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Non-Hermitian skin clusters from strong interactions

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Strong, non-perturbative interactions often lead to new exciting physics, as epitomized by emergent anyons from the Fractional Quantum Hall effect. Within the actively investigated domain of non-Hermitian physics, we discover a new family of states known as non-Hermitian skin clusters. Taking distinct forms as Vertex, Topological, Interface, Extended and Localized skin clusters, they generically originate from asymmetric correlated hoppings on a lattice, in the strongly interacting limit with quenched single-body energetics. Distinct from non-Hermitian skin modes which accumulate at boundaries, our skin clusters are predominantly translation invariant particle clusters. As purely interacting phenomena, they fall outside the purview of generalized Brillouin zone analysis, although our effective lattice formulation provides alternative analytic and topological characterization. Non-Hermitian skin clusters fundamentally originate from the fragmentation structure of the Hilbert space, and may thus be of significant interest in modern many-body contexts like the ETH and quantum scars.

INTRODUCTION

When many-body interactions dominate single-particle energetics, unexpected new physics often emerge [1–12]. A classic example is the appearance of anyonic quasiparticles in FQH systems, whose emergent statistics cannot be inferred from single-particle Chern topology alone [12–18]. Indeed, the effects of strong non-perturbative interactions are determined by the structure [19–24] of the many-body Hilbert space, not the first-quantized single particle description.

Lately, intense research has focused on non-Hermitian phenomena. Yet, the main avenues of non-Hermiticity - the non-Hermitian skin effect (NHSE) [25–45] and exceptional points [46–59] - are essentially single-particle mechanisms based on first-quantized notions like non-Bloch eigenstates and single-particle band structure [29, 60–66]. It is thus timely to ask if strong interactions can open the door to even more exotic phenomena. While several works have explored many-body effects and interactions in non-Hermitian settings [67–72], interactions have not been the dominant physical mechanism.

In this work, we report the discovery of novel non-Hermitian skin clusters [Table I] in the limit where interactions are much stronger than single-particle energetics. They are special translation invariant particle configurations, fundamentally distinct from known non-Hermitian skin modes, being shaped by the connectivity structure of the many-body Hilbert space [19–21, 73], not the real-space lattice, and do not even require physical boundaries. They possess new features like the loop gap which cannot be understood via the generalized Brillouin zone (GBZ), which has been highly successful in characterizing conventional skin modes [27–30, 33][74].

RESULTS

Emergent topology and non-locality in a minimal unbalanced two-body hopping model

To understand how non-perturbative non-Hermitian interactions can lead to skin cluster states, we introduce a minimal 1D bosonic model purely consisting of unbalanced two-body correlated hoppings. Single-particle energetics are assumed to have been quenched, analogous to the scenario of dispersionless Landau levels [75–80]. Our model contains the four simplest possible asymmetric two-body hoppings, such that one particle hops across one site and the other hops across two sites in the opposite direction: [Fig. 1(a)]:

$$H = \sum_i^L (t_1 + \gamma) c_{i+2}^\dagger c_i c_{i-1}^\dagger c_i + (t_1 - \gamma) c_i^\dagger c_{i+2} c_i^\dagger c_{i-1} \quad (1) \\ + (t_2 - \gamma) c_{i+1}^\dagger c_i c_{i-2}^\dagger c_i + (t_2 + \gamma) c_i^\dagger c_{i+1} c_i^\dagger c_{i-2}$$

where \hat{c}_i (\hat{c}_i^\dagger) is the bosonic annihilation (creation) operator at the i -th site. The non-Hermiticity is controlled by γ ; when $\gamma = 0$, each hopping process and its reverse occur with equal probability t_1 or t_2 , depending on the direction of center-of-mass translation. Although this Hamiltonian may superficially resemble NHSE models with asymmetric single-particle hoppings i.e. the Hatano-Nelson model [26, 37], it does not even act on single-particle states, implying that any skin cluster eigenstate must originate exclusively from particle-particle interactions.

The many-body Hilbert space of our model can be systematically dissected by identifying its various subsectors and how they are coupled [82]. We start by observing that each two-boson sector harbors hidden \mathbb{Z} topology associated with 1D chiral symmetry. To concretely see that, note that two bosons only interact if they are either on the same site or three sites apart [Fig. 1(a)]. Hence,

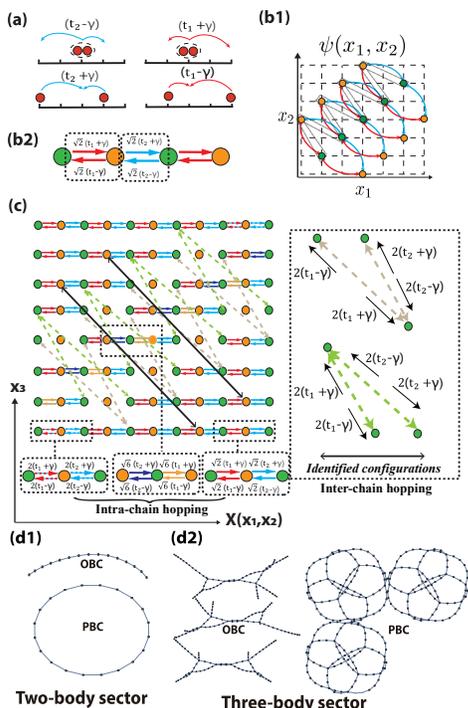


FIG. 1. (a) The four asymmetric two-body hoppings in our interacting model [Eq. 1], non-Hermitian whenever $\gamma \neq 0$. (b1) Two-boson configurations (x_1, x_2) which are non-trivially coupled form a 1D subspace represented by a non-Hermitian effective SSH model (b2) of length $2L - 5$, where L is the number of sites in the physical OBC chain. (c) Effective 2D lattice for a 3-boson OBC subspace, which comprises an array of non-Hermitian SSH chains that are non-locally coupled, with certain physically identical configurations $((X(x_1, x_2), x_3)$ and permutations) identified. See [81] for details. (d) The Hilbert space connectivity graphs of H in the 2 and 3-boson sectors for $L = 15$, demonstrating non-trivial PBC-to-OBC modifications to the graph structure beyond two bosons.

by restricting to only such configurations [Fig. 1(b1)], we can index the 2-boson subspace as an effective *single*-body 1D chain [Fig. 1(b2)] in configuration space. To see that, we take a two-boson state [83] $\psi(x_1, x_2)$ located at sites x_1 and $x_2 \geq x_1$, and introduce a relabeling $\Psi(X(x_1, x_2))$ for the *same state*, as given by

$$X(x_1, x_2) = \begin{cases} 2x_1 - 3 & \text{if } x_2 = x_1, \\ 2x_1 - 2 & \text{if } x_2 = x_1 - 3 \end{cases} \quad (2)$$

that captures configurations (x_1, x_2) acted by H . All other configurations decouple trivially. As illustrated in Fig. 1(b2) and elaborated in [81], the 2-boson sector reduces to an effective 1D non-Hermitian SSH chain with asymmetric couplings proportional to $t_1 \pm \gamma$ and $t_2 \pm \gamma$, since odd sites physically represent double bosonic occupancy and even sites represent bosons separated by three physical lattice spacings. We emphasize that generic models with asymmetric multi-particle hoppings often also contain variants of generalized SSH mod-

els [27, 84, 85].

The merits of our abovementioned basis relabeling become evident when we consider three or more particles. Key features of the many-body Hilbert space structure already emerge in a 3-boson sector. We write a three-boson state as $\psi(x_1, x_2, x_3) = \Psi(X(x_1, x_2), x_3)$, with permuted particle coordinates understood to correspond to the same state. A 3-boson sector then becomes a collection of coupled non-Hermitian SSH chains indexed by x_3 [Fig. 1c]. Their effective couplings come in two types: (i) dashed asymmetric couplings representing physical 2-body couplings between x_3 and x_1 or x_2 , weighted by boson degeneracy factors such as $\sqrt{2!}$, $\sqrt{2!^2}$ and $\sqrt{3!}$, and (ii) solid double arrows representing pairs of identified sites that correspond to the same physical site due to bosonic symmetry i.e. $(X(x_1, x_2), x_3)$ and $(X(x_2, x_3), x_1)$ for $x_2 = x_1 + 3 = x_3 - 3$. Also, due to bosonic symmetry, some asymmetric couplings become effectively non-local in the resultant lattice [81]. For instance, consider three bosons, two close together and acted by a local physical interaction, and the third far away from both. But due to symmetry under particle exchange, we have an equivalent description where the faraway boson is swapped with one of the nearby bosons, such that the local interaction appears as a non-local effective interaction.

Skin cluster states and Hilbert space fragmentation

Since interactions are reduced to ordinary hoppings on the effective lattice that represents the physically interesting many-body sector, they can be understood in terms of single-particle concepts. We show in Fig. 1(d1) and 1(d2) the Hilbert space connectivity under OBCs and PBCs, for 2-boson and 3-boson subspaces respectively (see [81] for that of 4 and 5-boson subspaces). While the 2-body subspaces simply form 1D chains, as previously explained, the 3-body spaces decouple into three sectors, each containing some states that are not easily reached, suggestive of Hilbert space fragmentation [19–22, 86–88]. Interestingly, for more than two particles, the Hilbert space connectivity graphs become much more complicated when going from OBCs to PBCs, much beyond merely “closing up” chains into loops. This hence hints of more exotic phenomena besides the conventional NHSE. Indeed, in the PBC case, there exists a hierarchy of smaller loops that are coupled only at a few selected states. When superimposed onto an effective lattice structure, these inter-loop couplings correspond to the previously mentioned effective non-local hoppings.

