \frac{1}{2} \tilde{\boldsymbol{\vartheta}}^T \boldsymbol{\vartheta}.
\end{aligned} \tag{17}$$

Taking the α derivative

$$\begin{aligned}
{}_0D_t^\alpha V(t, \mathbf{x}(t)) &\leq \sum_{p=1}^n \{^{(ijk)}s_p\} {}_0D_t^\alpha \{^{(ijk)}s_p\} + \frac{1}{m_1} \sum_{p=1}^n \{^{(ijk)}\tilde{r}_p^T\} {}_0D_t^\alpha \{^{(ijk)}\tilde{r}_p\} \\
&+ \frac{1}{m_2} \sum_{p=1}^n \{^{(ijk)}\tilde{q}_p^T\} {}_0D_t^\alpha \{^{(ijk)}\tilde{q}_p\} + \tilde{\boldsymbol{\theta}}^T {}_0D_t^\alpha \tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\beta}}^T {}_0D_t^\alpha \tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\vartheta}}^T {}_0D_t^\alpha \tilde{\boldsymbol{\vartheta}}.
\end{aligned} \tag{18}$$

Then substituting the Eq. (15) and the MSAUL (16) into Eq. (18), we obtain

$$\begin{aligned}
{}_0D_t^\alpha V(t, \mathbf{x}(t)) &\leq \sum_{i=1}^n {}^{(ijk)}s_i ({}^{(ijk)}\lambda_i) (-c_{ii} \mathbf{H}_i(z(t)) \tilde{\boldsymbol{\vartheta}} + c_{ii} \Delta h_i(z(t)) + c_{ii} \mu_i(z, t)) \\
&+ \sum_{j=1}^n {}^{(ijk)}s_j ({}^{(ijk)}\lambda_j) (b_{jj} \mathbf{G}_j(y(t)) \tilde{\boldsymbol{\beta}} - b_{jj} \Delta g_j(y(t)) - b_{jj} D_j(y, t)) \\
&+ \sum_{k=1}^n {}^{(ijk)}s_k ({}^{(ijk)}\lambda_k) (a_{kk} \mathbf{F}_k(x(t)) \tilde{\boldsymbol{\theta}} - a_{kk} \Delta f_k(x(t)) - a_{kk} d_k(x, t)) \\
&- \sum_{p=1}^n ({}^{(ijk)}\lambda_p) ({}^{(ijk)}\hat{r}_p + {}^{(ijk)}\hat{q}_p) \text{sign}({}^{(ijk)}s_p(t)) - \sum_{p=1}^n ({}^{(ijk)}\lambda_p) k_1 \tanh({}^{(ijk)}s_p) \\
&+ \tilde{\boldsymbol{\theta}}^T (-\mathbf{F}^T(x(t)) \boldsymbol{\gamma} - \varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}}) |\tilde{\boldsymbol{\theta}}|^w) \\
&+ \tilde{\boldsymbol{\beta}}^T (-\mathbf{G}^T(y(t)) \boldsymbol{\xi} - \varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}}) |\tilde{\boldsymbol{\beta}}|^w) \\
&+ \tilde{\boldsymbol{\vartheta}}^T (\mathbf{H}^T(z(t)) \boldsymbol{\varpi} - \varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}}) |\tilde{\boldsymbol{\vartheta}}|^w) \\
&+ \sum_{p=1}^n {}^{(ijk)}\tilde{r}_p^T ({}^{(ijk)}\lambda_p) (|{}^{(ijk)}s_p|) + \sum_{p=1}^n {}^{(ijk)}\tilde{q}_p^T ({}^{(ijk)}\lambda_p) (|{}^{(ijk)}s_p|).
\end{aligned} \tag{19}$$

Since

$$\begin{aligned}
&\sum_{j=1}^n {}^{(ijk)}s_j ({}^{(ijk)}\lambda_j) b_{jj} \mathbf{G}_j(y(t)) \tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}^T \mathbf{G}^T(y(t)) \boldsymbol{\xi}, \\
&\sum_{k=1}^n {}^{(ijk)}s_k ({}^{(ijk)}\lambda_k) a_{kk} \mathbf{F}_k(x(t)) \tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\theta}}^T \mathbf{F}^T(x(t)) \boldsymbol{\gamma}, \\
&\sum_{i=1}^n {}^{(ijk)}s_i ({}^{(ijk)}\lambda_i) c_{ii} \mathbf{H}_i(z(t)) \tilde{\boldsymbol{\vartheta}} = \tilde{\boldsymbol{\vartheta}}^T \mathbf{H}^T(z(t)) \boldsymbol{\varpi}.
\end{aligned} \tag{20}$$

Thus

$$\begin{aligned}
{}_0D_t^\alpha V(t, \mathbf{x}(t)) &\leq \sum_{i=1}^n {}^{(ijk)}s_i ({}^{(ijk)}\lambda_i) (c_{ii} \Delta h_i(z(t)) + c_{ii} \mu_i(z, t)) \\
&+ \sum_{j=1}^n {}^{(ijk)}s_j ({}^{(ijk)}\lambda_j) (-b_{jj} \Delta g_j(y(t)) - b_{jj} D_j(y, t)) \\
&+ \sum_{k=1}^n {}^{(ijk)}s_k ({}^{(ijk)}\lambda_k) (-a_{kk} \Delta f_k(x(t)) - a_{kk} d_k(x, t))
\end{aligned}$$

$$\begin{aligned}
& -\tilde{\boldsymbol{\theta}}^T(\varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}})|(\tilde{\boldsymbol{\theta}})|^w) - \tilde{\boldsymbol{\beta}}^T(\varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}})|(\tilde{\boldsymbol{\beta}})|^w) - \tilde{\boldsymbol{\vartheta}}^T(\varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}})|(\tilde{\boldsymbol{\vartheta}})|^w) \\
& - \sum_{p=1}^n ((ijk)\lambda_p)((ijk)\hat{r}_p + (ijk)\hat{q}_p) \text{sign}((ijk)s_p(t)) - \sum_{p=1}^n ((ijk)\lambda_p)k_1 \tanh((ijk)s_p) \\
& + \sum_{p=1}^n ((ijk)\lambda_p)((ijk)\tilde{r}_p^T)(|(ijk)s_p|) + \sum_{p=1}^n ((ijk)\lambda_p)((ijk)\tilde{q}_p^T)(|(ijk)s_p|) \\
& \leq \sum_{p=1}^n ((ijk)\lambda_p)((ijk)r_p)|^{(ijk)s_p} + \sum_{p=1}^n ((ijk)\lambda_p)((ijk)q_p)|^{(ijk)s_p} \\
& - \sum_{p=1}^n ((ijk)\lambda_p)((ijk)\hat{r}_p + (ijk)\hat{q}_p)|^{(ijk)s_p} + \sum_{p=1}^n ((ijk)\lambda_p)((ijk)\tilde{r}_p^T)(|(ijk)s_p|) \\
& + \sum_{p=1}^n ((ijk)\lambda_p)((ijk)\tilde{q}_p^T)(|(ijk)s_p|) - \tilde{\boldsymbol{\theta}}^T(\varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}})|(\tilde{\boldsymbol{\theta}})|^w) \\
& - \tilde{\boldsymbol{\beta}}^T(\varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}})|(\tilde{\boldsymbol{\beta}})|^w) - \tilde{\boldsymbol{\vartheta}}^T(\varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}})|(\tilde{\boldsymbol{\vartheta}})|^w) \\
& - \sum_{i=1}^n (ijk)s_p((ijk)\lambda_p)k_1 \tanh((ijk)s_p) \\
& = -\tilde{\boldsymbol{\theta}}^T(\varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}})|(\tilde{\boldsymbol{\theta}})|^w) - \tilde{\boldsymbol{\beta}}^T(\varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}})|(\tilde{\boldsymbol{\beta}})|^w) \\
& - \tilde{\boldsymbol{\vartheta}}^T(\varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}})|(\tilde{\boldsymbol{\vartheta}})|^w) - \sum_{i=1}^p (ijk)s_p k_1 \tanh(\lambda^{(ijk)}s_p).
\end{aligned} \tag{21}$$

Which establishes that the ${}_0D_t^\alpha V(t, \mathbf{x}(t))$ is negative definite. Thus, it follows from the FO Lyapunov stability theory that $\lim_{t \rightarrow \infty} (ijk)e_p = 0$ is achieved, i.e. we can say that the MSSMCS of the drive systems (4), (5) and response system (6) is accomplished in terms of $m = n$.

5 The synchronization of multi-switching FO system with different dimensions

In Section 5, the MSSMCS of FO chaotic systems with different dimension is formulated which means the dimension of D-R systems (4), (5) and (6) satisfies $m \neq n$. Thus, the scaling matrices A, B, C are given as non-diagonal matrices. We designed appropriate multi-switching adaptive controllers (MSAC) and some multi-switching adaptive updating laws (MSAUL) to realize the synchronization of the D-R system which are proved in Theorem 2.

When we choose $m \neq n$ and the non-diagonal matrices $A \in \mathbb{R}^{n \times m}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{n \times n}, C \neq 0$, the form of the error system (9) can be explained as:

$$\lim_{t \rightarrow \infty} {}^{(l w v)}e_p = \lim_{t \rightarrow \infty} \left[\sum_{i=1}^n (c_{li} z_i) - \left\{ \sum_{j=1}^m (b_{wj} y_j) + \sum_{k=1}^m (x_{vk} x_k) \right\} \right] = 0. \tag{22}$$

Assumption 2. Assumed that the external disturbances d_k, D_j, μ_i , uncertain non-linear vectors $\Delta f_k, \Delta g_j, \Delta h_i$ all have a bounded norm. Namely, there are suitable

positive constants $^{(l w v)}\rho_p, ^{(l w v)}\varrho_p$ that satisfy

$$\begin{aligned} \left| \sum_{i=1}^n c_{li} \mu_i - \left(\sum_{j=1}^m b_{wj} D_j + \sum_{k=1}^m x_{vk} d_k \right) \right| &\leq ^{(l w v)}\rho_p, \\ \left| \sum_{i=1}^n c_{li} \Delta h_i - \left(\sum_{j=1}^m b_{wj} \Delta g_j + \sum_{k=1}^m x_{vk} \Delta f_k \right) \right| &\leq ^{(l w v)}\varrho_p, \end{aligned} \quad (23)$$

where $p = 1, 2, \dots, n$.

Remark 8. The positive constants $^{(l w v)}\rho_p, ^{(l w v)}\varrho_p$ are unknown, $^{(l w v)}\hat{\rho}_p, ^{(l w v)}\hat{\varrho}_p$ are used to represent the estimation of parameter $^{(l w v)}\rho_p, ^{(l w v)}\varrho_p$.

According to the definition of the error vector (22), we get the FO error system as

$$\begin{aligned} {}_0 D_t^\alpha \{^{(l w v)}e_p\} &= \sum_{i=1}^n c_{li} \{ {}_0 D_t^\alpha z_i \} - \sum_{j=1}^m b_{wj} \{ {}_0 D_t^\alpha y_j \} - \sum_{k=1}^m x_{vk} \{ {}_0 D_t^\alpha x_k \} \\ &= \sum_{i=1}^n c_{li} \{ \mathbf{H}_i(z(t)) \boldsymbol{\vartheta} + h_i(z(t)) + \Delta h_i(z(t)) + \mu_i(z, t) + ^{(l w v)}u_p(t) \} \\ &\quad - \sum_{j=1}^m b_{wj} \{ \mathbf{G}_j(y(t)) \boldsymbol{\beta} + g_j(y(t)) + \Delta g_j(y(t)) + D_j(y, t) \} \\ &\quad - \sum_{k=1}^m x_{vk} \{ \mathbf{F}_k(x(t)) \boldsymbol{\theta} + f_k(x(t)) + \Delta f_k(x(t)) + d_k(x, t) \}. \end{aligned} \quad (24)$$

