Encoder Hurwitz Integers: The Hurwitz integers that have the “division with small remainder” property

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Encoder Hurwitz Integers: The Hurwitz integers that have the "division with small remainder" property

Ramazan Duran and Murat Guzeltepe

Abstract. The modulo function is used to construct signal constellations over high-dimensional vector spaces such as Gaussian integers, Einstein-Jacobi integers, and quaternion integers. There is a one-to-one relationship between the Euclid division and the modulo function. If the Euclid division works for any quaternion integers, then it has the "division with small remainder" property. The Hurwitz integers form a subring of ring of quaternions. Hurwitz constellations are constructed by using modulo with primitive Hurwitz integers whose norm is a prime integer. If the norm of Hurwitz integers is not a prime integer, then we can not set up an isomorphism between the residual class ring of ordinary integers and the left (right) equivalence class of Hurwitz integers since the size of sets is to be different. In this study, to solve this problem, we define a new set, named encoder Hurwitz integers, consisting of primitive Hurwitz integers that have the "division with small remainder" property as an alternative instead of the set of primitive Hurwitz integers. Also, we investigate their performance over additive Gaussian noise (AWGN) channel by means of constellation figure of merit (CFM), average energy, minimum square Euclidean distance, and signal-to-noise ratio (SNR).

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1. Introduction

In recent years, many researchers in coding theory have investigated some special code constructions over groups, fields or rings i.e. finite algebraic construction. A Gaussian integer is a complex number that real and imaginary parts are in $\mathbb{Z}$. The set of Gaussian integers is denoted by $\mathbb{Z}[i]$. It is shown

by $\mathbb{Z}[i] = \{\alpha = \alpha_1 + \alpha_2 i : \alpha_1, \alpha_2 \in \mathbb{Z}, i^2 = -1\}$. Gaussian integers form a commutative ring and a subset of the complex numbers field, since they are closed under addition and multiplication. Let $\alpha = \alpha_1 + \alpha_2 i$ be a Gaussian integer. The conjugate of α is equal to $\bar{\alpha} = \alpha_1 - \alpha_2 i$, the norm of α is equal to $N(\alpha) = \alpha_1^2 + \alpha_2^2$, and the inverse of α is equal to $\alpha^{-1} = \frac{\bar{\alpha}}{N(\alpha)}$ where it is a non-zero norm. A Gaussian integer is a prime Gaussian integer if its norm is a prime integer. A Gaussian integer is a primitive Gaussian integer just if the greatest common divisor (gcd) of all components is one i.e. $\gcd(\alpha_1, \alpha_2) = 1$. In [14], first study about code constructions over Gaussian integers is presented by Huber. In other words, Huber discovered a new way to construct codes for two dimensional signals by virtue of Gaussian integers, i.e. the integral points on the complex plane [14]. His original idea is to regard a finite field as a residue class of the Gaussian integer ring modulo a prime Gaussian integer and, by Euclidean division, to get a unique element of minimal norm in each residue class, which represents each element of a finite field. Therefore, each element of a finite field can be represented by a Gaussian integer with the minimal Galois norm in the residue class, and the residual class set of the selected Gaussian integer is called a signal constellation. The "signal constellation" is a communication term in engineering. In mathematics, it is equivalent to the residue class set as a term. Codes over Gaussian integers were studied in papers to [2, 7, 8, 14, 16].

The quaternions form a four-dimensional vector space that is an algebra over the set of the real numbers ($\mathbb{R}$), and a number system that extends the complex numbers ($\mathbb{C}$). The coding techniques in [14] have been generalized to codes over quaternion integers. In [15], Ozen and Guzeltepe studied codes over some finite fields by using commutative quaternion integers. Codes over rings of quaternion integers were studied in papers to [3, 15, 16, 17, 18]. A Hurwitz integer is called a quaternion whose components are all either integers ($\mathbb{Z}$) or half-integers ($\mathbb{Z} + \frac{1}{2}$). In [10], Guzeltepe studied classes of linear codes over Hurwitz integers equipped with a new metric that refer as the Hurwitz metric. In [19], Rohweder et al. presented a new algebraic construction of finite sets of Hurwitz integers by a respective modulo function, and investigated performance for transmission over the additive white Gaussian noise (AWGN) channel. Codes over Hurwitz integers were studied in papers to [10, 11, 12, 13, 19].

This work is organized as follows: In Section II, we give some basics information about quaternions, Hurwitz integers and modulo function. In Section III, we define a new set, named "encoder Hurwitz integers", consisting of primitive Hurwitz integers that have the "division with small remainder" property as an alternative instead of the set of primitive Hurwitz integers. In Section IV, we investigate the performances of Hurwitz constellations that lie on encoder Hurwitz integers for transmission over the additive white Gaussian noise by means of constellation figure of merit (CFM), average energy, and signal-noise-to ratio (SNR). Finally, we conclude the paper in Section V.

2. Basics

This section gives basic information about quaternions, Hurwitz integers, and modulo function.

Definition 2.1. [5, 2.5.1 Definition], The Hamilton quaternion algebra over $\mathbb{R}$, is the unital associative algebra given by the following representation:

- i. $\mathbb{H}(\mathbb{R})$ is the free $\mathbb{R}$ -module over the symbols $1, i, j, k$, that is: $\mathbb{H}(\mathbb{R}) = \{\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k : \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}\}$,
- ii. 1 is the multiplicative identity,
- iii. $i^2 = j^2 = k^2 = ijk = -1$,
- iv. $ij = -ji = k, jk = -kj = i, ki = -ik = j$.

The definition is natural, in the sense that any unital ring homomorphism $\mathbb{R}_1 \rightarrow \mathbb{R}_2$ extends to a unital ring homomorphism $\mathbb{H}(\mathbb{R}_1) \rightarrow \mathbb{H}(\mathbb{R}_2)$ by mapping 1 to 1 , i to i , j to j and k to k .

Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a quaternion. Here α_1 is the real part, and $\alpha_2 i + \alpha_3 j + \alpha_4 k$ is the imaginary part. $\alpha_1, \alpha_2, \alpha_3$, and α_4 are components of quaternion α . The commutative property of multiplication does not hold for quaternion integers. However, if the imaginary parts of quaternion integers are parallel to each other (for example, let α and β be quaternions and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. $\alpha = \alpha_1 + \alpha_2(i+j+k)$ and $\beta = \beta_1 + \beta_2(i+j+k)$ are parallel to each other), then their product is commutative. Also, multiplication of α and α^{-1} is commutative since $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$.

Definition 2.2. $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ is a Lipschitz integer just if $\alpha_1, \alpha_2, \alpha_3$, and α_4 are all integers. The set of all Lipschitz integers is denoted by $\mathcal{H}(\mathbb{Z})$. It is written by

$$\mathcal{H}(\mathbb{Z}) = \{\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k : \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}\}. \quad (2.1)$$

Definition 2.3. $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ is a Hurwitz integer just if either $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$ or $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} + \frac{1}{2}$. The set of all Hurwitz integers is denoted by $\mathcal{H}$. It is shown by

$$\begin{aligned} \mathcal{H} &= \{\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k : \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} \text{ or } \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} + \frac{1}{2}\} \\ &= \mathcal{H}(\mathbb{Z}) \cup \mathcal{H}(\mathbb{Z} + \frac{1}{2}) \end{aligned} \quad (2.2)$$

It is a subset of $\mathbb{H}(\mathbb{R})$. The ring of Hurwitz integers forms a subring of the ring of all quaternions because of closed under multiplication and addition. Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a Hurwitz integer. The conjugate of α is $\bar{\alpha} = \alpha_1 - \alpha_2 i - \alpha_3 j - \alpha_4 k$, the norm of α is $N(\alpha) = \alpha \cdot \bar{\alpha} = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2$, and the inverse of α is $\alpha^{-1} = \frac{\bar{\alpha}}{N(\alpha)}$ where it is a non-zero norm.

Definition 2.4. $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ a prime Hurwitz integer if its norm is a prime integer.

Definition 2.5. $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ is a primitive Hurwitz integer just if the greatest common divisor (gcd) $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is one. In particular, all components are integer.

Definition 2.6. A notation for rounding to the nearest integer is denoted by $\lfloor \cdot \rfloor$. It is defined as rounding a rational number to the integer closest to it. All components in a quaternion are separately rounding to the integer closest to it.

Note that the rounding is done in the direction $+\infty$ in this study.

Example 2.1. Let $\alpha = \frac{5}{4} + \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k \in \mathbb{H}(\mathbb{R})$. If we use the rounding notation for α , then, we obtain a Hurwitz integer whose components are all integers. That is,

$$\begin{aligned} \lfloor \alpha \rfloor &= \lfloor \frac{5}{4} + \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k \rfloor \\ &= \lfloor \frac{5}{4} \rfloor + \lfloor \frac{1}{2}i \rfloor + \lfloor -\frac{1}{2}j \rfloor + \lfloor -\frac{5}{2}k \rfloor \\ &= 1 + 1 \cdot i + 0 \cdot j + (-2) \cdot k \\ &= 1 + i - 2k. \end{aligned} \quad (2.3)$$

Definition 2.7. Let π be a Hurwitz integer. If there exist $\lambda \in \mathcal{H}$ such that $q_1 - q_2 = \lambda\pi$, then $q_1, q_2 \in \mathcal{H}$ are said to be right congruent modulo π . This relation is denoted by $q_1 \equiv_r q_2$.

This relation $q_1 \equiv_r q_2$ is an equivalence relation. The elements in the right ideal

$$\langle \pi \rangle = \{\lambda\pi : \lambda \in \mathcal{H}\} \quad (2.4)$$

define a normal subgroup of the additive group of the ring $\mathcal{H}$. The set of cosets to $\langle \pi \rangle$ in $\mathcal{H}$ defines Abelian group denoted by $\mathcal{H}_\pi = \mathcal{H} / \langle \pi \rangle$. Analogous results are valid for left congruences modulo π . In this study, we use left congruences modulo π . Note that we consider Hurwitz integers such that their norms are either a prime integer or not a prime integer in this study.

Definition 2.8. The modulo function $\mu : \mathbb{Z}_{N(\pi)} \rightarrow \mathcal{H}_\pi$ is defined by

$$\mu_\pi(z) = z \bmod \pi = z - \pi \cdot \lfloor \pi^{-1}z \rfloor = z - \pi \lfloor \frac{\bar{\pi}z}{N(\pi)} \rfloor \quad (2.5)$$

where π is a primitive Hurwitz integer and $z \in \mathbb{Z}_{N(\pi)}$.

