Encoder Hurwitz Integers: The Hurwitz integers that have the “division with small remainder” property

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Encoder Hurwitz Integers: The Hurwitz integers that have the "division with small remainder" property

Ramazan Duran and Murat Guzeltepe

Abstract. The residue class set of a Hurwitz integer is constructed by modulo function with primitive Hurwitz integer whose norm is a prime integer, i.e. prime Hurwitz integer. In this study, we consider primitive Hurwitz integer whose norm is both a prime integer and not a prime integer. If the norm of each element of the residue class set of a Hurwitz integer is less than the norm of the primitive Hurwitz integer used to construct the residue class set of the Hurwitz integer, then, the Euclid division algorithm works for this primitive Hurwitz integer. The Euclid division algorithm always works for prime Hurwitz integers. In other words, the prime Hurwitz integers and halves-integer primitive Hurwitz integers have the "division with small remainder" property. However, this property is ignored in some studies that have a constructed Hurwitz residue class set that lies on primitive Hurwitz integers that their norms are not a prime integer and their components are in integers set. In this study, we solve this problem by defining Hurwitz integers that have the "division with small remainder" property, namely, encoder Hurwitz integers set. Therefore, we can define appropriate metrics for codes over Lipschitz integers. Especially, Euclidean metric. Also, we investigate the performances of Hurwitz signal constellations (the left residue class set) obtained by modulo function with Hurwitz integers, which have the "division with small remainder" property, over the additive white Gaussian noise (AWGN) channel by means of the constellation figure of merit (CFM), average energy, and signal-to-noise ratio (SNR).

Keywords. Quaternion integer, Hurwitz integer, residue class, signal constellation, code construction.

1. Introduction

In recent years, many researchers in coding theory have investigated some special code constructions over groups, fields or rings i.e. finite algebraic

construction. Specially, they have studied code constructions over finite rings of integers and finite fields [1]-[8]. A Gaussian integer is a complex number that real and imaginary parts are in $\mathbb{Z}$. The set of Gaussian integers that denoted by $\mathbb{Z}[i]$ is shown by $\mathbb{Z}[i] = \{\alpha = \alpha_1 + \alpha_2 i : \alpha_1, \alpha_2 \in \mathbb{Z}, i^2 = -1\}$. Gaussian integers are a commutative ring and a subset of the complex numbers field, since they are closed under addition and multiplication. Let $\alpha = \alpha_1 + \alpha_2 i$ be a Gaussian integer. The conjugate of α is equal to $\bar{\alpha} = \alpha_1 - \alpha_2 i$, the norm of α is equal to $N(\alpha) = \alpha_1^2 + \alpha_2^2$, and the inverse of α is equal to $\alpha^{-1} = \frac{\bar{\alpha}}{N(\alpha)}$ where its norm is non-zero. A Gaussian integer is a prime Gaussian integer if its norm is a prime in $\mathbb{Z}$. A Gaussian integer is a primitive Gaussian integer just if greater common divisor (gcd) of all components is one i.e. $\gcd(\alpha_1, \alpha_2) = 1$. Hence, α_1 and α_2 are positive integers if $\alpha = \alpha_1 + \alpha_2 i$ is a primitive Gaussian integer. In [9], first study about code constructions over Gaussian integers is presented by Huber. In other words, Huber discovered a new way to construct codes for two dimensional signals by virtue of Gaussian integers, i.e. the integral points on the complex plane [9]. His original idea is to regard a finite field as a residue class of the Gaussian integer ring modulo a prime Gaussian integer and, by Euclidean division, to get a unique element of minimal norm in each residue class, which represents each element of a finite field. Therefore, each element of a finite field can be represented by a Gaussian integer with the minimal Galois norm in the residue class, and the residual class set of the selected Gaussian integer is called a signal constellation. The "signal constellation" is a communication term. Huber is used prime Gaussian integers such that $1 \equiv p \pmod{4}$ where $p = \alpha\bar{\alpha}$, and $\alpha_1 > \alpha_2 > 0$. In this study, we use primitive Gaussian integers that are both a prime integer and such that $\alpha_1 > \alpha_2 > 0$. Codes over rings of Gaussian integers were studied in papers to [9]-[13].

The quaternions are a four dimensional vector space that is an algebra over the set of the real numbers ($\mathbb{R}$), and a number system that extends the complex numbers ($\mathbb{C}$). The quaternions are a division algebra that is associative and non-commutative since the multiplication of quaternions has not commutative property. So, $\alpha\beta \neq \beta\alpha$ where α and β are quaternions. Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a quaternion where α_1 is real part and $\alpha_2 i + \alpha_3 j + \alpha_4 k$ is imaginary part. Multiplication of quaternions has commutative properties when $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$, and their imaginary parts are parallel to each other. The coding techniques in [9] have been generalized to codes over quaternion integers. In [14], Ozen and Guzeltepe study codes over some finite fields by using commutative quaternion integers. Codes over rings of quaternion integers were studied in papers to [13]-[17]. In this study, we consider Hurwitz integers, which are four dimensional signal constellations that are quotient rings. $\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ is a Hurwitz integer just if either all of $\alpha_1, \alpha_2, \alpha_3$, and α_4 are in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$. In [18], Guzeltepe studied classes of linear codes over Hurwitz integers equipped with a new metric that refer as the Hurwitz metric. In [19], Rohweder et al. presented a new algebraic construction of finite sets of Hurwitz integers by a respective modulo

function, and investigated performance for transmission over the additive white Gaussian noise (AWGN) channel, which is a noise channel model. The codes over Hurwitz integers given in [18-22].

This work is organized as follows: In the next section, we give some fundamental information about quaternions and Hurwitz integers. Also, we give the modulo function used to establish the notation and the notion of a residual class of Hurwitz integers ring with respect to primitive Hurwitz integers. In Section III, we define a set that consists of primitive Hurwitz integers that have the "division with small remainder" property. This set is named encoder Hurwitz integers set. In Section IV, we investigate the performance of Hurwitz constellations for transmission over the additive white Gaussian noise by means of constellation figure of merit (CFM), average energy, and signal-noise-to ratio (SNR). Finally, we conclude the paper in Section V.

2. Preliminaries

In this section, we give some fundamental information used throughout this paper.

Definition 2.1. The Hamilton quaternion algebra over $\mathbb{R}$, is the associative unital algebra given by the following representation:

- i. $\mathbb{H}(\mathbb{R})$ is the free $\mathbb{R}$ -module over the symbols $1, i, j, k$, that is: $\mathbb{H}(\mathbb{R}) = \{a_0 + a_1i + a_2j + a_3k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$,
- ii. 1 is the multiplicative identity,
- iii. $i^2 = j^2 = k^2 = ijk = -1$,
- iv. $ij = -ji = k, jk = -kj = i, ki = -ik = j$.

The definition is natural, in the sense that any unital ring homomorphism $\mathbb{R}_1 \rightarrow \mathbb{R}_2$ extends to a unital ring homomorphism $\mathbb{H}(\mathbb{R}_1) \rightarrow \mathbb{H}(\mathbb{R}_2)$ by mapping 1 to 1 , i to i , j to j and k to k [23, 2.5.1 Definition]. Let $\alpha = \alpha_1 + \alpha_2i + \alpha_3j + \alpha_4k$ be a quaternion. Here α_0 is a real part, and $\alpha_2i + \alpha_3j + \alpha_4k$ is an imaginary part. Also, $\alpha_1, \alpha_2, \alpha_3$, and α_4 are components of α quaternion. Multiplication of quaternions is non-commutative. But, if the imaginary parts of quaternions are parallel to each other, then multiplication of quaternions is commutative [14]. Also, multiplication of α and α^{-1} is commutative since $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$.

Definition 2.2. $\alpha = \alpha_1 + \alpha_2i + \alpha_3j + \alpha_4k$ is a Hurwitz integer just if either $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$ or $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} + \frac{1}{2}$. The set of all Hurwitz integers that denoted by $\mathcal{H}(\mathbb{Z})$ is shown by

$$\begin{aligned} \mathcal{H}(\mathbb{Z}) &= \{\alpha_1 + \alpha_2i + \alpha_3j + \alpha_4k : \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} \text{ or } \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z} + \frac{1}{2}\} \\ &= \mathbb{H}(\mathbb{Z}) \cup \mathbb{H}(\mathbb{Z} + \frac{1}{2}) \end{aligned} \quad (2.1)$$

The ring of Hurwitz integers $\mathcal{H}(\mathbb{Z})$ is forms a subring of the ring of all quaternions because of closed under multiplication and addition. Let $\alpha = \alpha_1 + \alpha_2i + \alpha_3j + \alpha_4k$ is a Hurwitz integer. The conjugate of α is $\bar{\alpha} = \alpha_1 - \alpha_2i - \alpha_3j - \alpha_4k$, the norm of α is $N(\alpha) = \alpha \cdot \bar{\alpha} = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2$, and the inverse of α is $\alpha^{-1} = \frac{\bar{\alpha}}{N(\alpha)}$ where its norm is non-zero. The units of $\mathcal{H}$ under multiplication

are $\left\{\pm 1, \pm i, \pm j, \pm k, \frac{\pm 1 \pm i \pm j \pm k}{2}\right\}$. So α is a unit of $\mathcal{H}$ such that $N(\alpha) = 1$. The set has 24 elements.

Definition 2.3. $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ a prime Hurwitz integer just if its norm is a rational prime integer.

Example 2.1. $\alpha = 2 - 3i + j + 3k$ and $\beta = \frac{3}{2} + \frac{5}{2}i - \frac{3}{2}j + \frac{7}{2}k$ are the prime Hurwitz integers because of $N(\alpha) = 2^2 + (-3)^2 + 1^2 + 3^2 = 23$ and $N(\beta) = \left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + \left(-\frac{3}{2}\right)^2 + \left(\frac{7}{2}\right)^2 = \frac{92}{4} = 23$.

Definition 2.4. $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ is a primitive Hurwitz integer just if greater common divisor of its components is equal to 1. That is, $\gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 1$.

Example 2.2. $\alpha = 3 + 4i + 2j + k$ is a primitive Hurwitz integer because of $\gcd(3, 4, 2, 1) = 1$. But it is not a prime Hurwitz integer because of $N(\alpha) = (3)^2 + (4)^2 + (2)^2 + (1)^2 = 30$.

Definition 2.5. The nearest integer rounding notation denoted by $\lfloor \cdot \rfloor$ is defined as rounding a rational number to the integer closest to its. For quaternions, each component of a quaternion is separately rounding to the integer closest to its. So, we obtain Hurwitz integers whose components are in $\mathbb{Z}$ from a quaternions. Note that the rounding is done in the direction $+\infty$ in this study.

Example 2.3. Let $\alpha = \frac{5}{4} + \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k \in \mathbb{H}(\mathbb{R})$. If we use nearest integer rounding notation for α , then we obtain a Hurwitz integer whose components are in $\mathbb{Z}$. That is,

$$\begin{aligned} \lfloor \alpha \rfloor &= \lfloor \frac{5}{4} + \frac{1}{2}i - \frac{1}{2}j - \frac{5}{2}k \rfloor \\ &= \lfloor \frac{5}{4} \rfloor + \lfloor \frac{1}{2}i \rfloor + \lfloor -\frac{1}{2}j \rfloor + \lfloor -\frac{5}{2}k \rfloor \\ &= 1 + 1 \cdot i + 0 \cdot j + (-2) \cdot k \\ &= 1 + i - 2k. \end{aligned} \tag{2.2}$$

The residue class set for codes over Gaussian integers that are two-dimensional signal space are constructed by the modulo function technique. Similarity Lipschitz constellation for codes over Lipschitz integers[25], we use this technique to construct Hurwitz constellation that lies on Hurwitz integers. This technique, known as the modulo function, is given by the following definition. Note that we consider the primitive Hurwitz integers whose norm is both a prime integer and not a prime integer, and left residue class set of primitive Hurwitz integer in this study.