For concrete analysis of our specific model, we decompose the effective lattice of Fig. 1c into regions that take different roles in forming the skin clusters [Fig. 2(a) for OBCs, 3(a) for PBCs]. Notably, there exists a “non-local” region around $x_3 \approx X(x_1, x_2)$ which is crossed by a large number of non-local effective hoppings across sev-

	Type of state	Location	OBC/PBC
1	Topological edge mode	Physical boundary	OBC
2	Skin edge mode	Physical boundary	OBC
3	Vertex skin cluster	Intersection	OBC
4	Topological skin cluster	Effective interface	OBC
5	Interface skin cluster	Effective interface	PBC
6	Extended skin cluster	Effective interface	PBC
7	Localized skin cluster	Non-local region	PBC

TABLE I. The 7 types of boundary states of our two-boson hopping model H , with the 5 new types of states (3-7) shaded. Vertex (3) and Topological (4) skin clusters require both effective and physical boundaries [Fig. 2(a) for OBCs]. Interface (5), Extended (6), and Localized (7) skin clusters are purely due to the effective interface in H [Fig. 3(a) for PBCs]. Only (1) and (2) have been reported in the literature.

eral effective sites (diagonal effective hoppings in Fig. 1c, which hop across the cyan “non-local” region in Fig. 2(a) or 3(a)). They arise from physical interactions between a two-boson sector and a third boson. The background “local” region experiences only nearest-neighbor SSH-type effective hoppings (white in Fig. 2(a) or 3(a)) from 2-boson processes only. Note that all effective hoppings originate from the one and only type of interaction term present in H ; the demarcation into “local” and “non-local” regions reflect the qualitatively different roles of processes involving different numbers of bosons. Non-local hoppings across the “non-local” region destroy the periodicity in the effective lattice, precluding any GBZ description that relates the distinct OBC and PBC spectra (Fig. 2(b), 3(b)), unlike in non-interacting models.

Importantly, effective interfaces emerge between the “local” and “non-local” regions, in addition to physical boundaries at the edges of the effective lattice, if any. This results in additional localization behavior beyond topological and skin edge localizations. We identified seven distinct types of boundary states [Table I], 5 of which are new. As effective interfaces are induced by particle statistics, not physical boundaries, our new states (Types 3 to 7) are named “skin cluster states”. Vertex skin clusters (3) are unique highly-localized states at point intersections (vertices) of effective and physical boundaries [Fig. 2(a)], and are elaborated in [81]. Topological (4), Interface (5), Extended (6), and Localized (7) skin clusters represent distinct new ways by which the effective interfaces exert topological and skin localization and will be elaborated below.

Although skin clusters also arise from asymmetric couplings, they differ from conventional NHSE skin states in a few important ways: (i) they only exist when two or more particles interact; (ii) they exist due to the inhomogeneity of effective couplings from particle statistics, *not* physical boundaries as in the NHSE [73]: in fact (5-7) exist under PBCs, not OBCs; (iii) while the NHSE usually gives rise to non-local responses [30, 31, 89], here the skin clusters themselves are already the consequence of

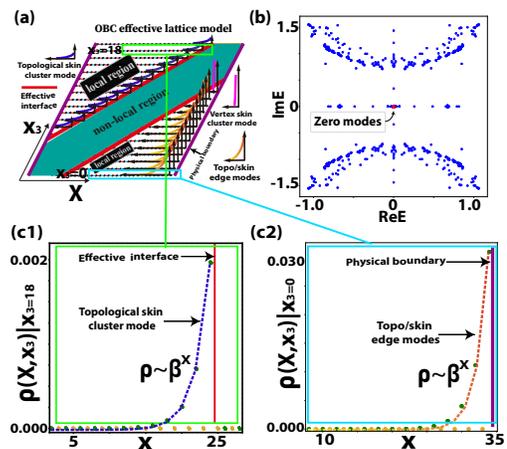


FIG. 2. (a) Schematic of the effective Hilbert space lattice for the 3-boson sector under OBCs. Regions with local (white) and non-local (cyan) effective hoppings are demarcated by the interaction-induced effective interfaces (red lines). Topological skin cluster modes accumulate against effective interfaces, while Vertex skin clusters are localized at intersections between effective interfaces and physical edges (purple), which also host ordinary skin/topo edge modes. (b) The three-boson OBC spectrum with parameters $t_1 = 1.0, t_2 = 0.2, \gamma = 0.8$, on a $(2L - 5) \times L$ effective lattice ($L = 20$). Zero modes correspond to either topological edge modes or skin clusters. (c) Spatial density profiles $\rho(X, x_3) = |\Psi(X, x_3)|^2$ of both types of topological modes, which clearly accumulate at either the effective interface or physical boundary with identical decay profiles $\rho(X, x_3) \sim \beta^X$, $\beta = (t_1 + \gamma)/(\gamma - t_2)$.

non-local effective couplings; (iv) due to the “non-local” region, the various types of skin clusters do not possess GBZ descriptions, unlike ordinary skin states.

Topological skin clusters

Topological skin clusters eigenstates (4) are topological zero modes at the effective interface between the “local” and “non-local” regions. Physically, they are manifestations of how the inherent SSH-type topology in a 2-boson sector interplays with the presence of a third boson. Interestingly, although effective interfaces exist identically across both OBC [Fig. 2] and PBC [Fig. 3] settings, Topological skin clusters only exist under OBCs. PBC skin clusters (5-7) differ significantly from them in both energetics [Figs. 2(b) vs. 3(b)] and spatial profile [Figs. 2(c) vs. 3(c)]. This enigma arises because of the non-local effect of local unbalanced hoppings: Under PBCs, the absence of physical boundaries (edges) leads to significant interference between the NHSE-like accumulation from either side of the non-local region, destroying “undiluted” topological states. That topological skin clusters (4) mathematically result from the same topological mechanism as ordinary topological edge states (1) can be seen from their vanishing energies [Fig. 2(b)],

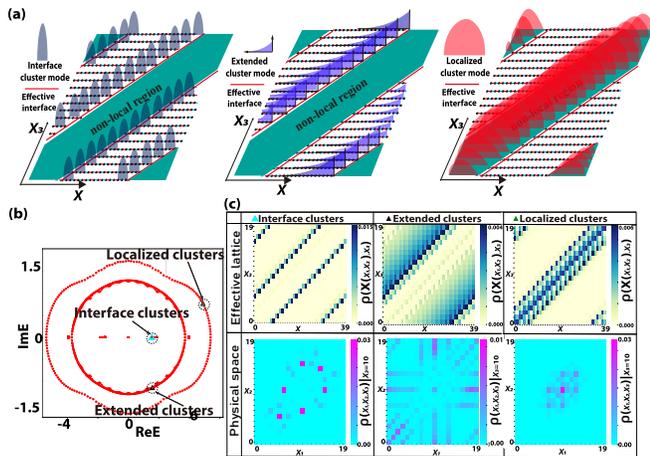


FIG. 3. (a) Schematics of the effective Hilbert space lattice for the 3-boson sector under PBCs, with each panel illustrating its respective type of PBC cluster mode. (b) The three-boson PBC spectrum with parameters $t_1 = 1.4, t_2 = 1.2, \gamma = 0.5, L = 20$. Generally, localized cluster states form an outer loop with the largest $|E|$, separated from the inner loop of Extended skin clusters via a *loop gap*. Interface clusters form isolated spectra with exact solutions [81]. (c) Numerically computed spatial density distribution of the three types of clusters. Top row: $\rho(X, x_3) = |\Psi(X(x_1, x_2), x_3)|^2$ in the effective lattice; Bottom row: $\rho = |\psi(x_1, x_2, x_3)|^2_{x_3=10}$ in the physical lattice. Interface clusters, while seemingly constrained at the effective interface, actually consist of bosons several sites from each other. Extended clusters favor physical configurations with two overlapping bosons and another far away. Localized clusters contain bosons that are physically localized within sites from each other.

and identical localization lengths [Fig. 2(c)] of $(\log \beta)^{-1}$, with $\beta = (t_1 + \gamma)/(\gamma - t_2)$ as is well-known for the non-Hermitian SSH model [27, 81].

Translation invariant skin clusters

Without physical boundaries (PBCs), the three types of skin clusters (5-7) that exist are of special non-topological origins. They are Interface (5), Extended (6) and Localized (7) skin cluster, so-called because they respectively exist at the interface between the “local” and “non-local” regions (5), are extended across the wide “local” region (6) or are localized within the relatively narrow “non-local” regions (7), as seen from $|\Psi(X(x_1, x_2), x_3)|^2$ density plots of Fig. 3(c). Although Interface clusters (2) appear the most localized of the three on the effective lattice, in physical coordinates, their density plot $|\psi(x_1, x_2, x_3)|^2$ (Fig. 3(c)) reveal strong correlations between particles a few sites apart. By contrast, Localized skin clusters (7) consist of 3 bosons that are indeed almost physically overlapping, “caged” by the non-local effective hoppings. The extended skin clusters (6) are the most delocalized but are still considered clus-

ters states because they are exponentially localized along the effective interface.

The eigenenergies of different types of skin cluster eigenstates generically form very distinct loci in the complex energy plane. As shown in Fig. 3(b), extended and localized cluster states (6,7) form two concentric spectral loops, while the interface states (5) form isolated points in the loop interiors. Under parameter tuning, these characteristic spectral loci remain largely robust, even though they may distort and cross each other [81]. *Structure of skin cluster states and the loop gap.*— The concentric rings of eigenenergies in the 3-body PBC spectrum [Fig. 3(b)] are robust across a large range of parameters, and can even persist even beyond three bosons, as elaborated in [81]. We name this unmistakable separation of concentric PBC spectral loops as a *loop gap*, in the spirit of point gaps for loops which encircle interior points [37]. To understand why the loop gap occurs, we note that the outer/inner spectral ring corresponds to localized/extended cluster states. In the fully localized limit where all bosons move rigidly like a single composite particle, the PBC spectrum traces out a large loop given by $E = \sum_j t_j e^{ipj}$, where $p \in [0, 2\pi)$ and t_j are the effective j -site hoppings acting on the composite particle. Compared to localized clusters, extended clusters must trace out smaller loops because at least one boson does not move in unison with the others [Fig. 3(c)], leading to “destructive interference”. Note that localized and extended clusters move freely as a whole, unlike interface clusters and topological edge modes which are constrained to specific physical locations. This causes the latter two to be spectrally enclosed within the localized and extended clusters [Fig. 4(a)] since, being confined, they cannot be significantly amplified by the asymmetric hoppings and must thus possess energies close to $E = 0$.

