A thermodynamic origin for the Cohen-Kaplan-Nelson bound

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Research Article

Keywords: CKN, Cohen-Kaplan-Nelson, Fischler-Susskind

Posted Date: November 10th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-1057939/v2

A thermodynamic origin for the Cohen-Kaplan-Nelson bound

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(Dated: 31-Oct-2021)

Abstract

The Cohen-Kaplan-Nelson bound is imposed on the grounds of logical consistency (with classical General Relativity) upon local quantum field theories. This paper puts the bound into the context of a thermodynamic principle applicable to a field with a particular equation of state in an expanding universe. This is achieved without overtly appealing to either a decreasing density of states or a minimum coupling requirement, though they might still be consistent with the results described.

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I. INTRODUCTION

The Cohen-Kaplan-Nelson [1] bound hints at a connection between the ultraviolet cutoff applicable to local effective field theory and the corresponding infrared cutoff, obtained by placing a limit on when gravitational effects should dominate the problem and collapse the relevant configuration into a black hole. The limit is actually ensured by stating, by observation, that gravitational effects do not actually appear to play a significant role. The bound therfore constrains the behavior of local quantum field theories in the universe. Further work by Banks and Draper [4] shows how to deduce the $\frac{3}{2}$-power of the entropy/length-scale dependence from an appropriately chosen ansatz for the density of states. Some other work [8] indicates how the bound might lead to a minimum coupling requirement akin to the weak gravity conjecture.

Some prior work, especially that of Fischler and Susskind [2] deduces various cosmological consequences of equations of state. It finds that the quadratic entropy/length-scale dependence required by holography is realized only with the equation of state $P = \rho$, which is achieved in the special case of a massless scalar field with no potential function, i.e., with only a kinetic term.

This work is motivated in order to find a common ground between the above sequence of thoughts as well as put the CKN bound in the context of a physical principle. To this end, we first re-derive the Fischler/Susskind principles (and the area-entropy connection), in the canonical ensemble, by minimizing the Helmholtz free energy. Then we apply the same condition to a radiation-dominated universe and deduce that the dependence of entropy on the length scale is indeed the $\frac{3}{2}$ power. This immediately allows us to deduce that the CKN bound is just equivalent to extremizing the free-energy in an expanding universe. We also confirm the Fischler-Susskind bound for the limiting case $P = \rho$.

II. THE FISCHLER-SUSSKIND BOUND

Let us study the thermodynamics of an entropy-saturating field in an expanding universe as in [2]. The authors impose the holographic condition that, operating with Planck units

$$\sigma R_H^D < [a(t) R_H]^{D-1} \tag{1}$$

where t is the cosmic time, $R_H = \int_0^t \frac{dt}{a(t)}$ is the horizon, σ is the (assumed) constant comoving entropy density and D is the number of spatial dimensions. Saturating the limit is achieved by setting $a(t) \sim t^p \rightarrow p = \frac{1}{D}$, which is consistent with an entropy-saturating equation of state ($P = w\rho$ with $w = 1$). In this limit, the R.H.S of Equation (1) is proportional to the square of $L = a(t)R_H$: in 3 spatial dimensions, i.e., the area.

We can derive these from applying the thermodynamic principle of extremizing the free energy as follows. We consider the following cases, for the radiation gas ($w = \frac{1}{3}$) and the scalar field with only kinetic term ($w = 1$).

We will use the following general formulas, *viz.* if $P = w\rho$, then, in D spatial dimensions, with t the cosmic time and $a(t)$ the scale factor

$$\begin{aligned} \rho &\sim \frac{1}{a^{D(1+w)}} \\ a(t) &\sim t^{\frac{2}{D(1+w)}} \\ T &\sim \frac{1}{a(t)^{Dw}} \end{aligned} \tag{2}$$

The causal horizon is defined as

$$\begin{aligned} R_H = \int_0^t \frac{dt}{a(t)} &\sim t^{1-\frac{2}{D(1+w)}} \\ L = a(t) \times R_H &= t \end{aligned} \tag{3}$$

In these terms, for a general equation of state,

$$\begin{aligned} \rho &\sim \frac{1}{L^2} \\ a(t) &\sim L^{\frac{2}{D(1+w)}} \\ T &\sim \frac{1}{L^{\frac{2w}{1+w}}} \\ \rho &\sim T^{\frac{1+w}{w}} \end{aligned} \tag{4}$$

In what follows, we will keep dimensions in check with appropriate powers of M_P, the Planck energy and work in natural units ($c = 1, \hbar = 1$).

III. THE CASE $w = \frac{1}{3}$

We define the Helmholtz free energy $F = \rho V - TS$ where ρ is the energy density, $V = L^3$ is the Hubble volume (where $L = a(t)R_H$) and T the cosmological temperature of the field in the universe. Extremizing F amounts to $\rho V \sim TS$. The temperature in a radiation-dominated universe, is $T \sim \frac{M_P}{a(t)} \sim \sqrt{\frac{M_P}{L}}$, while we assume that the entropy scales as $(M_P L)^r$ where r is yet to be determined. In such a universe, with a ultraviolet cutoff Λ, the total energy is $\rho V = L^3\big(\sigma T^4 + \Lambda^4\big)$. It is tempting to think that this term is dominated in the late(r) universe by the ultraviolet cutoff term, however, as we easily see, both the (bracketed) terms are of order $\sim \frac{M_P^2}{L^2}$ in the case (which will turn out to be the consistent solution)[3] where the entropy/length scaling is $\frac{3}{2}$. Now, straightforwardly applying the extremization relation, we obtain by inspection (of powers of L) that the solution to

$$\sigma\big(\frac{T}{M_P}\big)^4 + \big(\frac{\Lambda}{M_P}\big)^4 \sim \frac{1}{(M_P L)^{\frac{7}{2}-r}} \tag{5}$$

is that $r = \frac{3}{2}$.