Definition 2.6. The modulo function $\mu : \mathbb{Z}_{N(\pi)} \rightarrow \mathcal{H}_\pi$ is defined by

$$\mu_\pi(z) = z \bmod \pi = z - \pi \cdot \lfloor \alpha^{-1} z \rfloor = z - \pi \lfloor \frac{\bar{\pi} z}{N(\pi)} \rfloor \tag{2.3}$$

where π is a primitive Hurwitz integer and $z \in \mathbb{Z}_{N(\pi)}$. Here $\mathbb{Z}_{N(\pi)}$ is the well-known residual class ring of ordinary integers with $N(\pi)$ elements, $\mathcal{H}_\pi$ is the left residual class set of z modulo π , and $\mu_\pi(z)$ is given remainder

of z with respect to modulo π . We can also consider z as a Hurwitz integer such that its imaginary part is zero where $z \in \mathbb{Z}$. The quotient ring of the Hurwitz integers modulo this equivalence relation, which we denote as $\mathcal{H}_\pi = \{z \bmod \pi : z \in \mathbb{Z}_{N(\pi)}\}$. The $\mathcal{H}_\pi$ set contains $N(\pi)$ elements. If π is a prime Hurwitz integer, then the modulo function μ defines a bijective mapping from $\mathbb{Z}_{N(\pi)}$ into $\mathcal{H}_\pi$ which is a four-dimensional signal space. Therefore, the modulo function μ is a ring isomorphism between $\mathbb{Z}_{N(\pi)}$ and $\mathcal{H}_\pi$. Because there exists an inverse map [8] and we have $\mu(z_1 + z_2) = \mu(z_1) + \mu(z_2)$ and $\mu(z_1 z_2) = \mu(z_1)\mu(z_2)$ for any $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. If π is a primitive Hurwitz integer, the modulo function μ is a group isomorphism with respect to addition between $\mathbb{Z}_{N(\pi)}$ and $\mathcal{H}_\pi$. Because there exists an inverse map [8] and we have $\mu(z_1 + z_2) = \mu(z_1) + \mu(z_2)$ for any $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. After we define encoder Hurwitz integers set in the following section, we can define a ring isomorphism between $\mathbb{Z}_{N(\pi)}$ and $\mathcal{H}_\pi$ where π is an encoder Hurwitz integer.

In engineering, the "signal constellation" has been used as a communication term. In mathematics, the "signal constellation" means for residue class set. In the rest of this study, we use the "signal constellation" term instead of the "left residue class set" term. You can find more details which related to the arithmetics properties about arithmetic properties of quaternions and Hurwitz integers in [23-24].

3. Encoder Hurwitz Integers

The Euclid division algorithm says that there exists unique integers q and r such that $a = bq + r, 0 \leq |r| < |b|$ where $a, b \in \mathbb{Z}$. Here a is the dividend, b is the divisor, q is the quotient, r is the remainder, and $|\cdot|$ is the symbol for absolute value. If we generalize the Euclid division algorithm for Hurwitz integers whose components are in $\mathbb{Z}$, then there exists unique Hurwitz integers β and γ such that $\alpha = \pi\beta + \gamma, 0 \leq N(\gamma) \leq N(\pi)$ where $\alpha, \pi \in \mathcal{H}$ such that their components are in $\mathbb{Z}$. Therefore, the Euclid division algorithm does not work for Hurwitz integers whose components are in $\mathbb{Z}$ because of $0 \leq N(\gamma) \leq N(\pi)$. So, the primitive Hurwitz integers whose components are in $\mathbb{Z}$ do not have the "division with small remainder" property. If we generalize the Euclid division algorithm for Hurwitz integers whose components are in $\mathbb{Z} + \frac{1}{2}$, then there exists unique Hurwitz integers β and γ such that $\alpha = \pi\beta + \gamma, 0 \leq N(\gamma) < N(\pi)$ where $\alpha, \pi \in \mathcal{H}$ such that their components are in $\mathbb{Z} + \frac{1}{2}$. Therefore, the Euclid division algorithm works for Hurwitz integers whose components are in $\mathbb{Z} + \frac{1}{2}$ because of $0 \leq N(\gamma) < N(\pi)$. So, the primitive Hurwitz integers whose components are in $\mathbb{Z} + \frac{1}{2}$ have the "division with small remainder" property [24]. Also, the Euclid division algorithm generally works for prime Hurwitz integers. Because each element in the $\mathcal{H}_\pi$ has the minimal norm. So, the prime Hurwitz integers have the "division with small remainder" property. The following proposition and lemma imply that primitive Hurwitz integers whose components are in $\mathbb{Z}$ do not have the "division

with small remainder" property since not working Euclid division algorithm for primitive Hurwitz integers whose each component is an odd integer.

Proposition 3.1. *Let π is a primitive Hurwitz integer whose each component is an odd integer. Then,*

$$N(\mu_\pi(\frac{N(\pi)}{2})) = N(\pi) \quad (3.1)$$

with respect to equation (2.3).

Proof. Let $\pi = \pi_1 + \pi_2 i + \pi_3 j + \pi_4 k$ is a Hurwitz integer whose each component are an odd integer such that $\gcd(\pi_1, \pi_2, \pi_3, \pi_4) = 1$. The conjugate of π is $\bar{\pi} = \pi_1 - \pi_2 i - \pi_3 j - \pi_4 k$. We show that is $N(\mu_\pi(\frac{N(\pi)}{2})) = N(\pi)$ with respect to equation (2.3).

$$\begin{aligned} \mu_\pi(\frac{N(\pi)}{2}) &= \frac{N(\pi)}{2} - \pi \lfloor \frac{\bar{\pi} N(\pi)}{N(\pi)} \rfloor \\ &= \frac{N(\pi)}{2} - \pi \lfloor \frac{\bar{\pi}}{2} \rfloor \\ &= \frac{N(\pi)}{2} - (\pi_1 + \pi_2 i + \pi_3 j + \pi_4 k) \lfloor \frac{\pi_1 - \pi_2 i - \pi_3 j - \pi_4 k}{2} \rfloor. \end{aligned} \quad (3.2)$$

$\lfloor \frac{\pi_1}{2} \rfloor = ((\frac{\pi_1}{2}) + \frac{1}{2})$, $\lfloor \frac{-\pi_2}{2} \rfloor = ((\frac{-\pi_2}{2}) + \frac{1}{2})$, $\lfloor \frac{-\pi_3}{2} \rfloor = ((\frac{-\pi_3}{2}) + \frac{1}{2})$ and $\lfloor \frac{-\pi_4}{2} \rfloor = ((\frac{-\pi_4}{2}) + \frac{1}{2})$ since π_1, π_2, π_3 and π_4 are odd integers. Then,

$$\begin{aligned} \mu_\pi(\frac{N(\pi)}{2}) &= \frac{N(\pi)}{2} - (\pi_1 + \pi_2 i + \pi_3 j + \pi_4 k) [(\frac{\pi_1}{2} + \frac{1}{2}) + (\frac{-\pi_2}{2} + \frac{1}{2})i \\ &\quad + (\frac{-\pi_3}{2} + \frac{1}{2})j + (\frac{-\pi_4}{2} + \frac{1}{2})k] \\ &= \frac{N(\pi)}{2} - (\frac{\pi_1^2 + \pi_1}{2} + (\frac{-\pi_1 \pi_2 + \pi_1}{2})i + (\frac{-\pi_1 \pi_3 + \pi_1}{2})j \\ &\quad + (\frac{-\pi_1 \pi_4 + \pi_1}{2})k + (\frac{\pi_2 \pi_1 + \pi_2}{2})i + \frac{\pi_2^2 - \pi_2}{2} + (\frac{-\pi_2 \pi_3 + \pi_2}{2})k \\ &\quad + (\frac{\pi_2 \pi_4 - \pi_2}{2})j + (\frac{\pi_3 \pi_1 + \pi_3}{2})j + (\frac{\pi_3 \pi_2 - \pi_3}{2})k + (\frac{\pi_3^2 - \pi_3}{2}) \\ &\quad + (\frac{-\pi_3 \pi_4 + \pi_3}{2})i + (\frac{\pi_4 \pi_1 + \pi_4}{2})k + (\frac{-\pi_4 \pi_2 + \pi_4}{2})j \\ &\quad + (\frac{\pi_4 \pi_3 - \pi_4}{2})i + \frac{\pi_4^2 - \pi_4}{2}) \\ &= \frac{N(\pi)}{2} - \lfloor \frac{\pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2 + \pi_1 - \pi_2 - \pi_3 - \pi_4}{2} \\ &\quad + (\frac{-\pi_1 \pi_2 + \pi_1 + \pi_2 \pi_1 + \pi_2 - \pi_3 \pi_4 + \pi_3 + \pi_4 \pi_3 - \pi_4}{2})i \\ &\quad + (\frac{-\pi_1 \pi_3 + \pi_1 + \pi_2 \pi_4 - \pi_2 + \pi_3 \pi_1 + \pi_3 - \pi_4 \pi_2 + \pi_4}{2})j \\ &\quad + (\frac{-\pi_1 \pi_4 + \pi_1 - \pi_2 \pi_3 + \pi_2 + \pi_3 \pi_2 - \pi_3 + \pi_4 \pi_1 + \pi_4}{2})k \rfloor \\ \mu_\pi(\frac{N(\pi)}{2}) &= \frac{N(\pi)}{2} - \frac{N(\pi)}{2} + \frac{\pi_1 - \pi_2 - \pi_3 - \pi_4}{2} + (\frac{\pi_1 + \pi_2 + \pi_3 - \pi_4}{2})i \\ &\quad + (\frac{\pi_1 - \pi_2 + \pi_3 + \pi_4}{2})j + (\frac{\pi_1 + \pi_2 - \pi_3 + \pi_4}{2})k \\ &= \frac{\pi_1 - \pi_2 - \pi_3 - \pi_4}{2} + (\frac{\pi_1 + \pi_2 + \pi_3 - \pi_4}{2})i + (\frac{\pi_1 - \pi_2 + \pi_3 + \pi_4}{2})j \\ &\quad + (\frac{\pi_1 + \pi_2 - \pi_3 + \pi_4}{2})k. \end{aligned} \quad (3.3)$$

Therefore, we have

$$\begin{aligned} N(\mu_\pi(\frac{N(\pi)}{2})) &= N(\frac{\pi_1 - \pi_2 - \pi_3 - \pi_4}{2} + (\frac{\pi_1 + \pi_2 + \pi_3 - \pi_4}{2})i \\ &\quad + (\frac{\pi_1 - \pi_2 + \pi_3 + \pi_4}{2})j + (\frac{\pi_1 + \pi_2 - \pi_3 + \pi_4}{2})k) \\ &= \frac{4(\pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2)}{4} \\ &= \pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2 \end{aligned} \quad (3.4)$$

Since $N(\pi) = \pi_1^2 + \pi_2^2 + \pi_3^2 + \pi_4^2$, we have

$$N(\mu_\pi(\frac{N(\pi)}{2})) = N(\pi). \quad (3.5)$$

This completes the proof. $\square$

Let $\gamma = \mu_\pi(\frac{N(\pi)}{2})$. By proposition 3.1, we have $N(\mu_\pi(\frac{N(\pi)}{2})) = N(\gamma) = N(\pi)$. In other words, the norm of the remainder is equal to the norm of the divisor. We generalize proposition 3.1 for primitive Hurwitz integers whose norm is not a prime integer where its each component is in $\mathbb{Z}$ with the following lemma.

