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An EHL extension of the unsteady FBNS algorithm

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Abstract

Many engineering applications rely on lubricated gaps where the hydrodynamic pressure distribution is influenced by cavitation phenomena and elastic deformations. To obtain details about the conditions within the lubricated gap, solvers are required that can model cavitation and elastic deformation effects efficiently when a large amount of discretization cells is employed. The presented unsteady EHL-FBNS solver can compute the solution of such large problems that require the consideration of both mass-conserving cavitation and elastic deformation. The execution time of the presented algorithm scales almost with $N \log(N)$ where N is the number of computational grid points. A detailed description of the algorithm and the discretized equations is presented. The MATLAB[®] code is provided in the supplements along with a maintained version on GitHub to encourage its usage and further development. The output of the solver is compared to and validated with simulated and experimental results from the literature to provide a detailed comparison of different discretization schemes of the Couette term in presence of gap height discontinuities. As a final result, the most favourable scheme is identified for the unsteady study of surface textures in ball-on-disc tribometers under severe EHL conditions.

Keywords: Elasto-Hydrodynamic Lubrication (EHL),
Fischer-Burmeister-Newton-Schur (FBNS), Jakobsson-Floberg-Olsson (JFO),
Mass-conserving cavitation, Elastic half-space

1 Introduction

In the case of lubrication flows in narrow gaps, the Reynolds equation [1] is a handy tool to determine the hydrodynamic pressure distribution in a simpler way than by using the full Navier-Stokes Equations (NSE) [2, Ch. 7]. Since cavitation commonly occurs in lubrication flows, various models have been developed to describe this phenomenon [3]. Especially when it occurs within surface textures, mass-conserving properties of the cavitation model are required to properly describe the flow's transition from the cavitation region to the full-film region, because this full-film reformulation interface has a great effect on the extension of the cavitated area and the subsequent downstream rise in pressure within the full-film region [4]. The required mass-conserving properties can be taken into account with the Jakobsson-Floberg-Olsson (JFO) [5, 6] cavitation model [3]. Starting from the cavitation algorithm of Elrod [7], Giacomini *et al.* [8] developed a one-dimensional Finite Element Method (FEM) solver that couples the Reynolds equation with the mass-conserving JFO cavitation model through a complementarity formulation. This work was extended by Bertocchi *et al.* [9] to consider two-dimensional problems with compressible, piezoviscous and shear-thinning fluid behaviour. The arising complementarity problem was reformulated to be expressed by an unconstrained equation system by Woloszynski *et al.* [10], resulting in the Fischer-Burmeister-Newton-Schur (FBNS) algorithm. As demonstrated by Woloszynski *et al.*, the FBNS algorithm is of remarkable computational efficiency also for high spatial resolutions.

In many cases, the hydrodynamic pressure can deform the lubricated surfaces notably leading to the regime of Elasto-Hydrodynamic Lubrication (EHL) [11]. Various solvers have been developed to tackle EHL problems, some of the most prominent ones are the Finite Difference Method (FDM) Multi-grid solver of Venner and Lubrecht [12] and the FEM solver of Habchi [13]. Some algorithms are also capable of simulating surface contact along with the Reynolds equation [14–18]. Since the full-film reformulation interface is often not of relevance in EHL problems, many EHL solvers do not employ mass-conserving cavitation models. However, in some cases - such as starved lubrication - mass-conserving cavitation is crucial and has been considered in several works [19–21]. Among them, the coupling of the mass-conserving FBNS algorithm with an elastic deformation and a roughness asperity contact model was achieved by Ferretti [22]. In contrast to Ferretti's work, the FBNS algorithm is coupled with the elastic deformation of an elastic half-space within this paper, thus presenting the new EHL-FBNS algorithm. Due to the half-space assumption, the elastic deformation is a linear convolution of a kernel function

with the hydrodynamic pressure field. This allows exploiting the Fast Fourier Transformation (FFT) to speed up the computation of the elastic deformation [23]. Furthermore, a Proportional Integral Derivative (PID) controller is employed to meet the load balance equation through adjustment of the rigid body displacement as already introduced by Wang *et al.* [24]. Eventually, the EHL-FBNS algorithm is capable of efficiently computing the solution of large problems that require the consideration of both mass-conserving cavitation and elastic deformation at the same time.