Here $\mathbb{Z}_{N(\pi)}$ is the well-known residual class ring of ordinary integers with $N(\pi)$ elements and $\mu_\pi(z)$ is given remainder of $z(z \in \mathbb{Z}_{N(\pi)})$ with respect to modulo π . The set of quotient ring of the Hurwitz integers is denoted by $\mathcal{H}_\pi = \{z \bmod \pi : z \in \mathbb{Z}_{N(\pi)}\}$. The set $\mathcal{H}_\pi$ contains $N(\pi)$ elements. If π is a prime Hurwitz integer, then the modulo function μ defines a bijective mapping from $\mathbb{Z}_{N(\pi)}$ into $\mathcal{H}_\pi$ which is a four-dimensional signal space. Therefore, the modulo function μ is a ring isomorphism between $\mathbb{Z}_{N(\pi)}$ and $\mathcal{H}_\pi$. If π is a primitive Hurwitz integer, the modulo function μ is a group isomorphism with respect to addition between $\mathbb{Z}_{N(\pi)}$ and $\mathcal{H}_\pi$. Because there exists a inverse map [1] and we have $\mu(z_1 + z_2) = \mu(z_1) + \mu(z_2)$ and $\mu(z_1 z_2) = \mu(z_1)\mu(z_2)$ for any $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$.

In engineering, the "signal constellation" has been used as a communication term. In mathematics, the "signal constellation" is equivalent to the residue class set as a term. In the rest of this study, we use the "signal constellation" term instead of the "residue class set" term. You can find more

details which related to the arithmetics properties of quaternions and Hurwitz integers in [5],[4].

3. Encoder Hurwitz Integers

There is a one-to-one relationship between the Euclid division algorithm and the modulo function. It is easy to see the reasoning for that (Hint: the rounding notation). The Euclid division algorithm says that for every $a, b \in \mathbb{Z}$ there exist unique integers q and r such that $a = bq + r$, $0 \leq r < |b|$. Here a is the dividend, b is the divisor, q is the quotient, r is the remainder, and $|\cdot|$ is the symbol for absolute value. If we generalize the Euclid division algorithm for Hurwitz integers, then it sounds as follows. For every $\alpha, \pi \in \mathcal{H}$ such that their components are in $\mathbb{Z}$, there exists unique Hurwitz integers β and γ such that $\alpha = \pi\beta + \gamma$, $0 \leq N(\gamma) \leq N(\pi)$. For every $\alpha, \pi \in \mathcal{H}$ such that their components are in $\mathbb{Z} + \frac{1}{2}$, there exists unique Hurwitz integers β and γ such that $\alpha = \pi\beta + \gamma$, $0 \leq N(\gamma) < N(\pi)$. Namely, the Euclid division algorithm does not work for Hurwitz integers such that their components are in $\mathbb{Z}$ because of $N(\gamma) \leq N(\pi)$. On the other hand, the Euclid division algorithm works for Hurwitz integers such that their components are in $\mathbb{Z} + \frac{1}{2}$ because of $N(\gamma) < N(\pi)$. On the other hand, the Euclid division algorithm works for either odd Hurwitz integers or prime Hurwitz integers. Also, in this section, we will see that the Euclid division algorithm works for even Hurwitz integers that have a hold on some properties but do not work for some even Hurwitz integers. In general, the Euclid division algorithm does not work for Hurwitz integers. If the Euclid division algorithm works for a Hurwitz integer, then it has the "division with small remainder" property. So, the primitive Hurwitz integers such that their components are in $\mathbb{Z}$ do not have the "division with small remainder" property. On the other hand, the primitive Hurwitz integers such that their components are in $\mathbb{Z} + \frac{1}{2}$ have the "division with small remainder" property. If we want to construct a Hurwitz constellation with $N(\pi)$ elements, then the set of primitive Hurwitz integers can be cause some problems such as having less elements than $N(\pi)$ elements. This is an unwanted state to set up an isomorphism between $\mathbb{Z}_{N(\pi)}$ and $\mathcal{H}_\pi$. Because the modulo function is not an injective function. So, we will be better to study a different set instead of the set of primitive Hurwitz integers. Therefore, we define a new set, which is consist of primitive Hurwitz integers that have the "division with small remainder" property, in this study.

First, let us show why we shouldn't work on the set of primitive Hurwitz integers with the following lemma and proposition. Then, let us define a new set, which is consist of primitive Hurwitz integers that have the "division with small remainder" property. Note that the Euclid division algorithm works for Hurwitz integers such that all components are in $\mathbb{Z} + \frac{1}{2}$. The following lemma and proposition hint at why the Euclid division algorithm does not work for Hurwitz integers such that all components are in $\mathbb{Z}$. In particular, the

following proposition is help us to look for the Hurwitz integers which have the "division with small remainder" property.

Proposition 3.1. *Let π be a primitive Hurwitz integer whose components are all odd integers. Then,*

$$N(\mu_\pi(\frac{N(\pi)}{2})) = N(\pi). \quad (3.1)$$

Proof. Let $\pi = \pi_1 + \pi_2 i + \pi_3 j + \pi_4 k$ be a Hurwitz integer whose components are all odd integers such that $\gcd(\pi_1, \pi_2, \pi_3, \pi_4) = 1$. By equation 2.5,

$$\begin{aligned} \mu_\pi(\frac{N(\pi)}{2}) &= \frac{N(\pi)}{2} - \pi \lfloor \frac{\overline{\pi} N(\pi)}{N(\pi)} \rfloor \\ &= \frac{N(\pi)}{2} - \pi \lfloor \frac{\pi}{2} \rfloor \\ &= \frac{N(\pi)}{2} - (\pi_1 + \pi_2 i + \pi_3 j + \pi_4 k) \lfloor \frac{\pi_1 - \pi_2 i - \pi_3 j - \pi_4 k}{2} \rfloor. \end{aligned} \quad (3.2)$$

$\lfloor \frac{\pi_1}{2} \rfloor = ((\frac{\pi_1}{2}) + \frac{1}{2})$, $\lfloor \frac{-\pi_2}{2} \rfloor = ((\frac{-\pi_2}{2}) + \frac{1}{2})$, $\lfloor \frac{-\pi_3}{2} \rfloor = ((\frac{-\pi_3}{2}) + \frac{1}{2})$ and $\lfloor \frac{-\pi_4}{2} \rfloor = ((\frac{-\pi_4}{2}) + \frac{1}{2})$ since π_1, π_2, π_3 and π_4 are odd integers. Hereby,

$$\begin{aligned} \mu_\pi(\frac{N(\pi)}{2}) &= \frac{N(\pi)}{2} - (\pi_1 + \pi_2 i + \pi_3 j + \pi_4 k) [(\frac{\pi_1}{2} + \frac{1}{2}) + (\frac{-\pi_2}{2} + \frac{1}{2})i \\ &\quad + (\frac{-\pi_3}{2} + \frac{1}{2})j + (\frac{-\pi_4}{2} + \frac{1}{2})k] \\ &= \frac{N(\pi)}{2} - (\frac{\pi_1 + \pi_1}{2} + (\frac{-\pi_1 \pi_2 + \pi_1}{2})i + (\frac{-\pi_1 \pi_3 + \pi_1}{2})j \\ &\quad + (\frac{-\pi_1 \pi_4 + \pi_1}{2})k + (\frac{\pi_2 \pi_1 + \pi_2}{2})i + \frac{\pi_2^2 - \pi_2}{2} + (\frac{-\pi_2 \pi_3 + \pi_2}{2})k \\ &\quad + (\frac{\pi_2 \pi_4 - \pi_2}{2})j + (\frac{\pi_3 \pi_1 + \pi_3}{2})j + (\frac{\pi_3 \pi_2 - \pi_3}{2})k + (\frac{\pi_3^2 - \pi_3}{2}) \\ &\quad + (\frac{-\pi_3 \pi_4 + \pi_3}{2})i + (\frac{\pi_4 \pi_1 + \pi_4}{2})k + (\frac{-\pi_4 \pi_2 + \pi_4}{2})j \\ &\quad + (\frac{\pi_4 \pi_3 - \pi_4}{2})i + \frac{\pi_4^2 - \pi_4}{2} \\ &= \frac{N(\pi)}{2} - [\frac{\pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2 + \pi_1 - \pi_2 - \pi_3 - \pi_4}{2} \\ &\quad + (\frac{-\pi_1 \pi_2 + \pi_1 + \pi_2 \pi_1 + \pi_2 - \pi_3 \pi_4 + \pi_3 + \pi_4 \pi_3 - \pi_4}{2})i \\ &\quad + (\frac{-\pi_1 \pi_3 + \pi_1 + \pi_2 \pi_4 - \pi_2 + \pi_3 \pi_1 + \pi_3 - \pi_4 \pi_2 + \pi_4}{2})j \\ &\quad + (\frac{-\pi_1 \pi_4 + \pi_1 - \pi_2 \pi_3 + \pi_2 + \pi_3 \pi_2 - \pi_3 + \pi_4 \pi_1 + \pi_4}{2})k] \\ &= \frac{N(\pi)}{2} - \frac{N(\pi)}{2} - \frac{\pi_1 - \pi_2 - \pi_3 - \pi_4}{2} - (\frac{\pi_1 + \pi_2 + \pi_3 - \pi_4}{2})i \\ &\quad - (\frac{\pi_1 - \pi_2 + \pi_3 + \pi_4}{2})j - (\frac{\pi_1 + \pi_2 - \pi_3 + \pi_4}{2})k \\ &= -\frac{\pi_1 - \pi_2 - \pi_3 - \pi_4}{2} - (\frac{\pi_1 + \pi_2 + \pi_3 - \pi_4}{2})i - (\frac{\pi_1 - \pi_2 + \pi_3 + \pi_4}{2})j \\ &\quad - (\frac{\pi_1 + \pi_2 - \pi_3 + \pi_4}{2})k. \end{aligned}$$

Hereby,

$$\begin{aligned} N(\mu_\pi(\frac{N(\pi)}{2})) &= N(\frac{-\pi_1 + \pi_2 + \pi_3 + \pi_4}{2} + (\frac{-\pi_1 - \pi_2 - \pi_3 + \pi_4}{2})i \\ &\quad + (\frac{-\pi_1 + \pi_2 - \pi_3 - \pi_4}{2})j + (\frac{-\pi_1 - \pi_2 + \pi_3 - \pi_4}{2})k) \\ &= \frac{4(\pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2)}{4} \\ &= \pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2 \end{aligned} \quad (3.3)$$

Since $N(\pi) = \pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2$, then

$$N(\mu_\pi(\frac{N(\pi)}{2})) = N(\pi). \quad (3.4)$$

This completes the proof. $\square$

Lemma 3.1. *Let π be a primitive Hurwitz integer such that all components are in $\mathbb{Z}$, and α be a Hurwitz integer. Then,*

$$N(\mu_\pi(\alpha)) \leq N(\pi) \quad (3.5)$$

with respect to equation 2.5.