Lemma 3.1. *Let π be a primitive Hurwitz integer, and α be a Hurwitz integer. If π is a primitive Hurwitz integer whose all component are in $\mathbb{Z}$, then*

$$N(\mu_\pi(\alpha)) \leq N(\pi) \quad (3.6)$$

with respect to equation (2.3).

Proof. Let π be a primitive Hurwitz integer, and $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a Hurwitz integers which its norm is non-zero. If π is a primitive Hurwitz integer whose all component are in $\mathbb{Z}$, by equation (2.3)

$$\begin{aligned} \mu_\pi(\alpha) &= \alpha - \pi \lfloor \frac{\bar{\pi}\alpha}{N(\pi)} \rfloor \\ \pi^{-1}\mu_\pi(\alpha) &= \pi^{-1}\alpha - \pi^{-1}\pi \lfloor \frac{\bar{\pi}\alpha}{\pi\pi} \rfloor \\ &= \pi^{-1}\alpha - \lfloor \frac{\alpha}{\pi} \rfloor \\ &= \pi^{-1}(\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k) - \lfloor \frac{\alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k}{\pi} \rfloor \\ &= \pi^{-1}\alpha_1 + \pi^{-1}\alpha_2 i + \pi^{-1}\alpha_3 j + \pi^{-1}\alpha_4 k \\ &\quad - \lfloor \pi^{-1}\alpha_1 + \pi^{-1}\alpha_2 i + \pi^{-1}\alpha_3 j + \pi^{-1}\alpha_4 k \rfloor \\ &= \pi^{-1}\alpha_1 + \pi^{-1}\alpha_2 i + \pi^{-1}\alpha_3 j + \pi^{-1}\alpha_4 k \\ &\quad - \lfloor \pi^{-1}\alpha_1 + \pi^{-1}\alpha_2 i + \pi^{-1}\alpha_3 j + \pi^{-1}\alpha_4 k \rfloor. \\ &= \pi^{-1}\alpha_1 - \lfloor \pi^{-1}\alpha_1 \rfloor + (\pi^{-1}\alpha_2 - \lfloor \pi^{-1}\alpha_2 \rfloor)i \\ &\quad + (\pi^{-1}\alpha_3 - \lfloor \pi^{-1}\alpha_3 \rfloor)j + (\pi^{-1}\alpha_4 - \lfloor \pi^{-1}\alpha_4 \rfloor)k. \end{aligned} \quad (3.7)$$

Therefore, we have

$$\begin{aligned} |\pi^{-1}\alpha_1 - \lfloor \pi^{-1}\alpha_1 \rfloor| &\leq \frac{1}{2} \\ |\pi^{-1}\alpha_2 - \lfloor \pi^{-1}\alpha_2 \rfloor| &\leq \frac{1}{2} \\ |\pi^{-1}\alpha_3 - \lfloor \pi^{-1}\alpha_3 \rfloor| &\leq \frac{1}{2} \\ |\pi^{-1}\alpha_4 - \lfloor \pi^{-1}\alpha_4 \rfloor| &\leq \frac{1}{2}. \end{aligned} \quad (3.8)$$

Hereby,

$$\begin{aligned} (\pi^{-1}\alpha_1 - \lfloor \pi^{-1}\alpha_1 \rfloor)^2 &\leq (\frac{1}{2})^2 \\ (\pi^{-1}\alpha_2 - \lfloor \pi^{-1}\alpha_2 \rfloor)^2 &\leq (\frac{1}{2})^2 \\ (\pi^{-1}\alpha_3 - \lfloor \pi^{-1}\alpha_3 \rfloor)^2 &\leq (\frac{1}{2})^2 \\ (\pi^{-1}\alpha_4 - \lfloor \pi^{-1}\alpha_4 \rfloor)^2 &\leq (\frac{1}{2})^2 \end{aligned} \quad (3.9)$$

So, we have

$$\begin{aligned}
 N(\pi^{-1}\mu_\pi(\alpha)) &= (\pi^{-1}\alpha_1 - \lfloor \pi^{-1}\alpha_1 \rfloor)^2 + (\pi^{-1}\alpha_2 - \lfloor \pi^{-1}\alpha_2 \rfloor)^2 \\
 &\quad + (\pi^{-1}\alpha_3 - \lfloor \pi^{-1}\alpha_3 \rfloor)^2 + (\pi^{-1}\alpha_4 - \lfloor \pi^{-1}\alpha_4 \rfloor)^2 \\
 &\leq (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 \\
 N(\pi^{-1})N(\mu_\pi(\alpha)) &\leq \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} \\
 \frac{1}{N(\pi)}N(\mu_\pi(\alpha)) &\leq 1 \\
 N(\mu_\pi(\alpha)) &\leq N(\pi).
 \end{aligned} \tag{3.10}$$

This completes the proof. $\square$

Consequently, primitive Hurwitz integers, whose norm is not a prime integer, such that each component is in $\mathbb{Z}$, do not have the "division with small remainder" property. So, the Euclid division algorithm does not work for primitive Hurwitz integers, whose norm is not a prime integer, such that each component is in $\mathbb{Z}$. With the following example, given practices for proposition 3.1.

Example 3.1. Let $\pi = 3 + i + j + 3k$ be a Hurwitz integer. Also, π is a primitive Hurwitz integer because of $\gcd(3, 1, 1, 3) = 1$. By equation (2.3), π Hurwitz constellation is

$$\mathcal{H}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0, \mu_\pi(1) = 1, \mu_\pi(2) = 2, \mu_\pi(3) = 3, \\ \mu_\pi(4) = -2 - 2j, \mu_\pi(5) = -1 - 2j, \mu_\pi(6) = -2j, \\ \mu_\pi(7) = 1 - 2j, \mu_\pi(8) = 2 - 2j, \mu_\pi(9) = 3 - 2j, \\ \mu_\pi(10) = 1 - i - 3j - 3k, \mu_\pi(11) = -3 + 2j, \\ \mu_\pi(12) = -2 + 2j, \mu_\pi(13) = -1 + 2j, \mu_\pi(14) = 2j, \\ \mu_\pi(15) = 1 + 2j, \mu_\pi(16) = 2 + 2j, \mu_\pi(17) = -3, \\ \mu_\pi(18) = -2, \mu_\pi(19) = -1 \end{array} \right\}. \tag{3.11}$$

This set contains 20 elements because of $N(\pi) = 3^2 + 1^2 + 1^2 + 3^2 = 20$. Also, $N(\mu_\pi(10)) = N(\pi)$ because of $N(\mu_\pi(10)) = 3^2 + (1)^2 + (1)^2 + (3)^2 = 20$. The norm of other elements in the set (3.11) is less than the norm of π . Consequently, $\pi = 3 + i + j + 3k$ Hurwitz integer does not have the "division with small remainder" property. In addition, to be a Euclidean metric, the inequality $d(x', y') + d(y', z') \geq d(x', z')$ should be verified. Because the inequalities i) $d(x', y') = 0$ if and only if $x' = y'$ where $x', y' \in \mathcal{H}_\pi$, and ii) $d(x', y') = d(y', x')$ where $x', y' \in \mathcal{H}_\pi$, are supplied. We consider $x' = 1 - i - 3j - 3k$, $y' = -1$ and $z' = -2$ in (3.11). $d(x', y') = 23$ since $N(y - x) = N(-1 - 1 + i + 3j + 3k) = N(-2 + i + 3j + 3k) = 23$, $d(y', z') = 1$ since $N(z - y) = N(-2 + 1) = N(-1) = 1$, and $d(x', z') = 28$ since $N(z - y) = N(-2 - 1 + i + 3j + 3k) = N(-3 + i + 3j + 3k) = 28$. Hereby, we have

$$\begin{array}{ccc}
 d(x', y') + d(y', z') & \geq & d(x', z') \\
 23 + 1 & \geq & 28 \\
 24 & \geq & 28.
 \end{array} \tag{3.12}$$

But this is not true. Consequently, the Euclidean metric is not provide for the $\mathcal{H}_\pi$ constellation constructed by $\pi = 3 + i + j + 3k$ Hurwitz integer, and the Euclidean division algorithm is not work for $\pi = 3 + i + j + 3k$ Hurwitz integer.

We define a set that consists of the primitive Hurwitz integers that have the "division with small remainder" property with the following definition. This set is a subset of the primitive Hurwitz integers, and Hurwitz integers.

Definition 3.1. Let $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ be a Hurwitz integer. α Hurwitz integer is called an encoder Hurwitz integer if it is satisfying the following conditions. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}$,

- $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{Z}^+$,
- $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are not same parity, i.e. $\alpha_1, \alpha_2, \alpha_3$ and α_4 all together neither even nor odd integers,
- $\gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 1$.

Or,

- $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (\mathbb{Z} \cup \{0\}) + \frac{1}{2}$,

In other words, $\alpha = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$ primitive Hurwitz integer such that its all components are not odd integers, or $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (\mathbb{Z} \cup \{0\}) + \frac{1}{2}$, is called an encoder Hurwitz integer.

The definition 3.1 is a flexible definition. Namely, the elements of the encoder Hurwitz integers set are expandable or collapsible depending on the used modulo technique. In this study, we defined the above definition with respect to the modulo function defined in definition 2.6. Let now us show that the modulo function μ defined between $\mathbb{Z}_{N(\pi)}$ and $\mathcal{H}_\pi$ by equation (2.3) is a ring isomorphism with the following theorems.

Theorem 3.1. Let π be an encoder Hurwitz integer, and $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. $\mu : \mathbb{Z}_{N(\pi)} \rightarrow \mathcal{H}_\pi$ modulo function is a ring homomorphism with respect to $\mu_\pi(z_1 + z_2) = (\mu_\pi(z_1) + \mu_\pi(z_2)) \text{mod} \pi$ and $\mu_\pi(z_1 z_2) = (\mu_\pi(z_1) \mu_\pi(z_2)) \text{mod} \pi$.

Proof. Let π be an encoder Hurwitz integer and, $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. Define $\mu : \mathbb{Z}_{N(\pi)} \rightarrow \mathcal{H}_\pi$ modulo function by $\mu_\pi(z) = z \text{mod} \pi = z - \pi \lfloor \frac{\pi z}{N(\pi)} \rfloor$. Note that each component of λ_1 and λ_2 are in $\mathbb{Z}$ with respect to round notation. Therefore,

$$\begin{aligned} \mu_\pi(z_1 + z_2) &= (z_1 + z_2) \text{mod} \pi. \\ &= z_1 + z_2 - \pi \lfloor \frac{\pi(z_1 + z_2)}{N(\pi)} \rfloor. \end{aligned} \quad (3.13)$$

If $\mu_\pi(z_1) = z_1 \text{mod} \pi$ and $\mu_\pi(z_2) = z_2 \text{mod} \pi$, then, there exists λ_1 and λ_2 Hurwitz integers such that $z_1 = \pi \lambda_1 + \mu_\pi(z_1)$ and $z_2 = \pi \lambda_2 + \mu_\pi(z_2)$, respectively.