In the beginning of this paper, the basic equations and the general procedure of the EHL-FBNS algorithm are summarized. The algorithm is implemented in MATLAB[©] with the Finite Volume Method (FVM) and a generic order spatial discretization scheme for the Couette term of the Reynolds equation. The discretized equations are supplied in Appendix A. Then, the performance of the steady EHL-FBNS implementation is compared to the original FBNS algorithm of Woloszynski *et al.* [10]. Moreover, unsteady EHL-FBNS simulations are performed for the set-up of a single texture that passes through the EHL contact of a ball-on-disc tribometer. The results are compared to experimental and simulated data of Mourier *et al.* [25]. This allows to demonstrate the stability of the EHL-FBNS algorithm under severe EHL operating conditions with surface textures and to provide recommendations about the most suitable discretization scheme. Eventually, the EHL-FBNS algorithm is validated for surface texture investigations with ball-on-disc tribometers under unsteady EHL operating conditions.

The MATLAB[©] code, set-up and visualization scripts are provided in the supplements. The MATLAB[©] scripts are thoroughly commented to encourage their usage and further development. A maintained and publicly available version of the code can also be found on GitHub: <https://github.com/ErikHansenGit/EHL>.

2 Numerical Methods

2.1 Governing equations

Fig. 1 Schematic sketch of the lubricated gap.

The lubrication flow in a narrow gap (schematically depicted in Figure 1) is governed by the Reynolds equation considering mass-conserving cavitation with the JFO model [5, 6] at any set of spatial coordinates x_1 and x_2 and time t [9]:

$$\frac{\partial}{\partial x_1} \left(\frac{\rho_l h^3}{12\mu_l} \frac{\partial p}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{\rho_l h^3}{12\mu_l} \frac{\partial p}{\partial x_2} \right) - \frac{\partial}{\partial x_1} (\rho_l h u_m (1 - \theta)) - \frac{\partial}{\partial t} (\rho_l h (1 - \theta)) = 0. \quad (1)$$

In this equation, h denotes the gap height, ρ_l is the density of the liquid phase and μ_l describes the dynamic viscosity of the liquid phase. All of them can vary in space and time. The mean velocity $u_m = \frac{U_{up} + U_{low}}{2}$ is composed of U_{up} as the velocity of the upper surface and U_{low} as the velocity of the lower surface in x_1 -direction. The relative pressure $p = p_{hd} - p_{cav}$ and the cavity fraction $\theta = 1 - \frac{\rho}{\rho_l}$ are the solution variables, where ρ is the mixture density of the flow. The hydrodynamic pressure p_{hd} is prevented from falling below the cavitation pressure p_{cav} by adding the following complementary constraints [9]:

$$p \cdot \theta = 0, \quad p \geq 0, \quad \theta \geq 0. \quad (2)$$

Depending on whether the liquid phase is modelled as iso- or piezoviscous through the Roelands model, its dynamic viscosity reads [12, Ch. 1.3.3], [26]:

$$\mu_l = \begin{cases} \mu_0, \\ \mu_0 \exp \left((\ln(\mu_0) + 9.67) \cdot \left(-1 + \left(1 + \frac{(p_{hd} - p_{cav})}{p_{0,R}} \right)^{z_R} \right) \right), \end{cases} \quad (3)$$

where $z_R = \frac{\alpha_R p_{0,R}}{\ln(\mu_0 + 9.67)}$ [25] is the pressure viscosity index, α_R denotes the pressure viscosity coefficient and $p_{0,R}$ is a constant in the Roelands equation. The dynamic viscosity of the liquid phase at ambient pressure is μ_0 . Moreover, depending on whether the liquid phase is assumed to be of constant density or to be compressible according to the Dowson-Higginson model, the liquid phase density is given by [12, Ch. 1.3.4], [9, 26]:

$$\rho_l = \begin{cases} \rho_0, \\ \rho_0 \frac{C_1 + C_2(p_{hd} - p_{cav})}{C_1 + (p_{hd} - p_{cav})}, \end{cases} \quad (4)$$

where ρ_0 is the density of the liquid phase at ambient pressure and C_1 and C_2 are constants.

The gap height h can be constructed as a superposition of the rigid body displacement of the two surfaces h_d , the variation of the gap height due to the rigid geometry of the surfaces h_g and the elastic deformation of the gap height h_{el} due to the hydrodynamic pressure [2, Ch. 19.2]:

$$h = h_d + h_g + h_{el}. \quad (5)$$

Depending on whether the upper and lower surfaces are assumed to be rigid or elastic half-spaces, the elastic deformation of the gap height can be expressed as [12, Ch. 1.3.5]:

$$h_{el}(x_1, x_2) = \begin{cases} 0, \\ \frac{2}{\pi E'} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{p_{hd}(x'_1, x'_2)}{\sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2}} dx'_1 dx'_2, \end{cases} \quad (6)$$

where the reduced elastic modulus is stated as [12, Ch. 1.3.5]:

$$E' = \frac{2}{\frac{1 - \nu_{low}^2}{E_{low}} + \frac{1 - \nu_{up}^2}{E_{up}}}. \quad (7)$$

In this equation, E denotes the corresponding Young's modulus and ν the Poisson ratio of the upper and lower surface. If a constant rigid body displacement h_d is prescribed, the provided set of equations is sufficient to describe the EHL problem. If however a constant imposed normal load force $F_{N,imp}$ is prescribed, it needs to be satisfied by the normal force F_N resulting from the hydrodynamic pressure profile [27]:

$$F_{N,imp} = F_N = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p_{hd} - p_{amb} dx_1 dx_2, \quad (8)$$

where p_{amb} is the ambient pressure. The rigid body displacement h_d needs to be set such that the load balance Equation (8) is fulfilled.