The above arguments can be corroborated by the following 2 and 3-site density correlators, which reveal more insight into the real-space structure of the cluster states:

$$\begin{aligned} (a) \quad C_m(x_i, x_j) &= |\langle \psi_m | n_i n_j | \psi_m \rangle|, \quad (i \neq j) \\ (b) \quad C_m(x_i, x_j, x_k) &= |\langle \psi_m | n_i n_j n_k | \psi_m \rangle|, \end{aligned} \quad (3)$$

where ψ_m is the selected eigenstate and the density operator $n_i = c_i^\dagger c_i$ measures the occupancy at site i . These correlators are plotted in Fig. 4(b) for representative eigenstates ψ_m as specified in Fig. 4(a); to put their magnitudes in perspective, note that they scale respectively as L^{-2} and L^{-3} for a perfectly uniform state, but remain at unity for a perfectly clustered state. Comparing the 2-site correlator $C_m(x_i, x_j)$ in Fig. 4(b) with the single-site density plots in Fig. 3(c), both for the effective lattice, we observe similar trends despite the different model parameters and chosen states. However, $C_m(x_i, x_j)$ always reveals conspicuously strong spectral weights at the interfaces, suggesting that the bosons prefer to be separated by 3-5 sites regardless of the cluster type. The

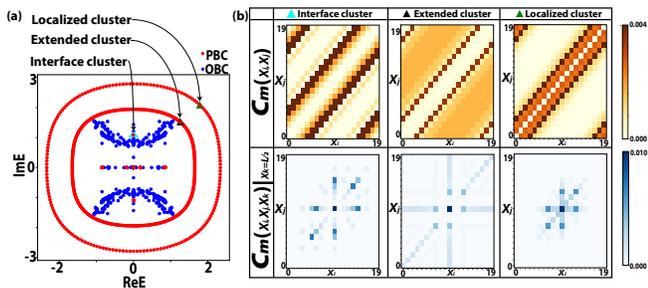


FIG. 4. (a) The 3-boson PBC spectrum (red) is characterized by a loop gap between concentric loops, which also enclose isolated PBC interface cluster energies and most of the OBC spectrum (blue). (b) Density correlators for representative PBC eigenstates ψ_m of the three cluster types from (a). 2-site correlators $C_m(x_i, x_j)$ for $i \neq j$ reveal that interface configurations $|x_i - x_j| = 3$ are favored in all cluster types. 3-site $C_m(x_i, x_j, x_k)|_{k=L/2}$, evaluated at $k = L/2$ WLOG due to translation symmetry, reveal that extended clusters are most likely to have a doubly occupied site, but that localized clusters possess the shortest correlation range. Parameters are $t_1 = 1.0, t_2 = 0.2, \gamma = 0.8$, with $L = 20$.

three site-correlator $C_m(x_i, x_j, x_k)$ further reveals that extended states tend to consist of 2 bosons on the same site, together with another arbitrarily far boson.

DISCUSSION

Our simple model with asymmetric two-body hoppings (Eq. 1) is a paradigmatic representation of non-reciprocal non-Hermitian physics in the strongly interacting limit where single-body energetics are quenched. While asymmetric couplings have been associated with the well-known single-particle NHSE, our purely interacting model exhibits various new, previously undiscovered clustering behavior [Table I (3-7)]. As a departure from conventional NHSE, these cluster states are mostly translation invariant, even if they require OBCs, as in topological skin clusters (4).

Fundamentally stemming from effective interfaces in a fragmented Hilbert space, our new states should exist in generic interacting lattices with (i) asymmetric couplings and (ii) quenched single-particle energetics. Their dynamical study constitutes a particularly promising direction, with Hilbert space fragmentation known to lead to emergent violations of the eigenstate thermalization hypothesis (ETH) [20, 86, 87, 90] and even quantum many-body scars [86, 91–95]. Appropriately generalized, our model may be physically realized with density-assisted tunneling in driven optical lattices [92, 96], quantum digital computer circuits with suitably arranged imaginary time evolution [97, 98], or, topoelectrical circuits [31, 44, 99–105] at the effective lattice level.

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SUPPLEMENTARY INFORMATION

The supplementary materials are organized according to the following sections:

1. Details of the effective lattice construction for our model, its topological properties, and its Hilbert space connectivity structure.
2. Details of the clustering properties of various types of 3 and 4-boson clusters, including analytic characterizations and alternative measures of clustering.

I. The effective lattice - construction details and connectivity structure

Our interacting model comprises exclusively of asymmetric two-boson correlated hoppings introduced in Eq. 1 of the main text:

$$H = \sum_i^L (t_1 + \gamma) c_{i+2}^\dagger c_i^\dagger c_{i-1}^\dagger c_i + (t_1 - \gamma) c_i^\dagger c_{i+2} c_i^\dagger c_{i-1} \quad (S1)$$

$$+ (t_2 - \gamma) c_{i+1}^\dagger c_i c_{i-2}^\dagger c_i + (t_2 + \gamma) c_i^\dagger c_{i+1} c_i^\dagger c_{i-2}$$

As a result, non-trivial physics requires at least two bosons, unlike usual single-particle non-Hermitian skin effect (NHSE) models. The main physics hosted by this interacting model appears whenever we have at least 3 particles. Below, we hence elaborate on its connectivity structure in the 2-boson and 3-boson sectors of its Hilbert space.

The two-boson sector as a non-Hermitian SSH model

According to the analysis in the main text, the Hilbert space graph of the two-body model is the effective SSH chain. Here we furnish more details of this effective SSH chain, which indirectly controls the physics of the two bosons in physical space.

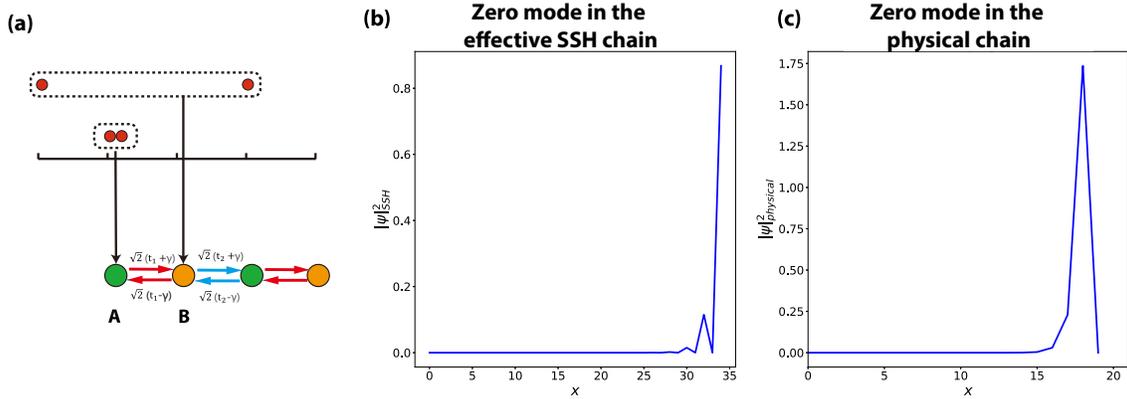


FIG. S1. (a) The Non-Hermitian effective SSH model corresponding to the physical two-boson model, and the corresponding two-boson state from the physical chain that maps to odd/even sites in the effective SSH chain. The squared amplitudes of the same zero mode in the effective chain (b) vs. the physical chain (c) with $t_1 = 1.2$, $t_2 = 0.2$, $\gamma = 1.0$, $L = 20$. The corresponding length of the effective SSH chain is $2L - 5$.

We consider the zero mode in the effective SSH chain, expressed as $\psi = (\psi_{1A}, \psi_{1B}, \dots, \psi_{n-1A}, \psi_{n-1B}, \psi_{nA})^T$ (A, B are the labels of the sublattices in Fig. S1). The sublattice wave function takes the exponential form $(\psi_{nA}, \psi_{nB})^T = \beta^n (\psi_{1A}, \psi_{1B})^T$ according to the generalized Brillouin zone ansatz [1, 2]. In the bulk, the eigen-equation satisfies

$$\begin{aligned} \sqrt{2}(t_1 - \gamma)\psi_{iB} + \sqrt{2}(t_2 + \gamma)\psi_{i-1B} &= E\psi_{iA} \\ \sqrt{2}(t_1 + \gamma)\psi_{iA} + \sqrt{2}(t_2 - \gamma)\psi_{i+1A} &= E\psi_{iB} \end{aligned} \quad (S2)$$

At the boundary, the OBC boundary condition gives

$$\begin{aligned}\sqrt{2}(t_1 - \gamma)\psi_{1B} &= E\psi_{1A} \\ \sqrt{2}(t_2 + \gamma)\psi_{n-1B} &= E\psi_{nA}.\end{aligned}\tag{S3}$$

From them, we can obtain the solution to the zero mode of SSH chain as $\psi_{1B} = 0, \beta = (t_1 + \gamma)/(\gamma - t_2)$. Fig.S1(b) presents a zero mode in the effective SSH chain while Fig. S1(c) presents its corresponding squared amplitude in physical space, as a two-boson system.

The three-boson sector

We now construct the effective lattice for the 3-boson sector of our model H (see Fig. S2 for the effective lattice under OBCs). In the following, we explain the details of mapping a state from the 3-boson Hilbert space to the effective lattice. We express the three-boson state in the physical Hilbert space as $|x_1, x_2, x_3\rangle$. According to the transformation Eq. 2 of the main text:

$$X(x_1, x_2) = \begin{cases} 2x_1 - 3 & \text{if } x_2 = x_1, \\ 2x_1 - 2 & \text{if } x_2 = x_1 - 3, \end{cases}\tag{S4}$$