Then, we have

$$\begin{aligned}
\mu_\pi(z_1 + z_2) &= \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2) \\
&\quad - \pi \left\lfloor \frac{\pi(\pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2))}{N(\pi)} \right\rfloor \\
&= \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2) \\
&\quad - \pi \left\lfloor \frac{\pi\pi\lambda_1 + \pi\mu_\pi(z_1) + \pi\pi\lambda_2 + \pi\mu_\pi(z_2)}{N(\pi)} \right\rfloor \\
&= \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2) \\
&\quad - \pi \left\lfloor \lambda_1 + \lambda_2 + \frac{\pi\mu_\pi(z_1) + \pi\mu_\pi(z_2)}{N(\pi)} \right\rfloor.
\end{aligned} \tag{3.14}$$

$\lfloor \lambda_1 + \lambda_2 \rfloor = \lambda_1 + \lambda_2$ since λ_1 and λ_2 are the Hurwitz integers whose each component is in $\mathbb{Z}$. Then, we have

$$\begin{aligned}
\mu_\pi(z_1 + z_2) &= \pi\lambda_1 + \mu_\pi(z_1) + \pi\lambda_2 + \mu_\pi(z_2) - \pi\lambda_1 - \pi\lambda_2 \\
&\quad - \pi \left\lfloor \frac{\pi\mu_\pi(z_1) + \pi\mu_\pi(z_2)}{N(\pi)} \right\rfloor \\
&= \mu_\pi(z_1) + \mu_\pi(z_2) - \pi \left\lfloor \frac{\pi(\mu_\pi(z_1) + \mu_\pi(z_2))}{N(\pi)} \right\rfloor.
\end{aligned} \tag{3.15}$$

So, we have

$$\mu_\pi(z_1 + z_2) = (\mu_\pi(z_1) + \mu_\pi(z_2)) \bmod \pi. \tag{3.16}$$

On the other hand,

$$\begin{aligned}
\mu_\pi(z_1 z_2) &= z_1 z_2 \bmod \pi \\
&= z_1 z_2 - \pi \left\lfloor \frac{\pi(z_1 z_2)}{N(\pi)} \right\rfloor.
\end{aligned} \tag{3.17}$$

If $\mu_\pi(z_1) = z_1 \bmod \pi$ and $\mu_\pi(z_2) = z_2 \bmod \pi$, then, there exists λ_1 and λ_2 Hurwitz integers such that $z_1 = \pi\lambda_1 + \mu_\pi(z_1)$ and $z_2 = \pi\lambda_2 + \mu_\pi(z_2)$, respectively. Then, we have

$$\begin{aligned}
\mu_\pi(z_1 z_2) &= (\pi\lambda_1 + \mu_\pi(z_1))(\pi\lambda_2 + \mu_\pi(z_2)) \\
&\quad - \pi \left\lfloor \frac{\pi(\pi\lambda_1 + \mu_\pi(z_1))(\pi\lambda_2 + \mu_\pi(z_2))}{N(\pi)} \right\rfloor \\
&= \pi\lambda_1 \pi\lambda_2 + \pi\lambda_1 \mu_\pi(z_2) + \mu_\pi(z_1) \pi\lambda_2 + \mu_\pi(z_1) \mu_\pi(z_2) \\
&\quad - \pi \left\lfloor \frac{\pi\pi\lambda_1 \pi\lambda_2 + \pi\pi\lambda_1 \mu_\pi(z_2) + \pi\mu_\pi(z_1) \pi\lambda_2 + \pi\mu_\pi(z_1) \mu_\pi(z_2)}{N(\pi)} \right\rfloor.
\end{aligned}$$

Since $N(\pi) = \pi\bar{\pi}$, we have

$$\begin{aligned}
\mu_\pi(z_1 z_2) &= \pi\lambda_1 \pi\lambda_2 + \pi\lambda_1 \mu_\pi(z_2) + \mu_\pi(z_1) \pi\lambda_2 + \mu_\pi(z_1) \mu_\pi(z_2) \\
&\quad - \pi \left\lfloor \lambda_1 \pi\lambda_2 + \lambda_1 \mu_\pi(z_2) + \frac{\pi\mu_\pi(z_1) \pi\lambda_2}{N(\pi)} + \frac{\pi\mu_\pi(z_1) \mu_\pi(z_2)}{N(\pi)} \right\rfloor.
\end{aligned}$$

Since $\lambda_1 \pi\lambda_2$, $\lambda_1 \mu_\pi(z_2)$ and $\frac{\pi\mu_\pi(z_1) \pi\lambda_2}{N(\pi)}$ are Hurwitz integers whose each component is in $\mathbb{Z}$, we have

$$\begin{aligned}
\mu_\pi(z_1 z_2) &= \pi\lambda_1 \pi\lambda_2 + \pi\lambda_1 \mu_\pi(z_2) + \mu_\pi(z_1) \pi\lambda_2 + \mu_\pi(z_1) \mu_\pi(z_2) - \pi\lambda_1 \pi\lambda_2 \\
&\quad - \pi\lambda_1 \mu_\pi(z_2) - \pi \frac{\pi\mu_\pi(z_1) \pi\lambda_2}{N(\pi)} - \pi \left\lfloor \frac{\pi\mu_\pi(z_1) \mu_\pi(z_2)}{N(\pi)} \right\rfloor \\
&= \mu_\pi(z_1) \mu_\pi(z_2) - \pi \left\lfloor \frac{\pi\mu_\pi(z_1) \mu_\pi(z_2)}{N(\pi)} \right\rfloor.
\end{aligned}$$

So, $\mu_\pi(z_1 z_2) = \mu_\pi(z_1) \mu_\pi(z_2) \bmod \pi$. Consequently, μ function is a ring homomorphism. This completes this proof. $\square$

Theorem 3.2. *Let π be an encoder Hurwitz integer. Then,*

$$\mathbb{Z}_{N(\pi)} \cong \mathcal{H}_\pi$$

with respect to equality (2.3).

Proof. Let π be an encoder Hurwitz integer and, $z_1, z_2 \in \mathbb{Z}_{N(\pi)}$. The modulo function defines a mapping from $\mathbb{Z}_{N(\pi)}$ to $\mathcal{H}_\pi$. This mapping is $\mu : \mathbb{Z}_{N(\pi)} \rightarrow \mathcal{H}_\pi$:

$$\mu_\pi(z) = z \bmod \pi = \gamma = z - \pi \lfloor \frac{\bar{\pi}z}{N(\pi)} \rfloor. \quad (3.18)$$

According to theorem 3.1, μ function is a ring homomorphism. This mapping is a surjective ring homomorphism because of $\text{Im}\mu = \{\mu_\pi(z) : z \in \mathbb{Z}_{N(\pi)}\} = \mathcal{H}_\pi$. If $z = 0$ where $z \in \mathbb{Z}_{N(\pi)}$, then we have

$$\begin{aligned} \mu_\pi(0) &= 0 - \pi \lfloor \frac{\bar{\pi}0}{N(\pi)} \rfloor \\ &= 0 - \pi \lfloor 0 \rfloor \\ &= 0 - \pi 0 \\ &= 0. \end{aligned} \quad (3.19)$$

If $z \neq 0$ where $z \in \mathbb{Z}_{N(\pi)}$, then $\mu_\pi(z)$ is to greater or equal than 1. Hereby, this mapping is a bijective ring homomorphism because of $\text{Ker}\mu = \{z \in \mathbb{Z}_{N(\pi)} : \mu_\pi(z) = 0\} = \{z \in \mathbb{Z}_{N(\pi)} : z = 0\} = \{0\}$. μ function is a ring isomorphism since it is both a surjective ring homomorphism and a bijective ring homomorphism, i.e. $\mathbb{Z}_{N(\pi)} \cong \mathcal{H}_\pi$. $\square$

The following proposition demonstrates that the encoder Hurwitz integers have the "division small remainder" property.

Proposition 3.2. *Let π is an encoder Hurwitz integer whose each component is not an odd integer. Then,*

$$N(\mu_\pi(z)) < N(\pi). \quad (3.20)$$

Proof. We shall analyze and prove this theorem case by case. Let $\pi = \pi_1 + \pi_2 i + \pi_3 j + \pi_4 k$ is an encoder Hurwitz integer.

Case : 1 Let π is an encoder Hurwitz integer such that π_1 is an even integer, and π_2, π_3 and π_4 are odd integers. So, $N(\pi)$ is a odd integer. By equation (2.3),

$$\mu_\pi(z) = z - \pi \lfloor \pi^{-1} z \rfloor \quad (3.21)$$

Hereby,

$$\begin{aligned} \mu_\pi(z) \pi^{-1} &= \pi^{-1} z - \pi^{-1} \pi \lfloor \pi^{-1} z \rfloor \\ &= \pi^{-1} z - \lfloor \pi^{-1} z \rfloor \\ &= \frac{\bar{\pi}z}{N(\pi)} - \lfloor \frac{\bar{\pi}z}{N(\pi)} \rfloor. \end{aligned} \quad (3.22)$$

Since $\bar{\pi} = \pi_1 - \pi_2 i - \pi_3 j - \pi_4 k$, we have

$$\begin{aligned} \mu_\pi(z) \pi^{-1} &= \frac{\pi_1 z}{N(\pi)} - (\frac{\pi_2 z}{N(\pi)})i - (\frac{\pi_3 z}{N(\pi)})j - (\frac{\pi_4 z}{N(\pi)})k \\ &\quad - \lfloor \frac{\pi_1 z}{N(\pi)} \rfloor - \lfloor \frac{\pi_2 z}{N(\pi)} \rfloor i - \lfloor \frac{\pi_3 z}{N(\pi)} \rfloor j - \lfloor \frac{\pi_4 z}{N(\pi)} \rfloor k \\ &= \frac{\pi_1 z}{N(\pi)} - \lfloor \frac{\pi_1 z}{N(\pi)} \rfloor - (\frac{\pi_2 z}{N(\pi)} - \lfloor \frac{\pi_2 z}{N(\pi)} \rfloor)i \\ &\quad - (\frac{\pi_3 z}{N(\pi)} - \lfloor \frac{\pi_3 z}{N(\pi)} \rfloor)j - (\frac{\pi_4 z}{N(\pi)} - \lfloor \frac{\pi_4 z}{N(\pi)} \rfloor)k \end{aligned} \quad (3.23)$$

Since π_1 is an even integer, π_2, π_3 and π_4 are odd integers, $N(\pi)$ is an odd integer, and $\frac{N(\pi)}{2}$ is not a integer,

$$\begin{aligned} 0 &\leq \left| \frac{\pi_1 z}{N(\pi)} - \left\lfloor \frac{\pi_1 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_2 z}{N(\pi)} - \left\lfloor \frac{\pi_2 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_3 z}{N(\pi)} - \left\lfloor \frac{\pi_3 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_4 z}{N(\pi)} - \left\lfloor \frac{\pi_4 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2}. \end{aligned} \quad (3.24)$$