The errors of unknown parameters $\boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\vartheta}$ have been defined in (12). For convenience, we define error of unknown constants $^{(l w v)}\rho_p, ^{(l w v)}\varrho_p$ as $^{(l w v)}\tilde{\rho}_p = ^{(l w v)}\hat{\rho}_p - ^{(l w v)}\rho_p, ^{(l w v)}\tilde{\varrho}_p = ^{(l w v)}\hat{\varrho}_p - ^{(l w v)}\varrho_p$. Thus, the sliding mode surface is designed as $^{(l w v)}s_p = ^{(l w v)}\lambda_p(^{(l w v)}e_p)$. We can get the following multi-switching adaptive controller (MSAC) (25) multi-switching adaptive updating laws (MSAUL) (28):

$$\begin{aligned} \sum_{i=1}^n c_{li} \{ ^{(l w v)}u_p \} &= - \sum_{i=1}^n c_{li} h_i + \sum_{j=1}^m b_{wj} g_j + \sum_{k=1}^m x_{vk} f_k + \sum_{k=1}^m x_{vk} \mathbf{F}_k(x(t)) \hat{\boldsymbol{\theta}} \\ &\quad + \sum_{j=1}^m b_{wj} \mathbf{G}_j(y(t)) \hat{\boldsymbol{\beta}} - \sum_{i=1}^n c_{li} \mathbf{H}_i(z(t)) \hat{\boldsymbol{\vartheta}} - (^{(l w v)}\hat{\rho}_p + ^{(l w v)}\hat{\varrho}_p) \text{sign} (^{(l w v)}s_p) \\ &\quad - k_1 \tanh (^{(l w v)}s_p), \end{aligned} \quad (25)$$

where $^{(l w v)}\lambda_p$ is a constant. Substituting (25) into Eq. (24), we obtain

$$\begin{aligned} {}_0 D_t^\alpha \{ ^{(l w v)}e_p \} &= \sum_{i=1}^n c_{li} \{ -\mathbf{H}_i(z(t)) \tilde{\boldsymbol{\vartheta}} + \Delta h_i(z(t)) + \mu_i(z, t) \} \\ &\quad + \sum_{j=1}^m b_{wj} \{ \mathbf{G}_j(y(t)) \tilde{\boldsymbol{\beta}} + \Delta g_j(y(t)) + D_j(y, t) \} \\ &\quad + \sum_{k=1}^m x_{vk} \{ \mathbf{F}_k(x(t)) \tilde{\boldsymbol{\theta}} + \Delta f_k(x(t)) + d_k(x, t) \} \end{aligned} \quad (26)$$

$$-({}^{(l w v)}\hat{\rho}_p + {}^{(l w v)}\hat{\varrho}_p)sign({}^{(l w v)}s_p) - k_1tanh({}^{(l w v)}s_p).$$

Thus, it follows from ${}^{(l w v)}s_p = {}^{(l w v)}\lambda_p({}^{(l w v)}e_p)$ and Eq. (26) that we obtain the following summation result.

$$\begin{aligned} & \sum_{p=1}^n \{({}^{(l w v)}s_p)\}_0 D_t^\alpha \{({}^{(l w v)}s_p)\} \\ &= \sum_{l=1}^n \sum_{i=1}^n ({}^{(l w v)}s_i)({}^{(l w v)}\lambda_i)c_{li} \{-\mathbf{H}_i(z(t))\tilde{\boldsymbol{\vartheta}} + \Delta h_i(z(t)) + \mu_i(z, t)\} \\ &+ \sum_{w=1}^n \sum_{j=1}^m ({}^{(l w v)}s_j)({}^{(l w v)}\lambda_j)b_{wj} \{\mathbf{G}_j(y(t))\tilde{\boldsymbol{\beta}} + \Delta g_j(y(t)) + D_j(y, t)\} \\ &+ \sum_{v=1}^n \sum_{k=1}^m ({}^{(l w v)}s_k)({}^{(l w v)}\lambda_k)x_{vk} \{\mathbf{F}_k(x(t))\tilde{\boldsymbol{\theta}} + \Delta f_k(x(t)) + d_k(x, t)\} \\ &- \sum_{p=1}^n ({}^{(l w v)}\lambda_p)({}^{(l w v)}\hat{\rho}_p + {}^{(l w v)}\hat{\varrho}_p)sign({}^{(l w v)}s_p(t)) - \sum_{p=1}^n ({}^{(l w v)}\lambda_p)k_1tanh({}^{(l w v)}s_p). \end{aligned} \quad (27)$$

The MSAUL with regard to unknown parameters $\boldsymbol{\theta}, \boldsymbol{\beta}, \boldsymbol{\vartheta}, {}^{(l w v)}\rho_p, {}^{(l w v)}\varrho_p$ are selected as

$$\begin{aligned} {}_0D_t^\alpha \tilde{\boldsymbol{\theta}} &= -\mathbf{F}^T(x(t))\boldsymbol{\gamma} - \varphi_1 sign(\tilde{\boldsymbol{\theta}})|(\tilde{\boldsymbol{\theta}})|^w, \\ {}_0D_t^\alpha \tilde{\boldsymbol{\beta}} &= -\mathbf{G}^T(y(t))\boldsymbol{\xi} - \varphi_2 sign(\tilde{\boldsymbol{\beta}})|(\tilde{\boldsymbol{\beta}})|^w, \\ {}_0D_t^\alpha \tilde{\boldsymbol{\vartheta}} &= \mathbf{H}^T(z(t))\boldsymbol{\varpi} - \varphi_3 sign(\tilde{\boldsymbol{\vartheta}})|(\tilde{\boldsymbol{\vartheta}})|^w, \\ {}_0D_t^\alpha \{({}^{(l w v)}\tilde{\rho}_p)\} &= m_1({}^{(l w v)}\lambda_p)|({}^{(l w v)}s_p(t))|, \\ {}_0D_t^\alpha \{({}^{(l w v)}\tilde{\varrho}_p)\} &= m_2({}^{(l w v)}\lambda_p)|({}^{(l w v)}s_p(t))|, \end{aligned} \quad (28)$$

where

$$\begin{aligned} \boldsymbol{\gamma} &= \left\{ \sum_{k=1}^m ({}^{(l w v)}\lambda_1)x_{vk}({}^{(l w v)}s_1), \sum_{k=1}^m ({}^{(l w v)}\lambda_2)x_{vk}({}^{(l w v)}s_2), \dots, \sum_{k=1}^m ({}^{(l w v)}\lambda_n)x_{vk}({}^{(l w v)}s_n) \right\}^T, \\ \boldsymbol{\xi} &= \left(\sum_{j=1}^m ({}^{(l w v)}\lambda_1)b_{wj}({}^{(l w v)}s_1), \sum_{j=1}^m ({}^{(l w v)}\lambda_2)b_{wj}({}^{(l w v)}s_2), \dots, \sum_{j=1}^m ({}^{(l w v)}\lambda_n)b_{wj}({}^{(l w v)}s_n) \right)^T, \\ \boldsymbol{\varpi} &= \left(\sum_{i=1}^n ({}^{(l w v)}\lambda_1)c_{li}({}^{(l w v)}s_1), \sum_{i=1}^n ({}^{(l w v)}\lambda_2)c_{li}({}^{(l w v)}s_2), \dots, \sum_{i=1}^n ({}^{(l w v)}\lambda_n)c_{li}({}^{(l w v)}s_n) \right)^T, \end{aligned}$$

m_1, m_2 are positive constants.

Theorem 2. For any given initial conditions $\mathbf{x}(0), \mathbf{y}(0), \mathbf{z}(0)$, if the Assumption 2 is hold, the synchronization error system (22) will achieve the multi-switching sliding mode combination synchronization (MSSMCS) via the multi-switching adaptive controller (MSAC) (25) and multi-switching adaptive updating laws (MSAUL) (28).

Proof. Adopting the Lyapunov function as:

$$V = \frac{1}{2} \sum_{p=1}^n {}^{(l w v)} s_p^2 + \frac{1}{2m_1} \sum_{p=1}^n \{ {}^{(l w v)} \tilde{\rho}_p^T \} \{ {}^{(l w v)} \tilde{\rho}_p \} + \frac{1}{2m_2} \sum_{p=1}^n \{ {}^{(l w v)} \tilde{\varrho}_p^T \} \{ {}^{(l w v)} \tilde{\varrho}_p \} + \frac{1}{2} \tilde{\boldsymbol{\theta}}^T \boldsymbol{\theta} + \frac{1}{2} \tilde{\boldsymbol{\beta}}^T \boldsymbol{\beta} + \frac{1}{2} \tilde{\boldsymbol{\vartheta}}^T \boldsymbol{\vartheta}. \quad (29)$$

Taking the α derivative

$$\begin{aligned} {}_0 D_t^\alpha V(t, \mathbf{x}(t)) &\leq \sum_{p=1}^n \{ {}^{(l w v)} s_p \} {}_0 D_t^\alpha \{ {}^{(l w v)} s_p \} + \frac{1}{m_1} \{ \sum_{p=1}^n {}^{(l w v)} \tilde{\rho}_p^T \} {}_0 D_t^\alpha \{ {}^{(l w v)} \tilde{\rho}_p \} \\ &\quad + \frac{1}{m_2} \sum_{p=1}^n \{ {}^{(l w v)} \tilde{\varrho}_p^T \} {}_0 D_t^\alpha \{ {}^{(l w v)} \tilde{\varrho}_p \} + \tilde{\boldsymbol{\theta}}^T {}_0 D_t^\alpha \tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\beta}}^T {}_0 D_t^\alpha \tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\vartheta}}^T {}_0 D_t^\alpha \tilde{\boldsymbol{\vartheta}}. \end{aligned} \quad (30)$$

Then substituting the (27) and MSAUL (28) into Eq. (30), we obtain

$$\begin{aligned} {}_0 D_t^\alpha V(t, \mathbf{x}(t)) &\leq \sum_{l=1}^n \sum_{i=1}^n {}^{(l w v)} s_i ({}^{(l w v)} \lambda_i) c_{li} \{ -\mathbf{H}_i(z(t)) \tilde{\boldsymbol{\vartheta}} + \Delta h_i(z(t)) + \mu_i(z, t) \} \\ &\quad + \sum_{w=1}^n \sum_{j=1}^m {}^{(l w v)} s_j ({}^{(l w v)} \lambda_j) b_{wj} \{ \mathbf{G}_j(y(t)) \tilde{\boldsymbol{\beta}} + \Delta g_j(y(t)) + D_j(y, t) \} \\ &\quad + \sum_{v=1}^n \sum_{k=1}^m {}^{(l w v)} s_k ({}^{(l w v)} \lambda_k) x_{vk} \{ \mathbf{F}_k(x(t)) \tilde{\boldsymbol{\theta}} + \Delta f_k(x(t)) + d_k(x, t) \} \\ &\quad - \sum_{p=1}^n ({}^{(l w v)} \lambda_p) ({}^{(l w v)} \hat{\rho}_p + {}^{(l w v)} \hat{\varrho}_p) \text{sign}({}^{(l w v)} s_p(t)) \\ &\quad - \sum_{p=1}^n ({}^{(l w v)} \lambda_p) k_1 \tanh({}^{(l w v)} s_p) + \tilde{\boldsymbol{\theta}}^T \{ -\mathbf{F}^T(x(t)) \boldsymbol{\gamma} - \varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}}) |(\tilde{\boldsymbol{\theta}})|^\omega \} \\ &\quad + \tilde{\boldsymbol{\beta}}^T \{ -\mathbf{G}^T(y(t)) \boldsymbol{\xi} - \varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}}) |(\tilde{\boldsymbol{\beta}})|^\omega \} \\ &\quad + \tilde{\boldsymbol{\vartheta}}^T \{ \mathbf{H}^T(z(t)) \boldsymbol{\varpi} - \varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}}) |(\tilde{\boldsymbol{\vartheta}})|^\omega \} \\ &\quad + \sum_{p=1}^n {}^{(l w v)} \tilde{\rho}_p^T ({}^{(l w v)} \lambda_p) (|{}^{(l w v)} s_p|) + \sum_{p=1}^n {}^{(l w v)} \tilde{\varrho}_p^T ({}^{(l w v)} \lambda_p) (|{}^{(l w v)} s_p|). \end{aligned} \quad (31)$$

Since

$$\begin{aligned} \sum_{v=1}^n \sum_{k=1}^m {}^{(l w v)} s_k ({}^{(l w v)} \lambda_k) x_{vk} \{ \mathbf{F}_k(x(t)) \tilde{\boldsymbol{\theta}} \} &= \tilde{\boldsymbol{\theta}}^T \mathbf{F}^T(x(t)) \boldsymbol{\gamma}, \\ \sum_{w=1}^n \sum_{j=1}^m {}^{(l w v)} s_j ({}^{(l w v)} \lambda_j) b_{wj} \{ \mathbf{G}_j(y(t)) \tilde{\boldsymbol{\beta}} \} &= \tilde{\boldsymbol{\beta}}^T \mathbf{G}^T(y(t)) \boldsymbol{\xi}, \\ \sum_{l=1}^n \sum_{i=1}^n {}^{(l w v)} s_i ({}^{(l w v)} \lambda_i) c_{li} \{ \mathbf{H}_i(z(t)) \tilde{\boldsymbol{\vartheta}} \} &= \tilde{\boldsymbol{\vartheta}}^T \mathbf{H}^T(z(t)) \boldsymbol{\varpi}. \end{aligned} \quad (32)$$

