On Harmonic Summability of Order α

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On Harmonic Summability of Order α

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Abstract. In this study, the concepts of lacunary $(H, 1)$ –summability of order α , lacunary strongly harmonically summability of order α , lacunary statistical logarithmic convergence of order α of sequences of real numbers are introduced and relations between these concepts are investigated.

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1. Introduction

In 1951, Steinhaus [28] and Fast [11] introduced the concept of statistical convergence. In 1959, Schoenberg [23] established some basic properties of statistical convergence and also studied the concept as a summability method. Altin et al. [2], Altinok et al. [3], Cakalli ([5],[6]), Et et al. [7], Colak [8], Connor [9], Di Maio and Kočinac [10], Fridy [13], Gungor et al. ([15],[16]), Isik et al. ([1],[17],[18]), Salat [21], Savas and Et [22], Sengul et al. ([24],[25],[26]) and many authors investigated some arguments related to this notion.

Let $\mathbb{N}$ be the set of all natural numbers, $K \subseteq \mathbb{N}$ and $K(n) = \{k \leq n : k \in K\}$. The natural density of K is defined by $\delta(K) = \lim_n \frac{1}{n} |K(n)|$ if $\lim_n \frac{1}{n} |K(n)|$ exists. The vertical bars indicate the number of the elements in enclosed set.

The number sequence $x = (x_i)$ is said to be statistically convergent to a if for every $\varepsilon > 0$ the set $K(\varepsilon) = \{i \leq n : |x_i - a| \geq \varepsilon\}$ has natural density zero.

An increasing integer sequence $\theta = (k_r)$ is said to be a lacunary sequence such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ is denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is abbreviated by q_r , and $q_1 = k_1$ for convenience. Lacunary convergence was studied in ([4],[12],[14]).

The harmonic mean of the sequence $x = (x_i)$ defined by

$$\tau_n = \frac{1}{\ell_n} \sum_{i=1}^n \frac{x_i}{i}, \text{ where } \ell_n = \sum_{i=1}^n \frac{1}{i} \approx \log n \text{ for } n = 1, 2, \dots$$

A sequence $x = (x_i)$ is said to be $(H, 1)$ –summable to a if

$$\lim_{n \rightarrow \infty} \frac{1}{\ell_n} \sum_{i=1}^n \frac{x_i}{i} = a.$$

The set of all $(H, 1)$ –summable real sequences will be denoted H . H^0 will denote the set of all real sequences which $(H, 1)$ –summable to 0. It is well known that ordinary convergence always implies harmonic summability, and that the converse implication holds only under additional conditions.

A sequence $x = (x_i)$ is said to be strongly harmonically summable to a , if

$$\lim_{n \rightarrow \infty} \frac{1}{\ell_n} \sum_{i=1}^n \frac{|x_i - a|}{i} = 0.$$

The set of all strongly harmonically summable real sequences will be denoted $[H]$. $[H^0]$ will denote the set of all real sequences strongly harmonically summable to 0. Harmonically summability was studied in ([19],[20],[27]).

Throughout the study, $\log$ means the natural logarithm. Furthermore, given two sequences $x = (x_i)$ and $y = (y_i)$ of positive numbers (except possibly a finite number of terms), we write $x_i \sim y_i$ if $\lim_{i \rightarrow \infty} x_i/y_i = 1$.

Let define lacunary harmonic of order α means of the sequence (x_i) by

$$T_r^\alpha = \frac{1}{(L_r)^\alpha} \sum_{i \in I_r} \frac{x_i}{i}, \text{ where } (L_r)^\alpha = \left(\sum_{i \in I_r} \frac{1}{i} \right)^\alpha, \quad r = 1, 2, \dots$$

2. Main Results

In this section, we will introduce the concepts of lacunary $(H, 1)$ –summability of order α , lacunary strongly harmonically summability of order α , lacunary statistical logarithmic convergence of order α of sequences of real numbers and investigate relations between these spaces.

Definition 2.1 Let $\theta = (k_r)$ be a lacunary sequence and α be a real number such that $0 < \alpha \leq 1$. A sequence $x = (x_i)$ of real numbers is said to be lacunary $(H, 1)$ –summable of order α to the number a , if there is a real number a such that

$$\lim_{r \rightarrow \infty} \frac{1}{(L_r)^\alpha} \sum_{i \in I_r} \frac{x_i}{i} = a.$$

In this case we write $H_\theta^\alpha - \lim x_i = a$. The set of all lacunary $(H, 1)$ –summable sequences of order α will be denoted H_θ^α . In the special case $\alpha = 1$ we shall write H_θ instead of H_θ^α and also in the special case $\theta = (2^r)$ we shall write H^α instead of H_θ^α .