Proof. Let π be a primitive Hurwitz integer such that all components are in $\mathbb{Z}$, and $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a Hurwitz integer. By equation 2.5,

$$\begin{aligned} \mu_\pi(\alpha) &= \alpha - \pi \lfloor \frac{\bar{\pi}\alpha}{N(\pi)} \rfloor \\ \pi^{-1}\mu_\pi(\alpha) &= \pi^{-1}\alpha - \pi^{-1}\pi \lfloor \frac{\bar{\pi}\alpha}{\pi\pi} \rfloor \\ &= \pi^{-1}\alpha - \lfloor \frac{\alpha}{\pi} \rfloor \\ &= \pi^{-1}(\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k) - \lfloor \frac{\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k}{\pi} \rfloor \\ &= \pi^{-1}\alpha_1 + \pi^{-1}\alpha_2 i + \pi^{-1}\alpha_3 j + \pi^{-1}\alpha_4 k \\ &\quad - \lfloor \pi^{-1}\alpha_1 + \pi^{-1}\alpha_2 i + \pi^{-1}\alpha_3 j + \pi^{-1}\alpha_4 k \rfloor \\ &= \pi^{-1}\alpha_1 + \pi^{-1}\alpha_2 i + \pi^{-1}\alpha_3 j + \pi^{-1}\alpha_4 k \\ &\quad - \lfloor \pi^{-1}\alpha_1 \rfloor - \lfloor \pi^{-1}\alpha_2 \rfloor i - \lfloor \pi^{-1}\alpha_3 \rfloor j - \lfloor \pi^{-1}\alpha_4 \rfloor k. \\ &= \pi^{-1}\alpha_1 - \lfloor \pi^{-1}\alpha_1 \rfloor + (\pi^{-1}\alpha_2 - \lfloor \pi^{-1}\alpha_2 \rfloor)i \\ &\quad + (\pi^{-1}\alpha_3 - \lfloor \pi^{-1}\alpha_3 \rfloor)j + (\pi^{-1}\alpha_4 - \lfloor \pi^{-1}\alpha_4 \rfloor)k. \end{aligned} \quad (3.6)$$

Therefore,

$$\begin{aligned} |\pi^{-1}\alpha_1 - \lfloor \pi^{-1}\alpha_1 \rfloor| &\leq \frac{1}{2} \\ |\pi^{-1}\alpha_2 - \lfloor \pi^{-1}\alpha_2 \rfloor| &\leq \frac{1}{2} \\ |\pi^{-1}\alpha_3 - \lfloor \pi^{-1}\alpha_3 \rfloor| &\leq \frac{1}{2} \\ |\pi^{-1}\alpha_4 - \lfloor \pi^{-1}\alpha_4 \rfloor| &\leq \frac{1}{2}. \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} (\pi^{-1}\alpha_1 - \lfloor \pi^{-1}\alpha_1 \rfloor)^2 &\leq \left(\frac{1}{2}\right)^2 \\ (\pi^{-1}\alpha_2 - \lfloor \pi^{-1}\alpha_2 \rfloor)^2 &\leq \left(\frac{1}{2}\right)^2 \\ (\pi^{-1}\alpha_3 - \lfloor \pi^{-1}\alpha_3 \rfloor)^2 &\leq \left(\frac{1}{2}\right)^2 \\ (\pi^{-1}\alpha_4 - \lfloor \pi^{-1}\alpha_4 \rfloor)^2 &\leq \left(\frac{1}{2}\right)^2 \end{aligned} \quad (3.8)$$

Hereby,

$$\begin{aligned} N(\pi^{-1}\mu_\pi(\alpha)) &= (\pi^{-1}\alpha_1 - \lfloor \pi^{-1}\alpha_1 \rfloor)^2 + (\pi^{-1}\alpha_2 - \lfloor \pi^{-1}\alpha_2 \rfloor)^2 \\ &\quad + (\pi^{-1}\alpha_3 - \lfloor \pi^{-1}\alpha_3 \rfloor)^2 + (\pi^{-1}\alpha_4 - \lfloor \pi^{-1}\alpha_4 \rfloor)^2 \\ &\leq \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ N(\pi^{-1})N(\mu_\pi(\alpha)) &\leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \\ \frac{1}{N(\pi)}N(\mu_\pi(\alpha)) &\leq 1 \\ N(\mu_\pi(\alpha)) &\leq N(\pi). \end{aligned} \quad (3.9)$$

This completes the proof. $\square$

Example 3.1. Let $\pi = 3 + i + j + 3k$ be a Hurwitz integer. Also, π is a primitive Hurwitz integer because of $\gcd(3, 1, 1, 3) = 1$. By equation 2.5,

Hurwitz constellation is

$$\mathcal{H}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0, \mu_\pi(1) = 1, \mu_\pi(2) = 2, \mu_\pi(3) = 3, \\ \mu_\pi(4) = -2 - 2j, \mu_\pi(5) = -1 - 2j, \mu_\pi(6) = -2j, \\ \mu_\pi(7) = 1 - 2j, \mu_\pi(8) = 2 - 2j, \mu_\pi(9) = 3 - 2j, \\ \mu_\pi(10) = 1 - i - 3j - 3k, \mu_\pi(11) = -3 + 2j, \\ \mu_\pi(12) = -2 + 2j, \mu_\pi(13) = -1 + 2j, \mu_\pi(14) = 2j, \\ \mu_\pi(15) = 1 + 2j, \mu_\pi(16) = 2 + 2j, \mu_\pi(17) = -3, \\ \mu_\pi(18) = -2, \mu_\pi(19) = -1 \end{array} \right\}. \quad (3.10)$$

This set contains 20 elements because of $N(\pi) = 3^2 + 1^2 + 1^2 + 3^2 = 20$. Also, $N(\mu_\pi(10)) = N(\pi)$ because of $N(\mu_\pi(10)) = 3^2 + (1)^2 + (1)^2 + (3)^2 = 20$. Consequently, Hurwitz integer $\pi = 3 + i + j + 3k$ does not have the "division with small remainder" property. And so on.

Below definition 3.1, we define with respect to modulo function in equation 2.5. Also, note that the Hurwitz integers whose components are all half-integers are a natural encoder Hurwitz integers.

Definition 3.1. Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a Hurwitz integer. α Hurwitz integer is called an encoder Hurwitz integer if it is satisfying the following conditions;

- $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}^+$,
- $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are not same parity, i.e. $\alpha_1, \alpha_2, \alpha_3$ and α_4 all together neither even integers nor odd integers,
- $\gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 1$.

Or,

- $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (\mathbb{Z}^+ \cup \{0\}) + \frac{1}{2}$,

In other words, $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ primitive Hurwitz integer such that its all components are not odd integers, or $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (\mathbb{Z} \cup \{0\}) + \frac{1}{2}$, is called an encoder Hurwitz integer.

The definition 3.1 is a flexible definition. Namely, the set of encoder Hurwitz integers are expandable or collapsible depending on the used modulo technique. We show that the modulo function μ defined between $\mathbb{Z}_{N(\pi)}$ and $\mathcal{H}_\pi$ by equation 2.5 is a ring isomorphism with the following theorems.

Theorem 3.1. Let π be an encoder Hurwitz integer, and $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. The modulo function $\mu : \mathbb{Z}_{N(\pi)} \rightarrow \mathcal{H}_\pi$ is a ring homomorphism with respect to $\mu_\pi(z_1 + z_2) = (\mu_\pi(z_1) + \mu_\pi(z_2)) \bmod \pi$ and $\mu_\pi(z_1 z_2) = (\mu_\pi(z_1) \mu_\pi(z_2)) \bmod \pi$.

Proof. Let π be an encoder Hurwitz integer and, $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. Define $\mu : \mathbb{Z}_{N(\pi)} \rightarrow \mathcal{H}_\pi$ modulo function by $\mu_\pi(z) = z \bmod \pi = z - \pi \lfloor \frac{\bar{\pi} z}{N(\pi)} \rfloor$. By equation 2.5,

$$\mu_\pi(z_1 + z_2) = z_1 + z_2 - \pi \lfloor \frac{\bar{\pi}(z_1 + z_2)}{N(\pi)} \rfloor. \quad (3.11)$$

If $\mu_\pi(z_1) = z_1 \bmod \pi$ and $\mu_\pi(z_2) = z_2 \bmod \pi$, then, there exist the Hurwitz integers λ_1 and λ_2 such that $z_1 = \pi\lambda_1 + \mu_\pi(z_1)$ and $z_2 = \pi\lambda_2 + \mu_\pi(z_2)$, respectively. Therefore,

$$\begin{aligned}\mu_\pi(z_1 + z_2) &= \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2) - \pi \left\lfloor \frac{\overline{\pi\lambda_1 + \mu_\pi(z_1)} + \overline{\pi\lambda_2 + \mu_\pi(z_2)}}{N(\pi)} \right\rfloor \\ &= \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2) - \pi \left\lfloor \frac{\overline{\pi\lambda_1} + \overline{\pi\mu_\pi(z_1)} + \overline{\pi\lambda_2} + \overline{\pi\mu_\pi(z_2)}}{N(\pi)} \right\rfloor \\ &= \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2) - \pi \left\lfloor \lambda_1 + \lambda_2 + \frac{\overline{\pi\mu_\pi(z_1)} + \overline{\pi\mu_\pi(z_2)}}{N(\pi)} \right\rfloor.\end{aligned}\tag{3.12}$$

$\lfloor \lambda_1 \rfloor = \lambda_1$ and $\lfloor \lambda_2 \rfloor = \lambda_2$ since λ_1 and λ_2 are the Hurwitz integers whose all components are in $\mathbb{Z}$. Hereby,

$$\begin{aligned}\mu_\pi(z_1 + z_2) &= \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2) - \pi\lambda_1 - \pi\lambda_2 - \pi \left\lfloor \frac{\overline{\pi\mu_\pi(z_1)} + \overline{\pi\mu_\pi(z_2)}}{N(\pi)} \right\rfloor \\ &= \mu_\pi(z_1) + \mu_\pi(z_2) - \pi \left\lfloor \frac{\overline{\pi\mu_\pi(z_1)} + \overline{\pi\mu_\pi(z_2)}}{N(\pi)} \right\rfloor.\end{aligned}\tag{3.13}$$