As it turns out, this solution can also be obtained by counting states in a self-consistent manner. If we count the states in the quantum field theory, we'd obtain (from a standard local field theory calculation), and approximate the solution for Λ by ignoring the non-vacuum energy part,

$$\begin{aligned}(M_P L)^r = (\textit{dimensionless constants}) \times L^3 \int_{\frac{1}{L}}^{\Lambda} dk\, k^2 \sim L^3\Lambda^3 = (M_P L)^3 (\frac{\Lambda}{M_P})^3 \\ \to (M_P L)^r \sim (M_P L)^3 \frac{1}{(M_P L)^{\frac{3}{4}(\frac{7}{2}-r)}} \\ \to r = \frac{3}{2}\end{aligned} \tag{6}$$

which again indicates that the entropy scales as the 3/2 power of the Hubble length scale, as is known. Note that we have derived this from an application of the minimization of free energy principle rather than the CKN condition, though, as may be noted. Equation (5), with this value of r, is indeed the CKN condition in disguise, if we dropped the non-vacuum part and also set $r = \frac{3}{2}$.

To demonstrate the generality of this principle, we next apply it to the case of the "perfect" gas, i.e., with $w = 1$, exemplified by the scalar field with only a kinetic term.

IV. THE CASE $w = 1$

In this case, the total energy again is written as the sum of the field energy plus the vacuum energy upto the ultraviolet cutoff, i.e., $\rho V = L^3(\alpha T^2 + \Lambda^4)$. In this case, however, the temperature term in the universe dominated by this type of radiation is $T \sim \frac{M_P}{a(t)^3} \sim \frac{1}{L}$ (derived under the constant entropy density assumption [6]). Again, assuming that the entropy $S \sim (M_P L)^r$, we derive the equation

$$\big(\alpha\big(\frac{T}{M_P}\big)^2 + \big(\frac{\Lambda}{M_P}\big)^4\big) \sim \frac{1}{(M_P L)^{4-r}} \tag{7}$$

A similar observation to the above is in order. In this case, if we neglect the radiation energy term, $\Lambda^4 \sim \frac{M_P^r}{L^{4-r}}$, while the first term is of order $\sim \frac{M_P^2}{L^2}$. If, as we expect, $r = 2$ in this case, we will find that both terms are of the same order. Again, by inspection of

$$\big(\alpha\big(\frac{T}{M_P}\big)^2 + \big(\frac{\Lambda}{M_P}\big)^4\big) \sim \frac{1}{(M_P L)^{4-r}}$$
$$\rightarrow r = 2 \tag{8}$$

is a self-consistent solution. We thus find, in agreement with Fischler and Susskind [2], that the entropy/area relation is reached for the case $P = \rho$.

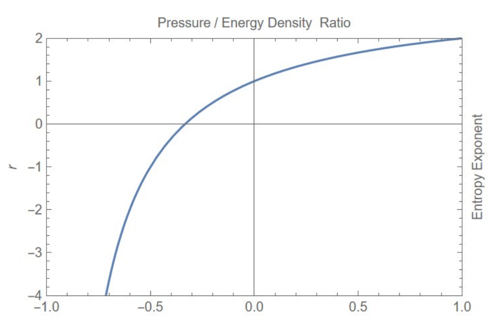
Note, however, that if we use the self-consistency check in the previous section, setting $\Lambda \sim \frac{1}{(M_P L)^{1-\frac{r}{4}}}$, we find an inconsistent solution, i.e., if $r = 2$ is the entropy exponent, the number of states still scales as $\frac{3}{2}$. Using other methods of reducing the density of states leads to an unphysical exponent. It appears that the free-energy minimization condition appears to work well in this circumstance. It is conceivable that more research would be useful in this respect.

All this can be generalized to D spatial dimensions, i.e.,

$$r = D - \frac{2}{1+w} \tag{9}$$

In the terminology of Banks [4], the power dependence $L(\epsilon) \sim \frac{1}{\epsilon^m}$ and we will find $m = 1 + \frac{2}{3}(D - \frac{2}{1+w})$.

FIG. 1. Entropy Exponent vs. Pressure/Energy Density Ratio



V. CONCLUSIONS

We have established a simple free-energy minimization principle in the canonical ensemble to derive the CKN bound. To some extent, these calculations use a blend of classical gravitational physics with standard quantum field theory. They are similar in spirit to the entropic views of gravity, though they immediately shed light on entropy-length scale relations. We extend the method to compute the bound applicable to other equations of state and demonstrate that one can deduce (in the radiation-dominated case) the entropy/length-scale relations without amending the density of states of the underlying quantum field theory. It is quite possible that the procedure itself, akin to focusing attention on equilibrium states already has this thinning of density of states built in, this is being further studied.

VI. DATA DISCLOSURE

Data sharing is not applicable to this article as no new data were created or analyzed in this study. No conflicts exist.

VII. ACKNOWLEDGMENTS

Detailed discussions with Tom Banks and Scott Thomas were especially useful. An especially lucid colloquium by Patrick Draper motivated this study and discussions with him after are also gratefully acknowledged. The hospitality of the Rutgers Physics Department and the NHETC is also acknowledged.

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