Thus

$${}_0 D_t^\alpha V(t, \mathbf{x}(t)) \leq \sum_{l=1}^n \sum_{i=1}^n {}^{(l w v)} s_i ({}^{(l w v)} \lambda_i) c_{li} \{ \Delta h_i(z(t)) + \mu_i(z, t) \}$$

$$\begin{aligned}
& + \sum_{w=1}^n \sum_{j=1}^m {}^{(l w v)} s_j ({}^{(l w v)} \lambda_j) b_{w j} \{ \Delta g_j(y(t)) + D_j(y, t) \} \\
& + \sum_{v=1}^n \sum_{k=1}^m {}^{(l w v)} s_k ({}^{(l w v)} \lambda_k) x_{v k} \{ \Delta f_k(x(t)) + d_k(x, t) \} \\
& + \tilde{\boldsymbol{\theta}}^T \{ -\mathbf{F}^T(x(t)) \boldsymbol{\gamma} - \varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}}) |(\tilde{\boldsymbol{\theta}})|^\omega \} \\
& + \tilde{\boldsymbol{\beta}}^T \{ -\mathbf{G}^T(y(t)) \boldsymbol{\xi} - \varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}}) |(\tilde{\boldsymbol{\beta}})|^\omega \} \\
& + \tilde{\boldsymbol{\vartheta}}^T \{ \mathbf{H}^T(z(t)) \boldsymbol{\varpi} - \varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}}) |(\tilde{\boldsymbol{\vartheta}})|^\omega \} - \sum_{p=1}^n ({}^{(l w v)} \lambda_p) ({}^{(l w v)} \hat{\rho}_p \\
& + ({}^{(l w v)} \hat{\rho}_p) \text{sign}({}^{(l w v)} s_p(t)) + \sum_{p=1}^n ({}^{(l w v)} \tilde{\rho}_p^T ({}^{(l w v)} \lambda_p) (|{}^{(l w v)} s_p|) \\
& + \sum_{p=1}^n ({}^{(l w v)} \tilde{\rho}_p^T ({}^{(l w v)} \lambda_p) (|{}^{(l w v)} s_p|) - \sum_{p=1}^n ({}^{(l w v)} \lambda_p) k_1 \tanh({}^{(l w v)} s_p) \quad (33) \\
& \leq \sum_{p=1}^n ({}^{(l w v)} \rho_p ({}^{(l w v)} \lambda_p) (|{}^{(l w v)} s_p|) + \sum_{p=1}^n ({}^{(l w v)} \rho_p ({}^{(l w v)} \lambda_p) (|{}^{(l w v)} s_p|) \\
& - \sum_{p=1}^n ({}^{(l w v)} \lambda_p) ({}^{(l w v)} \hat{\rho}_p + ({}^{(l w v)} \hat{\rho}_p) \text{sign}({}^{(l w v)} s_p(t)) \\
& + \sum_{p=1}^n ({}^{(l w v)} \tilde{\rho}_p^T ({}^{(l w v)} \lambda_p) (|{}^{(l w v)} s_p|) + \sum_{p=1}^n ({}^{(l w v)} \tilde{\rho}_p^T ({}^{(l w v)} \lambda_p) (|{}^{(l w v)} s_p|) \\
& - \tilde{\boldsymbol{\theta}}^T (\varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}}) |(\tilde{\boldsymbol{\theta}})|^\omega) - \tilde{\boldsymbol{\beta}}^T (\varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}}) |(\tilde{\boldsymbol{\beta}})|^\omega) \\
& - \tilde{\boldsymbol{\vartheta}}^T (\varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}}) |(\tilde{\boldsymbol{\vartheta}})|^\omega) - \sum_{p=1}^n ({}^{(l w v)} \lambda_p) k_1 \tanh({}^{(l w v)} s_p) \\
& = -\tilde{\boldsymbol{\theta}}^T (\varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}}) |(\tilde{\boldsymbol{\theta}})|^\omega) - \tilde{\boldsymbol{\beta}}^T (\varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}}) |(\tilde{\boldsymbol{\beta}})|^\omega) \\
& - \tilde{\boldsymbol{\vartheta}}^T (\varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}}) |(\tilde{\boldsymbol{\vartheta}})|^\omega) - \sum_{p=1}^n ({}^{(l w v)} \lambda_p) k_1 \tanh({}^{(l w v)} s_p).
\end{aligned}$$

Which establishes that the ${}_0 D_t^\alpha V(t, \mathbf{x}(t))$ is negative definite. Thus, it follows from the FO Lyapunov stability theory that $\lim_{t \rightarrow \infty} {}^{(l w v)} e_p = 0$ is achieved. On the other hand, we can say that the MSSMCS of the drive systems (4), (5) and response system (6) is accomplished in terms of $m \neq n$.

The following corollaries are successfully analyzed from Theorem 2 and their proofs are omitted here. By the way, the Theorem 1 has the same theory, we are not going to describe.

Corollary 1. If the matrices $A \neq 0, B = 0, C \neq 0$, then the drive systems (4) achieve the MSSMCS with the response systems (6) provided the following controller,

$$\begin{aligned}
\sum_{i=1}^n c_{li} \{ {}^{(l w v)} u_p \} = & - \sum_{i=1}^n c_{li} h_i + \sum_{k=1}^m x_{v k} f_k + \sum_{k=1}^m x_{v k} \mathbf{F}_k(x(t)) \hat{\boldsymbol{\theta}} - \sum_{i=1}^n c_{li} \mathbf{H}_i(z(t)) \hat{\boldsymbol{\vartheta}} \\
& - ({}^{(l w v)} \hat{\rho}_p + ({}^{(l w v)} \hat{\rho}_p) \text{sign}({}^{(l w v)} s_p) - k_1 \tanh({}^{(l w v)} s_p).
\end{aligned}$$

And the adaptive updating laws,

$$\begin{aligned} {}_0D_t^\alpha \tilde{\boldsymbol{\theta}} &= -\mathbf{F}^T(x(t))\boldsymbol{\gamma} - \varphi_1 \text{sign}(\tilde{\boldsymbol{\theta}})|(\tilde{\boldsymbol{\theta}})|^w, \\ {}_0D_t^\alpha \tilde{\boldsymbol{\vartheta}} &= \mathbf{H}^T(z(t))\boldsymbol{\varpi} - \varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}})|(\tilde{\boldsymbol{\vartheta}})|^w, \\ {}_0D_t^\alpha \{^{(l w v)}\tilde{\rho}_p\} &= m_1(^{(l w v)}\lambda_p)|^{(l w v)}s_p(t)|, \\ {}_0D_t^\alpha \{^{(l w v)}\tilde{\varrho}_p\} &= m_2(^{(l w v)}\lambda_p)|^{(l w v)}s_p(t)|, \end{aligned}$$

where the explanation of $\boldsymbol{\gamma}$, $\boldsymbol{\varpi}$ can be seen (28).

Corollary 2. If the matrices $A = 0, B \neq 0, C \neq 0$, then the drive systems (5) achieve the MSSMCS with the response systems (6) provided the following controller,

$$\begin{aligned} \sum_{i=1}^n c_{li} \{^{(l w v)}u_p\} &= - \sum_{i=1}^n c_{li} h_i + \sum_{j=1}^m b_{wj} g_j + \sum_{j=1}^m b_{wj} \mathbf{G}_j(y(t)) \hat{\boldsymbol{\beta}} - \sum_{i=1}^n c_{li} \mathbf{H}_i(z(t)) \hat{\boldsymbol{\vartheta}} \\ &\quad - (^{(l w v)}\hat{\rho}_p + ^{(l w v)}\hat{\varrho}_p) \text{sign}(^{(l w v)}s_p) - k_1 \tanh(^{(l w v)}s_p). \end{aligned}$$

And the adaptive updating laws,

$$\begin{aligned} {}_0D_t^\alpha \tilde{\boldsymbol{\beta}} &= -\mathbf{G}^T(y(t))\boldsymbol{\xi} - \varphi_2 \text{sign}(\tilde{\boldsymbol{\beta}})|(\tilde{\boldsymbol{\beta}})|^w, \\ {}_0D_t^\alpha \tilde{\boldsymbol{\vartheta}} &= \mathbf{H}^T(z(t))\boldsymbol{\varpi} - \varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}})|(\tilde{\boldsymbol{\vartheta}})|^w, \\ {}_0D_t^\alpha \{^{(l w v)}\tilde{\rho}_p\} &= m_1(^{(l w v)}\lambda_p)|^{(l w v)}s_p(t)|, \\ {}_0D_t^\alpha \{^{(l w v)}\tilde{\varrho}_p\} &= m_2(^{(l w v)}\lambda_p)|^{(l w v)}s_p(t)|, \end{aligned}$$

where the explanation of $\boldsymbol{\xi}$, $\boldsymbol{\varpi}$ can be seen (28).

Corollary 3. If the matrices $A = 0, B = 0, C \neq 0$, then the equilibrium point $(0, 0, 0, 0)$ of response systems (6) is asymptotically stable provided the following controller,

$$\begin{aligned} \sum_{i=1}^n c_{li} \{^{(l w v)}u_p\} &= - \sum_{i=1}^n c_{li} h_i - \sum_{i=1}^n c_{li} \mathbf{H}_i(z(t)) \hat{\boldsymbol{\vartheta}} \\ &\quad - (^{(l w v)}\hat{\rho}_p + ^{(l w v)}\hat{\varrho}_p) \text{sign}(^{(l w v)}s_p) - k_1 \tanh(^{(l w v)}s_p), \end{aligned}$$

And the adaptive updating laws,

$$\begin{aligned} {}_0D_t^\alpha \tilde{\boldsymbol{\vartheta}} &= \mathbf{H}^T(z(t))\boldsymbol{\varpi} - \varphi_3 \text{sign}(\tilde{\boldsymbol{\vartheta}})|(\tilde{\boldsymbol{\vartheta}})|^w, \\ {}_0D_t^\alpha \{^{(l w v)}\tilde{\rho}_p\} &= m_1(^{(l w v)}\lambda_p)|^{(l w v)}s_p(t)|, \\ {}_0D_t^\alpha \{^{(l w v)}\tilde{\varrho}_p\} &= m_2(^{(l w v)}\lambda_p)|^{(l w v)}s_p(t)|, \end{aligned}$$

where the explanation of $\boldsymbol{\varpi}$ can be seen (28).

6 Numerical simulation

This section mainly demonstrates the reliability and validity of the suggested multi-switching sliding mode combination synchronization scheme. For the D-R systems (4), (5) and (6) with same dimension, we selected two error states to elaborate the method, namely, $i \neq j \neq k$ and $i \neq j = k$. For the D-R systems (4), (5) and (6) with different dimensions, we selected $l \neq w \neq v$ and $l \neq w = v$. In each case, we have given the specific form of controller and parameter adapting rate via the specific FO hyper-chaotic or chaotic systems. .