so that the three-boson state can be written on the effective lattice as

$$|x_1, x_2, x_3\rangle = |X(x_1, x_2), x_3\rangle\tag{S5}$$

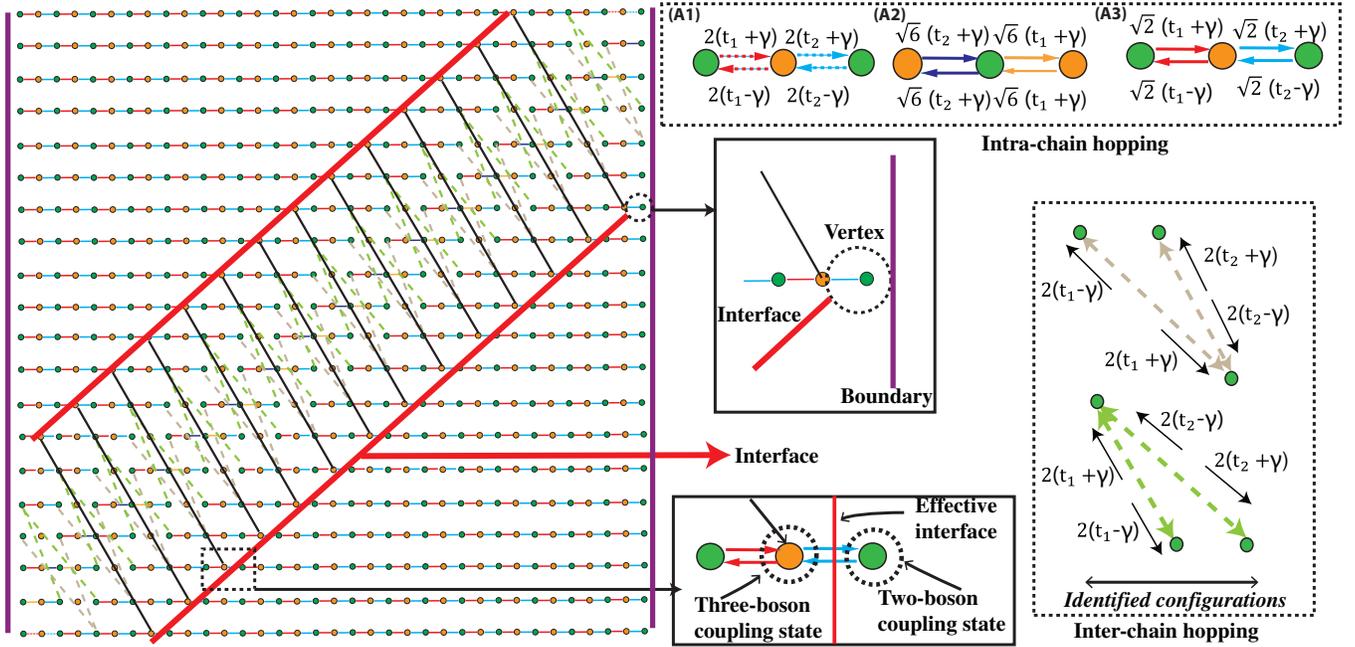


FIG. S2. Schematic of the OBC effective model for three bosons, shown for an effective lattice of size $(2L - 5) \times L$ ($L = 20$). The combined effects of the boson-boson couplings and bosonic statistics give rise to the inter-chain hoppings. The effective interface appears between the central “non-local region” between the two red diagonal lines with many inter-chain hoppings, and the “local region” with only intra-chain hoppings. In the non-local region, the inter-chain hoppings depend on the positions of all three bosons, unlike the local region where the effective SSH hoppings depend only on the positions of two of the bosons.

As a result, the two-body hoppings in the physical space becomes effective hoppings in the effective 2D lattice. In Fig. S2, there are two types of hoppings in the effective lattice. The intra-chain hoppings (A1)(A2)(A3) come from

the two-body hoppings (if the boundary is present, $|1, f(1, 4)\rangle \neq |f(1, 1), 4\rangle$)

$$\begin{aligned}
 (A1): \quad & c_1^\dagger c_2 c_4^\dagger c_2 |1, f(2, 2)\rangle = 2 |1, 1, 4\rangle = 2 |1, f(1, 4)\rangle \\
 (A2): \quad & c_{i-1}^\dagger c_i c_{i+2}^\dagger c_i |X(x_i, x_i), x_i\rangle = \sqrt{6} |X(x_{i-1}, x_{i+2}), x_i\rangle \\
 (A3): \quad & c_{i-1}^\dagger c_i c_{i+2}^\dagger c_i |X(x_i, x_i), x_j\rangle = \sqrt{2} |X(x_{i-1}, x_{i+2}), x_j\rangle, \quad (|x_j - x_i| > 3)
 \end{aligned} \tag{S6}$$

The inter-chain hoppings come from

$$\begin{aligned}
 c_{i-1}^\dagger c_i c_{i+2}^\dagger c_i |X(x_i, x_i), x_{i+2}\rangle &= 2 |x_{i-1}, x_{i+2}, x_{i+2}\rangle \\
 &= 2 |X(x_{i+2}, x_{i+2}), x_{i-1}\rangle
 \end{aligned} \tag{S7}$$

Notably, there exists equivalent labels of states like $|X(x_{i-2}, x_i), x_{i+2}\rangle = |x_{i-2}, X(x_i, x_{i+2})\rangle$, which are indicated as "Identified configurations" in Fig. S2.

To check the validity of our effective model, we compare the PBC and OBC spectrum of the physical and effective models, and the spectrum results are all in agreement.

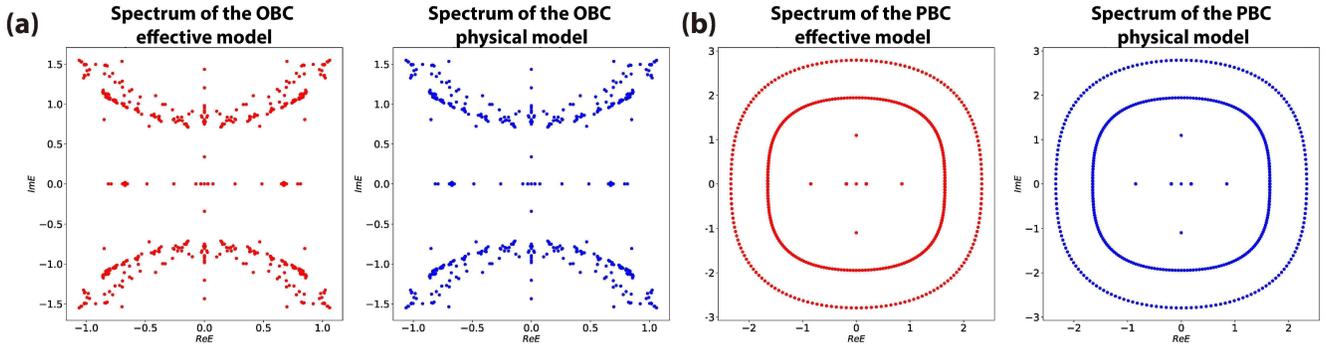


FIG. S3. Comparison of the 3-boson OBC and PBC spectra between our physical (red) and effective (blue) models, with exact agreement as demonstrated. Parameters are $t_1 = 1, t_2 = 0.2, \gamma = 0.8, L = 20$.

Hilbert space structure and fragmentation

Here, we present the connectivity structure of the Hilbert space of our system in sectors with a various number of particles. This reveals the geometric symmetry of our model that is not apparent when drawn in terms of an array of coupled effective SSH chains.

In our connectivity graphs, the tunneling between the two states, irrespective of actual amplitude or type, contributes to one effective hopping (See Fig. S4 (a1)-(b2)). The connectivity graphs reveal some degree of Hilbert space fragmentation induced by the interactions and particle statistics in our model for three or more particles (The Hilbert space graph consists of three or four disconnected subgraphs). In addition, it also reveals the nontrivial effects of boundary conditions: The Hilbert space structure of PBC vs. OBC graphs can be substantially different due to the non-local nature of some of the effective hoppings.

As a comparison, we also plot the Hilbert space connectivity graph (Fig. S4 (c1)-(c2)) of the half-filled Bose-Hubbard model with nearest hopping t , on-site interaction U , and potential μ : $H_{\text{Bose-Hubbard}} = -t \sum_{\langle i,j \rangle} \hat{c}_i^\dagger \hat{c}_j + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i$.

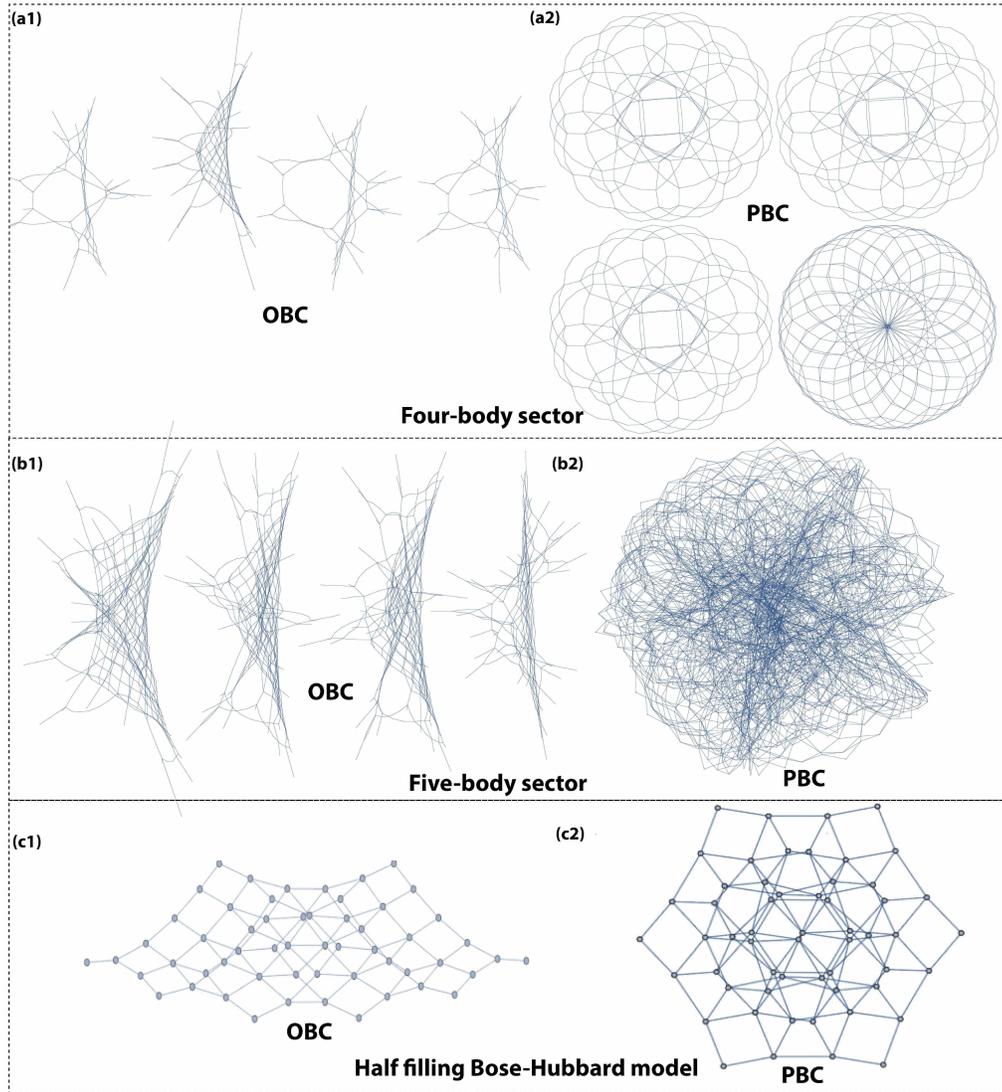


FIG. S4. The Hilbert space connectivity graphs of our interacting Hamiltonian H with N bosons: (a1) OBC, $L=12$, $N=4$, (a2) PBC, $L=12$, $N=4$, (b1) OBC, $L=12$, $N=5$, (b2) PBC, $L=12$, $N=5$. For reference, also plotted are the graphs for the half-filled Bose-Hubbard model: (c1) OBC, $L=6$, (c2) PBC, $L=6$.