Definition 2.2 Let $\theta = (k_r)$ be a lacunary sequence and α be a real number such that $0 < \alpha \leq 1$. A sequence $x = (x_i)$ of real numbers is said to be lacunary strongly harmonically summable of order α to the number a , if there is a real number a such that

$$\lim_{r \rightarrow \infty} \frac{1}{(L_r)^\alpha} \sum_{i \in I_r} \frac{|x_i - a|}{i} = 0.$$

In this case we write $[H_\theta]^\alpha - \lim x_i = a$. The set of all lacunary strongly harmonically summable sequences of order α will be denoted $[H_\theta]^\alpha$. In the special case $\alpha = 1$ we shall write $[H_\theta]$ instead of $[H_\theta]^\alpha$ and also in the special case $\theta = (2^r)$ we shall write $[H]^\alpha$ instead of $[H_\theta]^\alpha$. If $a = 0$, then we shall write $[H_\theta^0]^\alpha$ instead of $[H_\theta]^\alpha$.

Theorem 2.3 Let $\theta = (k_r)$ be a lacunary sequence and α be a real number such that $0 < \alpha \leq 1$. Then $[H_\theta]^\alpha$ and H_θ^α are linear spaces.

Proof. Omitted.

Theorem 2.4 Let $\theta = (k_r)$ be a lacunary sequence and α be a real number such that $0 < \alpha \leq 1$. The sequence space $[H]^\alpha$ is a Banach space normed by

$$\|x\| = \sup_n \frac{1}{\ell_n^\alpha} \sum_{i=1}^n \frac{|x_i|}{i}. \quad (1)$$

Proof. It is a routine verification that $[H]^\alpha$ is a normed space, normed by (1). Let (x^s) be a Cauchy sequence such that $x^s = (x_i^s)_i = (x_1^s, x_2^s, \dots) \in [H]^\alpha$ for each $s \in \mathbb{N}$. Then, we have

$$\|x^s - x^t\| \rightarrow 0 \text{ as } s, t \rightarrow \infty.$$

Let $\varepsilon > 0$ be given, then there exists a positive integer $n_0(\varepsilon)$ such that $\|x^s - x^t\| \leq \varepsilon$ for all $s, t > n_0$. So we have

$$\frac{1}{\ell_n^\alpha} \sum_{i=1}^n \frac{|x_i^s - x_i^t|}{i} \leq \varepsilon, \text{ for all } s, t > n_0.$$

Hence, we get $|x_i^s - x_i^t| \rightarrow 0$ as $s, t \rightarrow \infty$, and for all $i \in \mathbb{N}$. This implies that the sequence $(x_i^s)_i = (x_i^1, x_i^2, \dots)$ is a Cauchy in $\mathbb{R}$ for each $i \in \mathbb{N}$. Also, it is convergent, because $\mathbb{R}$ is complete. Assume that $\lim_s x_i^s = x_i$ for each $i \in \mathbb{N}$. As (x^s) is a Cauchy sequence, then for every $\varepsilon > 0$ there exists a number $n_0 = n_0(\varepsilon)$ such that

$$\|x^s - x^t\| < \varepsilon$$

for all $s, t \geq n_0$. Hence, for all $n \in \mathbb{N}$ and all $s, t \geq n_0$, we have

$$\frac{1}{\ell_n^\alpha} \sum_{i=1}^n \frac{|x_i^s - x_i^t|}{i} < \varepsilon.$$

Taking limit as $t \rightarrow \infty$ in the last inequalities, we have

$$\lim_t \frac{1}{\ell_n^\alpha} \sum_{i=1}^n \frac{|x_i^s - x_i^t|}{i} = \frac{1}{\ell_n^\alpha} \sum_{i=1}^n \frac{|x_i^s - x_i|}{i} < \varepsilon$$

for $s \geq n_0$ and all $n \in \mathbb{N}$. This implies that

$$\|x^s - x\| = \sup_n \frac{1}{\ell_n^\alpha} \sum_{i=1}^n \frac{|x_i^s - x_i|}{i} \rightarrow 0,$$

and hence $x^s \rightarrow x$ as $s \rightarrow \infty$. As

$$\frac{1}{\ell_n^\alpha} \sum_{i=1}^n \frac{|x_i - a|}{i} \leq \frac{1}{\ell_n^\alpha} \sum_{i=1}^n \frac{|x_i^{n_0} - a|}{i} + \frac{1}{\ell_n^\alpha} \sum_{i=1}^n \frac{|x_i^{n_0} - x_i|}{i},$$

we get $x \in [H]^\alpha$.