Therefore, we have

$$\begin{aligned} &N\left(\frac{\pi_1 z}{N(\pi)} - \left\lfloor \frac{\pi_1 z}{N(\pi)} \right\rfloor\right) + N\left(\frac{\pi_2 z}{N(\pi)} - \left\lfloor \frac{\pi_2 z}{N(\pi)} \right\rfloor\right) + N\left(\frac{\pi_3 z}{N(\pi)} - \left\lfloor \frac{\pi_3 z}{N(\pi)} \right\rfloor\right) \\ &+ N\left(\frac{\pi_4 z}{N(\pi)} - \left\lfloor \frac{\pi_4 z}{N(\pi)} \right\rfloor\right) < \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1. \end{aligned} \quad (3.25)$$

Hereby, we have

$$N(\mu_\pi(z)\pi^{-1}) = N(\mu_\pi(z))N(\pi^{-1}) = N(\mu_\pi(z))\frac{1}{N(\pi)} < 1. \quad (3.26)$$

Consequently,

$$N(\mu_\pi(z)) < N(\pi). \quad (3.27)$$

Case : 2 Let π is an encoder Hurwitz integer such that π_1 and π_2 are even integers, and π_3 and π_4 are odd integers. Firstly, we should check whether to verify or not equation 3.1 in preposition 3.1 in case of $N(\pi)$ is an even integer. Let $z = \frac{N(\pi)}{2}$. By equation (2.3), we have

$$\mu_\pi\left(\frac{N(\pi)}{2}\right) = \frac{N(\pi)}{2} - \pi \left\lfloor \pi^{-1} \frac{N(\pi)}{2} \right\rfloor. \quad (3.28)$$

Hereby,

$$\begin{aligned} \mu_\pi\left(\frac{N(\pi)}{2}\right)\pi^{-1} &= \pi^{-1} \frac{N(\pi)}{2} - \pi^{-1} \pi \left\lfloor \pi^{-1} \frac{N(\pi)}{2} \right\rfloor \\ &= \frac{\pi}{2} - \left\lfloor \frac{\pi}{2} \right\rfloor \end{aligned} \quad (3.29)$$

Since $\bar{\pi} = \pi_1 - \pi_2 i - \pi_3 j - \pi_4 k$,

$$\begin{aligned} \mu_\pi\left(\frac{N(\pi)}{2}\right)\pi^{-1} &= \frac{\pi_1}{2} - \left(\frac{\pi_2}{2}\right)i - \left(\frac{\pi_3}{2}\right)j - \left(\frac{\pi_4}{2}\right)k \\ &\quad - \left\lfloor \frac{\pi_1}{2} \right\rfloor + \left\lfloor \frac{\pi_2}{2} \right\rfloor i + \left\lfloor \frac{\pi_3}{2} \right\rfloor j + \left\lfloor \frac{\pi_4}{2} \right\rfloor k \\ &= \frac{\pi_1}{2} - \left\lfloor \frac{\pi_1}{2} \right\rfloor - \left(\frac{\pi_2}{2} - \left\lfloor \frac{\pi_2}{2} \right\rfloor\right)i \\ &\quad - \left(\frac{\pi_3}{2} - \left\lfloor \frac{\pi_3}{2} \right\rfloor\right)j - \left(\frac{\pi_4}{2} - \left\lfloor \frac{\pi_4}{2} \right\rfloor\right)k \end{aligned} \quad (3.30)$$

$\left\lfloor \frac{\pi_1}{2} \right\rfloor = \frac{\pi_1}{2}$ and $\left\lfloor \frac{\pi_2}{2} \right\rfloor = \frac{\pi_2}{2}$ since π_2 and π_3 are even integers. Also, $\left\lfloor \frac{\pi_3}{2} \right\rfloor = \frac{\pi_3+1}{2}$ and $\left\lfloor \frac{\pi_4}{2} \right\rfloor = \frac{\pi_4+1}{2}$ since π_3 and π_4 are odd integers. So,

$$\begin{aligned} \mu_\pi\left(\frac{N(\pi)}{2}\right)\pi^{-1} &= \frac{\pi_1}{2} - \frac{\pi_1}{2} - \left(\frac{\pi_2}{2} - \frac{\pi_2}{2}\right)i \\ &\quad - \left(\frac{\pi_3}{2} - \frac{\pi_3+1}{2}\right)j - \left(\frac{\pi_4}{2} - \frac{\pi_4+1}{2}\right)k \\ &= \left(\frac{1}{2}\right)j + \left(\frac{1}{2}\right)k. \end{aligned} \quad (3.31)$$

Hereby, we have

$$\begin{aligned} N(\mu_\pi\left(\frac{N(\pi)}{2}\right)\pi^{-1}) &= N\left(\left(\frac{1}{2}\right)j + \left(\frac{1}{2}\right)k\right) \\ N(\mu_\pi\left(\frac{N(\pi)}{2}\right))N(\pi^{-1}) &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ N(\mu_\pi\left(\frac{N(\pi)}{2}\right))\left(\frac{1}{N(\pi)}\right) &= \frac{1}{4} + \frac{1}{4} \\ N(\mu_\pi\left(\frac{N(\pi)}{2}\right)) &= \frac{N(\pi)}{2}. \end{aligned} \quad (3.32)$$

Therefore,

$$N(\mu_\pi(\frac{N(\pi)}{2})) < N(\pi). \quad (3.33)$$

Let $z \neq \frac{N(\pi)}{2}$. By equation (2.3), we have

$$\mu_\pi(z) = z - \pi \lfloor \pi^{-1} z \rfloor. \quad (3.34)$$

Hereby,

$$\begin{aligned} \mu_\pi(z)\pi^{-1} &= \pi^{-1}z - \pi^{-1}\pi \lfloor \pi^{-1}z \rfloor \\ &= \pi^{-1}z - \lfloor \pi^{-1}z \rfloor \\ &= \frac{\bar{\pi}z}{N(\pi)} - \lfloor \frac{\bar{\pi}z}{N(\pi)} \rfloor. \end{aligned} \quad (3.35)$$

Since $\bar{\pi} = \pi_1 - \pi_2i - \pi_3j - \pi_4k$,

$$\begin{aligned} \mu_\pi(z)\pi^{-1} &= \frac{\pi_1z}{N(\pi)} - (\frac{\pi_2z}{N(\pi)})i - (\frac{\pi_3z}{N(\pi)})j - (\frac{\pi_4z}{N(\pi)})k \\ &\quad - \lfloor \frac{\pi_1z}{N(\pi)} \rfloor - \lfloor \frac{\pi_2z}{N(\pi)} \rfloor i - \lfloor \frac{\pi_3z}{N(\pi)} \rfloor j - \lfloor \frac{\pi_4z}{N(\pi)} \rfloor k \\ &= \frac{\pi_1z}{N(\pi)} - \lfloor \frac{\pi_1z}{N(\pi)} \rfloor - (\frac{\pi_2z}{N(\pi)} - \lfloor \frac{\pi_2z}{N(\pi)} \rfloor)i \\ &\quad - (\frac{\pi_3z}{N(\pi)} - \lfloor \frac{\pi_3z}{N(\pi)} \rfloor)j - (\frac{\pi_4z}{N(\pi)} - \lfloor \frac{\pi_4z}{N(\pi)} \rfloor)k. \end{aligned} \quad (3.36)$$

Since $z \neq \frac{N(\pi)}{2}$,

$$\begin{aligned} 0 &\leq \left| \frac{\pi_1z}{N(\pi)} - \lfloor \frac{\pi_1z}{N(\pi)} \rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_2z}{N(\pi)} - \lfloor \frac{\pi_2z}{N(\pi)} \rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_3z}{N(\pi)} - \lfloor \frac{\pi_3z}{N(\pi)} \rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_4z}{N(\pi)} - \lfloor \frac{\pi_4z}{N(\pi)} \rfloor \right| < \frac{1}{2}. \end{aligned} \quad (3.37)$$

Hereby, we have

$$\begin{aligned} &N(\frac{\pi_1z}{N(\pi)} - \lfloor \frac{\pi_1z}{N(\pi)} \rfloor) + N(\frac{\pi_2z}{N(\pi)} - \lfloor \frac{\pi_2z}{N(\pi)} \rfloor) + N(\frac{\pi_3z}{N(\pi)} - \lfloor \frac{\pi_3z}{N(\pi)} \rfloor) \\ &+ N(\frac{\pi_4z}{N(\pi)} - \lfloor \frac{\pi_4z}{N(\pi)} \rfloor) < (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 = 1 \end{aligned} \quad (3.38)$$

Therefore,

$$N(\mu_\pi(z)\pi^{-1}) = N(\mu_\pi(z))N(\pi^{-1}) = N(\mu_\pi(z))\frac{1}{N(\pi)} < 1 \quad (3.39)$$

Consequently,

$$N(\mu_\pi(z)) < N(\pi). \quad (3.40)$$

Case : 3 Let π is an encoder Hurwitz integer such that π_1, π_2 and π_3 are even integers, and π_4 is an odd integer. So, $N(\pi)$ is an odd integer. By equation (2.3), we have

$$\mu_\pi(z) = z - \pi \lfloor \pi^{-1} z \rfloor \quad (3.41)$$

Hereby,

$$\begin{aligned} \mu_\pi(z)\pi^{-1} &= \pi^{-1}z - \pi^{-1}\pi \lfloor \pi^{-1}z \rfloor \\ &= \pi^{-1}z - \lfloor \pi^{-1}z \rfloor \\ &= \frac{\bar{\pi}z}{N(\pi)} - \lfloor \frac{\bar{\pi}z}{N(\pi)} \rfloor. \end{aligned} \quad (3.42)$$

Since $\bar{\pi} = \pi_1 - \pi_2 i - \pi_3 j - \pi_4 k$,

$$\begin{aligned} \mu_\pi(z)\pi^{-1} &= \frac{\pi_1 z}{N(\pi)} - \left(\frac{\pi_2 z}{N(\pi)}\right)i - \left(\frac{\pi_3 z}{N(\pi)}\right)j - \left(\frac{\pi_4 z}{N(\pi)}\right)k \\ &\quad - \left\lfloor \frac{\pi_1 z}{N(\pi)} \right\rfloor i - \left\lfloor \frac{\pi_2 z}{N(\pi)} \right\rfloor j - \left\lfloor \frac{\pi_3 z}{N(\pi)} \right\rfloor k \\ &= \frac{\pi_1 z}{N(\pi)} - \left\lfloor \frac{\pi_1 z}{N(\pi)} \right\rfloor - \left(\frac{\pi_2 z}{N(\pi)} - \left\lfloor \frac{\pi_2 z}{N(\pi)} \right\rfloor\right)i \\ &\quad - \left(\frac{\pi_3 z}{N(\pi)} - \left\lfloor \frac{\pi_3 z}{N(\pi)} \right\rfloor\right)j - \left(\frac{\pi_4 z}{N(\pi)} - \left\lfloor \frac{\pi_4 z}{N(\pi)} \right\rfloor\right)k \end{aligned} \quad (3.43)$$

Since π_4 is an odd integer, π_1, π_2 and π_3 are even integers, and $N(\pi)$ is an odd integer,

$$\begin{aligned} 0 &\leq \left| \frac{\pi_1 z}{N(\pi)} - \left\lfloor \frac{\pi_1 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_2 z}{N(\pi)} - \left\lfloor \frac{\pi_2 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_3 z}{N(\pi)} - \left\lfloor \frac{\pi_3 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2} \\ 0 &\leq \left| \frac{\pi_4 z}{N(\pi)} - \left\lfloor \frac{\pi_4 z}{N(\pi)} \right\rfloor \right| < \frac{1}{2}. \end{aligned} \quad (3.44)$$

Hereby, we have

$$\begin{aligned} &N\left(\frac{\pi_1 z}{N(\pi)} - \left\lfloor \frac{\pi_1 z}{N(\pi)} \right\rfloor\right) + N\left(\frac{\pi_2 z}{N(\pi)} - \left\lfloor \frac{\pi_2 z}{N(\pi)} \right\rfloor\right) + N\left(\frac{\pi_3 z}{N(\pi)} - \left\lfloor \frac{\pi_3 z}{N(\pi)} \right\rfloor\right) \\ &+ N\left(\frac{\pi_4 z}{N(\pi)} - \left\lfloor \frac{\pi_4 z}{N(\pi)} \right\rfloor\right) < \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1 \end{aligned} \quad (3.45)$$

Therefore,

$$N(\mu_\pi(z)\pi^{-1}) = N(\mu_\pi(z))N(\pi^{-1}) = N(\mu_\pi(z))\frac{1}{N(\pi)} < 1 \quad (3.46)$$

Consequently,

$$N(\mu_\pi(z)) < N(\pi). \quad (3.47)$$

This completes the proof. $\square$

With the following examples, giving an example for each case in the proposition 3.2.