II. Clustering behavior in 3-boson and 4-boson clusters

Cluster states of topological origin

In our effective lattice for 3 bosons under OBCs, there exists both physical boundaries and effective interfaces. When $\gamma \neq 0$, localization occurs at both the two boundaries. Fig. S5 presents the state amplitude distribution in the effective lattice of a selected OBC zero mode with $t_1 = 1.2, t_2 = 0.2, \gamma = 0.5, L = 20$.

To show that the localizations near the interface and the physical boundary have the same topological origin, we perform the curve fitting of these decays and compare the analytical result $\beta = (t_1 + \gamma)/(\gamma - t_2)$ (the solution of the effective non-Hermitian SSH model) with the curve fitted results from numerical diagonalization of the 3-body system. Indeed, for the zero modes of the three-body sector, the decays at the physical boundary (β') and effective interfaces (β'') all agree viz. $\beta'', \beta' \approx \beta$ ($\rho(x) \approx \beta^x$).

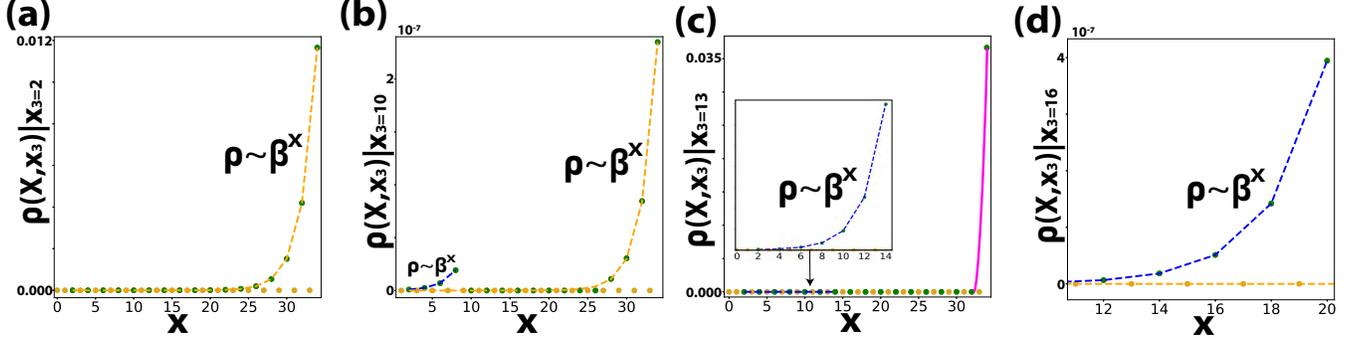


FIG. S5. The 2D cross-sectional density $\rho = |\Psi(X, x_3)|^2$ in the OBC effective lattice, with fixed $x_3 = 2(a), 10(b), 13(c), 16(d)$ of selected states with $t_1 = 1.0, t_2 = 0.2, \gamma = 2.0, L = 20$. For the cases $x_3 = 2, 10$, there is skin localization at the physical boundary (the yellow curves). For the cases $x_3 = 10, 13, 16$, there is skin localization at the interface (the blue curves). Also present are the trivial mode and the vertex localization (the pink curve in (c)) at the intersection between the physical boundary and the interface in the effective lattice.

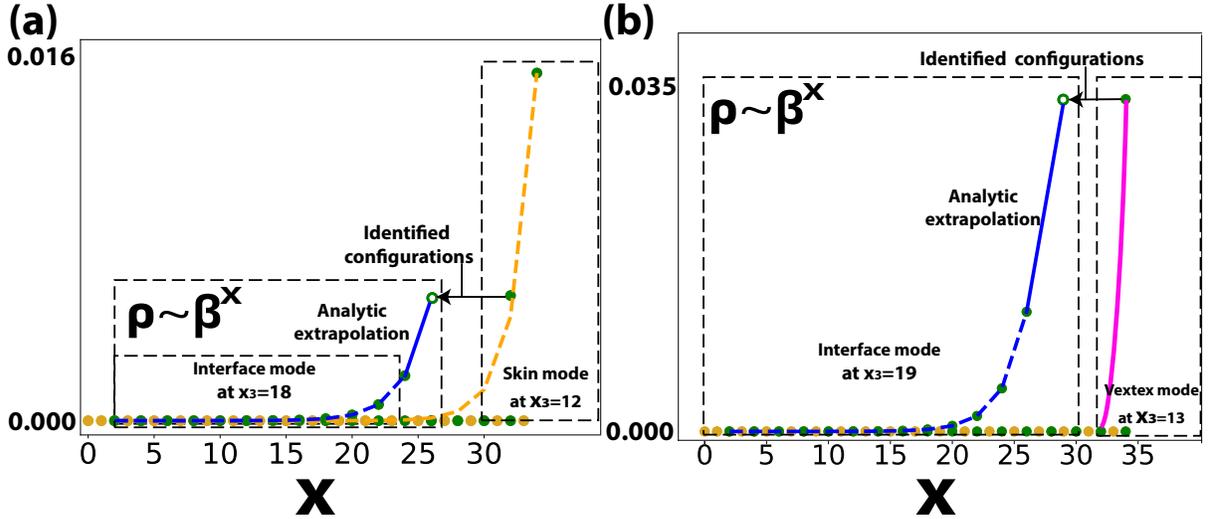


FIG. S6. Bosonic symmetry leads to equivalent “identified configurations” in the effective lattice, and hence skin cluster states on it that can be pieced together like a jigsaw-puzzle. (a) The yellow curve: the skin mode of $\rho = |\Psi(X, x_3)|^2_{x_3=12}$; The blue curve: the interface mode of $\rho = |\Psi(X, x_3)|^2_{x_3=18}$. The “Identified configurations” between sites allows us to move a section (the dashed rectangle) of the skin mode to the nearest site of the interface mode (the green hollow circle). We perform an analytic extrapolation according to the non-Bloch scaling controlled by β (the solid blue line) to link the two ends. (b) The pink curve: the vertex mode of $\rho = |\Psi(X, x_3)|^2_{x_3=13}$; The blue curve: the interface mode of $\rho = |\Psi(X, x_3)|^2_{x_3=19}$.

The interface within the effective lattice is demarcated by “identified configurations” arising from bosonic statistics

(See the structure of the lattice in Fig. S2). So as to explain these interface localizations under OBC, we plot skin (vertex) modes and the interface mode connected by the "Identified configurations" together (Fig. S6). These results suggest that the interface mode is a branch of the skin (vertex) mode, such that we can use analytic extrapolation to attach the interface mode to a section of the skin (vertex) mode.

Analytic ansatz for Interface clusters

Here we elaborate on the structure of special isolated interface clusters under PBCs (blue triangular dots in Fig. S7). We shall approximate them with the compact state ansatz $|i, j, k\rangle = c_i^\dagger c_j^\dagger c_k^\dagger |0\rangle$, which exist under PBCs:

$$|\Psi\rangle_{com} = \sum_i^L (\psi_{-1}|i-1, i-1, i+3\rangle + \psi_0|i-3, i, i+3\rangle + \psi_1|i-3, i+2, i+2\rangle). \quad (S8)$$

Substituting it into our Hamiltonian H (Eq. S1), we have

$$\begin{aligned} (t_2 - \gamma)(\psi_1 + \psi_{-1}) &= E\psi_0 \\ (t_2 + \gamma)\psi_0 &= E\psi_1 \\ (t_2 + \gamma)\psi_0 &= E\psi_{-1}. \end{aligned} \quad (S9)$$

We get the solution

$$\psi_1^2 = \psi_{-1}^2, \quad E = \pm\sqrt{2(t_2^2 - \gamma^2)}, \quad (S10)$$

which turns out to be an excellent approximation of our interface cluster states. In the case shown in Fig. S7, $E_{numerical} \approx \pm 1.0955491i$ for the numerical interface cluster, and $E_{analytical} = \pm\sqrt{2(t_2^2 - \gamma^2)} \approx \pm 1.0954451i$ for the compact state ansatz $|\Psi\rangle_{com} \approx |\Psi\rangle_{interface}$.

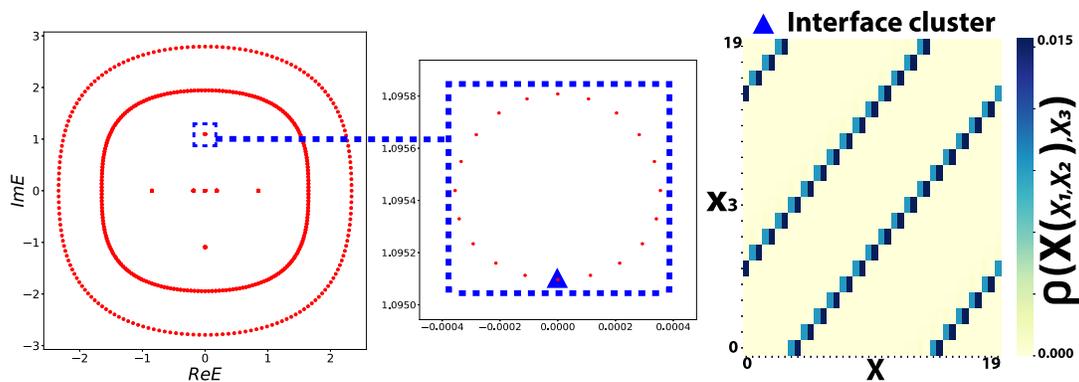


FIG. S7. Representative 3-boson PBC interface cluster states with $t_1 = 1.0, t_2 = 0.2, \gamma = 0.8, L = 20$, which are almost degenerate in spectrum, and all localized along the interfaces in the effective lattice. Their average eigenenergy and spatial profile agrees excellently with the compact state ansatz.

Comparison between various types of cluster states

In Fig. S8 below, we provide more detailed comparisons between representative states of distinct types, plotted in both the physical and effective lattices.