Theorem 2.5 Let $\theta = (k_r)$ be a lacunary sequence and α be a real number such that $0 < \alpha \leq 1$. $\liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} > 1$ if and only if $[H]^\alpha \subseteq [H_\theta]^\alpha$.

Proof. Suppose that $\liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} > 1$; then there exists a $\delta > 0$ such that $\frac{\ell_{k_r}}{\ell_{k_{r-1}}} \geq 1 + \delta$ for sufficiently large $r \geq 1$, which implies that

$$\frac{\ell_{k_r}}{L_r} \leq \frac{1 + \delta}{\delta} \implies \left(\frac{\ell_{k_r}}{L_r} \right)^\alpha \leq \left(\frac{1 + \delta}{\delta} \right)^\alpha \quad \text{and} \quad \frac{\ell_{k_{r-1}}}{L_r} \leq \frac{1}{\delta} \implies \left(\frac{\ell_{k_{r-1}}}{L_r} \right)^\alpha \leq \left(\frac{1}{\delta} \right)^\alpha$$

where $(L_r)^\alpha = (\ell_{k_r} - \ell_{k_{r-1}})^\alpha$.

If $x \in [H^0]^\alpha$, then for every $\varepsilon > 0$ and for sufficiently large r , we obtain

$$\begin{aligned} T_r^\alpha &= \frac{1}{(L_r)^\alpha} \sum_{i \in I_r} \frac{|x_i|}{i} = \frac{1}{(L_r)^\alpha} \left(\sum_{i=1}^{k_r} \frac{|x_i|}{i} - \sum_{i=1}^{k_{r-1}} \frac{|x_i|}{i} \right) \\ &= \frac{\ell_{k_r}^\alpha}{(L_r)^\alpha} \left(\frac{1}{\ell_{k_r}^\alpha} \sum_{i=1}^{k_r} \frac{|x_i|}{i} \right) - \frac{\ell_{k_{r-1}}^\alpha}{(L_r)^\alpha} \left(\frac{1}{\ell_{k_{r-1}}^\alpha} \sum_{i=1}^{k_{r-1}} \frac{|x_i|}{i} \right), \end{aligned}$$

this proves the sufficiency.

Now assume that $\liminf_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}} = 1$. Since θ is a lacunary sequence, we can select a subsequence (k_{r_j}) of θ satisfying

$$\frac{\ell_{k_{r_j}}}{\ell_{k_{r_j-1}}} < \frac{1}{j} \quad \text{and} \quad \frac{\ell_{k_{r_j-1}}^\alpha}{\ell_{k_{r_j-1}}} > j$$

where $r_j \geq r_{j-1} + 2$.

Define $x = (x_i)$ by

$$x_i = \begin{cases} 1, & \text{if } i \in I_{r_j}, \text{ for some } j = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

Then, for any real a ,

$$\frac{1}{(L_{r_j})^\alpha} \sum_{i \in I_{r_j}} \frac{|x_i - a|}{i} = \frac{L_{r_j}}{(L_{r_j})^\alpha} |1 - a| \rightarrow \infty, \quad \text{as } r_j \rightarrow \infty$$

and for $r \neq r_j$

$$\frac{1}{(L_r)^\alpha} \sum_{i \in I_r} \frac{|x_i - a|}{i} = \frac{L_r}{(L_r)^\alpha} |a| \rightarrow \infty, \quad \text{as } r \rightarrow \infty.$$

Therefore $x \notin [H_\theta]^\alpha$. However, $x \in [H^0]^\alpha$ since t any sufficiently large integer we can find the unique j for which $\ell_{k_{r_j-1}} < \ell_t \leq \ell_{k_{r_j+1}-1}$ and write

$$\frac{1}{\ell_t^\alpha} \sum_{i=1}^t \frac{|x_i|}{i} \leq \frac{\ell_{k_{r_j-1}} + L_{r_j}}{\ell_{k_{r_j-1}}^\alpha} < \frac{\ell_{k_{r_j-1}}}{\ell_{k_{r_j-1}}^\alpha} + \frac{\ell_{k_{r_j}}}{\ell_{k_{r_j-1}}^\alpha} < \frac{2}{j}.$$

As $t \rightarrow \infty$ it follows that also $j \rightarrow \infty$. Hence $x \in [H^0]^\alpha$.