6.1 Numerical simulations for FO chaotic system with same dimension

As an example we choose FO hyper-chaotic Lorenz, Chen systems as the drive systems, and the FO hyper-chaotic Lü system as the response system. Adding SD to the D-R systems, we obtain

$$\begin{pmatrix} {}_0D_t^\alpha x_1 \\ {}_0D_t^\alpha x_2 \\ {}_0D_t^\alpha x_3 \\ {}_0D_t^\alpha x_4 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} x_4 + \Delta f_1 \\ -x_1 x_3 - x_2 + \Delta f_2 \\ x_1 x_2 + \Delta f_3 \\ -x_2 x_3 + \Delta f_4 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}, \quad (34)$$

$$\begin{pmatrix} {}_0D_t^\alpha y_1 \\ {}_0D_t^\alpha y_2 \\ {}_0D_t^\alpha y_3 \\ {}_0D_t^\alpha y_4 \end{pmatrix} = \begin{pmatrix} y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & y_1 & 0 \\ 0 & -y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_4 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \\ r \end{pmatrix} + \begin{pmatrix} y_4 + \Delta g_1 \\ -y_1 y_3 + \Delta g_2 \\ y_1 y_2 + \Delta g_3 \\ y_2 y_3 + \Delta g_4 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix}, \quad (35)$$

$$\begin{pmatrix} {}_0D_t^\alpha z_1 \\ {}_0D_t^\alpha z_2 \\ {}_0D_t^\alpha z_3 \\ {}_0D_t^\alpha z_4 \end{pmatrix} = \begin{pmatrix} z_2 - z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & -z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix} + \begin{pmatrix} z_4 + \Delta h_1 \\ -z_1 z_3 + \Delta h_2 \\ z_1 z_2 + \Delta h_3 \\ z_1 z_3 + \Delta h_4 \end{pmatrix} + \begin{pmatrix} \mu_1 + u_1 \\ \mu_2 + u_2 \\ \mu_3 + u_3 \\ \mu_4 + u_4 \end{pmatrix}. \quad (36)$$

Choosing the parameters are $a_1 = 10, b_1 = 28, c_1 = 8/3, d_1 = -1, a_2 = 35, b_2 = 3, c_2 = 12, d_2 = 7, r = 0.5, a_3 = 36, b_3 = 20, c_3 = 3, d_3 = 0.5$. The initial conditions take as $x(0) = (2, -2, 1, 1), y(0) = (1, 1, 2, 2), z(0) = (-20, 3, 1, 3)$. When $\Delta g = 0, \Delta f = 0, \Delta h = 0, d(t) = 0, D(t) = 0, \mu(t) = 0$ and $\alpha = 0.97$, we can obtain the attractor graphs of the FO hyper-chaotic Lorenz, Chen, and Lü system using the FO prediction-correction method which are presented in Fig.1 The multi-switching error state modes between the drive systems (34), (35) and the response system (36) are listed.

$$i \neq j \neq k \left\{ \begin{array}{llllll} (123)e_1, & (132)e_2, & (124)e_3, & (142)e_4, & (134)e_5, & (143)e_6, \\ (213)e_7, & (231)e_8, & (214)e_9, & (241)e_{10}, & (234)e_{11}, & (243)e_{12}, \\ (321)e_{13}, & (312)e_{14}, & (324)e_{15}, & (314)e_{16}, & (341)e_{17}, & (342)e_{18}, \\ (412)e_{19}, & (421)e_{20}, & (413)e_{21}, & (423)e_{22}, & (432)e_{23}, & (431)e_{24}. \end{array} \right.$$

$$i \neq j = k \left\{ \begin{array}{llllll} (122)e_{25}, & (133)e_{26}, & (144)e_{27}, & (211)e_{28}, & (233)e_{29}, & (244)e_{30}, \\ (311)e_{31}, & (322)e_{32}, & (344)e_{33}, & (411)e_{34}, & (422)e_{35}, & (433)e_{36}. \end{array} \right.$$

$$i = j \neq k \left\{ \begin{array}{llllll} (112)e_{37}, & (113)e_{38}, & (114)e_{39}, & (221)e_{40}, & (223)e_{41}, & (224)e_{42}, \\ (331)e_{43}, & (332)e_{44}, & (334)e_{45}, & (441)e_{46}, & (442)e_{47}, & (443)e_{48}. \end{array} \right.$$

$$i = k \neq j \left\{ \begin{array}{llllll} (121)e_{49}, & (131)e_{50}, & (141)e_{51}, & (212)e_{52}, & (232)e_{53}, & (242)e_{54}, \\ (313)e_{55}, & (323)e_{56}, & (343)e_{57}, & (414)e_{58}, & (424)e_{59}, & (434)e_{60}. \end{array} \right.$$

Fig.1. The 3D phase plot of FO hyper-chaotic Lorenz, Chen, Lü systems indicating in sub-pictures (a)-(c) respectively.

$$i = k = j \left\{ \begin{array}{ll} {}^{(111)}e_{61}, & {}^{(222)}e_{62}, \\ {}^{(333)}e_{63}, & {}^{(444)}e_{64}. \end{array} \right.$$

In this chapter, we will select the appropriate switching error variable for the situation of $i \neq j \neq k$, namely, $\mathbf{e} = ({}^{(124)}e_3, {}^{(243)}e_{12}, {}^{(312)}e_{14}, {}^{(431)}e_{24})^T$, $i \neq j = k$ namely, $\mathbf{e} = ({}^{(122)}e_{25}, {}^{(244)}e_{30}, {}^{(311)}e_{31}, {}^{(433)}e_{36})^T$ and obtain the following two switching modes for numerical simulation:

$$\text{Switch-1} : \left(\begin{array}{l} {}^{(124)}e_3 = c_{11}z_1 - (b_{22}y_2 + x_{44}x_4) \\ {}^{(243)}e_{12} = c_{22}z_2 - (b_{44}y_4 + x_{33}x_3) \\ {}^{(312)}e_{14} = c_{33}z_3 - (b_{11}y_1 + x_{22}x_2) \\ {}^{(431)}e_{24} = c_{44}z_4 - (b_{33}y_3 + x_{11}x_1) \end{array} \right), \quad (37)$$

$$Switch-2 : \begin{pmatrix} {}^{(122)}e_{25} = c_{11}z_1 - (b_{22}y_2 + x_{22}x_2) \\ {}^{(244)}e_{30} = c_{22}z_2 - (b_{44}y_4 + x_{44}x_4) \\ {}^{(311)}e_{31} = c_{33}z_3 - (b_{11}y_1 + x_{11}x_1) \\ {}^{(433)}e_{36} = c_{44}z_4 - (b_{33}y_3 + x_{33}x_3) \end{pmatrix}. \quad (38)$$

In order to confirm the robustness and reliability of the investigated MSAC (13) and MSAUL (16), the nonlinear uncertainty and external disturbance are selected as follows:

$$\begin{aligned} \mathbf{d}(\mathbf{x}(t)) &= (-0.1\cos(t), -0.2\cos(2t), 0.3\sin(3t), 0.4\sin(4t))^T, \\ \mathbf{D}(\mathbf{y}(t)) &= (-0.1\sin(t), -0.2\sin(2t), 0.3\cos(3t), 0.4\cos(4t))^T, \\ \boldsymbol{\mu}(\mathbf{z}(t)) &= (0.1\cos(5t), 0.2\cos(6t), 0.3\sin(7t), 0.4\sin(8t))^T, \\ \Delta\mathbf{f}(\mathbf{x}(t)) &= (0.1\cos(2tx_1), 0.2\cos(2tx_2), 0.3\sin(2tx_3), 0.4\sin(2tx_4))^T, \\ \Delta\mathbf{g}(\mathbf{y}(t)) &= (0.1\sin(2\pi t), 0.2\sin(\sin(y_2)), 0.3\cos(2\pi y_3), 0.4\cos(4tx_8))^T, \\ \Delta\mathbf{h}(\mathbf{z}(t)) &= (0.1\sin(2t\sin(z_1)), 0.2\sin(2t\sin(z_2)), 0.3\sin(3t), 0.4\sin(4t))^T. \end{aligned} \quad (39)$$

6.1.1 Switch-1

It follows from switch-1 (37) that the FO error dynamic system is expressed as:

$$\begin{aligned} {}_0D_t^\alpha \{ {}^{(124)}e_3 \} &= c_{11} \{ {}_0D_t^\alpha z_1 \} - b_{22} \{ {}_0D_t^\alpha y_2 \} - a_{44} \{ {}_0D_t^\alpha x_4 \} \\ &= c_{11} \{ (z_2 - z_1)a_3 + z_4 + \Delta h_1 + \mu_1 + {}^{(124)}u_3 \} \\ &\quad - b_{22} \{ y_2c_2 + d_2y_1 - y_1y_3 + \Delta g_2 + D_2 \} \\ &\quad - a_{44} \{ x_4d_1 - x_2x_3 + \Delta f_4 + d_4 \}, \\ {}_0D_t^\alpha \{ {}^{(243)}e_{12} \} &= c_{22} \{ {}_0D_t^\alpha z_2 \} - b_{44} \{ {}_0D_t^\alpha y_4 \} - a_{33} \{ {}_0D_t^\alpha x_3 \} \\ &= c_{22} \{ z_2b_3 - z_1z_3 + \Delta h_2 + \mu_2 + {}^{(243)}u_{12} \} \\ &\quad - b_{44} \{ y_4r + y_2y_3 + \Delta g_4 + D_4 \} \\ &\quad - a_{33} \{ -x_3c_1 + x_1x_2 + \Delta f_3 + d_3 \}, \\ {}_0D_t^\alpha \{ {}^{(312)}e_{14} \} &= c_{33} \{ {}_0D_t^\alpha z_3 \} - b_{11} \{ {}_0D_t^\alpha y_1 \} - a_{22} \{ {}_0D_t^\alpha x_2 \} \\ &= c_{33} \{ -z_3c_3 + z_1z_2 + \Delta h_3 + \mu_3 + {}^{(312)}u_{14} \} \\ &\quad - b_{11} \{ (y_2 - y_1)a_2 + y_4 + \Delta g_1 + D_1 \} \\ &\quad - a_{22} \{ x_1b_1 - x_1x_3 - x_2 + \Delta f_2 + d_2 \}, \\ {}_0D_t^\alpha \{ {}^{(431)}e_{24} \} &= c_{44} \{ {}_0D_t^\alpha z_4 \} - b_{33} \{ {}_0D_t^\alpha y_3 \} - a_{11} \{ {}_0D_t^\alpha x_1 \} \\ &= c_{44} \{ z_4d_3 + z_1z_3 + \Delta h_4 + \mu_4 + {}^{(431)}u_{24} \} \\ &\quad - b_{33} \{ (-y_3b_2 + y_1y_2 + \Delta g_3 + D_3) \} \\ &\quad - a_{11} \{ (x_2 - x_1)a_1 + x_4 + \Delta f_1 + d_1 \}. \end{aligned}$$

It follows from the form of MSAC (13) and MSAUL (16) that the controller is designed as follows:

$$\begin{aligned} {}^{(124)}u_3 &= -z_4 - (z_2 - z_1)\hat{a}_3 + \frac{b_{22}}{c_{11}}(-y_1y_3 + y_2\hat{c}_2 + \hat{d}_2y_1) + \frac{a_{44}}{c_{11}}(-x_2x_3 + x_4\hat{d}_1) \\ &\quad - ({}^{(124)}\hat{r}_3 + {}^{(124)}\hat{q}_3)\sin({}^{(124)}s_3) - k_1\tanh({}^{(124)}s_3), \end{aligned}$$

$$\begin{aligned}
^{(243)}u_{12} &= z_1 z_3 - z_2 \hat{b}_3 + \frac{b_{44}}{c_{22}}(y_2 y_3 + y_4 \hat{r}) + \frac{a_{33}}{c_{22}}(x_2 x_1 - x_3 \hat{c}_1) \\
&\quad - (^{(243)}\hat{r}_{12} + ^{(243)}\hat{q}_{12}) \text{sign} (^{(243)}s_{12}) - k_1 \tanh (^{(243)}s_{12}), \\
^{(312)}u_{14} &= -z_1 z_2 + z_3 \hat{c}_3 + \frac{b_{11}}{c_{33}}(y_4 + (y_2 - y_1)\hat{a}_2) + \frac{a_{22}}{c_{33}}(x_1 x_3 + x_2 + x_1 \hat{b}_1) \\
&\quad - (^{(312)}\hat{r}_{14} + ^{(312)}\hat{q}_{14}) \text{sign} (^{(312)}s_{14}) - k_1 \tanh (^{(312)}s_{14}), \\
^{(431)}u_{24} &= -z_1 z_3 - z_4 \hat{d}_3 + \frac{b_{33}}{c_{44}}(y_1 y_2 - y_3 \hat{b}_2) + \frac{a_{11}}{c_{44}}(x_4 + (x_2 - x_1)\hat{a}_1) \\
&\quad - (^{(431)}\hat{r}_{24} + ^{(431)}\hat{q}_{24}) \text{sign} (^{(431)}s_{24}) - k_1 \tanh (^{(431)}s_{24}).
\end{aligned}$$

and the parameters updating laws are designed as follows:

$$\begin{aligned}
\hat{a}_1 &= -(x_2 - x_1)\{^{(431)}s_{24}\} - k_2 \text{sign}(\tilde{a}_1)|\tilde{a}_1|^\omega, & \hat{b}_1 &= -x_1\{^{(312)}s_{14}\} - k_2 \text{sign}(\tilde{b}_1)|\tilde{b}_1|^\omega, \\
\hat{c}_1 &= x_3\{^{(243)}s_{12}\} - k_2 \text{sign}(\tilde{c}_1)|\tilde{c}_1|^\omega, & \hat{d}_1 &= -x_4\{^{(124)}s_3\} - k_2 \text{sign}(\tilde{d}_1)|\tilde{d}_1|^\omega, \\
\hat{a}_2 &= -(y_2 - y_1)\{^{(312)}s_{14}\} - k_3 \text{sign}(\tilde{a}_2)|\tilde{a}_2|^\omega, & \hat{b}_2 &= y_3\{^{(431)}s_{24}\} - k_3 \text{sign}(\tilde{b}_2)|\tilde{b}_2|^\omega, \\
\hat{c}_2 &= -y_2\{^{(124)}s_3\} - k_3 \text{sign}(\tilde{c}_2)|\tilde{c}_2|^\omega, & \hat{d}_2 &= -y_1\{^{(124)}s_3\} - k_3 \text{sign}(\tilde{d}_2)|\tilde{d}_2|^\omega, \\
\hat{r} &= -y_4\{^{(243)}s_{12}\} - k_3 \text{sign}(\tilde{r})|\tilde{r}|^\omega, & \hat{a}_3 &= (z_2 - z_1)\{^{(124)}s_3\} - k_4 \text{sign}(\tilde{a}_3)|\tilde{a}_3|^\omega, \\
\hat{b}_3 &= z_2\{^{(243)}s_{12}\} - k_4 \text{sign}(\tilde{b}_3)|\tilde{b}_3|^\omega, & \hat{c}_3 &= -z_3\{^{(312)}s_{14}\} - k_4 \text{sign}(\tilde{c}_3)|\tilde{c}_3|^\omega, \\
\hat{d}_3 &= z_4\{^{(431)}s_{24}\} - k_4 \text{sign}(\tilde{d}_3)|\tilde{d}_3|^\omega, & ^{(124)}\hat{r}_3 &= m_1\{^{(124)}\lambda_3\}|^{(124)}s_3|, \\
^{(243)}\hat{r}_{12} &= m_1\{^{(243)}\lambda_{12}\}|^{(243)}s_{12}|, & ^{(312)}\hat{r}_{14} &= m_1\{^{(312)}\lambda_{14}\}|^{(312)}s_{14}|, \\
^{(431)}\hat{r}_{24} &= m_1\{^{(431)}\lambda_{24}\}|^{(431)}s_{24}|, & ^{(124)}\hat{q}_3 &= m_1\{^{(124)}\lambda_3\}|^{(124)}s_3|, \\
^{(243)}\hat{q}_{12} &= m_2\{^{(243)}\lambda_{12}\}|^{(243)}s_{12}|, & ^{(312)}\hat{q}_{14} &= m_2\{^{(312)}\lambda_{14}\}|^{(312)}s_{14}|, \\
^{(431)}\hat{q}_{24} &= m_2\{^{(431)}\lambda_{24}\}|^{(431)}s_{24}|.
\end{aligned}$$

6.1.2 Switch-2

It follows from switch-2 (38) that the FO error dynamic system is expressed as:

$$\begin{aligned}
{}_0D_t^\alpha\{^{(122)}e_{25}\} &= c_{11}\{{}_0D_t^\alpha z_1\} - b_{22}\{{}_0D_t^\alpha y_2\} - a_{220}\{D_t^\alpha x_2\} \\
&= c_{11}\{(z_2 - z_1)a_3 + z_4 + \Delta h_1 + \mu_1 + ^{(122)}u_{25}\} \\
&\quad - b_{22}\{y_2 c_2 + d_2 y_1 - y_1 y_3 + \Delta g_2 + D_2\} \\
&\quad - a_{22}\{x_1 b_1 - x_1 x_3 - x_2 + \Delta f_2 + d_2\} \\
{}_0D_t^\alpha\{^{(244)}e_{30}\} &= c_{22}\{{}_0D_t^\alpha z_2\} - b_{44}\{{}_0D_t^\alpha y_4\} - a_{44}\{{}_0D_t^\alpha x_4\} \\
&= c_{22}\{z_2 b_3 - z_1 z_3 + \Delta h_2 + \mu_2 + ^{(244)}u_{30}\} \\
&\quad - b_{44}\{y_4 r + y_2 y_3 + \Delta g_4 + D_4\} \\
&\quad - a_{44}\{x_4 d_1 - x_2 x_3 + \Delta f_4 + d_4\} \\
{}_0D_t^\alpha\{^{(311)}e_{31}\} &= c_{33}\{{}_0D_t^\alpha z_3\} - b_{11}\{{}_0D_t^\alpha y_1\} - a_{11}\{{}_0D_t^\alpha x_1\} \\
&= c_{33}\{-z_3 c_3 + z_1 z_2 + \Delta h_3 + \mu_3 + ^{(311)}u_{31}\} \\
&\quad - b_{11}\{(y_2 - y_1)a_2 + y_4 + \Delta g_1 + D_1\} \\
&\quad - a_{11}\{(x_2 - x_1)a_1 + x_4 + \Delta f_1 + d_1\} \\
{}_0D_t^\alpha\{^{(433)}e_{36}\} &= c_{44}\{{}_0D_t^\alpha z_4\} - b_{33}\{{}_0D_t^\alpha y_3\} - a_{33}\{{}_0D_t^\alpha x_3\} \\
&= c_{44}\{z_4 d_3 + z_1 z_3 + \Delta h_4 + \mu_4 + ^{(433)}u_{36}\}
\end{aligned}$$

$$\begin{aligned} & -b_{33}\{-y_3b_2 + y_1y_2 + \Delta g_3 + D_3\} \\ & -a_{33}\{-x_3c_1 + x_1x_2 + \Delta f_3 + d_3\} \end{aligned}$$

It follows from the form of MSAC (13) and MSAUL (16) that the controller is designed as follows:

$$\begin{aligned} {}^{(122)}u_{25} &= -z_4 - (z_2 - z_1)\hat{a}_3 + \frac{b_{22}}{c_{11}}(y_1y_3 + y_2\hat{c}_2 + \hat{d}_2y_1) + \frac{a_{22}}{c_{11}}(-x_1x_3 - x_2 + x_1\hat{b}_1) \\ & \quad - ({}^{(122)}\hat{r}_{25} + {}^{(122)}\hat{q}_{25})\text{sign}({}^{(122)}s_{25}) - k_1\tanh({}^{(122)}s_{25}), \\ {}^{(244)}u_{30} &= z_1z_3 - z_2\hat{b}_3 + \frac{b_{44}}{c_{22}}(y_2y_3 + y_4\hat{r}) + \frac{a_{44}}{c_{22}}(-x_2x_3 + x_4\hat{d}_1) \\ & \quad - ({}^{(244)}\hat{r}_{30} + {}^{(244)}\hat{q}_{30})\text{sign}({}^{(244)}s_{30}) - k_1\tanh({}^{(244)}s_{30}), \\ {}^{(311)}u_{31} &= -z_1z_2 + z_3\hat{c}_3 + \frac{b_{11}}{c_{33}}(y_4 + (y_2 - y_1)\hat{a}_2) + \frac{a_{11}}{c_{33}}(x_4 + (x_2 - x_1)\hat{a}_1) \\ & \quad - ({}^{(311)}\hat{r}_{31} + {}^{(311)}\hat{q}_{31})\text{sign}({}^{(311)}s_{31}) - k_1\tanh({}^{(311)}s_{31}), \\ {}^{(433)}u_{36} &= -z_1z_3 - z_4\hat{d}_3 + \frac{b_{33}}{c_{44}}(y_1y_2 - y_3\hat{b}_2) + \frac{a_{33}}{c_{44}}(x_1x_2 - x_3\hat{c}_1) \\ & \quad - ({}^{(433)}\hat{r}_{36} + {}^{(433)}\hat{q}_{36})\text{sign}({}^{(433)}s_{36}) - k_1\tanh({}^{(433)}s_{36}). \end{aligned}$$

and the parameters updating laws are designed as follows:

$$\begin{aligned} \hat{a}_1 &= -(x_2 - x_1)({}^{(311)}s_{31}) - k_2\text{sign}(\tilde{a}_1)|\tilde{a}_1|^\omega, & \hat{b}_1 &= -x_1({}^{(122)}s_{25}) - k_2\text{sign}(\tilde{b}_1)|\tilde{b}_1|^\omega, \\ \hat{c}_1 &= x_3({}^{(433)}s_{36}) - k_2\text{sign}(\tilde{c}_1)|\tilde{c}_1|^\omega, & \hat{d}_1 &= -x_4({}^{(244)}s_{30}) - k_2\text{sign}(\tilde{d}_1)|\tilde{d}_1|^\omega, \\ \hat{a}_2 &= -(y_2 - y_1)({}^{(311)}s_{31}) - k_3\text{sign}(\tilde{a}_2)|\tilde{a}_2|^\omega, & b_2 &= y_3({}^{(433)}s_{36}) - k_3\text{sign}(\tilde{b}_2)|\tilde{b}_2|^\omega, \\ \hat{c}_2 &= -y_2({}^{(122)}s_{25}) - k_3\text{sign}(\tilde{c}_2)|\tilde{c}_2|^\omega, & \hat{d}_2 &= -y_1({}^{(122)}s_{25}) - k_3\text{sign}(\tilde{d}_2)|\tilde{d}_2|^\omega, \\ \hat{r} &= -y_4({}^{(244)}s_{30}) - k_3\text{sign}(\tilde{r})|\tilde{r}|^\omega, & \hat{a}_3 &= (z_2 - z_1)({}^{(122)}s_{25}) - k_4\text{sign}(\tilde{a}_3)|\tilde{a}_3|^\omega, \\ \hat{b}_3 &= z_2({}^{(244)}s_{30}) - k_4\text{sign}(\tilde{b}_3)|\tilde{b}_3|^\omega, & \hat{c}_3 &= -z_3({}^{(311)}s_{31}) - k_4\text{sign}(\tilde{c}_3)|\tilde{c}_3|^\omega, \\ \hat{d}_3 &= z_4({}^{(433)}s_{36}) - k_4\text{sign}(\tilde{d}_3)|\tilde{d}_3|^\omega, & {}^{(122)}\hat{r}_{25} &= m_1({}^{(122)}\lambda_{25})|{}^{(122)}s_{25}|, \\ {}^{(244)}\hat{r}_{30} &= m_1\{{}^{(244)}\lambda_{30}\}|{}^{(244)}s_{30}|, & {}^{(311)}\hat{r}_{31} &= m_1\{{}^{(311)}\lambda_{31}\}|{}^{(311)}s_{31}|, \\ {}^{(433)}\hat{r}_{36} &= m_1\{{}^{(433)}\lambda_{36}\}|{}^{(433)}s_{36}|, & {}^{(122)}\hat{q}_{25} &= m_2({}^{(122)}\lambda_{25})|{}^{(122)}s_{25}|, \\ {}^{(244)}\hat{q}_{30} &= m_2\{{}^{(244)}\lambda_{30}\}|{}^{(244)}s_{30}|, & {}^{(311)}\hat{q}_{31} &= m_2\{{}^{(311)}\lambda_{31}\}|{}^{(311)}s_{31}|, \\ {}^{(433)}\hat{q}_{36} &= m_2\{{}^{(433)}\lambda_{36}\}|{}^{(433)}s_{36}|. \end{aligned}$$

In this two numerical simulations, we adopt the matrices A, B, C as identity matrices. The initial values of the drive systems (34), (35) and response system (36) are selected as $x(0) = (2, -2, 1, 1)$, $y(0) = (1, 1, 2, 2)$, $z(0) = (-20, 3, 1, 3)$. The initial conditions of the unknown parameter in D-R system and upper bound of SD are selected as $(a_1(0), b_1(0), c_1(0), d_1(0)) = (2, 1, -2, 1)$, $(a_2(0), b_2(0), c_2(0), d_2(0), r(0)) = (1, -3, 1, 1, -5)$, $(a_3(0), b_3(0), c_3(0), d_3(0)) = (1, 1, -2, -2)$, $(r_1(0), r_2(0), r_3(0), r_4(0)) = (-2.5, 1, -3, 1)$, $(q_1(0), q_2(0), q_3(0), q_4(0)) = (1, -2.5, -5, 1)$. The constants ${}^{(ijk)}\lambda_p = 2$, $m_1 = 10$, $m_2 = 10$, $k_1 = 10$, $k_2 = 10$, $k_3 = 10$, $k_4 = 10$, $\omega = 0.5$. The trajectories about the error variables $\mathbf{e} = ({}^{(124)}e_3, {}^{(243)}e_{12}, {}^{(312)}e_{14}, {}^{(431)}e_{24})^T$ and $\mathbf{e} = ({}^{(122)}e_{25}, {}^{(244)}e_{30}, {}^{(311)}e_{31}, {}^{(433)}e_{36})^T$ are drawn in figures 2 and 6. The synchronization for the state trajectories of drive systems (34), (35) and response system (36) are drawn in figures 3 and 7. The trajectories of estimations $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{v}}$ are drawn in figures 4 and 8, ${}^{(ijk)}r_p$, and ${}^{(ijk)}q_p$ are drawn in figures 5 and 9. Motivated by the numerical simulation

Fig.2. The synchronization errors $^{(124)}e_3, ^{(243)}e_{12}, ^{(312)}e_{14}, ^{(431)}e_{24}$ change with time t .

results, it can reveal that the drive systems (4), (5) and response system (6) achieve MSSMCS. Therefore, the multi-switching adaptive controllers (MSAC) (13) and some suitable multi-switching adaptive updating laws (MSAUL) (16) are effective.