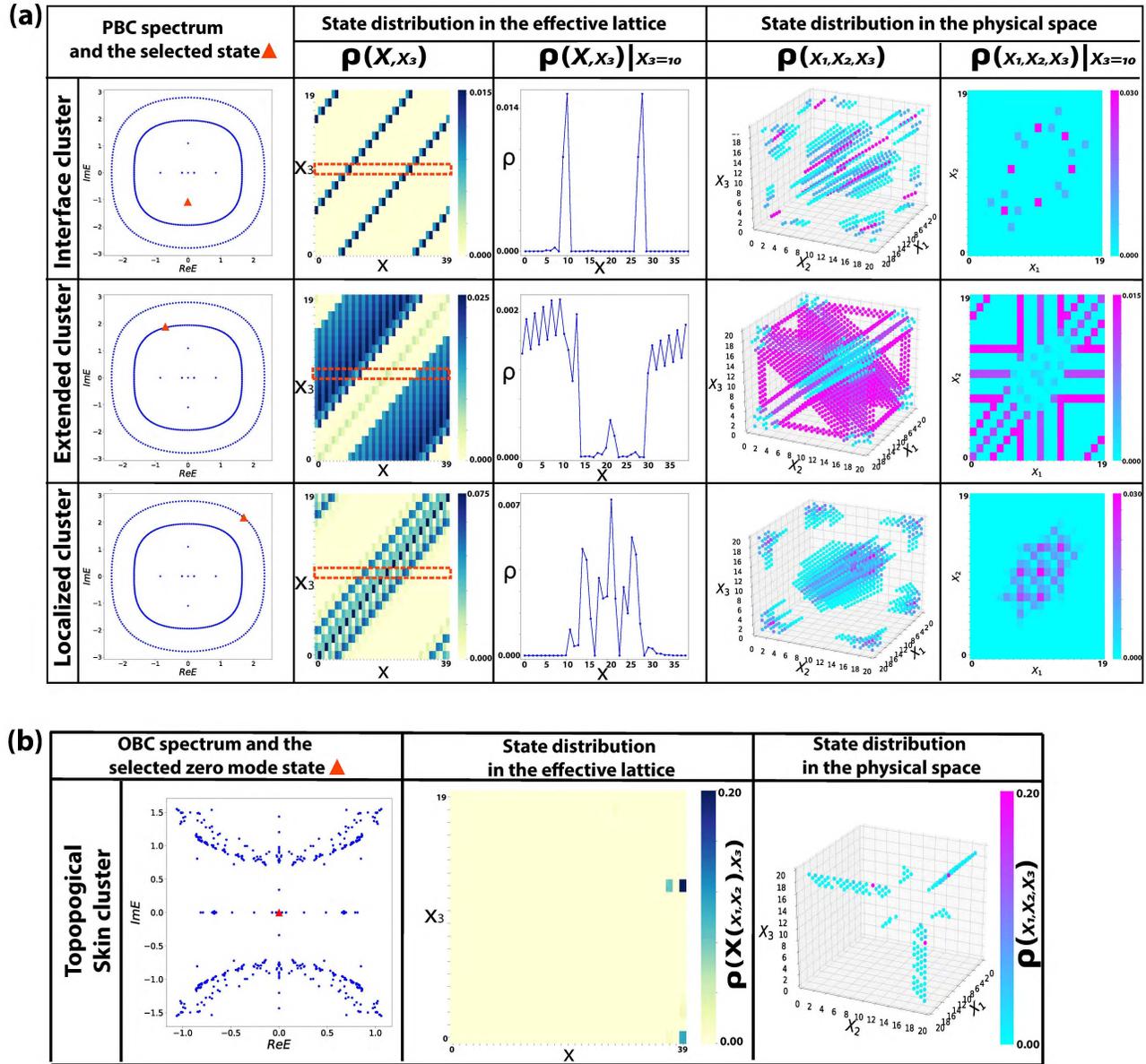


FIG. S8. 3-boson skin cluster states of H with parameters $t_1 = 1.0, t_2 = 0.2, \gamma = 0.8, L = 20$. (a) PBC case, from left to right: The PBC spectrum; the state distribution in the effective lattice $\rho(X, x_3)$ of the selected state (orange triangle), and its density plot $\rho(X, x_3)|_{x_3=10}$ along a cross-section; its spatial distribution $\rho(x_1, x_2, x_3)$ in physical space (the sites with $\rho(x_1, x_2, x_3) < 0.0001$ are not plotted for clarity), and the same data along the cross-section $\rho(x_1, x_2, x_3)|_{x_3=10}$. Indeed, localized clusters are the most tightly clustered; interface clusters comprises configurations at the literal interface of localized clusters; and extended clusters frequently involve one arbitrarily far boson. (b) OBC case, from left to right: the OBC spectrum; the state distribution in the effective lattice $\rho(X, x_3)$ of the selected zero mode state (orange triangle); its spatial distribution $\rho(x_1, x_2, x_3)$ in physical space, revealing essentially no dependence on one of the particle coordinates i.e. it is a two-body phenomenon.

PBC spectral loop structure for skin cluster states

Under a wide range of t_1, t_2, γ parameters, the PBC spectrum in the three-body sector exhibits distinct concentric loops with states on the outer and inner loops exhibiting utterly different behavior. The exact shapes of these loops depend on their parameters (see Fig. S9), but their clear separation i.e. loop gap is rather robust. Previously, we have already classified these states into localized clusters (outer loop), extended clusters (inner loop), and interface clusters (isolated state hosted by Eq. (S10)).

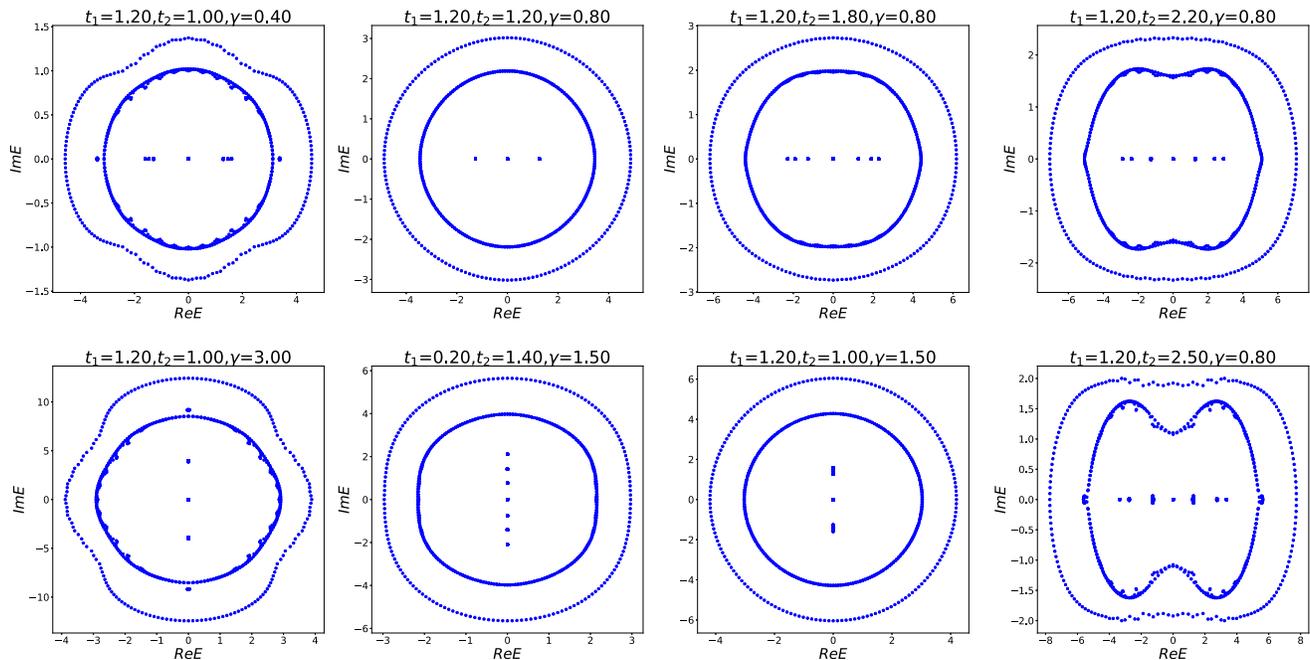


FIG. S9. 3-boson PBC spectra with different t_1, t_2, γ . The two concentric loops always remain, even though their detailed shape changes. The length of the physical chain is $L = 20$.

To further probe the robustness of the loop gap, we add physical nearest neighbor Hermitian hoppings to our original interacting Hamiltonian H :

$$H' = H + t \sum_i (c_{i+1}^\dagger c_i + h.c.) \quad (\text{S11})$$

In other words, we perturb our original model H with single-body dynamics with strength t . Fig. S10 presents the PBC spectrum of three boson H' with increasing t . We find that when increasing t , some states move towards the zero energy origin, blurring the inner extended clusters loop. At sufficiently large t , the isolated interface clusters acquire nontrivial dynamics and start to merge with the extended clusters. However, the localized clusters (outer loop) remain stable even when t is comparable to the original interaction terms in H . This can be further verified in the spatial distribution of the localized clusters upon increasing t (Fig. S11), which indeed remain almost unchanged.

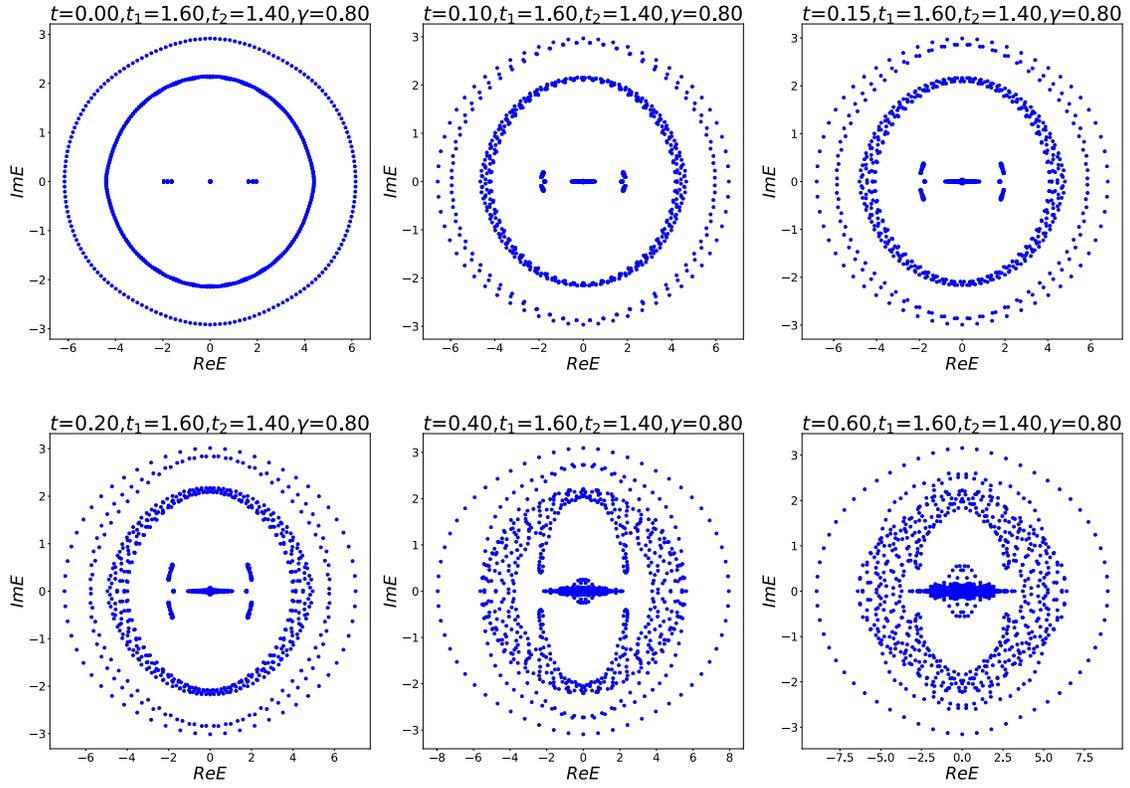


FIG. S10. The three-body PBC spectrum of H' with fixed t_1, t_2, γ and different values of single-body dispersion t . The length of the physical chain is $L = 20$. Even though the inner loop (extended clusters) and isolated inner states (interface clusters) start to merge at sufficiently large t , the outer loop (localized clusters) remains untouched, separated from the other (inner) states by a robust loop gap.