Example 3.2. Case 1 $\pi = 1 + 3i + 2j + k$ is an encoder Hurwitz integer. By equation (2.3), the π Hurwitz constellation is

$$\mathcal{L}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0, \mu_\pi(1) = 1, \mu_\pi(2) = 2, \mu_\pi(3) = i + j - 2k, \\ \mu_\pi(4) = -1 + 2j + k, \mu_\pi(5) = 2j + k, \mu_\pi(6) = 1 + 2j + k, \\ \mu_\pi(7) = 2 + 2j + k, \mu_\pi(8) = -2 - 2j - k, \mu_\pi(9) = -1 - 2j - k, \\ \mu_\pi(10) = -2j - k, \mu_\pi(11) = 1 - 2j - k, \mu_\pi(12) = -i - j + 2k, \\ \mu_\pi(13) = -2, \mu_\pi(14) = -1, \end{array} \right\}. \quad (3.48)$$

The set contains 15 elements because of $N(\pi) = 1^2 + 3^2 + 2^2 + 1^2 = 15$. The norm of each element in the set is less than the norm of π .

Example 3.3. Case 2 $\pi = 2 + 3i + j + 2k$ is an encoder Hurwitz integer. By equation (2.3), the π Hurwitz constellation is

$$\mathcal{L}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0, \mu_\pi(1) = 1, \mu_\pi(2) = 2, \mu_\pi(3) = 3, \\ \mu_\pi(4) = 1 + 2i + 2j - k, \mu_\pi(5) = -2 - 2j - k, \mu_\pi(6) = -1 - 2j - k, \\ \mu_\pi(7) = -2j - k, \mu_\pi(8) = 1 - 2j - k, \mu_\pi(9) = 2 - 2j - k, \\ \mu_\pi(10) = -1 + 2j + k, \mu_\pi(11) = 2j + k, \mu_\pi(12) = 1 + 2j + k, \\ \mu_\pi(13) = 2 + 2j + k, \mu_\pi(14) = -1 - 2i - 2j + k, \\ \mu_\pi(15) = -2i - 2j + k, \mu_\pi(16) = -2, \mu_\pi(17) = -1 \end{array} \right\}. \quad (3.49)$$

The set contains 18 elements because of $N(\pi) = 2^2 + 3^2 + 1^2 + 2^2 = 18$. The norm of each element in the set is less than the norm of π .

Example 3.4. Case 3 $\pi = 2 + 3i + 2j + 2k$ is an encoder Hurwitz integer. By equation (2.3), the π Hurwitz constellation is

$$\mathcal{L}_\pi = \left\{ \begin{array}{l} \mu_\pi(0) = 0, \mu_\pi(1) = 1, \mu_\pi(2) = 2, \mu_\pi(3) = 3, \\ \mu_\pi(4) = 1 + 2i + 2j - 2k, \mu_\pi(5) = 2 + 2i + 2j - 2k, \\ \mu_\pi(6) = 3 - i - j + k, \mu_\pi(7) = -2 - i - j + k, \\ \mu_\pi(8) = -1 - i - j + k, \mu_\pi(9) = -i - j + k, \\ \mu_\pi(10) = 1 - i - j + k, \mu_\pi(11) = -1 + i + j - k, \\ \mu_\pi(12) = i + j - k, \mu_\pi(13) = 1 + i + j - k, \\ \mu_\pi(14) = 2 + i + j - k, \mu_\pi(15) = 3 + i + j - k, \\ \mu_\pi(16) = -2 - 2i - 2j + 2k, \mu_\pi(17) = -1 - 2i - 2j + 2k, \\ \mu_\pi(18) = -3, \mu_\pi(19) = -2, \mu_\pi(20) = -1 \end{array} \right\}. \quad (3.50)$$

The set contains 21 elements because of $N(\pi) = 2^2 + 3^2 + 2^2 + 2^2 = 21$. The norm of each element in the set is less than the norm of π .

Example 3.2, example 3.3, and example 3.4 are verified all the conditions for it to be an Euclidean metric. Also the Euclidean division algorithm works for the Hurwitz integers in example 3.2, example 3.3, and example 3.4. As a result of these examples, we represent the following proposition.

In the following example, we show that it does not have the "division with small remainder" property of a Hurwitz integer used to obtain the Hurwitz constellation constructed with a different technique by Rohweder et al..

Example 3.5. In [19], Rohweder et al. presented the new construction method for Hurwitz integers by

$$\mu_\pi(z) = z - \lfloor \frac{z\bar{\pi}}{N(\pi)} \rfloor \pi \quad (3.51)$$

where π is a primitive Hurwitz integer and $z \in \mathbb{Z}_{N(\pi)}$. They proposed four-dimensional Hurwitz integer signal constellations are obtained from the following mapping

$$\mathcal{H}_\pi = \mathcal{L}_\pi \bigcup \mathcal{O}_\pi, \quad (3.52)$$

where $\mathcal{L}_\pi$ is the subset of Lipschitz integers, which are quaternions whose all components are in $\mathbb{Z}$, which can be evaluated by

$$\mathcal{L}_\pi = \{\mu_\pi(a + bj) : a, b \in \mathbb{Z}_{N(\pi)}\} \quad (3.53)$$

where $\mathbb{Z}_{N(\pi)}$ denotes the ring of integers modulo $N(\pi)$. Also, $\mathcal{O}_\pi$ in equation (3.52) is the corresponding coset of half-integers, which can be calculated by

$$\mathcal{O}_\pi = \{\mu_\pi(h + w) : h \in \mathcal{L}_\pi\}, \quad (3.54)$$

where $w = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$. This set must be 100 elements. The proposition 3.1 says that if π is a Hurwitz integer whose each component is an odd integer and, $N(\pi)$ is an even integer, then we should check the $\frac{N(\pi)}{2}$ integer at the construction method used for Hurwitz integers. We consider $3 + i$ in [19,

Table I]. Hereby, $N(\pi) = 3^2 + 1^2 = 10$. Let be $a = \frac{N(\pi)}{2} = \frac{10}{2} = 5$ and $b = \frac{N(\pi)}{2} = \frac{10}{2} = 5$. So, we consider $\mu_\pi(5 + 5j)$ in $\mathcal{L}_\pi$. By equation (3.51),

$$\begin{aligned}
 \mu_\pi(5 + 5j) &= 5 + 5j - \left\lfloor \frac{(5+5j)(3-i)}{10} \right\rfloor (3+i) \\
 &= 5 + 5j - \left\lfloor \frac{15-5i+15j+5k}{10} \right\rfloor (3+i) \\
 &= 5 + 5j - (2 + 2j + k)(3+i) \\
 &= 5 + 5j - 6 - 2i - 7j - k \\
 &= -1 - 2i - 2j - k.
 \end{aligned} \tag{3.55}$$

Hereby, $N(\mu_\pi(5 + 5j)) = N(\pi)$ because of $N(\mu_\pi(5 + 5j)) = (-1)^2 + (-2)^2 + (-2)^2 + (-1)^2 = 10$. So, $5 + 5j \equiv 0 \pmod{3+i}$. If $a = 0$ and $b = 0$, then $\mu_\pi(0) = 0$. Thereby, the $\mathcal{L}_\pi$ has elements less than 100 elements since $\mu_\pi(0) = \mu_\pi(5 + 5j) \equiv 0 \pmod{3+i}$. This contradicts to the size of the $\mathcal{O}_\pi$ constellation, which is with 100 elements. Similarly, we can not construct $\mathcal{O}_\pi$ constellation with 100 elements by $2 + 2i + j + k$ primitive Lipschitz integer, too. Consequently, we can say that $3 + i$ and $2 + 2i + j + k$ do not have the "division with small remainder" property with respect to the method in [19]. The method in [19] inappropriate to construct the Hurwitz constellation with 100 elements.

The technique in [19] is more appropriate for the Lipschitz integers whose norm is an odd number but inappropriate for the Lipschitz integers whose norm is an even number. Note that the definition 3.1 is a flexible definition. According to the modulo technique in [19], we can define the set of the encoder Hurwitz integers with "The Hurwitz integer whose the greatest common divisor of its components is one and its norm is an odd number is called an encoder Hurwitz integer.". The set of encoder Hurwitz integers is the set of the Hurwitz integers remaining with taking out the Hurwitz integers providing proposition 3.1 from the Hurwitz integers set, in general. We refer to proposition 3.1 and example 3.5 for this general definition. Note that we consider the definition 3.1 in this study. Note that we consider definition 3.1 in this study. We can come to a conclusion that the Euclid division algorithm works for the elements of the encoder Hurwitz integers set. So, we can construct well-defined Hurwitz constellations in terms of algebraic constructions for codes over Hurwitz integers. Consequently, we should use proposition 3.1 to check whether the Hurwitz integers used to construct Hurwitz constellations have the "division with small remainder" property or not. Consequently, we should use proposition 3.1 to check whether the Hurwitz integers used to construct Hurwitz constellations have the "division with small remainder" property or not.