So,

$$\mu_\pi(z_1 + z_2) = (\mu_\pi(z_1) + \mu_\pi(z_2)) \bmod \pi.\tag{3.14}$$

On the other hand,

$$\begin{aligned}\mu_\pi(z_1 z_2) &= z_1 z_2 \bmod \pi \\ &= z_1 z_2 - \pi \left\lfloor \frac{\overline{z_1 z_2}}{N(\pi)} \right\rfloor.\end{aligned}\tag{3.15}$$

If $\mu_\pi(z_1) = z_1 \bmod \pi$ and $\mu_\pi(z_2) = z_2 \bmod \pi$, then, there exists the Hurwitz integers λ_1 and λ_2 such that $z_1 = \pi\lambda_1 + \mu_\pi(z_1)$ and $z_2 = \pi\lambda_2 + \mu_\pi(z_2)$, respectively. Then,

$$\begin{aligned}\mu_\pi(z_1 z_2) &= (\pi\lambda_1 + \mu_\pi(z_1))(\pi\lambda_2 + \mu_\pi(z_2)) - \pi \left\lfloor \frac{\overline{(\pi\lambda_1 + \mu_\pi(z_1))(\pi\lambda_2 + \mu_\pi(z_2))}}{N(\pi)} \right\rfloor \\ &= \pi\lambda_1\pi\lambda_2 + \pi\lambda_1\mu_\pi(z_2) + \mu_\pi(z_1)\pi\lambda_2 + \mu_\pi(z_1)\mu_\pi(z_2) \\ &\quad - \pi \left\lfloor \frac{\overline{\pi\lambda_1\pi\lambda_2} + \overline{\pi\lambda_1\mu_\pi(z_2)} + \overline{\pi\mu_\pi(z_1)\pi\lambda_2} + \overline{\pi\mu_\pi(z_1)\mu_\pi(z_2)}}{N(\pi)} \right\rfloor \\ &= \pi\lambda_1\pi\lambda_2 + \pi\lambda_1\mu_\pi(z_2) + \mu_\pi(z_1)\pi\lambda_2 + \mu_\pi(z_1)\mu_\pi(z_2) \\ &\quad - \pi \left\lfloor \lambda_1\pi\lambda_2 + \lambda_1\mu_\pi(z_2) + \frac{\overline{\pi\mu_\pi(z_1)\pi\lambda_2}}{N(\pi)} + \frac{\overline{\pi\mu_\pi(z_1)\mu_\pi(z_2)}}{N(\pi)} \right\rfloor.\end{aligned}$$

Since $\lambda_1\pi\lambda_2$, $\lambda_1\mu_\pi(z_2)$ and $\frac{\overline{\pi\mu_\pi(z_1)\pi\lambda_2}}{N(\pi)}$ are Hurwitz integers whose components are all integers, then

$$\begin{aligned}\mu_\pi(z_1 z_2) &= \pi\lambda_1\pi\lambda_2 + \pi\lambda_1\mu_\pi(z_2) + \mu_\pi(z_1)\pi\lambda_2 + \mu_\pi(z_1)\mu_\pi(z_2) - \pi\lambda_1\pi\lambda_2 \\ &\quad - \pi\lambda_1\mu_\pi(z_2) - \pi \frac{\overline{\pi\mu_\pi(z_1)\pi\lambda_2}}{N(\pi)} - \pi \left\lfloor \frac{\overline{\pi\mu_\pi(z_1)\mu_\pi(z_2)}}{N(\pi)} \right\rfloor \\ &= \mu_\pi(z_1)\mu_\pi(z_2) - \pi \left\lfloor \frac{\overline{\pi\mu_\pi(z_1)\mu_\pi(z_2)}}{N(\pi)} \right\rfloor.\end{aligned}$$

So, $\mu_\pi(z_1 z_2) = \mu_\pi(z_1)\mu_\pi(z_2) \bmod \pi$. Consequently, μ function is a ring homomorphism. This completes this proof. $\square$

Theorem 3.2. *Let π be an encoder Hurwitz integer. Then,*

$$\mathbb{Z}_{N(\pi)} \cong \mathcal{H}_\pi$$

with respect to equality 2.5.

Proof. Let π be an encoder Hurwitz integer and, $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. The modulo function defines a mapping from $\mathbb{Z}_{N(\pi)}$ to $\mathcal{H}_\pi$. This mapping is $\mu : \mathbb{Z}_{N(\pi)} \rightarrow \mathcal{H}_\pi$:

$$\mu_\pi(z) = z \bmod \pi = \gamma = z - \pi \lfloor \frac{\bar{\pi}z}{N(\pi)} \rfloor. \quad (3.16)$$

According to theorem 3.1, μ function is a ring homomorphism. This mapping is a surjective ring homomorphism because of $\text{Im}\mu = \{\mu_\pi(z) : z \in \mathbb{Z}_{N(\pi)}\} = \mathcal{H}_\pi$. If $z = 0$ where $z \in \mathbb{Z}_{N(\pi)}$, then

$$\begin{aligned} \mu_\pi(0) &= 0 - \pi \lfloor \frac{\bar{\pi}0}{N(\pi)} \rfloor \\ &= 0 - \pi \lfloor 0 \rfloor \\ &= 0 - \pi 0 \\ &= 0. \end{aligned} \quad (3.17)$$

If $z \neq 0$ where $z \in \mathbb{Z}_{N(\pi)}$, then $\mu_\pi(z)$ is to greater or equal than 1. Hereby, this mapping is an injective ring homomorphism because of $\text{Ker}\mu = \{z \in \mathbb{Z}_{N(\pi)} : \mu_\pi(z) = 0\} = \{z \in \mathbb{Z}_{N(\pi)} : z = 0\} = \{0\}$. μ function is a ring isomorphism since it is both a surjective ring homomorphism and an injective ring homomorphism, i.e. $\mathbb{Z}_{N(\pi)} \cong \mathcal{H}_\pi$. $\square$

The following proposition hint at why the encoder Hurwitz integers have the "division small remainder" property.

Proposition 3.2. *Let π be an encoder Hurwitz integer. Then,*

$$N(\mu_\pi(z)) < N(\pi). \quad (3.18)$$

Proof. We shall prove this theorem by cases. Let $\pi = \pi_1 + \pi_2 i + \pi_3 j + \pi_4 k$ be an encoder Hurwitz integer.

Case : 1 Let $\pi = \pi_1 + \pi_2 i + \pi_3 j + \pi_4 k$ be an encoder Hurwitz integer such that π_1 is an even integer, and π_2, π_3 and π_4 are odd integers. So, $N(\pi)$ is an odd integer. By equation 2.5,

$$\begin{aligned} \mu_\pi(z)\pi^{-1} &= \pi^{-1}z - \pi^{-1}\pi \lfloor \pi^{-1}z \rfloor \\ &= \pi^{-1}z - \lfloor \pi^{-1}z \rfloor \\ &= \frac{\bar{\pi}z}{N(\pi)} - \lfloor \frac{\bar{\pi}z}{N(\pi)} \rfloor. \end{aligned} \quad (3.19)$$

Then,

$$\begin{aligned} \mu_\pi(z)\pi^{-1} &= \frac{\pi_1 z}{N(\pi)} - (\frac{\pi_2 z}{N(\pi)})i - (\frac{\pi_3 z}{N(\pi)})j - (\frac{\pi_4 z}{N(\pi)})k \\ &\quad - \lfloor \frac{\pi_1 z}{N(\pi)} \rfloor - \lfloor \frac{\pi_2 z}{N(\pi)} \rfloor i - \lfloor \frac{\pi_3 z}{N(\pi)} \rfloor j - \lfloor \frac{\pi_4 z}{N(\pi)} \rfloor k \\ &= \frac{\pi_1 z}{N(\pi)} - \lfloor \frac{\pi_1 z}{N(\pi)} \rfloor - (\frac{\pi_2 z}{N(\pi)} - \lfloor \frac{\pi_2 z}{N(\pi)} \rfloor)i \\ &\quad - (\frac{\pi_3 z}{N(\pi)} - \lfloor \frac{\pi_3 z}{N(\pi)} \rfloor)j - (\frac{\pi_4 z}{N(\pi)} - \lfloor \frac{\pi_4 z}{N(\pi)} \rfloor)k \end{aligned} \quad (3.20)$$

Since π_1 is an even integer, π_2, π_3 and π_4 are odd integers, $N(\pi)$ is an odd integer, and $\frac{N(\pi)}{2}$ is not an integer, then

$$\begin{aligned} 0 &\leq \left| \frac{\pi_1 z}{N(\pi)} - \lfloor \frac{\pi_1 z}{N(\pi)} \rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_2 z}{N(\pi)} - \lfloor \frac{\pi_2 z}{N(\pi)} \rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_3 z}{N(\pi)} - \lfloor \frac{\pi_3 z}{N(\pi)} \rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_4 z}{N(\pi)} - \lfloor \frac{\pi_4 z}{N(\pi)} \rfloor \right| < \frac{1}{2}. \end{aligned} \quad (3.21)$$

Therefore,

$$N\left(\frac{\pi_1 z}{N(\pi)} - \left\lfloor \frac{\pi_1 z}{N(\pi)} \right\rfloor\right) + N\left(\frac{\pi_2 z}{N(\pi)} - \left\lfloor \frac{\pi_2 z}{N(\pi)} \right\rfloor\right) + N\left(\frac{\pi_3 z}{N(\pi)} - \left\lfloor \frac{\pi_3 z}{N(\pi)} \right\rfloor\right) + N\left(\frac{\pi_4 z}{N(\pi)} - \left\lfloor \frac{\pi_4 z}{N(\pi)} \right\rfloor\right) < \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1. \quad (3.22)$$

Hereby,

$$N(\mu_\pi(z)\pi^{-1}) = N(\mu_\pi(z))N(\pi^{-1}) = N(\mu_\pi(z))\frac{1}{N(\pi)} < 1. \quad (3.23)$$