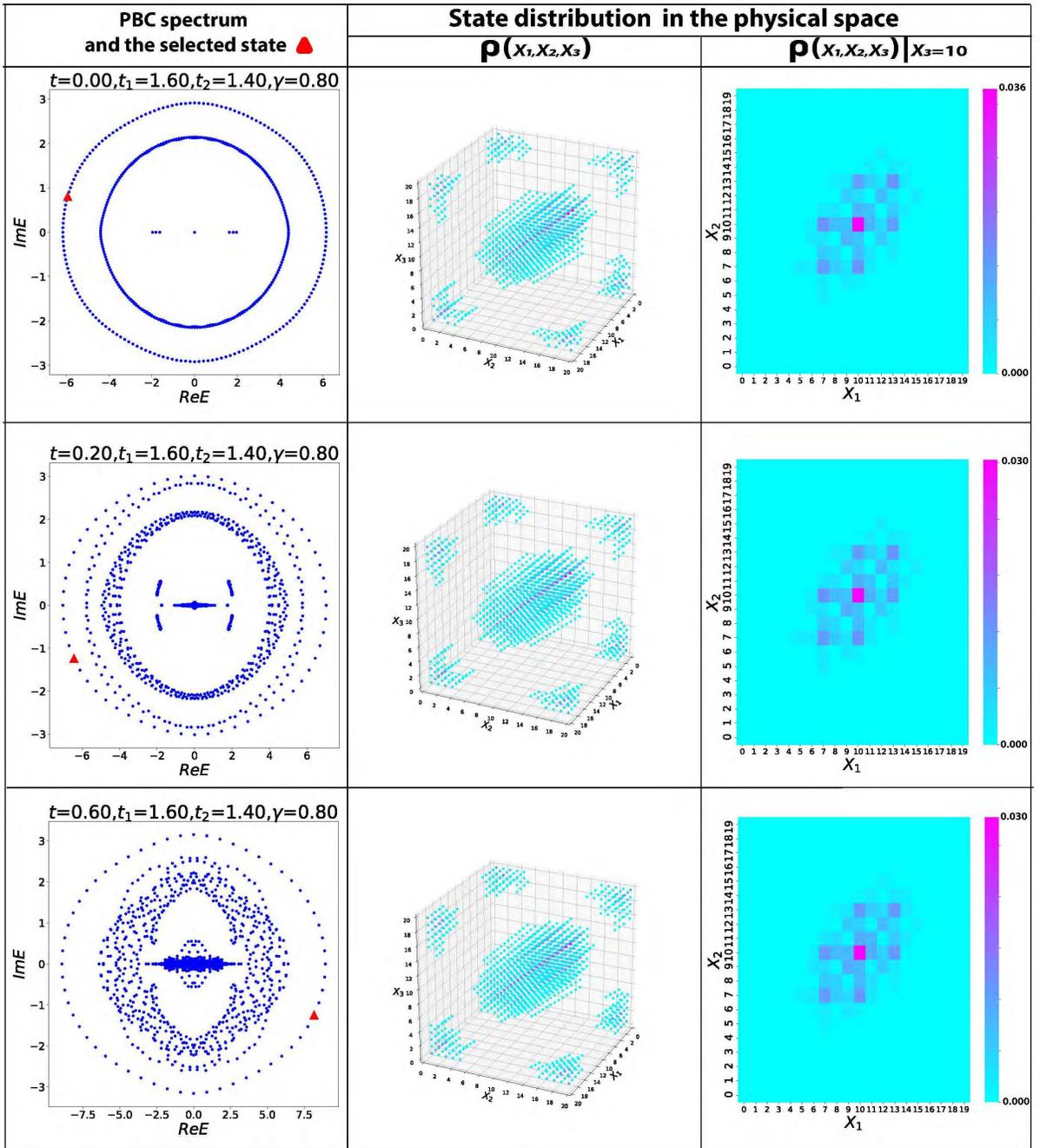


FIG. S11. Robustness of the localized clusters on the outer spectral loop, subject to different t . Left: The three-body PBC spectrum with $t = 0, 0.2, 0.6$; $t_1 = 1.6, t_2 = 1.4, \gamma = 0.8$. The length of the physical chain is $L = 20$. Middle: The physical spatial distribution of the selected state (red triangle), and Right: its corresponding cross-section at fixed $X_3 = 10$. In the 3D state distribution plot, the sites with $\rho(x_1, x_2, x_3) < 0.0001$ are left out for clarity. Clearly, the localized clusters remain almost invariant across different t .

Correlation density functions and the interaction-induced loop gap

In our two-body correlated hopping model H , the clustering properties of the bosonic eigenstates constitute the most important results. The extent of pair clustering of a given state ψ_m can be measured by the density-density correlation function

$$C_m(x_i, x_j) = \left| \langle \psi_m | c_i^\dagger c_i c_j^\dagger c_j | \psi_m \rangle \right|. \quad (\text{S12})$$

Since H obviously requires two bosons to occupy the same site either before or after they hop, we trivially have strong peaks in $C_m(x_i, x_j)$ for $i = j$. To better illustrate what happens when $x_i \neq x_j$, we shall omit plotting $C_m(x_i, x_i)$ in the following plots.

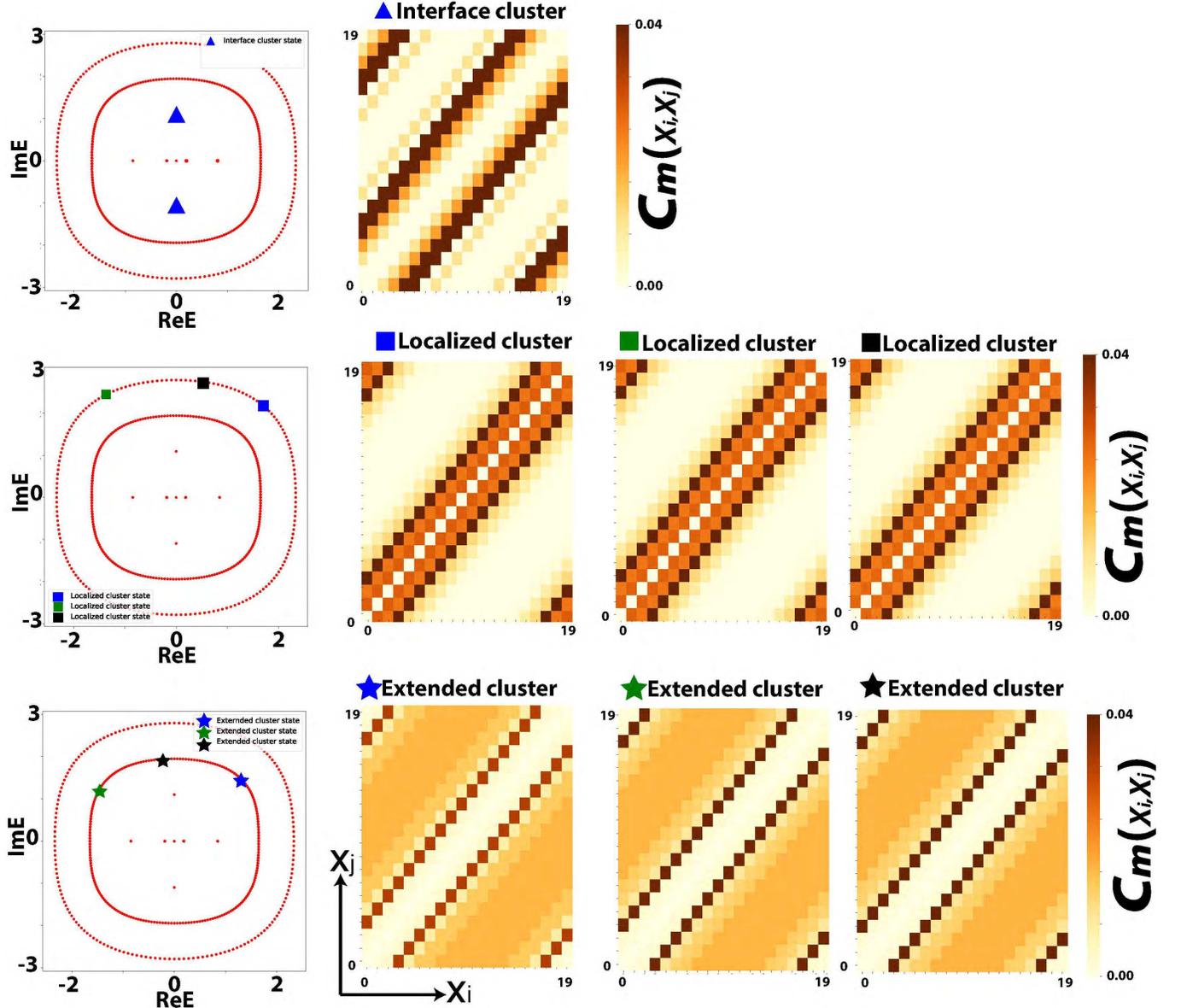


FIG. S12. The density-density correlation function $C_m(x_i, x_j)$ for illustrative PBC 3-boson clusters of various types, with $t_1 = 1.0, t_2 = 0.2, \gamma = 0.8, L = 20$. There is little variation between the two-body clustering behaviors of different states of the same type, even though their eigenenergies can be completely different (but still lying within the same loop).