4. Performances of Hurwitz Constellations for Transmission over AWGN channel

In this section, we are first giving some distance and performance measures, and then we investigate the performances of Hurwitz constellations that lies on encoder Hurwitz integers for transmission over AWGN channel by agency

of average energy, CFM, and SNR gains. Note that we investigate the performances of Hurwitz constellations constructed with Hurwitz integers whose components are in $\mathbb{Z} + \frac{1}{2}$ by using the technique used for Gaussian constellations in [9] and Lipschitz integers in [25] in this study. Because the Hurwitz constellations constructed of primitive Hurwitz integers which their components are in $\mathbb{Z}$ show the same performances with Lipschitz constellations in [25]. Therefore, we give set partitioning property on larger Hurwitz integers namely, proposed Hurwitz integers, since the Hurwitz constellations that have the same size with Gaussian constellations are almost shown the same performances for transmission over the AWGN channel. We follow the procedures in [25] for some distance, performance measures and set partitioning property. The average energy of a constellation denoted by $\mathcal{E}_\pi$ is computed by

$$\mathcal{E}_\pi = \frac{1}{N(\pi)} \sum_{z=0}^{N(\pi)-1} N(\mu_\pi(z)). \quad (4.1)$$

The squared Euclidean distance of two Hurwitz integers is defined as

$$d_E(\alpha, \beta) = N(\beta - \alpha) \quad (4.2)$$

and the minimum squared Euclidean distance of the constellation is

$$\delta_\lambda^2 = \min_{\alpha \neq \beta} d_E(\alpha, \beta). \quad (4.3)$$

where $\alpha, \beta \in \mathcal{H}_\pi$. In [26], Forney and Wei proposed the constellation figure of merit (CFM) to compare signal constellations of different dimensions. The CFM is the ratio of the minimum squared Euclidean distance and the average energy per two-dimensions. So, the CFM of a M -dimensional constellation is computed by

$$CFM = \frac{M\delta_\pi^2}{2\mathcal{E}_\pi}. \quad (4.4)$$

A higher CFM leads to a better performance for transmission over an AWGN channel [24]. Asymptotic coding gain means for higher signal to noise ratio (SNR) [9]. The SNR of M -dimensional constellation is computed by

$$SNR = -10 \cdot \log_{10}(\text{CFM of signal constellation}). \quad (4.5)$$

The SNR gains of a Hurwitz constellation over the AWGN channel is

$$SNR = -10 \cdot \log_{10}\left(\frac{\text{CFM of Hurwitz signal constellation}}{\text{CFM of Gaussian noise constellation}}\right). \quad (4.6)$$

Note that the number of elements of the Hurwitz constellation and Gaussian constellation should be the same to compare performances over the AWGN channel. A residue class ring of Hurwitz integers $\mathcal{H}_\pi$ arises from the residue class ring of integers $\mathbb{Z}_{N(\pi)} = \{0, 1, \dots, N(\pi) - 1\}$ for an integer $N(\pi)$. If $N(\pi)$ is not a prime integer, then we can partition the set $\mathbb{Z}_{N(\pi)}$ into subsets of equal size. Let $N = c \cdot d$ where N is the elements number of Hurwitz constellation. We can partition the set $\mathcal{H}_\pi$ into c subsets $\mathcal{H}_\pi^{(0)}, \dots, \mathcal{H}_\pi^{(c-1)}$ each

with d elements. The subsets correspond to the integer sets $\mathbb{Z}_\pi^{(0)}, \dots, \mathbb{Z}_\pi^{(c-1)}$, where

$$\mathbb{Z}_{N(\pi)}^{(0)} = \{0, c, 2c, \dots, (d-1)c\} \quad (4.7)$$

and $\mathbb{Z}_\pi^{(1)}, \dots, \mathbb{Z}_\pi^{(c-1)}$ are the cosets of $\mathbb{Z}_\pi^{(0)}$, i.e. $\mathbb{Z}_\pi^{(l)} = \{z : z - l \in \mathbb{Z}_\pi^{(0)}\}$. Note that the number of elements of the Hurwitz constellation and Gaussian constellation should be the equal size to compare performances over the AWGN channel but proposed Hurwitz constellations should not be. Hence, we can apply set partitioning property on proposed primitive Hurwitz constellations. The SNR gains of a proposed Hurwitz constellation over the AWGN channel is computed by

$$\begin{aligned} SNR &= -10 \cdot \log_{10} \left(\frac{\text{CFM of } \mathcal{H}_\pi^{(0)}}{\text{CFM of Gaussian constellation}} \right) \\ &\quad + 10 \cdot \log_{10} \left(\frac{\text{CFM of Hurwitz constellation}}{\text{CFM of Gaussian constellation}} \right) \\ &= -10(\log_{10}(\text{CFM of } \mathcal{H}_\pi^{(0)}) \\ &\quad - \log_{10}(\text{CFM of Hurwitz constellation})) \\ &\quad + 10 \cdot \log_{10}(\text{CFM of Gaussian constellation}) \\ &\quad - 10 \cdot \log_{10}(\text{CFM of Gaussian constellation}) \\ &= -10(\log_{10}(\text{CFM of } \mathcal{H}_\pi^{(0)}) \\ &\quad - \log_{10}(\text{CFM of Hurwitz constellation})) \\ &= -10 \cdot \log_{10} \left(\frac{\text{CFM of } \mathcal{H}_\pi^{(0)}}{\text{CFM of Hurwitz constellation}} \right) \end{aligned} \quad (4.8)$$

where Hurwitz signal constellation $\mathcal{H}_\pi^{(0)}$ and the Hurwitz constellation are the equal size. Guzeltepe [18], and Rohweder et al. [19] separately presented different techniques for Hurwitz constellations. Guzeltepe [18] investigated performances of the Hurwitz constellations with $N(\pi)^2$ elements where π is a primitive Hurwitz integer, over the AWGN channel by using isomorphism between $\mathcal{H}_{\pi^2}$ and $\mathcal{Z}_{\pi^2}$. You can see example 3.5, or [19] for the technique of Rohweder et al.. Note that we use isomorphism between $\mathcal{H}_\pi$ and $\mathcal{Z}_\pi$ in this study.

Example 4.1. In Table I, we present the performance of the Hurwitz constellation constructed by encoder Hurwitz integers whose each component is in $\mathbb{Z} + \frac{1}{2}$ over the AWGN channel by means of average energy, CFM, and SNR coding gains. In Table I, the Hurwitz constellations obtained from the modulo function technique in this study have almost similar properties as Lipschitz constellations in the paper of Freudenberg et al. in [25]. The performance of Hurwitz constellations over the AWGN channel in Table I is not so good but better than nothing with respect to the performance of the Hurwitz constellations whose components is in $\mathbb{Z}$ over the AWGN channel. You can see [26] for the performance of the Hurwitz constellations whose components is in $\mathbb{Z}$ over the AWGN channel. Because the performances of Hurwitz constellations whose components is in $\mathbb{Z}$ and the performances of Lipschitz constellations are the same. Similarity, the performances of proposed Hurwitz constellations whose components is in $\mathbb{Z}$ are the same with the performances of proposed Lipschitz constellations in [25, Table I].

TABLE 1. Table of CFM, Energy and SNR of Hurwitz constellations constructed by Hurwitz integers whose coefficients are in $\mathbb{Z} + \frac{1}{2}$ (d:The number of elements (norm) of Hurwitz signal constellation)

d	Primitive Hurwitz Integer	Constellations				SNR gain
		CFM		Energy		
		Gauss	Hurwitz	Gauss	Hurwitz	
5	$\frac{3}{2} + \frac{3}{2}i + \frac{1}{2}j + \frac{1}{2}k$	1.2500	1.6667	0.8000	1.200	1.25
13	$\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.4643	0.5200	2.1538	3.8462	0.49
17	$\frac{5}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.3542	0.3864	2.8235	5.1765	0.38
25	$\frac{9}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{1}{2}k$	0.2404	0.2551	4.1600	7.8400	0.26
29	$\frac{9}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k$	0.2071	0.2180	4.8276	9.1724	0.22
37	$\frac{11}{2} + \frac{5}{2}i + \frac{1}{2}j + \frac{1}{2}k$	0.1623	0.1690	6.1622	11.8378	0.18
41	$\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.1464	0.1519	6.8293	13.1707	0.16
53	$\frac{13}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.1132	0.1165	8.8302	17.1698	0.12
61	$\frac{15}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{1}{2}k$	0.0984	0.1008	10.1639	19.8361	0.10
65	$\frac{11}{2} + \frac{11}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.0923	0.0945	10.8308	21.1692	0.10
73	$\frac{17}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$	0.0822	0.0839	12.1644	23.8356	0.09
85	$\frac{13}{2} + \frac{13}{2}i + \frac{1}{2}j + \frac{1}{2}k$	0.0706	0.0719	14.1467	27.8353	0.08
89	$\frac{17}{2} + \frac{7}{2}i + \frac{3}{2}j + \frac{3}{2}k$	0.0674	0.0686	14.8315	29.1685	0.08
97	$\frac{19}{2} + \frac{5}{2}i + \frac{1}{2}j + \frac{1}{2}k$	0.0619	0.0628	16.1649	31.8351	0.06

Example 4.2. In Table II, we present the performance of the proposed Hurwitz constellation constructed by encoder Hurwitz integers whose each component is in $\mathbb{Z} + \frac{1}{2}$ over the AWGN channel by means of average energy, CFM, and SNR coding gains. The proposed Hurwitz constellations in Table II have advantage performances for transmission over the AWGN channel by set partitioning property.

There also exist different proposed primitive Hurwitz integers used to construct proposed Hurwitz constellations that have higher CFM and lower average energy in equal size. You can examine the following examples. The below examples are given clues about the construction of tables.

Example 4.3. We consider proposed Hurwitz constellation(s) with $N = 3 \cdot 13 = 39$ elements. There exist four different proposed primitive Hurwitz integers used to construct proposed Hurwitz constellations with $N = 39$. These

TABLE 2. Table of average energies, CFMs and SNR coding gains of proposed Hurwitz constellation constructed by proposed Hurwitz integers whose coefficients are in $\mathbb{Z} + \frac{1}{2}$ (N : The number of elements (norm) of a proposed Hurwitz signal constellation, c: The number of subsets of a proposed Hurwitz signal constellation, d:The number of elements (norm) of subset of a proposed Hurwitz signal constellation)

N	c	d	Proposed Primitive Hurwitz Integers	Constellations				SNR gain [dB]
				CFM			Energy	
				Gauss	Hurwitz	Proposed Hurwitz	Proposed Hurwitz	
15	3	5	$\frac{5}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k$	1.2500	1.6667	1.2500	4.8000	-1.25
39	3	13	$\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k$	0.4643	0.5200	1.5600	11.5385	4.77
51	3	17	$\frac{11}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{3}{2}k$	0.3542	0.3864	1.1591	15.5294	4.77
75	3	25	$\frac{13}{2} + \frac{9}{2}i + \frac{7}{2}j + \frac{1}{2}k$	0.2404	0.2551	0.7653	23.5200	4.77
87	3	29	$\frac{13}{2} + \frac{11}{2}i + \frac{7}{2}j + \frac{3}{2}k$	0.2071	0.2180	0.6541	27.5172	4.77
185	5	37	$\frac{21}{2} + \frac{13}{2}i + \frac{9}{2}j + \frac{7}{2}k$	0.1673	0.1690	0.8447	59.1982	6.99
205	5	41	$\frac{19}{2} + \frac{17}{2}i + \frac{11}{2}j + \frac{7}{2}k$	0.1464	0.1519	0.7593	65.8537	6.99
265	5	53	$\frac{23}{2} + \frac{19}{2}i + \frac{13}{2}j + \frac{1}{2}k$	0.1132	0.1165	0.5824	85.8491	6.99
427	7	61	$\frac{33}{2} + \frac{21}{2}i + \frac{13}{2}j + \frac{3}{2}k$	0.0984	0.1008	0.7058	138.8520	8.45
455	7	65	$\frac{27}{2} + \frac{25}{2}i + \frac{21}{2}j + \frac{5}{2}k$	0.0923	0.0945	0.4717	148.4000	6.98
511	7	73	$\frac{33}{2} + \frac{21}{2}i + \frac{17}{2}j + \frac{15}{2}k$	0.0822	0.0839	0.5874	166.8490	8.45
595	7	85	$\frac{37}{2} + \frac{29}{2}i + \frac{13}{2}j + \frac{1}{2}k$	0.0706	0.0719	0.5030	194.8470	8.45
623	7	89	$\frac{35}{2} + \frac{33}{2}i + \frac{13}{2}j + \frac{3}{2}k$	0.0674	0.0686	0.4800	204.1800	8.45
873	9	97	$\frac{41}{2} + \frac{35}{2}i + \frac{19}{2}j + \frac{15}{2}k$	0.0619	0.0628	0.5654	286.5150	9.54