So,

$$N(\mu_\pi(z)) < N(\pi). \quad (3.24)$$

Case : 2 Let π be an encoder Hurwitz integer such that π_1 and π_2 are even integers and, π_3 and π_4 are odd integers. Firstly, we should check whether to verify or not equation 3.1 in proposition 3.1 in case of $N(\pi)$ is an even integer. Let $z = \frac{N(\pi)}{2}$. By equation 2.5,

$$\mu_\pi\left(\frac{N(\pi)}{2}\right) = \frac{N(\pi)}{2} - \pi \left\lfloor \pi^{-1} \frac{N(\pi)}{2} \right\rfloor. \quad (3.25)$$

Then,

$$\begin{aligned} \mu_\pi\left(\frac{N(\pi)}{2}\right)\pi^{-1} &= \pi^{-1} \frac{N(\pi)}{2} - \pi^{-1} \pi \left\lfloor \pi^{-1} \frac{N(\pi)}{2} \right\rfloor \\ &= \frac{\bar{\pi}}{2} - \left\lfloor \frac{\bar{\pi}}{2} \right\rfloor. \end{aligned} \quad (3.26)$$

Since $\bar{\pi} = \pi_1 - \pi_2 i - \pi_3 j - \pi_4 k$, then

$$\begin{aligned} \mu_\pi\left(\frac{N(\pi)}{2}\right)\pi^{-1} &= \frac{\pi_1}{2} - \left(\frac{\pi_2}{2}\right)i - \left(\frac{\pi_3}{2}\right)j - \left(\frac{\pi_4}{2}\right)k \\ &\quad - \left\lfloor \frac{\pi_1}{2} \right\rfloor + \left\lfloor \frac{\pi_2}{2} \right\rfloor i + \left\lfloor \frac{\pi_3}{2} \right\rfloor j + \left\lfloor \frac{\pi_4}{2} \right\rfloor k \\ &= \frac{\pi_1}{2} - \left\lfloor \frac{\pi_1}{2} \right\rfloor - \left(\frac{\pi_2}{2} - \left\lfloor \frac{\pi_2}{2} \right\rfloor\right)i \\ &\quad - \left(\frac{\pi_3}{2} - \left\lfloor \frac{\pi_3}{2} \right\rfloor\right)j - \left(\frac{\pi_4}{2} - \left\lfloor \frac{\pi_4}{2} \right\rfloor\right)k. \end{aligned} \quad (3.27)$$

$\left\lfloor \frac{\pi_1}{2} \right\rfloor = \frac{\pi_1}{2}$ and $\left\lfloor \frac{\pi_2}{2} \right\rfloor = \frac{\pi_2}{2}$ since π_2 and π_3 are even integers. Also, $\left\lfloor \frac{\pi_3}{2} \right\rfloor = \frac{\pi_3+1}{2}$ and $\left\lfloor \frac{\pi_4}{2} \right\rfloor = \frac{\pi_4+1}{2}$ since π_3 and π_4 are odd integers. Therefore,

$$\begin{aligned} \mu_\pi\left(\frac{N(\pi)}{2}\right)\pi^{-1} &= \frac{\pi_1}{2} - \frac{\pi_1}{2} - \left(\frac{\pi_2}{2} - \frac{\pi_2}{2}\right)i \\ &\quad - \left(\frac{\pi_3}{2} - \frac{\pi_3+1}{2}\right)j - \left(\frac{\pi_4}{2} - \frac{\pi_4+1}{2}\right)k \\ &= \left(\frac{1}{2}\right)j + \left(\frac{1}{2}\right)k. \end{aligned} \quad (3.28)$$

Hereby,

$$\begin{aligned} N(\mu_\pi\left(\frac{N(\pi)}{2}\right)\pi^{-1}) &= N\left(\left(\frac{1}{2}\right)j + \left(\frac{1}{2}\right)k\right) \\ N(\mu_\pi\left(\frac{N(\pi)}{2}\right))N(\pi^{-1}) &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ N(\mu_\pi\left(\frac{N(\pi)}{2}\right))\left(\frac{1}{N(\pi)}\right) &= \frac{1}{4} + \frac{1}{4} \\ N(\mu_\pi\left(\frac{N(\pi)}{2}\right)) &= \frac{N(\pi)}{2}. \end{aligned} \quad (3.29)$$

So,

$$N(\mu_\pi\left(\frac{N(\pi)}{2}\right)) < N(\pi). \quad (3.30)$$

On the other hand, let $z \neq \frac{N(\pi)}{2}$. By equation 2.5,

$$\mu_\pi(z) = z - \pi \left\lfloor \pi^{-1} z \right\rfloor. \quad (3.31)$$

Then,

$$\begin{aligned}\mu_\pi(z)\pi^{-1} &= \pi^{-1}z - \pi^{-1}\pi\lfloor\pi^{-1}z\rfloor \\ &= \pi^{-1}z - \lfloor\pi^{-1}z\rfloor \\ &= \frac{\bar{\pi}z}{N(\pi)} - \lfloor\frac{\bar{\pi}z}{N(\pi)}\rfloor.\end{aligned}\tag{3.32}$$

Since $\bar{\pi} = \pi_1 - \pi_2i - \pi_3j - \pi_4k$, then

$$\begin{aligned}\mu_\pi(z)\pi^{-1} &= \frac{\pi_1z}{N(\pi)} - (\frac{\pi_2z}{N(\pi)})i - (\frac{\pi_3z}{N(\pi)})j - (\frac{\pi_4z}{N(\pi)})k \\ &\quad - \lfloor\frac{\pi_1z}{N(\pi)}\rfloor - \lfloor\frac{\pi_2z}{N(\pi)}\rfloor i - \lfloor\frac{\pi_3z}{N(\pi)}\rfloor j - \lfloor\frac{\pi_4z}{N(\pi)}\rfloor k \\ &= \frac{\pi_1z}{N(\pi)} - \lfloor\frac{\pi_1z}{N(\pi)}\rfloor - (\frac{\pi_2z}{N(\pi)} - \lfloor\frac{\pi_2z}{N(\pi)}\rfloor)i \\ &\quad - (\frac{\pi_3z}{N(\pi)} - \lfloor\frac{\pi_3z}{N(\pi)}\rfloor)j - (\frac{\pi_4z}{N(\pi)} - \lfloor\frac{\pi_4z}{N(\pi)}\rfloor)k.\end{aligned}\tag{3.33}$$

Since $z \neq \frac{N(\pi)}{2}$, then

$$\begin{aligned}0 &\leq \left| \frac{\pi_1z}{N(\pi)} - \lfloor\frac{\pi_1z}{N(\pi)}\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_2z}{N(\pi)} - \lfloor\frac{\pi_2z}{N(\pi)}\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_3z}{N(\pi)} - \lfloor\frac{\pi_3z}{N(\pi)}\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_4z}{N(\pi)} - \lfloor\frac{\pi_4z}{N(\pi)}\rfloor \right| < \frac{1}{2}.\end{aligned}\tag{3.34}$$

Therefore,

$$\begin{aligned}N(\frac{\pi_1z}{N(\pi)} - \lfloor\frac{\pi_1z}{N(\pi)}\rfloor) + N(\frac{\pi_2z}{N(\pi)} - \lfloor\frac{\pi_2z}{N(\pi)}\rfloor) + N(\frac{\pi_3z}{N(\pi)} - \lfloor\frac{\pi_3z}{N(\pi)}\rfloor) \\ + N(\frac{\pi_4z}{N(\pi)} - \lfloor\frac{\pi_4z}{N(\pi)}\rfloor) < (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 = 1\end{aligned}\tag{3.35}$$

Hereby,

$$N(\mu_\pi(z)\pi^{-1}) = N(\mu_\pi(z))N(\pi^{-1}) = N(\mu_\pi(z))\frac{1}{N(\pi)} < 1\tag{3.36}$$

So,

$$N(\mu_\pi(z)) < N(\pi).\tag{3.37}$$

Case : 3 Let π be an encoder Hurwitz integer such that π_1, π_2 and π_3 are even integers, and π_4 is an odd integer. So, $N(\pi)$ is an odd integer. By equation 2.5,

$$\mu_\pi(z) = z - \pi\lfloor\pi^{-1}z\rfloor\tag{3.38}$$

Then,

$$\begin{aligned}\mu_\pi(z)\pi^{-1} &= \pi^{-1}z - \pi^{-1}\pi\lfloor\pi^{-1}z\rfloor \\ &= \pi^{-1}z - \lfloor\pi^{-1}z\rfloor \\ &= \frac{\bar{\pi}z}{N(\pi)} - \lfloor\frac{\bar{\pi}z}{N(\pi)}\rfloor.\end{aligned}\tag{3.39}$$

Since $\bar{\pi} = \pi_1 - \pi_2i - \pi_3j - \pi_4k$, then

$$\begin{aligned}\mu_\pi(z)\pi^{-1} &= \frac{\pi_1z}{N(\pi)} - (\frac{\pi_2z}{N(\pi)})i - (\frac{\pi_3z}{N(\pi)})j - (\frac{\pi_4z}{N(\pi)})k \\ &\quad - \lfloor\frac{\pi_1z}{N(\pi)}\rfloor - \lfloor\frac{\pi_2z}{N(\pi)}\rfloor i - \lfloor\frac{\pi_3z}{N(\pi)}\rfloor j - \lfloor\frac{\pi_4z}{N(\pi)}\rfloor k \\ &= \frac{\pi_1z}{N(\pi)} - \lfloor\frac{\pi_1z}{N(\pi)}\rfloor - (\frac{\pi_2z}{N(\pi)} - \lfloor\frac{\pi_2z}{N(\pi)}\rfloor)i \\ &\quad - (\frac{\pi_3z}{N(\pi)} - \lfloor\frac{\pi_3z}{N(\pi)}\rfloor)j - (\frac{\pi_4z}{N(\pi)} - \lfloor\frac{\pi_4z}{N(\pi)}\rfloor)k\end{aligned}\tag{3.40}$$

Since π_4 is an odd integer, π_1, π_2 and π_3 are even integers, and $N(\pi)$ is an odd integer, then

$$\begin{aligned} 0 &\leq \left| \frac{\pi_1 z}{N(\pi)} - \left\lfloor \frac{\pi_1 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_2 z}{N(\pi)} - \left\lfloor \frac{\pi_2 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_3 z}{N(\pi)} - \left\lfloor \frac{\pi_3 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_4 z}{N(\pi)} - \left\lfloor \frac{\pi_4 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2}. \end{aligned} \quad (3.41)$$

Therefore,

$$\begin{aligned} &N\left(\frac{\pi_1 z}{N(\pi)} - \left\lfloor \frac{\pi_1 z}{N(\pi)} \right\rfloor\right) + N\left(\frac{\pi_2 z}{N(\pi)} - \left\lfloor \frac{\pi_2 z}{N(\pi)} \right\rfloor\right) + N\left(\frac{\pi_3 z}{N(\pi)} - \left\lfloor \frac{\pi_3 z}{N(\pi)} \right\rfloor\right) \\ &+ N\left(\frac{\pi_4 z}{N(\pi)} - \left\lfloor \frac{\pi_4 z}{N(\pi)} \right\rfloor\right) < \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1 \end{aligned} \quad (3.42)$$

Hereby,

$$N(\mu_\pi(z)\pi^{-1}) = N(\mu_\pi(z))N(\pi^{-1}) = N(\mu_\pi(z))\frac{1}{N(\pi)} < 1 \quad (3.43)$$

So,

$$N(\mu_\pi(z)) < N(\pi). \quad (3.44)$$

This completes the proof. $\square$

With the following examples, giving an example for each case in the proposition 3.2.