6.2 Numerical simulations for FO chaotic system with different dimensions

As an example we choose FO Lorenz, Chen chaotic system as the drive system, and the FO hyper-chaotic Lü system as the response system. Adding SD to the D-R systems, we obtain

$$\begin{pmatrix} {}_0D_t^\alpha x_1 \\ {}_0D_t^\alpha x_2 \\ {}_0D_t^\alpha x_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} 0 + \Delta f_1 \\ -x_1 x_3 - x_2 + \Delta f_2 \\ x_1 x_2 + \Delta f_3 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, \quad (40)$$

$$\begin{pmatrix} {}_0D_t^\alpha y_1 \\ {}_0D_t^\alpha y_2 \\ {}_0D_t^\alpha y_3 \end{pmatrix} = \begin{pmatrix} y_2 - y_1 & 0 & 0 \\ -y_1 & 0 & y_1 + y_2 \\ 0 & -y_3 & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 + \Delta g_1 \\ -y_1 y_3 + \Delta g_2 \\ y_1 y_2 + \Delta g_3 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}, \quad (41)$$

$$\begin{pmatrix} {}_0D_t^\alpha z_1 \\ {}_0D_t^\alpha z_2 \\ {}_0D_t^\alpha z_3 \\ {}_0D_t^\alpha z_4 \end{pmatrix} = \begin{pmatrix} z_2 - z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & -z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{pmatrix} \begin{pmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{pmatrix} + \begin{pmatrix} z_4 + \Delta h_1 \\ -z_1 z_3 + \Delta h_2 \\ z_1 z_2 + \Delta h_3 \\ z_1 z_3 + \Delta h_4 \end{pmatrix} + \begin{pmatrix} \mu_1 + u_1 \\ \mu_2 + u_2 \\ \mu_3 + u_3 \\ \mu_4 + u_4 \end{pmatrix}, \quad (42)$$

Choosing the parameters are $a_1 = 10, b_1 = 28, c_1 = 8/3, a_2 = 35, b_2 = 3, c_2 = 28, a_3 = 36, b_3 = 20, c_3 = 3, d_3 = 0.5$. The initial conditions take as $x(0) = (2, -2, 1), y(0) =$

Fig.3. The synchronization for state variable z_1 and $x_4 + y_2$, z_2 and $x_3 + y_4$, z_3 and $x_2 + y_1$, z_4 and $x_1 + y_3$ of drive systems (34), (35) and response system (36) indicating in sub-pictures (a)-(d) respectively.

$(1, -1, 3), z(0) = (-2, 3, 1, 3)$. When $\Delta g = 0$, $\Delta f = 0$, $\Delta h = 0$, $d(t) = 0$, $D(t) = 0$, $\mu(t) = 0$ and $\alpha = 0.97$, we can obtain the attractor graphs of the FO Lorenz, Chen chaotic system which are presented in figure 10. The attractor graphs of hyper-chaotic Lü system can be seen figure 1. Assume

$$A = B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (43)$$

The following multi-switching error state modes between the drive systems (40), (41) and the response system (42) are listed.

(a)

(b)

(c)

Fig.4. The estimation of parameters $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1$ of drive system (34) (a), $\hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2, \hat{r}$ of drive system (35) (b), $\hat{a}_3, \hat{b}_3, \hat{c}_3, \hat{d}_3$ of response system (36) (c).

(a)

(b)

Fig.5. The estimation of parameters $^{(124)}\hat{r}_3, ^{(243)}\hat{r}_{12}, ^{(312)}\hat{r}_{14}, ^{(431)}\hat{r}_{24}$ and $^{(124)}\hat{q}_3, ^{(243)}\hat{q}_{12}, ^{(312)}\hat{q}_{14}, ^{(431)}\hat{q}_{24}$ shown in sub-pictures (a) and (b).

Fig.6. The synchronization errors $(^{122})e_{25}, (^{244})e_{30}, (^{311})e_{31}, (^{433})e_{36}$ change with time t .

$$l \neq w \neq v \left\{ \begin{array}{llllll} (^{123})e_1, & (^{132})e_2, & (^{124})e_3, & (^{142})e_4, & (^{134})e_5, & (^{143})e_6, \\ (^{213})e_7, & (^{231})e_8, & (^{214})e_9, & (^{241})e_{10}, & (^{234})e_{11}, & (^{243})e_{12}, \\ (^{321})e_{13}, & (^{312})e_{14}, & (^{324})e_{15}, & (^{314})e_{16}, & (^{341})e_{17}, & (^{342})e_{18}, \\ (^{412})e_{19}, & (^{421})e_{20}, & (^{413})e_{21}, & (^{423})e_{22}, & (^{432})e_{23}, & (^{431})e_{24}. \end{array} \right.$$

$$l \neq w = v \left\{ \begin{array}{llllll} (^{122})e_{25}, & (^{133})e_{26}, & (^{144})e_{27}, & (^{211})e_{28}, & (^{233})e_{29}, & (^{244})e_{30}, \\ (^{311})e_{31}, & (^{322})e_{32}, & (^{344})e_{33}, & (^{411})e_{34}, & (^{422})e_{35}, & (^{433})e_{36}. \end{array} \right.$$

$$l = w \neq v \left\{ \begin{array}{llllll} (^{112})e_{37}, & (^{113})e_{38}, & (^{114})e_{39}, & (^{221})e_{40}, & (^{223})e_{41}, & (^{224})e_{42}, \\ (^{331})e_{43}, & (^{332})e_{44}, & (^{334})e_{45}, & (^{441})e_{46}, & (^{442})e_{47}, & (^{443})e_{48}. \end{array} \right.$$

$$l = v \neq w \left\{ \begin{array}{llllll} (^{121})e_{49}, & (^{131})e_{50}, & (^{141})e_{51}, & (^{212})e_{52}, & (^{232})e_{53}, & (^{242})e_{54}, \\ (^{313})e_{55}, & (^{323})e_{56}, & (^{343})e_{57}, & (^{414})e_{58}, & (^{424})e_{59}, & (^{434})e_{60}. \end{array} \right.$$

$$l = w = v \left\{ \begin{array}{ll} (^{111})e_{61}, & (^{222})e_{62}, \\ (^{333})e_{63}, & (^{444})e_{64}. \end{array} \right.$$

In this Section, we will select the appropriate switching error variable for the situation of $l \neq w \neq v$, namely, $\mathbf{e} = (^{142})e_4, (^{234})e_{12}, (^{321})e_{13}, (^{413})e_{21})^T$, $l \neq w = v$, namely, $\mathbf{e} = (^{122})e_{25}, (^{244})e_{30}, (^{311})e_{31}, (^{433})e_{36})^T$ and obtain the following two switching modes for numerical simulation:

(a)

(b)

(c)

(d)

Fig.7. The synchronization for state variable z_1 and $x_2 + y_2$, z_2 and $x_4 + y_4$, z_3 and $x_1 + y_1$, z_4 and $x_3 + y_3$ of drive systems (34), (35) and response system (36) indicating in sub-pictures (a)-(d) respectively.

$$\text{Switch} - 1: \left\{ \begin{array}{l} {}^{(142)}e_4 = \left[\sum_{i=1}^4 (c_{1i} z_i) - \left\{ \sum_{j=1}^3 (b_{4j} y_j) + \sum_{k=1}^3 (x_{2k} x_k) \right\} \right], \\ {}^{(234)}e_{12} = \left[\sum_{i=1}^4 (c_{2i} z_i) - \left\{ \sum_{j=1}^3 (b_{3j} y_j) + \sum_{k=1}^3 (x_{4k} x_k) \right\} \right], \\ {}^{(321)}e_{13} = \left[\sum_{i=1}^4 (c_{3i} z_i) - \left\{ \sum_{j=1}^3 (b_{2j} y_j) + \sum_{k=1}^3 (x_{1k} x_k) \right\} \right], \\ {}^{(413)}e_{21} = \left[\sum_{i=1}^4 (c_{4i} z_i) - \left\{ \sum_{j=1}^3 (b_{1j} y_j) + \sum_{k=1}^3 (x_{3k} x_k) \right\} \right], \end{array} \right. \quad (44)$$

Fig.8. The estimation of parameters $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1$ of drive system (34) (a), $\hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2, \hat{r}$ of drive system (35) (b), $\hat{a}_3, \hat{b}_3, \hat{c}_3, \hat{d}_3$ of response system (36) (c).

$$\text{Switch-2 : } \left\{ \begin{array}{l}
 {}^{(122)}e_{25} = \left[\sum_{i=1}^4 (c_{1i}z_i) - \left\{ \sum_{j=1}^3 (b_{2j}y_j) + \sum_{k=1}^3 (x_{2k}x_k) \right\} \right], \\
 {}^{(244)}e_{30} = \left[\sum_{i=1}^4 (c_{2i}z_i) - \left\{ \sum_{j=1}^3 (b_{4j}y_j) + \sum_{k=1}^3 (x_{4k}x_k) \right\} \right], \\
 {}^{(311)}e_{31} = \left[\sum_{i=1}^4 (c_{3i}z_i) - \left\{ \sum_{j=1}^3 (b_{1j}y_j) + \sum_{k=1}^3 (x_{1k}x_k) \right\} \right], \\
 {}^{(433)}e_{36} = \left[\sum_{i=1}^4 (c_{4i}z_i) - \left\{ \sum_{j=1}^3 (b_{3j}y_j) + \sum_{k=1}^3 (x_{3k}x_k) \right\} \right],
 \end{array} \right. \quad (45)$$

In order to exhibit the effectiveness of the designed MSAC (25) and the MSAUL (28),

(a)

(b)

Fig.9. The estimation of parameters $^{(122)}\hat{r}_{25}$, $^{(244)}\hat{r}_{30}$, $^{(311)}\hat{r}_{31}$, $^{(433)}\hat{r}_{36}$ and $^{(122)}\hat{q}_{25}$, $^{(244)}\hat{q}_{30}$, $^{(311)}\hat{q}_{31}$, $^{(433)}\hat{q}_{36}$ shown in sub-pictures (a) and (b).

(a)

(b)

Fig.10. The 3D phase plot of FO Lorenz chaotic system (a), FO Chen chaotic system (b).

the nonlinear uncertainty and external disturbance are assumed as follows:

$$\begin{aligned}
 \mathbf{d}(\mathbf{x}(t)) &= (-0.1\cos(t), -0.2\cos(2t), 0.3\sin(3t))^T, \\
 \mathbf{D}(\mathbf{y}(t)) &= (-0.1\sin(t), -0.2\sin(2t), 0.3\cos(3t))^T, \\
 \boldsymbol{\mu}(\mathbf{z}(t)) &= (0.1\cos(5t), 0.2\cos(6t), 0.3\sin(7t), 0.4\sin(8t))^T, \\
 \Delta \mathbf{f}(\mathbf{x}(t)) &= (0.1\cos(2tx_1), 0.2\cos(2tx_2), 0.3\sin(2tx_3))^T, \\
 \Delta \mathbf{g}(\mathbf{y}(t)) &= (0.1\sin(2\pi t), 0.2\sin(\text{sign}(y_2)), 0.3\cos(2\pi y_3))^T, \\
 \Delta \mathbf{h}(\mathbf{z}(t)) &= (0.1\sin(2t\text{sign}(z_1)), 0.2\sin(2t\text{sign}(z_2)), 0.3\sin(3t), 0.4\sin(4t))^T.
 \end{aligned}$$

Fig.11. The synchronization errors $^{(142)}e_4, ^{(234)}e_{12}, ^{(321)}e_{13}, ^{(413)}e_{21}$ change with time t .