proposed primitive Hurwitz integers are $\frac{7}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{3}{2}k$, $\frac{9}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{5}{2}k$, $\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k$, and $\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k$. There is no Gaussian constellation that has an equal size with the proposed Hurwitz constellation that has $N = 39$ elements. Note that proposed primitive Hurwitz integers are not to be the same size as primitive Gaussian integers to apply set partitioning property for proposed primitive Hurwitz integers. Firstly, we consider $\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k$, and $\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k$ proposed primitive Hurwitz

integers. For $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation, the minimum squared Euclidean distance, CFM and average energy are 1, 12.5128 and 0.1598, respectively. The $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation is partition the $c = 3$ different subsets with each set $d = 13$ elements. So, the SNR coding gain of $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation is computed by using a subset of $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation. This subset is the set $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}^{(0)}$ with 13 elements. Because, the minimum squared Euclidean distance of $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}^{(0)}$ constellation is larger than the minimum squared Euclidean distance of $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}^{(1)}$ and $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}^{(2)}$ constellations. The minimum squared Euclidean distance, CFM and average energy of $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}^{(0)}$ Hurwitz signal constellation that is a subset of $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ Hurwitz constellation are 9, 11.5385 and 1.5600, respectively. Also, the primitive Hurwitz integers used to construct the Hurwitz constellation that is the equivalence $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}^{(0)}$ constellation with 13 elements are $\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k$, $\frac{5}{2}+\frac{5}{2}i+\frac{1}{2}j+\frac{1}{2}k$, and $\frac{7}{2}+\frac{1}{2}i+\frac{1}{2}j+\frac{1}{2}k$. The minimum squared Euclidean distance, CFM and average energy of these Hurwitz constellations are 1, 0.5200, and 3.8462. The minimum squared Euclidean distance, CFM and average energy of the Gaussian constellation $\mathcal{G}_{3+2i}$ with 13 elements are 1, 2.1539 and 0.4643, respectively. We consider $\mathcal{H}_{\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k}$ primitive Hurwitz integer calculating SNR coding gain. Therefore, SNR coding gain of $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation is

$$\text{SNR} = -10 \log \left(\frac{\text{CFM of } \mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}^{(0)}}{\text{CFM of } \mathcal{H}_{\frac{5}{2}+\frac{3}{2}i+\frac{3}{2}j+\frac{3}{2}k}} \right) = -10 \log \left(\frac{1.5600}{0.5200} \right) = 4.77. \quad (4.9)$$

For $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation, the minimum squared Euclidean distance, CFM and average energy are 1, 12.5128 and 0.1598, respectively. The $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation is partition the $c = 3$ different subsets with each set $d = 13$ elements. So, the SNR coding gain of $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation is computed by using a subset of $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation. This subset is the set $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}^{(0)}$ with 13 elements. Because, the minimum squared Euclidean distance of $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}^{(0)}$ is larger than the minimum squared Euclidean distance of $\mathcal{L}^{(1)}$ and $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}^{(2)}$. The minimum squared Euclidean distance, CFM and average energy of $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}^{(0)}$ Hurwitz signal constellation that is a subset of $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellation are 9, 11.5385 and 1.5600, respectively. The average energy, CFM and minimum squared Euclidean distance of $\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k}$ and $\mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellations have the same. So,

$$\mathcal{H}_{\frac{9}{2}+\frac{7}{2}i+\frac{5}{2}j+\frac{1}{2}k} \text{ and } \mathcal{H}_{\frac{11}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$$

proposed Hurwitz constellations have the same performances for transmission over AWGN channel. Lastly, we consider $\frac{7}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{3}{2}k$, and $\frac{9}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{5}{2}k$ proposed primitive Hurwitz integers. For both constellations, the minimum squared Euclidean distance, CFM and average energy are 1, 12.5128 and 0.1598. The subsets of these proposed Hurwitz constellations are $\mathcal{H}_{\frac{7}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{3}{2}k}^{(0)}$, and $\mathcal{H}_{\frac{9}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{5}{2}k}^{(0)}$. The average energy, CFM and minimum squared Euclidean distance of these subsets are 3, 11.5385 and 0.5200, respectively. Therefore, SNR coding gain of these proposed Hurwitz constellations is

$$\text{SNR} = -10 \log \left(\frac{\text{CFM of } \mathcal{H}_{\frac{7}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{3}{2}k}^{(0)}}{\text{CFM of } \mathcal{H}_{\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k}} \right) = -10 \log \left(\frac{0.5200}{0.5200} \right) = 0. \quad (4.10)$$

Consequently, the constellations that have higher CFM and larger minimum square Euclidean distance are $\mathcal{H}_{\frac{9}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{1}{2}k}$ and $\mathcal{H}_{\frac{11}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}$ proposed Hurwitz constellations. We choose $\frac{5}{2} + \frac{3}{2}i + \frac{3}{2}j + \frac{3}{2}k$ proposed primitive Hurwitz integer to represent in Table I, and $\frac{9}{2} + \frac{7}{2}i + \frac{5}{2}j + \frac{1}{2}k$ primitive Hurwitz integer to represent in Table II to create regular tables that are not crowded.

Example 4.4. We consider proposed Hurwitz constellation(s) with $N = 3 \cdot 29 = 87$ elements. There exist eight different proposed primitive Hurwitz integers used to construct proposed Hurwitz constellations with $N = 87$. These proposed primitive Hurwitz integers are $\frac{11}{2} + \frac{11}{2}i + \frac{9}{2}j + \frac{5}{2}k$, $\frac{13}{2} + \frac{9}{2}i + \frac{7}{2}j + \frac{7}{2}k$, $\frac{13}{2} + \frac{11}{2}i + \frac{7}{2}j + \frac{3}{2}k$, $\frac{13}{2} + \frac{13}{2}i + \frac{3}{2}j + \frac{1}{2}k$, $\frac{15}{2} + \frac{7}{2}i + \frac{7}{2}j + \frac{5}{2}k$, $\frac{15}{2} + \frac{11}{2}i + \frac{1}{2}j + \frac{1}{2}k$, $\frac{17}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{3}{2}k$, and $\frac{17}{2} + \frac{7}{2}i + \frac{3}{2}j + \frac{1}{2}k$. There is no Gaussian constellation that has an equal size with the proposed Hurwitz constellation that has $N = 87$ elements. The minimum squared Euclidean distance, CFM and average energy of proposed Hurwitz constellations constructed by these proposed primitive Hurwitz integers are 1, 28.5057 and 0.0702, respectively. These proposed Hurwitz constellations are partition the $c = 3$ different subsets with each set $d = 29$ elements. So, the SNR coding gains of these proposed Hurwitz constellation is separately computed by using a subset of these proposed Hurwitz constellations. We consider $\mathcal{H}_{\frac{13}{2} + \frac{11}{2}i + \frac{7}{2}j + \frac{3}{2}k}^{(0)}$, and $\mathcal{H}_{\frac{17}{2} + \frac{7}{2}i + \frac{53}{2}j + \frac{1}{2}k}^{(0)}$ subsets with 29 elements of $\mathcal{H}_{\frac{13}{2} + \frac{11}{2}i + \frac{7}{2}j + \frac{3}{2}k}$, and $\mathcal{H}_{\frac{17}{2} + \frac{7}{2}i + \frac{3}{2}j + \frac{1}{2}k}$, respectively. The minimum square Euclidean distance of these subsets is larger than others. The minimum square Euclidean distance of these subsets is 9 but the others is 6. Also average energy and CFM of these subsets are 27.5172 and 0.6541 but the others are 27.5172 and 0.4361, respectively. Also, the primitive Hurwitz integers used to construct the Hurwitz constellations that is the equivalence $\mathcal{H}_{\frac{13}{2} + \frac{11}{2}i + \frac{7}{2}j + \frac{3}{2}k}^{(0)}$, and $\mathcal{H}_{\frac{17}{2} + \frac{7}{2}i + \frac{53}{2}j + \frac{1}{2}k}^{(0)}$ constellations with 29 elements are $\frac{7}{2} + \frac{7}{2}i + \frac{3}{2}j + \frac{3}{2}k$, and $\frac{9}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k$. The minimum squared Euclidean distance, CFM and average energy of $\mathcal{H}_{\frac{7}{2} + \frac{5}{2}i + \frac{5}{2}j + \frac{1}{2}k}$ Hurwitz constellation are 1, 0.2525, and 7.9200. The minimum squared Euclidean distance, CFM and average energy of $\mathcal{H}_{\frac{7}{2} + \frac{7}{2}i + \frac{3}{2}j + \frac{3}{2}k}$, and $\mathcal{H}_{\frac{9}{2} + \frac{5}{2}i + \frac{3}{2}j + \frac{1}{2}k}$ Hurwitz constellations are

1, 0.2180, and 9.1724. We consider $\mathcal{H}_{\frac{9}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ and $\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k$ primitive Hurwitz integers calculating SNR coding gain. The minimum squared Euclidean distance, CFM and average energy of the Gaussian constellation $\mathcal{G}_{5+2i}$ with 29 elements are 1, 4.8276 and 0.2071, respectively. Therefore, SNR coding gain of $\mathcal{H}_{\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k}$ and proposed Hurwitz constellation is

$$\text{SNR} = -10 \log\left(\frac{\text{CFM of } \mathcal{H}_{\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k}^{(0)}}{\text{CFM of } \mathcal{H}_{\frac{9}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k}}\right) = -10 \log\left(\frac{0.6541}{0.2180}\right) = 4.77. \quad (4.11)$$

Consequently, the constellations that have higher CFM and larger minimum square Euclidean distance are $\mathcal{H}_{\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k}$ and $\mathcal{H}_{\frac{17}{2}+\frac{7}{2}i+\frac{3}{2}j+\frac{1}{2}k}$ proposed Hurwitz constellations. We choose $\frac{13}{2}+\frac{11}{2}i+\frac{7}{2}j+\frac{3}{2}k$ proposed primitive Hurwitz integer to represent in Table I, and $\frac{9}{2}+\frac{5}{2}i+\frac{3}{2}j+\frac{1}{2}k$ primitive Hurwitz integer to represent in Table II to create regular tables that are not crowded.

5. Conclusion

In this study, we investigated Hurwitz integers that have "division with small remainder" property and defined a new set, which is formed Hurwitz integers that have "division with small remainder" property, named encoder Hurwitz integers. So, we can define a Euclidean metric for Hurwitz constellations that lies on the Hurwitz integers or, different appropriate metrics for codes over the rings of Hurwitz integers. We showed that the Euclid division algorithm whether works or not for Hurwitz integers whose coefficients are in $\mathbb{Z}$ with proposition 3.1 and proposition 3.2. Whichever technique we use, we can check whether the Euclidean division algorithm works for Hurwitz integers with these propositions (see example 3.5). In addition, we examined the performances of the Hurwitz integers whose components are in $\mathbb{Z} + \frac{1}{2}$ of transmission over the AWGN channel. Also, the modulo function defined in this paper shows an inappropriate technique for Hurwitz constellations. New techniques can improvement such as Rohweder et al.[19], and Guzeltepe [18]. In our forward study, we will be investigated the performances of Hurwitz constellations for transmission over the AWGN channel by a new technique such as technique in [18]. Therefore, this paper is written to be a reference in our following study and colleagues' forward studies.

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