Example 3.2. Case 1 Let $\pi = 1 + 3i + 2j + k$ be an encoder Hurwitz integer. By equation 2.5, the Hurwitz constellation π is

$$\mathcal{L}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0, \mu_\pi(1) = 1, \mu_\pi(2) = 2, \mu_\pi(3) = i + j - 2k, \\ \mu_\pi(4) = -1 + 2j + k, \mu_\pi(5) = 2j + k, \mu_\pi(6) = 1 + 2j + k, \\ \mu_\pi(7) = 2 + 2j + k, \mu_\pi(8) = -2 - 2j - k, \mu_\pi(9) = -1 - 2j - k, \\ \mu_\pi(10) = -2j - k, \mu_\pi(11) = 1 - 2j - k, \mu_\pi(12) = -i - j + 2k, \\ \mu_\pi(13) = -2, \mu_\pi(14) = -1, \end{array} \right\}. \quad (3.45)$$

This set contains 15 elements because of $N(\pi) = 1^2 + 3^2 + 2^2 + 1^2 = 15$. The norm of each element in the set 3.45 is less than the norm of π .

Example 3.3. Case 2 Let $\pi = 2 + 3i + j + 2k$ be an encoder Hurwitz integer. By equation 2.5, the Hurwitz constellation π is

$$\mathcal{L}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0, \mu_\pi(1) = 1, \mu_\pi(2) = 2, \mu_\pi(3) = 3, \\ \mu_\pi(4) = 1 + 2i + 2j - k, \mu_\pi(5) = -2 - 2j - k, \mu_\pi(6) = -1 - 2j - k, \\ \mu_\pi(7) = -2j - k, \mu_\pi(8) = 1 - 2j - k, \mu_\pi(9) = 2 - 2j - k, \\ \mu_\pi(10) = -1 + 2j + k, \mu_\pi(11) = 2j + k, \mu_\pi(12) = 1 + 2j + k, \\ \mu_\pi(13) = 2 + 2j + k, \mu_\pi(14) = -1 - 2i - 2j + k, \\ \mu_\pi(15) = -2i - 2j + k, \mu_\pi(16) = -2, \mu_\pi(17) = -1 \end{array} \right\}. \quad (3.46)$$

This set contains 18 elements because of $N(\pi) = 2^2 + 3^2 + 1^2 + 2^2 = 18$. The norm of each element in the set 3.46 is less than the norm of π .

Example 3.4. Case 3 Let $\pi = 2 + 3i + 2j + 2k$ be an encoder Hurwitz integer. By equation 2.5, the Hurwitz constellation π is

$$\mathcal{L}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0, \mu_\pi(1) = 1, \mu_\pi(2) = 2, \mu_\pi(3) = 3, \\ \mu_\pi(4) = 1 + 2i + 2j - 2k, \mu_\pi(5) = 2 + 2i + 2j - 2k, \\ \mu_\pi(6) = 3 - i - j + k, \mu_\pi(7) = -2 - i - j + k, \\ \mu_\pi(8) = -1 - i - j + k, \mu_\pi(9) = -i - j + k, \\ \mu_\pi(10) = 1 - i - j + k, \mu_\pi(11) = -1 + i + j - k, \\ \mu_\pi(12) = i + j - k, \mu_\pi(13) = 1 + i + j - k, \\ \mu_\pi(14) = 2 + i + j - k, \mu_\pi(15) = 3 + i + j - k, \\ \mu_\pi(16) = -2 - 2i - 2j + 2k, \mu_\pi(17) = -1 - 2i - 2j + 2k, \\ \mu_\pi(18) = -3, \mu_\pi(19) = -2, \mu_\pi(20) = -1 \end{array} \right\}. \quad (3.47)$$

This set contains 21 elements because of $N(\pi) = 2^2 + 3^2 + 2^2 + 2^2 = 21$. The norm of each element in the set 3.47 is less than the norm of π .

In the following example, we show that it does not have the "division with small remainder" property of a Hurwitz integer used to obtain the Hurwitz constellation constructed with a different technique by Rohweder et al..

Example 3.5. In [19], Rohweder et al. presented the new construction method for Hurwitz integers by

$$\mu_\pi(z) = z - \lfloor \frac{z\bar{\pi}}{N(\pi)} \rfloor \pi \quad (3.48)$$

where π is a primitive Hurwitz integer and $z \in \mathbb{Z}_{N(\pi)}$. They proposed four-dimensional Hurwitz signal constellations are obtained from the following mapping

$$\mathcal{H}_\pi = \mathcal{L}_\pi \bigcup \mathcal{O}_\pi, \quad (3.49)$$

where $\mathcal{L}_\pi$ is the subset of Lipschitz integers, which are quaternions whose all components are in $\mathbb{Z}$, which can be evaluated by

$$\mathcal{L}_\pi = \{\mu_\pi(a + bj) : a, b \in \mathbb{Z}_{N(\pi)}\} \quad (3.50)$$

where $\mathbb{Z}_{N(\pi)}$ denotes the ring of integers modulo $N(\pi)$. Also, $\mathcal{O}_\pi$ in equation 3.49 is the corresponding coset of half-integers, which can be calculated by

$$\mathcal{O}_\pi = \{\mu_\pi(h + w) : h \in \mathcal{L}_\pi\}, \quad (3.51)$$

where $w = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$. The proposition 3.1 says that if π is a Hurwitz integer whose components are all odd integers, and $N(\pi)$ is an even integer, then we should check the $\frac{N(\pi)}{2}$ integer at the construction method used for Hurwitz integers. We consider $3 + i$ in [19, Table I]. Hereby, $N(\pi) = 3^2 + 1^2 = 10$. Let $a = \frac{N(\pi)}{2} = \frac{10}{2} = 5$ and $b = \frac{N(\pi)}{2} = \frac{10}{2} = 5$. So, we consider $\mu_\pi(5 + 5j)$ in $\mathcal{L}_\pi$. By equation 3.48,

$$\begin{aligned} \mu_\pi(5 + 5j) &= 5 + 5j - \lfloor \frac{(5+5j)(3-i)}{10} \rfloor (3 + i) \\ &= 5 + 5j - \lfloor \frac{15-5i+15j+5k}{10} \rfloor (3 + i) \\ &= 5 + 5j - (2 + 2j + k)(3 + i) \\ &= 5 + 5j - 6 - 2i - 7j - k \\ &= -1 - 2i - 2j - k. \end{aligned} \quad (3.52)$$

Hereby, $N(\mu_\pi(5+5j)) = N(\pi)$ because of $N(\mu_\pi(5+5j)) = (-1)^2 + (-2)^2 + (-2)^2 + (-1)^2 = 10$. Then we consider $\mu_\pi(h = (-1-2i-2j-k) + (w = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k)) = \mu_\pi(-\frac{1}{2} - \frac{3}{2}i - \frac{3}{2}j - \frac{1}{2}k)$ in $\mathcal{O}_\pi$ where $h = (-1-2i-2j-k) \in \mathcal{L}_\pi$. By equation 3.48

$$\begin{aligned} \mu_\pi(-\frac{1}{2} - \frac{3}{2}i - \frac{3}{2}j - \frac{1}{2}k) &= -\frac{1}{2} - \frac{3}{2}i + \frac{5}{2}j - \frac{1}{2}k - \lfloor \frac{(-\frac{1}{2} - \frac{3}{2}i + \frac{5}{2}j - \frac{1}{2}k)(3-i)}{10} \rfloor (3+i) \\ &= -\frac{1}{2} - \frac{3}{2}i + \frac{5}{2}j - \frac{1}{2}k - \lfloor \frac{-3-4i-4j-3k}{10} \rfloor (3+i) \\ &= -\frac{1}{2} - \frac{3}{2}i + \frac{5}{2}j - \frac{1}{2}k - (0)(3+i) \\ &= -\frac{1}{2} - \frac{3}{2}i + \frac{5}{2}j - \frac{1}{2}k \end{aligned} \tag{3.53}$$

Hereby, $N(\mu_\pi(-\frac{1}{2} - \frac{3}{2}i - \frac{3}{2}j - \frac{1}{2}k)) = N(\pi)$ because of $N(\mu_\pi(-\frac{1}{2} - \frac{3}{2}i - \frac{3}{2}j - \frac{1}{2}k)) = (-\frac{1}{2})^2 + (-\frac{3}{2})^2 + (-\frac{3}{2})^2 + (-\frac{1}{2})^2 = 10$. Consequently, we say that $3+i$ does not have the "division with small remainder" property since the norm of $-1-2i-2j-k$ and $-\frac{1}{2} - \frac{3}{2}i + \frac{5}{2}j - \frac{1}{2}k$ is equal to the norm of $3+i$. For a primitive Hurwitz integer such that its norm is 10, we propose $2+2i+j+k$ instead of $3+i$.