Theorem 2.6 Let $0 < \alpha \leq 1$ and $\theta = (k_r)$ be a lacunary sequence. If $\limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}^\alpha} < \infty$, then $[H_\theta] \subset [H]^\alpha$.

Proof. Let $\limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}^\alpha} < \infty$, then there exists a constant $M > 0$ such that $\frac{\ell_{k_r}}{\ell_{k_{r-1}}^\alpha} < M$ for all $r \geq 1$. Now let $x \in [H_\theta^0]$ and $\varepsilon > 0$, then we can then find $R > 0$ and $K > 0$ such that $\sup_{i \geq R} T_i < \varepsilon$ and $T_i < K$ for all $i = 1, 2, \dots$. Then if t is any integer with $\ell_{k_{r-1}} < \ell_t \leq \ell_{k_r}$, where $r > R$, we can write

$$\begin{aligned} \frac{1}{\ell_t^\alpha} \sum_{i=1}^t \frac{|x_i|}{i} &\leq \frac{1}{\ell_{k_{r-1}}^\alpha} \sum_{i=1}^{k_r} \frac{|x_i|}{i} = \frac{1}{\ell_{k_{r-1}}^\alpha} \left(\sum_{I_1} \frac{|x_i|}{i} + \sum_{I_2} \frac{|x_i|}{i} + \dots + \sum_{I_r} \frac{|x_i|}{i} \right) \\ &= \frac{\ell_{k_1}}{\ell_{k_{r-1}}^\alpha} T_1 + \frac{\ell_{k_2} - \ell_{k_1}}{\ell_{k_{r-1}}^\alpha} T_2 + \dots + \frac{\ell_{k_R} - \ell_{k_{R-1}}}{\ell_{k_{r-1}}^\alpha} T_R + \frac{\ell_{k_{R+1}} - \ell_{k_R}}{\ell_{k_{r-1}}^\alpha} T_{R+1} + \dots + \frac{\ell_{k_r} - \ell_{k_{r-1}}}{\ell_{k_{r-1}}^\alpha} T_r \\ &\leq \left(\sup_{i \geq 1} T_i \right) \frac{\ell_{k_R}}{\ell_{k_{r-1}}^\alpha} + \left(\sup_{i \geq R} T_i \right) \frac{\ell_{k_r} - \ell_{k_R}}{\ell_{k_{r-1}}^\alpha} < K \frac{\ell_{k_R}}{\ell_{k_{r-1}}^\alpha} + \varepsilon M \end{aligned}$$

Hence $x \in [H^0]^\alpha$.

Theorem 2.7 If $x \in [H_\theta]^\alpha \cap [H]^\alpha$ and $\limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}^\alpha} < \infty$, then the limit value of the sequence $x = (x_i)$ in $[H_\theta]^\alpha$ and $[H]^\alpha$ is the same.

Proof. Let $[H_\theta]^\alpha - \lim x_i = a$ and $[H]^\alpha - \lim x_i = a'$, and suppose that $a \neq a'$. Since $\limsup_r \frac{\ell_{k_r}}{\ell_{k_{r-1}}^\alpha} < \infty$ by Theorem 2.6 we have $[H_\theta^0] \subset [H^0]^\alpha$. Since $(x - a) \in [H_\theta^0]$, it follows that

$(x - a) \in [H^0]^\alpha$ and therefore $\frac{1}{\ell_t^\alpha} \sum_{i=1}^t \frac{|x_i - a|}{i} \rightarrow 0$. Then we have

$$\frac{1}{\ell_t^\alpha} \sum_{i=1}^t \frac{|x_i - a'|}{i} + \frac{1}{\ell_t^\alpha} \sum_{i=1}^t \frac{|x_i - a|}{i} \geq \frac{1}{\ell_t^\alpha} \frac{|a - a'|}{i} > 0,$$

which yields a contradiction, since both terms on the left converge to 0.