6.2.1 Switch-1

It follows from switch-1 (44) that the FO error dynamic system is expressed as:

$$\begin{aligned}
{}_0D_t^\alpha \{^{(142)}e_4\} &= \left[\sum_{i=1}^4 c_{1i}({}_0D_t^\alpha z_i) - \left\{ \sum_{j=1}^3 b_{4j}({}_0D_t^\alpha y_j) + \sum_{k=1}^3 x_{2k}({}_0D_t^\alpha x_k) \right\} \right] \\
&= \{ (z_2 - z_1)a_3 + z_4 + \Delta h_1 + \mu_1 + ^{(142)}u_4 \} \\
&\quad - \{ (y_2 - y_1)a_2 - y_3b_2 + y_1y_2 + \Delta g_1 + \Delta g_3 + D_1 + D_3 \} \\
&\quad - \{ 2(x_2 - x_1)a_1 + x_1b_1 - x_1x_3 - x_2 + 2\Delta f_1 + \Delta f_2 + 2d_1 + d_2 \}, \\
{}_0D_t^\alpha \{^{(234)}e_{12}\} &= \left[\sum_{i=1}^4 c_{2i}({}_0D_t^\alpha z_i) - \left\{ \sum_{j=1}^3 b_{3j}({}_0D_t^\alpha y_j) + \sum_{k=1}^3 x_{4k}({}_0D_t^\alpha x_k) \right\} \right] \\
&= \{ z_2b_3 - z_1z_3 + \Delta h_2 + \mu_2 + ^{(234)}u_{12} \} - \{ -y_1a_2 - 2y_3b_2 \\
&\quad + (y_2 + y_1)c_2 - y_1y_3 + 2y_1y_2 + \Delta g_2 + 2\Delta g_3 + D_2 + 2D_3 \} \\
&\quad - \{ (x_2 - x_1)a_1 - x_3c_1 + x_1x_2 + \Delta f_1 + \Delta f_3 + d_1 + d_3 \}, \\
{}_0D_t^\alpha \{^{(321)}e_{13}\} &= \left[\sum_{i=1}^4 c_{3i}({}_0D_t^\alpha z_i) - \left\{ \sum_{j=1}^3 b_{2j}({}_0D_t^\alpha y_j) + \sum_{k=1}^3 x_{1k}({}_0D_t^\alpha x_k) \right\} \right] \\
&= \{ -z_3c_3 + z_1z_2 + \Delta h_3 + \mu_3 + ^{(321)}u_{13} \} - \{ 2(y_2 - 2y_1)a_2 \\
&\quad + (y_1 + y_2)c_2 - y_1a_2 - y_1y_3 + 2\Delta g_1 + \Delta g_2 + 2D_1 + D_2 \} \\
&\quad - \{ (x_2 - x_1)a_1 - x_3c_1 - x_1x_2 + \Delta f_1 + \Delta f_3 + d_1 + d_3 \}, \\
{}_0D_t^\alpha \{^{(413)}e_{21}\} &= \left[\sum_{i=1}^4 c_{4i}({}_0D_t^\alpha z_i) - \left\{ \sum_{j=1}^3 b_{1j}({}_0D_t^\alpha y_j) + \sum_{k=1}^3 x_{3k}({}_0D_t^\alpha x_k) \right\} \right]
\end{aligned}$$

Fig.12. The synchronization for state variable z_1 and $2x_1 + x_2 + y_1 + y_3$, z_2 and $x_1 + x_3 + y_2 + 2y_3$, z_3 and $x_1 + x_3 + 2y_1 + y_2$, z_4 and $x_2 + 2x_3 + y_1 + y_3$ of drive systems (40), (41) and response system (42) indicating in sub-pictures (a)-(d) respectively.

$$\begin{aligned}
&= \{z_4 d_3 + z_1 z_3 + \Delta h_4 + \mu_4 + {}^{(413)}u_{21}\} \\
&- \{(y_2 - y_1)a_2 - y_3 b_2 + y_1 y_2 + \Delta g_1 + \Delta g_3 + D_1 + D_3\} \\
&- \{x_1 b_1 - 2x_3 c_1 - x_1 x_3 - x_2 + 2x_1 x_2 + \Delta f_2 + 2\Delta f_3 + d_2 + 2d_3\}.
\end{aligned}$$

It follows from the form of MSAC (25) and MSAUL (28) that the controller is designed as follows:

$$\begin{aligned}
{}^{(142)}u_4 &= -z_4 + y_1 y_2 - x_1 x_3 - x_2 - \hat{a}_3(z_2 - z_1) + \hat{a}_2(y_2 - y_1) - \hat{b}_1 y_3 \\
&\quad + 2\hat{a}_1(x_2 - x_1) + \hat{b}_1 x_1 - ({}^{(142)}\hat{\rho}_4 + {}^{(142)}\hat{\varrho}_4) \text{sign}({}^{(142)}s_4) - k_1 \tanh({}^{(142)}s_4), \\
{}^{(234)}u_{12} &= z_1 z_3 - y_1 y_3 + 2y_1 y_2 + x_1 x_2 - \hat{b}_3 z_2 - \hat{a}_2 y_1 - 2\hat{b}_2 y_3 + \hat{c}_2(y_1 + y_2) \\
&\quad + \hat{a}_1(x_2 - x_1) - \hat{c}_1 x_3 - ({}^{(234)}\hat{\rho}_{12} + {}^{(234)}\hat{\varrho}_{12}) \text{sign}({}^{(234)}s_{12}) - k_1 \tanh({}^{(234)}s_{12}), \\
{}^{(321)}u_{13} &= -z_1 z_2 - y_1 y_3 + x_1 x_2 + \hat{c}_3 z_3 - \hat{a}_2 y_3 + 2\hat{a}_2(y_2 - 2y_1) + \hat{c}_2(y_1 + y_2) \\
&\quad + \hat{a}_1(x_2 - x_1) - \hat{c}_1 x_3 - ({}^{(321)}\hat{\rho}_{13} + {}^{(321)}\hat{\varrho}_{13}) \text{sign}({}^{(321)}s_{13}) - k_1 \tanh({}^{(321)}s_{13}), \\
{}^{(413)}u_{21} &= -z_1 z_3 + y_1 y_2 - x_1 x_3 - x_2 + 2x_1 x_2 - d_3 z_4 + a_2(y_2 - y_1) - b_2 y_3
\end{aligned}$$

$$+b_1x_1 - 2c_1x_3 - \binom{(413)}{\hat{\rho}_{21}} + \binom{(413)}{\hat{\rho}_{21}} \text{sign}(\binom{(413)}{s_{21}}) - k_1 \tanh(\binom{(413)}{s_{21}}),$$

and the parameters updating laws are designed as follows:

$$\begin{aligned}\hat{b}_1 &= -x_1\{(\binom{(142)}{s_4}) + (\binom{(413)}{s_{21}})\} - k_2 \text{sign}(\tilde{b}_1)(\text{abs}(\tilde{b}_1)^\omega), \\ \hat{c}_1 &= x_3\{(\binom{(234)}{s_{12}}) + (\binom{(321)}{s_{13}}) + 2(\binom{(142)}{s_4})\} - k_2 \text{sign}(\tilde{c}_1)(\text{abs}(\tilde{c}_1)^\omega), \\ \hat{b}_2 &= y_3\{(\binom{(142)}{s_4}) + 2(\binom{(234)}{s_{12}}) + (\binom{(413)}{s_{21}})\} - k_3 \text{sign}(\tilde{b}_2)(\text{abs}(\tilde{b}_2)^\omega), \\ \hat{c}_2 &= -(y_1 + y_2)\{(\binom{(234)}{s_{12}}) + (\binom{(321)}{s_{13}})\} - k_3 \text{sign}(\tilde{c}_2)(\text{abs}(\tilde{c}_2)^\omega), \\ \hat{a}_3 &= (z_2 - z_1)(\binom{(142)}{s_4}) - k_4 \text{sign}(\tilde{a}_3)(\text{abs}(\tilde{a}_3)^\omega), \\ \hat{b}_3 &= z_2(\binom{(234)}{s_{12}}) - k_4 \text{sign}(\tilde{b}_3)(\text{abs}(\tilde{b}_3)^\omega), \\ \hat{c}_3 &= -z_3(\binom{(321)}{s_{13}}) - k_4 \text{sign}(\tilde{c}_3)(\text{abs}(\tilde{c}_3)^\omega), \\ \hat{d}_3 &= z_4(\binom{(413)}{s_{21}}) - k_4 \text{sign}(\tilde{d}_3)(\text{abs}(\tilde{d}_3)^\omega), \\ \hat{a}_1 &= -(x_2 - x_1)\{(\binom{(321)}{s_{13}}) + (\binom{(234)}{s_{12}}) + 2(\binom{(413)}{s_{21}})\} - k_2 \text{sign}(\tilde{a}_1)(\text{abs}(\tilde{a}_1)^\omega), \\ \hat{a}_2 &= -\{(y_2 - y_1)((\binom{(142)}{s_4}) + (\binom{(413)}{s_{21}}) + 2(\binom{(321)}{s_{13}})) - y_3((\binom{(234)}{s_{12}}) + (\binom{(321)}{s_{13}}))\} \\ &\quad - k_1 \text{sign}(\tilde{a}_2)(\text{abs}(\tilde{a}_2)^\omega),\end{aligned}$$

$$\begin{aligned}\binom{(142)}{\hat{\rho}_4} &= m_1(\binom{(142)}{\lambda_4})|\binom{(142)}{s_4}|, & \binom{(234)}{\hat{\rho}_{12}} &= m_1\{(\binom{(234)}{\lambda_{12}})|\binom{(234)}{s_{12}}|, \\ \binom{(321)}{\hat{\rho}_{13}} &= m_1\{(\binom{(321)}{\lambda_{13}})|\binom{(321)}{s_{13}}|, & \binom{(413)}{\hat{\rho}_{21}} &= m_1\{(\binom{(413)}{\lambda_{21}})|\binom{(413)}{s_{21}}|, \\ \binom{(142)}{\hat{\rho}_4} &= m_2(\binom{(142)}{\lambda_4})|\binom{(142)}{s_4}|, & \binom{(234)}{\hat{\rho}_{12}} &= m_2(\binom{(234)}{\lambda_{12}})|\binom{(234)}{s_{12}}|, \\ \binom{(321)}{\hat{\rho}_{13}} &= m_2(\binom{(321)}{\lambda_{13}})|\binom{(321)}{s_{13}}|, & \binom{(413)}{\hat{\rho}_{21}} &= m_2(\binom{(413)}{\lambda_{21}})|\binom{(413)}{s_{21}}|.\end{aligned}$$

6.2.2 Switch-2

It follows from switch-2 (45) that the FO error dynamic system is expressed as:

$$\begin{aligned}{}_0D_t^\alpha\{^{(122)}e_{25}\} &= [\sum_{i=1}^4 c_{1i}({}_0D_t^\alpha z_i) - \{\sum_{j=1}^3 b_{2j}({}_0D_t^\alpha y_j) + \sum_{k=1}^3 x_{2k}({}_0D_t^\alpha x_k)\}] \\ &= \{(z_2 - z_1)a_3 + z_4 + \Delta h_1 + \mu_1 + ^{(122)}u_{25}\} - \{2(y_2 - y_1)a_2 \\ &\quad - y_1a_2 + (y_1 + y_2)c_2 - y_1y_3 + 2\Delta g_1 + \Delta g_2 + 2D_1 + D_2\} \\ &\quad - \{2(x_2 - x_1)a_1 + x_1b_1 - x_1x_3 - x_2 + 2\Delta f_1 + \Delta f_2 + 2d_1 + d_2\}, \\ {}_0D_t^\alpha\{^{(244)}e_{30}\} &= [\sum_{i=1}^4 c_{2i}({}_0D_t^\alpha z_i) - \{\sum_{j=1}^3 b_{4j}({}_0D_t^\alpha y_j) + \sum_{k=1}^3 x_{4k}({}_0D_t^\alpha x_k)\}] \\ &= \{z_2b_3 - z_1z_3 + \Delta h_2 + \mu_2 + ^{(244)}u_{30}\} \\ &\quad - \{(y_2 - y_1)a_2 - y_3b_2 + y_1y_2 + \Delta g_1 + \Delta g_3 + D_1 + D_3\} \\ &\quad - \{(x_2 - x_1)a_1 - x_3c_1 + x_1x_2 + \Delta f_1 + \Delta f_3 + d_1 + d_3\}, \\ {}_0D_t^\alpha\{^{(311)}e_{31}\} &= [\sum_{i=1}^4 c_{3i}({}_0D_t^\alpha z_i) - \{\sum_{j=1}^3 b_{1j}({}_0D_t^\alpha y_j) + \sum_{k=1}^3 x_{1k}({}_0D_t^\alpha x_k)\}] \\ &= \{-z_3c_3 + z_1z_2 + \Delta h_3 + \mu_3 + ^{(311)}u_{31}\} \\ &\quad - \{(y_2 - y_1)a_2 - y_3b_2 + y_1y_2 + \Delta g_1 + \Delta g_3 + D_1 + D_3\} \\ &\quad - \{(x_2 - x_1)a_1 - x_3c_1 + x_1x_2 + \Delta f_1 + \Delta f_3 + d_1 + d_3\},\end{aligned}$$