4. Performances of Hurwitz Constellations for Transmission over AWGN channel

In this section, we give distance and performance measures. Then, we investigate the performances of Hurwitz constellations for transmission over the AWGN channel by means of average energy, CFM, and SNR coding gain. Note that we investigate the performances of Hurwitz constellations that lie on Hurwitz integers whose components are all half-integers ($\mathbb{Z} + \frac{1}{2}$). Because the Hurwitz constellations constructed using primitive Hurwitz integers whose components are all integers ($\mathbb{Z}$) have the same performances as Lipschitz constellations in [9] (Every Lipschitz integer is a Hurwitz integer, but the converse may not be true). Also, we give set partitioning property on larger Hurwitz integers, named proposed Hurwitz integers. Because the Hurwitz constellations with the same size as Gaussian constellations are almost shown the same performances for transmission over the AWGN channel. We follow the procedures in [9] for distance, performance measures, and set partitioning property. The average energy of a constellation is denoted by $\mathcal{E}_\pi$. It is computed by

$$\mathcal{E}_\pi = \frac{1}{N(\pi)} \sum_{z=0}^{N(\pi)-1} N(\mu_\pi(z)). \tag{4.1}$$

The squared Euclidean distance of the Hurwitz constellation is computed by

$$d_E(\alpha, \beta) = N(\beta - \alpha), \tag{4.2}$$

and the minimum squared Euclidean distance of the Hurwitz constellation is computed by

$$\delta_\lambda^2 = \min_{\alpha \neq \beta} d_E(\alpha, \beta). \tag{4.3}$$

where $\alpha, \beta \in \mathcal{H}_\pi$. In [6], Forney and Wei proposed the constellation figure of merit (CFM) to compare signal constellations of different dimensions. The CFM of a M -dimensional constellation is computed by

$$\text{CFM} = \frac{M\delta_\pi^2}{2\mathcal{E}_\pi}. \quad (4.4)$$

A higher CFM leads to a better performance for transmission over an AWGN channel [4]. Asymptotic coding gain means for higher signal to noise ratio (SNR) [14]. The SNR coding gain of M -dimensional constellation is computed by

$$\text{SNR} = 10 \cdot \log_{10} \left(\frac{\text{CFM of signal constellation}}{\text{CFM of noise signal constellation}} \right). \quad (4.5)$$

where the noise signal constellation is the Gaussian constellation. The SNR coding gain of a Hurwitz constellation over the AWGN channel is

$$\text{SNR} = 10 \cdot \log_{10} \left(\frac{\text{CFM of } \mathcal{H}_\pi}{\text{CFM of } \mathcal{G}_\pi} \right). \quad (4.6)$$

where Hurwitz constellation $\mathcal{H}_\pi$ and Gaussian constellation $\mathcal{G}_\pi$ are the equal size. A residue class ring of Hurwitz integers $\mathcal{H}_\pi$ arises from the residue class ring of integers $\mathbb{Z}_{N(\pi)} = \{0, 1, \dots, N(\pi) - 1\}$ where $N(\pi)$ is an integer. If $N(\pi)$ is not a prime integer, then we can partition the set $\mathbb{Z}_{N(\pi)}$ into subsets with equal size. Let $N = c \cdot d$ be the number of elements in Hurwitz constellation, c is the number of subsets, and d is the number of elements in each subset. We can partition the set $\mathcal{H}_\pi$ into c subsets $(\mathcal{H}_\pi^{(0)}, \dots, \mathcal{H}_\pi^{(c-1)})$, each set having d elements. The subsets correspond to the integer sets $\mathbb{Z}_\pi^{(0)}, \dots, \mathbb{Z}_\pi^{(c-1)}$, where

$$\mathbb{Z}_{N(\pi)}^{(0)} = \{0, c, 2c, \dots, (d-1)c\}, \quad (4.7)$$

and $\mathbb{Z}_\pi^{(1)}, \dots, \mathbb{Z}_\pi^{(c-1)}$ are the cosets of $\mathbb{Z}_\pi^{(0)}$, i.e. $\mathbb{Z}_\pi^{(l)} = \{z : z - l \in \mathbb{Z}_\pi^{(0)}\}$. Note that the number of elements in the Hurwitz constellation and Gaussian constellation should be equal in size to compare performances over the AWGN channel. However, the number of elements in the proposed Hurwitz constellations should not be the same as the Gaussian constellation. Therefore, we can apply set partitioning property on proposed primitive Hurwitz constellations. The SNR coding gain of a proposed Hurwitz constellation over the AWGN channel is computed by

$$\begin{aligned} \text{SNR} &= 10 \cdot \log_{10} \left(\frac{\text{CFM of } \mathcal{H}_\pi^{(0)}}{\text{CFM of } \mathcal{G}_\pi} \right) - 10 \cdot \log_{10} \left(\frac{\text{CFM of } \mathcal{H}_\pi}{\text{CFM of } \mathcal{G}_\pi} \right) \\ &= 10 \cdot \log_{10} \left(\text{CFM of } \mathcal{H}_\pi^{(0)} \right) - 10 \cdot \log_{10} \left(\text{CFM of } \mathcal{G}_\pi \right) \\ &\quad - 10 \cdot \log_{10} \left(\text{CFM of } \mathcal{H}_\pi \right) + 10 \cdot \log_{10} \left(\text{CFM of } \mathcal{G}_\pi \right) \\ &= 10 \cdot \log_{10} \left(\text{CFM of } \mathcal{H}_\pi^{(0)} \right) - 10 \cdot \log_{10} \left(\text{CFM of } \mathcal{H}_\pi \right) \\ &= 10 \cdot \log_{10} \left(\frac{\text{CFM of } \mathcal{H}_\pi^{(0)}}{\text{CFM of } \mathcal{H}_\pi} \right). \end{aligned} \quad (4.8)$$

Rohweder et al. [19] presented a new techniques for Hurwitz constellations. See in example 3.5, or [19] for the modulo technique of Rohweder et al..

TABLE 1. Table of CFM, average energy, and SNR coding gain of Hurwitz constellations constructed using primitive Hurwitz integers whose components are in $\mathbb{Z} + \frac{1}{2}$ (d : The number of elements in the Hurwitz constellation)

d	Primitive Hurwitz Integer	Constellations				SNR_{dB} gain
		CFM		Energy		
		Gauss	Hurwitz	Gauss	Hurwitz	
5	$\frac{3}{2} + \frac{3}{2}i + \frac{1}{2}j + \frac{1}{2}k$	1.2500	1.6667	0.8000	1.200	1.25
13	$\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.4643	0.5200	2.1538	3.8462	0.49
17	$\frac{5}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.3542	0.3864	2.8235	5.1765	0.38
25	$\frac{9}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{1}{2}k$	0.2404	0.2551	4.1600	7.8400	0.26
29	$\frac{9}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k$	0.2071	0.2180	4.8276	9.1724	0.22
37	$\frac{11}{2} + \frac{5}{2}i + \frac{1}{2}j + \frac{1}{2}k$	0.1623	0.1690	6.1622	11.8378	0.18
41	$\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.1464	0.1519	6.8293	13.1707	0.16
53	$\frac{13}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.1132	0.1165	8.8302	17.1698	0.12
61	$\frac{15}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{1}{2}k$	0.0984	0.1008	10.1639	19.8361	0.10
65	$\frac{11}{2} + \frac{11}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.0923	0.0945	10.8308	21.1692	0.10
73	$\frac{17}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$	0.0822	0.0839	12.1644	23.8356	0.09
85	$\frac{13}{2} + \frac{13}{2}i + \frac{1}{2}j + \frac{1}{2}k$	0.0706	0.0719	14.1467	27.8353	0.08
89	$\frac{17}{2} + \frac{7}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.0674	0.0686	14.8315	29.1685	0.08
97	$\frac{19}{2} + \frac{5}{2}i + \frac{1}{2}j + \frac{1}{2}k$	0.0619	0.0628	16.1649	31.8351	0.06

Example 4.1. In Table I, we present the performances of the Hurwitz constellations constructed using encoder Hurwitz integers whose components are all integers over the AWGN channel by means of average energy, CFM, and SNR coding gain. The performances of Hurwitz constellations over the AWGN channel in Table I are not so good but better than nothing according to the performances of Hurwitz constellations constructed using Hurwitz integers whose components are all integers. See [6] for the performances of the Hurwitz constellations constructed using Hurwitz integers whose components are all integers over the AWGN channel. The performances of proposed Hurwitz constellations constructed using Hurwitz integers whose components are all integers are the same as the proposed Lipschitz constellations in [9, Table I].

TABLE 2. Table of average energy, CFM, and SNR coding gain of the proposed Hurwitz constellation constructed using proposed Hurwitz integers whose components are in $\mathbb{Z} + \frac{1}{2}$ (N : The number of elements in the proposed Hurwitz constellation, c : The number of subsets of the proposed Hurwitz constellation, d : The number of elements in the each subset of the proposed Hurwitz constellation)

N	c	d	Proposed Primitive Hurwitz Integers	Constellations				SNR_{dB}
				CFM			Energy	
				Gauss	Hurwitz	Proposed Hurwitz	Proposed Hurwitz	
15	3	5	$\frac{5}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k$	1.2500	1.6667	1.2500	4.8000	-1.25
39	3	13	$\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k$	0.4643	0.5200	1.5600	11.5385	4.77
51	3	17	$\frac{11}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{3}{2}k$	0.3542	0.3864	1.1591	15.5294	4.77
75	3	25	$\frac{13}{2} + \frac{9}{2}i + \frac{7}{2}j + \frac{1}{2}k$	0.2404	0.2551	0.7653	23.5200	4.77
87	3	29	$\frac{13}{2} + \frac{11}{2}i + \frac{7}{2}j + \frac{3}{2}k$	0.2071	0.2180	0.6541	27.5172	4.77
185	5	37	$\frac{21}{2} + \frac{13}{2}i + \frac{9}{2}j + \frac{7}{2}k$	0.1673	0.1690	0.8447	59.1982	6.99
205	5	41	$\frac{19}{2} + \frac{17}{2}i + \frac{11}{2}j + \frac{7}{2}k$	0.1464	0.1519	0.7593	65.8537	6.99
265	5	53	$\frac{23}{2} + \frac{19}{2}i + \frac{13}{2}j + \frac{1}{2}k$	0.1132	0.1165	0.5824	85.8491	6.99
427	7	61	$\frac{33}{2} + \frac{21}{2}i + \frac{13}{2}j + \frac{3}{2}k$	0.0984	0.1008	0.7058	138.8520	8.45
455	7	65	$\frac{27}{2} + \frac{25}{2}i + \frac{21}{2}j + \frac{5}{2}k$	0.0923	0.0945	0.4717	148.4000	6.98
511	7	73	$\frac{33}{2} + \frac{21}{2}i + \frac{17}{2}j + \frac{15}{2}k$	0.0822	0.0839	0.5874	166.8490	8.45
595	7	85	$\frac{37}{2} + \frac{29}{2}i + \frac{13}{2}j + \frac{1}{2}k$	0.0706	0.0719	0.5030	194.8470	8.45
623	7	89	$\frac{35}{2} + \frac{33}{2}i + \frac{13}{2}j + \frac{3}{2}k$	0.0674	0.0686	0.4800	204.1800	8.45
873	9	97	$\frac{41}{2} + \frac{35}{2}i + \frac{19}{2}j + \frac{15}{2}k$	0.0619	0.0628	0.5654	286.5150	9.54

Example 4.2. In Table II, we present the performances of the proposed Hurwitz constellation constructed using encoder Hurwitz integers whose components are in $\mathbb{Z} + \frac{1}{2}$ over the AWGN channel by means of average energy, CFM, and SNR coding gain. The proposed Hurwitz constellations in Table II have advantage performances for transmission over the AWGN channel.