Theorem 2.8 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$. Then we have

(i) If

$$\liminf_{r \rightarrow \infty} \frac{(L_r)^\alpha}{(L'_r)^\beta} > 0 \quad (2)$$

holds then $[H_{\theta'}]^\beta \subset [H_\theta]^\alpha$,

(ii) If

$$\lim_{r \rightarrow \infty} \frac{L'_r}{(L_r)^\beta} = 1 \quad (3)$$

holds and $x \in \ell_\infty$ then $[H_\theta]^\alpha \subset [H_{\theta'}]^\beta$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $L_r = \ell_{k_r} - \ell_{k_{r-1}}$, $L'_r = \ell'_{s_r} - \ell'_{s_{r-1}}$.

Proof. (i) Let $x \in [H_{\theta'}]^\beta$ and suppose that (2) holds. We obtain

$$\begin{aligned} \frac{1}{(L'_r)^\beta} \sum_{i \in J_r} \frac{|x_i - a|}{i} &\geq \frac{1}{(L'_r)^\beta} \sum_{i \in I_r} \frac{|x_i - a|}{i} \\ &\geq \frac{(L_r)^\alpha}{(L'_r)^\beta} \frac{1}{(L_r)^\alpha} \sum_{i \in I_r} \frac{|x_i - a|}{i} \end{aligned}$$

for all $r \in \mathbb{N}$. Therefore $[H_{\theta'}]^\beta \subset [H_\theta]^\alpha$.

(ii) Let $x \in [H_\theta]^\alpha$ and assume that (3) holds. Since $x \in \ell_\infty$ then there exists some $M > 0$ such that $\frac{|x_i - a|}{i} \leq |x_i - a| \leq M$ for all k . Now, since $I_r \subseteq J_r$ and $L_r \leq L'_r$ for all $r \in \mathbb{N}$, we may write

$$\begin{aligned} \frac{1}{(L'_r)^\beta} \sum_{i \in J_r} \frac{|x_i - a|}{i} &= \frac{1}{(L'_r)^\beta} \sum_{i \in J_r - I_r} \frac{|x_i - a|}{i} + \frac{1}{(L'_r)^\beta} \sum_{i \in I_r} \frac{|x_i - a|}{i} \\ &\leq \frac{L'_r - L_r}{(L'_r)^\beta} M + \frac{1}{(L'_r)^\beta} \sum_{i \in I_r} \frac{|x_i - a|}{i} \\ &\leq \frac{L'_r - (L_r)^\beta}{(L_r)^\beta} M + \frac{1}{(L_r)^\alpha} \sum_{i \in I_r} \frac{|x_i - a|}{i} \\ &\leq \left(\frac{L'_r}{(L_r)^\beta} - 1 \right) M + \frac{1}{(L_r)^\alpha} \sum_{i \in I_r} \frac{|x_i - a|}{i} \end{aligned}$$

for every $r \in \mathbb{N}$. Therefore $\ell_\infty \cap [H_\theta]^\alpha \subset [H_{\theta'}]^\beta$.

Thus in the light of Theorem 2.8, we have the following result:

Corollary 2.9 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.

If (2) holds then,

(i) $[H_{\theta'}] \subset [H_\theta]^\alpha$ for $0 < \alpha \leq 1$,

(ii) $[H_{\theta'}] \subset [H_\theta]$.

If (3) holds and $x \in \ell_\infty$ then,

(i) $[H_\theta]^\alpha \subset [H_{\theta'}]$ for $0 < \alpha \leq 1$,

$$(ii) [H_\theta] \subset [H_{\theta'}].$$

Definition 2.10 Let $\theta = (k_r)$ be a lacunary sequence and α be a real number such that $0 < \alpha \leq 1$. A sequence $x = (x_i)$ of real numbers is said to be lacunary statistically logarithmic convergent of order α to the number a , if there is a real number a such that for every $\varepsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{(L_r)^\alpha} \left| \left\{ i \in I_r : \frac{1}{i} |x_i - a| \geq \varepsilon \right\} \right| = 0.$$

In this case we write $[SH_\theta]^\alpha - \lim x_i = a$. The set of all lacunary statistically logarithmic convergent of order α sequences will be denoted $[SH_\theta]^\alpha$. In the special case $\alpha = 1$ we shall write $[SH_\theta]$ instead of $[SH_\theta]^\alpha$ and also in the special case $\theta = (2^r)$ we shall write $[SH]^\alpha$ instead of $[SH_\theta]^\alpha$.