From Fig. S12 above, what differentiates the outer (localized cluster states) and the inner loop states (extended cluster states) is their clustering properties. Qualitatively, the localized states are contributed more by short-range correlations, while the extended states are contributed more by the long-range correlations. To better quantify that, we introduce the pair correlation length of a given state ψ_k :

$$L_{\text{correlation}} = \sum_{i,j} |\langle \psi_k | (\hat{x}_i - \hat{x}_j) \hat{n}_i \hat{n}_j | \psi_k \rangle| \quad (\text{S13})$$

where the \hat{x}_i is the position operator and the $\hat{n}_i = c_i^\dagger c_i$ is the occupation number operator for the i -th site. Fig. S13 shows that the pair correlation length is clearly longer for the inner spectral loop (extended clusters), both for 3-boson and 4-boson clusters. The magnitude of $L_{\text{correlation}}$ scales like $\langle \hat{n}_i \hat{n}_j \Delta x \rangle$, and is largest if the state contains two dense “clumps” that are widely separated.

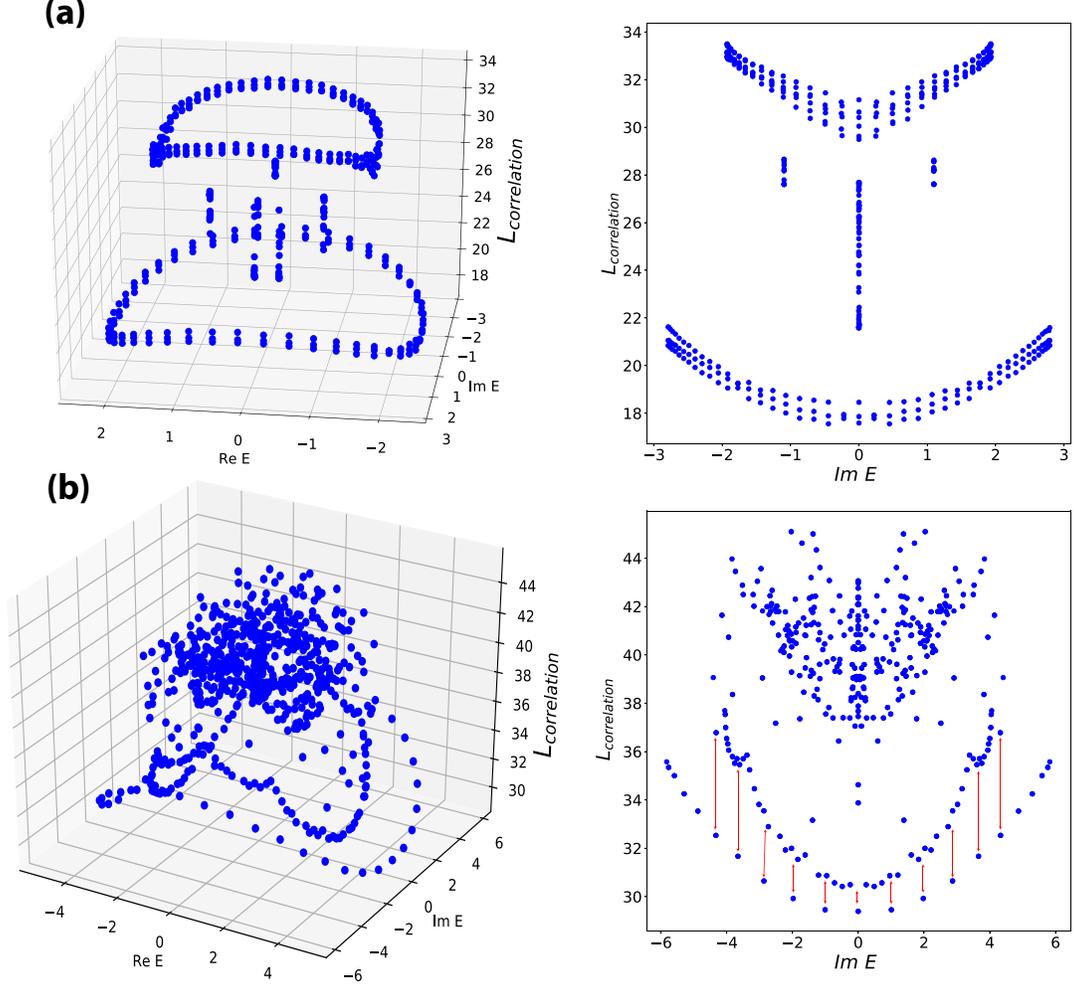


FIG. S13. Plots of the pair correlation lengths $L_{\text{correlation}}$ of all eigenstates as a function of the complex energy (Left), and as a function of the imaginary energy i.e. with collapsed real energy (Right). Shown are (a) 3-boson clusters and (b) 4-boson clusters, both corresponding to parameters $t_1 = 1, t_2 = 0.2, \gamma = 0.8$, but with $L = 15$ and $L = 10$ respectively. The contrast in $L_{\text{correlation}}$ is clear for the 3-boson clusters; for 4-boson clusters, it is also quite evident when viewed along the imaginary energy axis. The outer loop splits into two loops with slightly different pair correlation lengths, as indicated by the red vertical lines.

As a complementary measure of clustering, we also introduce the cluster density ρ_k for a given state ψ_k :

$$\rho_k = \sum_i \langle \psi_k | n_i^2 | \psi_k \rangle. \quad (\text{S14})$$

It measures the density of clusters for a particular state, and will be maximal for states where more bosons overlap at the same site. Fig. S14 presents the distribution of the cluster density for three-boson and four-boson PBC clusters. There is a clear gap from the outer to inner loops, indicating significantly more "clusters" on the outer loop cluster states. Intermediate states emerge between the outer and inner spectral loops in the four-boson case, delineating a saddle surface. Overall, the cluster density is evidently larger for outer loop states with large energies, consistent with the discussion in the main text.

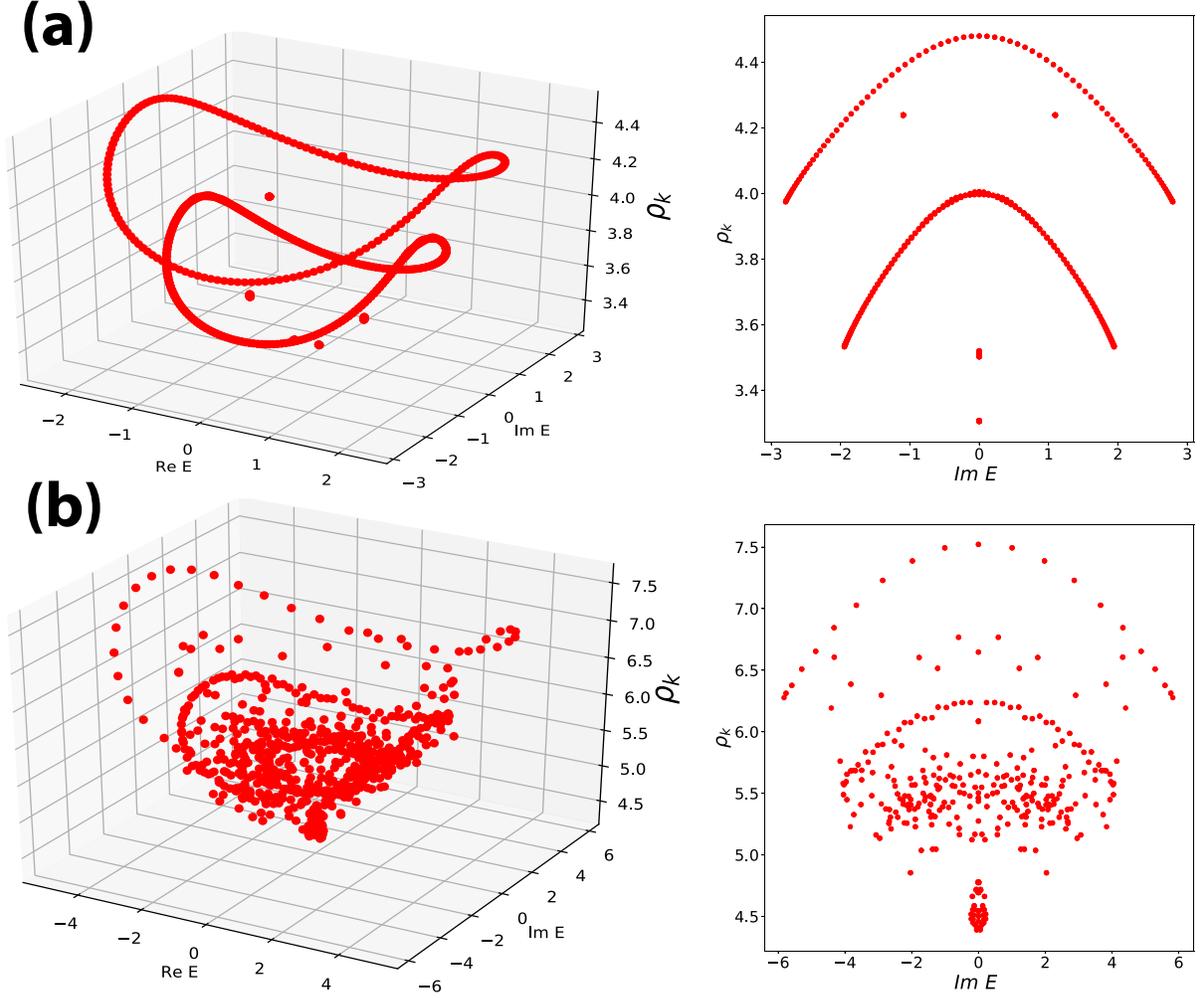


FIG. S14. The cluster density ρ_k plotted in the complex energy plane (Left) and also the imaginary energy (Right), so as to resolve the different loops more distinctively. Plotted in (a) and (b) are the 3-boson and 4-boson cases with $L = 20$ and $L = 10$ respectively, both with parameters $t_1 = 1.0, t_2 = 0.2, \gamma = 0.8$.

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- [1] Shunyu Yao and Zhong Wang, "Edge states and topological invariants of non-hermitian systems," *Physical review letters* **121**, 086803 (2018).
 [2] Ching Hua Lee and Ronny Thomale, "Anatomy of skin modes and topology in non-hermitian systems," *Physical Review B* **99**, 201103 (2019).