There also exist different proposed Hurwitz integers used to construct proposed Hurwitz constellations that have higher CFM and lower average

energy. See the following examples. The following examples hint at how the tables are constructed.

Example 4.3. Let us consider proposed Hurwitz constellation(s) with $N = 3 \cdot 13 = 39$ elements. For $N = 39$, there exist four different proposed Hurwitz integers used to construct proposed Hurwitz constellations. These proposed Hurwitz integers are $\frac{7}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{3}{2}k$, $\frac{9}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{5}{2}k$, $\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k$, and $\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k$. There is no Gaussian constellation with an equal size to the proposed Hurwitz constellation with $N = 39$ elements. Note that, for applying set partitioning property on proposed Hurwitz integers, the proposed Hurwitz integers are not the same size as primitive Gaussian integers. Firstly, we consider the proposed Hurwitz integers $\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k$, and $\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k$. For the proposed Hurwitz constellation $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}$, the minimum squared Euclidean distance, CFM, and average energy are 1, 12.5128 and 0.1598, respectively. The proposed Hurwitz constellation $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}$ partitions into three ($c = 3$) different subsets, each set having 13 ($d = 13$) elements. So, the SNR coding gain of the proposed Hurwitz constellation $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}$ is computed using a subset of the proposed Hurwitz constellation $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}^{(0)}$. This subset is the set $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}^{(0)}$ with 13 elements. Because, the minimum squared Euclidean distance of the Hurwitz constellation $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}^{(0)}$ is larger than the minimum squared Euclidean distance of the Hurwitz constellations $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}^{(1)}$ and $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}^{(2)}$. The minimum squared Euclidean distance, CFM, and average energy of the Hurwitz constellation $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}^{(0)}$ are 9, 11.5385 and 1.5600, respectively. Also, the Hurwitz constellation $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}^{(0)}$ with 13 elements is equivalent to the primitive Hurwitz integers $\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k$, $\frac{5}{2} + \frac{5}{2}i + \frac{1}{2}j + \frac{1}{2}k$, and $\frac{7}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$. The minimum squared Euclidean distance, CFM, and average energy of these Hurwitz constellations are 1, 0.5200, and 3.8462. The minimum squared Euclidean distance, CFM, and average energy of the Gaussian constellation $\mathcal{G}_{3+2i}$ with 13 elements are 1, 2.1539 and 0.4643, respectively. We consider the primitive Hurwitz integer $\mathcal{H}_{\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k}$ to compute SNR coding gain of the proposed Hurwitz constellation $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}$. Therefore, SNR coding gain of it is

$$\text{SNR} = 10 \log \left(\frac{\text{CFM of } \mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}^{(0)}}{\text{CFM of } \mathcal{H}_{\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k}} \right) = 10 \log \left(\frac{1.5600}{0.5200} \right) = 4.77. \quad (4.9)$$

For the proposed Hurwitz constellation $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}$, the minimum squared Euclidean distance, CFM, and average energy are 1, 12.5128 and 0.1598, respectively. The proposed Hurwitz constellation $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}$ partitions into $c = 3$ different subsets, each having $d = 13$ elements. So, the SNR coding gain of the proposed Hurwitz constellation $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}$ is computed by using a subset of the proposed Hurwitz constellation $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}^{(0)}$. This subset

is the Hurwitz constellation $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}^{(0)}$ with 13 elements. Because, the minimum squared Euclidean distance of $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}^{(0)}$ is larger than the minimum squared Euclidean distance of $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}^{(1)}$ and $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}^{(2)}$. The minimum squared Euclidean distance, CFM, and average energy of the Hurwitz constellation $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}^{(0)}$ are 9, 11.5385 and 1.5600, respectively. The proposed Hurwitz constellations $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}$ and $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}$ have the same values for the average energy, CFM, and minimum squared Euclidean distance. So, the proposed Hurwitz constellations $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}$ and $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}$ have the same performances for transmission over AWGN channel. Lastly, we consider the proposed primitive Hurwitz integers $\frac{7}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{3}{2}k$, and $\frac{9}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{5}{2}k$. For both constellations, the minimum squared Euclidean distance, CFM, and average energy are 1, 12.5128 and 0.1598, respectively. The subsets of these proposed Hurwitz constellations are $\mathcal{H}_{\frac{7}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{3}{2}k}^{(0)}$, and $\mathcal{H}_{\frac{9}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{5}{2}k}^{(0)}$. The average energy, CFM, and minimum squared Euclidean distance of these subsets are 3, 11.5385 and 0.5200, respectively. Therefore, SNR coding gain of these proposed Hurwitz constellations is

$$\text{SNR} = 10 \log\left(\frac{\text{CFM of } \mathcal{H}_{\frac{7}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{3}{2}k}^{(0)}}{\text{CFM of } \mathcal{H}_{\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k}}\right) = 10 \log\left(\frac{0.5200}{0.5200}\right) = 0. \quad (4.10)$$

Consequently, the constellations that have higher CFM and larger minimum square Euclidean distance are the proposed Hurwitz constellations $\mathcal{H}_{\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k}$ and $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}$. We choose the primitive Hurwitz integer $\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k$ to represent in Table I, and the proposed Hurwitz integer $\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k$ to represent in Table II to create regular tables that are not crowded.

Example 4.4. Let us consider proposed Hurwitz constellation(s) with $N = 3 \cdot 29 = 87$ elements. For $N = 87$, There exist eight different proposed Hurwitz integers used to construct proposed Hurwitz constellations. These proposed Hurwitz integers are $\frac{11}{2} + \frac{11}{2}i + \frac{9}{2}j + \frac{5}{2}k$, $\frac{13}{2} + \frac{9}{2}i + \frac{7}{2}j + \frac{7}{2}k$, $\frac{13}{2} + \frac{11}{2}i + \frac{7}{2}j + \frac{3}{2}k$, $\frac{13}{2} + \frac{13}{2}i + \frac{3}{2}j + \frac{1}{2}k$, $\frac{15}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{5}{2}k$, $\frac{15}{2} + \frac{11}{2}i + \frac{1}{2}j + \frac{1}{2}k$, $\frac{17}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{3}{2}k$, and $\frac{17}{2} + \frac{7}{2}i + \frac{3}{2}j + \frac{1}{2}k$. For $N = 87$, there is no Gaussian constellation with an equal size to the proposed Hurwitz constellation. The minimum squared Euclidean distance, CFM, and average energy of these proposed Hurwitz constellations are 1, 28.5057 and 0.0702, respectively. They partition into $c = 3$ different subsets, each set having $d = 29$ elements. We consider the subsets $\mathcal{H}_{\frac{13}{2} + \frac{11}{2}i + \frac{7}{2}j + \frac{3}{2}k}^{(0)}$, and $\mathcal{H}_{\frac{17}{2} + \frac{7}{2}i + \frac{3}{2}j + \frac{1}{2}k}^{(0)}$ of $\mathcal{H}_{\frac{13}{2} + \frac{11}{2}i + \frac{7}{2}j + \frac{3}{2}k}$, and $\mathcal{H}_{\frac{17}{2} + \frac{7}{2}i + \frac{3}{2}j + \frac{1}{2}k}$, respectively. Because, the minimum square Euclidean distances of these subsets are larger than others. The minimum square Euclidean distance of these subsets is 9, but the other constellations are 6. Also average energy and CFM of these subsets are 27.5172 and 0.6541 but the other constellations are 27.5172 and 0.4361, respectively. The minimum squared Euclidean distance,

CFM, and average energy of the Hurwitz constellation $\mathcal{H}_{\frac{7}{2}+\frac{5}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ are 1, 0.2525, and 7.9200. The minimum squared Euclidean distance, CFM, and average energy of the Hurwitz constellations $\mathcal{H}_{\frac{7}{2}+\frac{7}{2}i+\frac{3}{2}j+\frac{3}{2}k}$, and $\mathcal{H}_{\frac{9}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ are 1, 0.2180, and 9.1724, respectively. We consider the primitive Hurwitz integers $\mathcal{H}_{\frac{9}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ and $\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k$. The minimum squared Euclidean distance, CFM, and average energy of the Gaussian constellation $\mathcal{G}_{5+2i}$ are 1, 4.8276 and 0.2071, respectively. SNR coding gain of the proposed Hurwitz constellation $\mathcal{H}_{\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k}$ is

$$\text{SNR} = 10 \log \left(\frac{\text{CFM of } \mathcal{H}_{\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k}^{(0)}}{\text{CFM of } \mathcal{H}_{\frac{9}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}} \right) = 10 \log \left(\frac{0.6541}{0.2180} \right) = 4.77. \quad (4.11)$$

Consequently, the proposed Hurwitz constellations with higher CFM and larger minimum square Euclidean distance are $\mathcal{H}_{\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k}$ and $\mathcal{H}_{\frac{17}{2}+\frac{7}{2}i+\frac{3}{2}j+\frac{1}{2}k}$. We choose the proposed Hurwitz integer $\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k$ to represent in Table II, and the primitive Hurwitz integer $\frac{9}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k$ to represent in Table I to create regular tables that are not crowded.

5. Conclusion

In this study, we investigated Hurwitz integers with the "division with small remainder" property. We defined a new set consisting of primitive Hurwitz integers with the "division with small remainder" property, named encoder Hurwitz integers. This set helps us establish an isomorphism between $\mathbb{Z}_n$ and $\mathcal{H}_\pi$, unlike the set of primitive Hurwitz integers. We presented a proposition (Proposition 3). It controls whether the Euclid division algorithm works or not in case Hurwitz integers such that all components are in $\mathbb{Z}$. In addition, we proposed new Hurwitz constellations for transmission over AWGN channel (see Table I, and Table II). In particular, the Hurwitz constellations constructed by Hurwitz integers whose components are all half-integers ($\mathbb{Z} + \frac{1}{2}$). Also, we investigated the performances of them for transmission over the AWGN channel.

6. Data Availability Statement

Conflict of interest The author states that there is no conflict of interest and no funding has been used to execute this research.

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