We give the following three theorem without proof.

Theorem 2.11 Let $\theta = (k_r)$ be a lacunary sequence and α be a real number such that $0 < \alpha \leq 1$. Then $[SH_\theta]^\alpha$ is linear space.

Theorem 2.12 Let $0 < \alpha \leq \beta \leq 1$. Then $[H_\theta]^\alpha \subseteq [SH_\theta]^\beta$.

Theorem 2.13 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$. If (2) holds then $[SH_{\theta'}]^\beta \subset [SH_\theta]^\alpha$.

The following results are a consequence of Theorem 2.13.

Corollary 2.14 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$. If (2) holds then,

$$(i) [SH_{\theta'}] \subset [SH_\theta]^\alpha \text{ for } 0 < \alpha \leq 1,$$

$$(ii) [SH_{\theta'}] \subset [SH_\theta].$$

Theorem 2.15 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$.

$$(i) \text{ Let (2) holds, then } [H_{\theta'}]^\beta \subset [SH_\theta]^\alpha,$$

$$(ii) \text{ Let (3) holds and } x = (x_i) \text{ be a bounded sequence, then } [SH_\theta]^\alpha \subset [H_{\theta'}]^\beta.$$

Proof. (i) Omitted.

(ii) Assume that $x \in [SH_\theta]^\alpha$ and $x \in \ell_\infty$, and there exists $M > 0$ such that $\frac{|x_i - a|}{i} \leq M$ for all i , then for every $\varepsilon > 0$ we obtain

$$\begin{aligned}
\frac{1}{(L'_r)^\beta} \sum_{i \in J_r} \frac{|x_i - a|}{i} &= \frac{1}{(L'_r)^\beta} \sum_{i \in J_r - I_r} \frac{|x_i - a|}{i} + \frac{1}{(L'_r)^\beta} \sum_{i \in I_r} \frac{|x_i - a|}{i} \\
&\leq \left(\frac{L'_r - L_r}{(L'_r)^\beta} \right) M + \frac{1}{(L'_r)^\beta} \sum_{i \in I_r} \frac{|x_i - a|}{i} \\
&\leq \left(\frac{L'_r - (L_r)^\beta}{(L_r)^\beta} \right) M + \frac{1}{(L'_r)^\beta} \sum_{i \in I_r} \frac{|x_i - a|}{i} \\
&\leq \left(\frac{L'_r}{(L_r)^\beta} - 1 \right) M + \frac{1}{(L_r)^\alpha} \sum_{\substack{i \in I_r \\ \frac{|x_i - a|}{i} \geq \varepsilon}} \frac{|x_i - a|}{i} + \frac{1}{(L_r)^\beta} \sum_{\substack{i \in I_r \\ \frac{|x_i - a|}{i} < \varepsilon}} \frac{|x_i - a|}{i} \\
&\leq \left(\frac{L'_r}{(L_r)^\beta} - 1 \right) M + \frac{M}{(L_r)^\alpha} \left| \left\{ i \in I_r : \frac{|x_i - a|}{i} \geq \varepsilon \right\} \right| + \frac{L_r}{(L_r)^\beta} \varepsilon \\
&\leq \left(\frac{L'_r}{(L_r)^\beta} - 1 \right) M + \frac{M}{(L_r)^\alpha} \left| \left\{ i \in I_r : \frac{|x_i - a|}{i} \geq \varepsilon \right\} \right| + \frac{L'_r}{(L_r)^\beta} \varepsilon
\end{aligned}$$

Consequently $[SH_\theta]^\alpha \subset [H_{\theta'}]^\beta$ for $0 < \alpha \leq \beta \leq 1$.

Thus in the light of Theorem 2.15, we have the following result:

Corollary 2.16 Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$.

If (2) holds then,

(i) $[H_{\theta'}] \subset [SH_\theta]^\alpha$ for $0 < \alpha \leq 1$,

(ii) $[H_{\theta'}] \subset [SH_\theta]$.

If (3) holds then,

(i) $[SH_\theta]^\alpha \subset [H_{\theta'}]$ for $0 < \alpha \leq 1$,

(ii) $[SH_\theta] \subset [H_{\theta'}]$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

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