Fig.13. The estimation of parameters $\hat{a}_1, \hat{b}_1, \hat{c}_1$ of drive system (40) (a), $\hat{a}_2, \hat{b}_2, \hat{c}_2$ of drive system (41) (b), $\hat{a}_3, \hat{b}_3, \hat{c}_3, \hat{d}_3$ of drive system (42) (c).

$$\begin{aligned}
{}_0D_t^\alpha \{^{(433)}e_{36}\} &= \left[\sum_{i=1}^4 c_{4i}({}_0D_t^\alpha z_i) - \left\{ \sum_{j=1}^3 b_{3j}({}_0D_t^\alpha y_j) + \sum_{k=1}^3 x_{3k}({}_0D_t^\alpha x_k) \right\} \right] \\
&= \{z_4 d_3 + z_1 z_3 + \Delta h_4 + \mu_4 + ^{(433)}u_{36}\} - \{-y_1 a_2 - 2y_3 b_2 \\
&\quad + (y_2 + y_1)c_2 - y_1 y_3 + 2y_1 y_2 + \Delta g_2 + 2\Delta g_3 + D_2 + 2D_3\} \\
&\quad - \{x_1 b_1 - 2x_3 c_1 - x_1 x_3 - x_2 + 2x_1 x_2 + \Delta f_2 + 2\Delta f_3 + d_2 + 2d_3\}.
\end{aligned}$$

It follows from the form of MSAC (25) and MSAUL (28) that the controller is designed as follows:

$$\begin{aligned}
^{(122)}u_{25} &= -z_4 - y_1 y_3 - x_1 x_3 - x_2 - \hat{a}_3(z_2 - z_1) + 2\hat{a}_2(y_2 - 2y_1) + \hat{c}_2(y_1 + y_2) - \hat{a}_2 y_1 \\
&\quad + 2\hat{a}_1(x_2 - x_1) + \hat{b}_1 x_1 - (^{(122)}\hat{\rho}_{25} + ^{(122)}\hat{\varrho}_{25})\text{sign}(^{(122)}s_{25}) - k_1 \tanh(^{(122)}s_{25}), \\
^{(244)}u_{30} &= z_1 z_3 + y_1 y_2 + x_1 x_2 - \hat{b}_3 z_2 + \hat{a}_2(y_2 - y_1) - \hat{b}_1 y_3 \\
&\quad + \hat{a}_1(x_2 - x_1) - \hat{c}_1 x_3 - (^{(244)}\hat{\rho}_{30} + ^{(244)}\hat{\varrho}_{30})\text{sign}(^{(244)}s_{30}) - k_1 \tanh(^{(244)}s_{30}), \\
^{(311)}u_{31} &= -z_1 z_2 + y_1 y_2 + x_1 x_2 + \hat{c}_3 z_3 + a_2(y_2 - y_1) - b_2 y_3
\end{aligned}$$

Fig.14. The estimation of parameters $^{(142)}\hat{\rho}_4$, $^{(234)}\hat{\rho}_{12}$, $^{(321)}\hat{\rho}_{13}$, $^{(413)}\hat{\rho}_{21}$ and $^{(142)}\hat{\varrho}_4$, $^{(234)}\hat{\varrho}_{12}$, $^{(321)}\hat{\varrho}_{13}$, $^{(413)}\hat{\varrho}_{21}$ shown in sub-pictures (a) and (b).

Fig.15. The synchronization errors $^{(122)}e_{25}$, $^{(244)}e_{30}$, $^{(311)}e_{31}$, $^{(433)}e_{36}$ change with time t .

$$\begin{aligned}
& +\hat{a}_1(x_2 - x_1) - \hat{c}_1x_3 - (^{(311)}\hat{\rho}_{31} + ^{(311)}\hat{\varrho}_{31})\text{sign}(^{(311)}s_{31}) - k_1\tanh(^{(311)}s_{31}), \\
^{(433)}u_{36} = & -z_1z_3 - y_1y_3 + 2y_1y_2 - x_1x_3 - x_2 + 2x_1x_2 - d_3z_4 - \hat{a}_2y_1 - 2\hat{b}_2y_3 + \hat{c}_2(y_1 + y_2) \\
& +b_1x_1 - 2c_1x_3 - (^{(433)}\hat{\rho}_{36} + ^{(433)}\hat{\varrho}_{36})\text{sign}(^{(433)}s_{36}) - k_1\tanh(^{(433)}s_{36}).
\end{aligned}$$

and the parameters updating laws are designed as follows:

$$\begin{aligned}
^{(122)}\hat{\rho}_{25} &= m_1(^{(122)}\lambda_{25})|^{(122)}s_{25}|, & ^{(244)}\hat{\rho}_{30} &= m_1(^{(244)}\lambda_{30})|^{(244)}s_{30}|, \\
^{(311)}\hat{\rho}_{31} &= m_1(^{(311)}\lambda_{31})|^{(311)}s_{31}|, & ^{(433)}\hat{\rho}_{36} &= m_1(^{(433)}\lambda_{36})|^{(433)}s_{36}|, \\
^{(122)}\hat{\varrho}_{25} &= m_2(^{(122)}\lambda_{25})|^{(122)}s_{25}|, & ^{(244)}\hat{\varrho}_{30} &= m_2(^{(244)}\lambda_{30})|^{(244)}s_{30}|, \\
^{(311)}\hat{\varrho}_{31} &= m_2(^{(311)}\lambda_{31})|^{(311)}s_{31}|, & ^{(433)}\hat{\varrho}_{36} &= m_2(^{(433)}\lambda_{36})|^{(433)}s_{36}|,
\end{aligned}$$

Fig.16. The synchronization for state variable z_1 and $2x_1 + x_2 + 2y_1 + y_2$, z_2 and $x_1 + x_3 + y_1 + y_3$, z_3 and $x_1 + x_3 + y_1 + y_3$, z_4 and $x_2 + 2x_3 + y_2 + 2y_3$ of drive systems (40), (41) and response system (42) indicating in sub-pictures (a)-(d) respectively.

$$\begin{aligned}
\hat{a}_1 &= -(x_2 - x_1)\{^{(311)}s_{31} + ^{(244)}s_{30} + 2(^{(433)}s_{36})\} - k_2 \text{sign}(\tilde{a}_1)(\text{abs}(\tilde{a}_1)^\omega), \\
\hat{b}_1 &= -x_1\{^{(122)}s_{25} + ^{(433)}s_{36}\} - k_2 \text{sign}(\tilde{b}_1)(\text{abs}(\tilde{b}_1)^\omega), \\
\hat{c}_1 &= x_3\{^{(244)}s_{30} + ^{(311)}s_{31} + 2(^{(122)}s_{25})\} - k_1 \text{sign}(\tilde{c}_1)(\text{abs}(\tilde{c}_1)^\omega), \\
\hat{a}_2 &= -((y_2 - y_1)\{^{(244)}s_{30} + 2(^{(433)}s_{36}) + ^{(311)}s_{31}\} - y_1\{^{(122)}s_{25} + ^{(433)}s_{36}\}) \\
&\quad - k_3 \text{sign}(\tilde{a}_2)(\text{abs}(\tilde{a}_2)^\omega), \\
\hat{b}_2 &= y_3\{^{(244)}s_{30} + 2(^{(122)}s_{25}) + ^{(311)}s_{31}\} - k_3 \text{sign}(\tilde{b}_2)(\text{abs}(\tilde{b}_2)^\omega), \\
\hat{c}_2 &= -(y_1 + y_2)(^{(122)}s_{25} + ^{(433)}s_{36}) - k_3 \text{sign}(\tilde{c}_2)(\text{abs}(\tilde{c}_2)^\omega), \\
\hat{a}_3 &= (z_2 - z_1)(^{(122)}s_{25}) - k_4 \text{sign}(\tilde{a}_3)(\text{abs}(\tilde{a}_3)^\omega), \\
\hat{b}_3 &= z_2(^{(244)}s_{30}) - k_4 \text{sign}(\tilde{b}_3)(\text{abs}(\tilde{b}_3)^\omega), \\
\hat{c}_3 &= -z_3(^{(311)}s_{31}) - k_4 \text{sign}(\tilde{c}_3)(\text{abs}(\tilde{c}_3)^\omega),
\end{aligned} \tag{46}$$

Fig.17. The estimation of parameters $\hat{a}_1, \hat{b}_1, \hat{c}_1$ of drive system (40) (a), $\hat{a}_2, \hat{b}_2, \hat{c}_2$ of drive system (41) (b), $\hat{a}_3, \hat{b}_3, \hat{c}_3, \hat{d}_3$ of drive system (43) (c).

$$\hat{d}_3 = z_4({}^{(433)}s_{36}) - k_4 \text{sign}(\tilde{d}_3)(\text{abs}(\tilde{d}_3)^\omega).$$

In this two numerical simulations, the definition of matrices A, B, C are shown in (43). The initial conditions of the drive systems (40), (41) and response system (42) are selected as $x(0) = (2, -2, 1, 1)$, $y(0) = (1, 1, 2, 2)$, $z(0) = (-20, 3, 1, 3)$. The initial values of the unknown parameter in D-R system and upper bound of SD are selected as $(a_1(0), b_1(0), c_1(0), d_1(0)) = (2, 1, -2, 1)$, $(a_2(0), b_2(0), c_2(0), d_2(0), r(0)) = (1, -3, 1, 1, -5)$, $(a_3(0), b_3(0), c_3(0), d_3(0)) = (1, 1, -2, -2)$, $(r_1(0), r_2(0), r_3(0), r_4(0)) = (-2.5, 1, -3, 1)$, $(q_1(0), q_2(0), q_3(0), q_4(0)) = (1, -2.5, -5, 1)$. The constants ${}^{(l w v)}\lambda_p = 2$, $m_1 = 10$, $m_2 = 10$, $k_1 = 10$, $k_2 = 10$, $k_3 = 10$, $k_4 = 10$, $\omega = 0.5$. The trajectories about the error variables $\mathbf{e} = ({}^{(142)}e_4, {}^{(234)}e_{12}, {}^{(321)}e_{13}, {}^{(413)}e_{21})^T$ and $\mathbf{e} = ({}^{(122)}e_{25}, {}^{(244)}e_{30}, {}^{(311)}e_{31}, {}^{(433)}e_{36})^T$ are drawn in figures 11 and 15. The synchronization for the state trajectories drive systems (40), (41) and response system (42) are drawn in figures 12 and 16. The trajectories of estimations $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\vartheta}}$ are drawn in figures 13 and 17, ${}^{(l w v)}\rho_p$, and ${}^{(l w v)}\varrho_p$ are drawn in figures 14 and 18. Motivated by the numerical simulation results, it can reveal that the drive systems (4), (5) and response system (6)

Fig.18. The estimation of parameters $(122)\hat{\rho}_{25}$, $(244)\hat{\rho}_{30}$, $(311)\hat{\rho}_{31}$, $(433)\hat{\rho}_{36}$ and $(122)\hat{\rho}_{25}$, $(244)\hat{\rho}_{30}$, $(311)\hat{\rho}_{31}$, $(433)\hat{\rho}_{36}$ shown in sub-pictures (a) and (b).

achieve MSSMCS. Therefore, the multi-switching adaptive controllers (MSAC) (25) and some suitable multi-switching adaptive updating laws (MSAUL) (28) are effective.

7 Conclusion

In this article, we investigated the multi-switching sliding mode combination synchronization (MSSMCS) of fractional order (FO) non-identical chaotic systems with unknown parameters under double stochastic disturbances (SD). In the theoretical parts, the FO chaotic systems with the different (or same) dimension are considered. Our idea for this topic is that with the help of the Lyapunov theory, adaptive control technology and sliding mode control technique, we put forward the fractional order sliding surface, multi-switching adaptive controllers (MSAC) and multi-switching adaptive updating laws (MSAUL) which can achieve the the state variables of the drive systems are synchronized with the different state variables of the response systems. Meanwhile, the unknown parameters are identified and upper bound of stochastic disturbances are examined accurately. What's more, the combination drive systems and single response system we chose are very general. The different description of the scale matrices can make the multi-switching projection synchronization, multi-switching complete synchronization, multi-switching anti-synchronization etc, become the special cases of MSSMCS. Motivated by the numerical simulation results, it is clear that the different error variables quickly converges to the equilibrium point. Therefore, the multi-switching adaptive controllers (MSAC) are effective and have strong robustness. Next, we will concentrate on the fractional order multi-switching synchronization of time-delay systems under multiple stochastic disturbances, and the parameters of system are still unknown.

Declarations

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