2.2 EHL-FBNS algorithm

The set of equations described above can be solved numerically with the EHL-FBNS algorithm presented in the following. This new algorithm is based on the FBNS algorithm developed by Woloszynski et al. [10] and extends it by taking elastic surface deformation and the load balance equation into account. First of all, the dimensionless Reynolds equation considering mass-conserving cavitation is defined as:

$$G = \underbrace{\frac{\partial}{\partial x_1^*} \left(\xi_{Po}^* \frac{\partial p^*}{\partial x_1^*} \right) + \left(\frac{x_{1,ref}}{x_{2,ref}} \right)^2 \frac{\partial}{\partial x_2^*} \left(\xi_{Po}^* \frac{\partial p^*}{\partial x_2^*} \right)}_{\text{Poiseuille term}} - \underbrace{\frac{\partial}{\partial x_1^*} (\xi_{Co}^* (1 - \theta))}_{\text{Couette term}} - \underbrace{\frac{\partial}{\partial t^*} (\xi_{Ti}^* (1 - \theta))}_{\text{Unsteady term}}. \quad (9)$$

It can be derived by inserting the following non-dimensional quantities (indicated by *) and reference quantities (denoted by the index $_{ref}$) into and reformulating Equation (1):

$$\begin{aligned} x_1 &= x_1^* x_{1,ref}, & x_2 &= x_2^* x_{2,ref}, & t &= t^* t_{ref}, & \rho_l &= \rho^* \rho_{ref}, \\ \mu_l &= \mu^* \mu_{ref}, & h &= h^* h_{ref}, & u_m &= u_m^* u_{m,ref}, & p &= p^* p_{ref}. \end{aligned} \quad (10)$$

Similar to Venner and Lubrecht [12, Ch. 6.3], the coefficients of the Poiseuille, Couette and unsteady term within the dimensionless Reynolds equation can be consolidated as:

$$\xi_{Po}^* = \frac{\rho^* h^{*3}}{\mu^*}, \quad \xi_{Co}^* = 12 \frac{x_{1,ref} u_{m,ref} \mu_{ref}}{h_{ref}^2 p_{ref}} \rho^* h^* u_m^*, \quad \xi_{Ti}^* = 12 \frac{x_{1,ref}^2 \mu_{ref}}{t_{ref} h_{ref}^2 p_{ref}} \rho^* h^*. \quad (11)$$

The complimentary constraints (2) are replaced by the Fischer-Burmeister equation in non-dimensional form [10]:

$$F = p^* + \theta - \sqrt{p^{*2} + \theta^2}. \quad (12)$$

The EHL-FBNS algorithm uses the Newton-Raphson method to determine the values of p^* and θ such that G and F get sufficiently close to 0, thus solving the dimensionless Reynolds Equation (9) and the dimensionless Fischer-Burmeister Equation (12). By evaluating the discretized form of G and F at each discrete position, an equation system is created. The discretized equations are obtained through the Finite Volume Method (FVM), where the second order midpoint rule is applied to evaluate surface and volume integrals. The required values and derivatives of the Poiseuille term are discretized with a second order central scheme, of the Couette term by either the first order upwind interpolation (UI) or the third order quadratic upwind interpolation (QUICK) and the unsteady term with the first order Euler implicit scheme [28, Ch. 3.3, 4, 6.3.2]. The discretized expressions of Equations (9) and (12) are provided in Appendix A.1. The set of discrete dimensionless Reynolds equations $\vec{G}$ can be expressed in matrix-vector notation through the pressure coefficients

contributed by the Poiseuille term A_{Po} , the discretized dimensionless relative pressures $\vec{p}^*$, the cavity fraction coefficients contributed by the Couette and unsteady term B , the discretized cavity fractions $\vec{\theta}$ and the remaining constants from the Couette and unsteady term $\vec{c}$ [10]:

$$\vec{G} = A_{Po}\vec{p}^* + B\vec{\theta} + \vec{c}. \quad (13)$$

The set of discrete dimensionless Fischer-Burmeister Equations (12) are denoted by $\vec{F}(p^*, \vec{\theta})$. The non-dimensional properties μ^* and ρ^* at each discrete point are computed according to the respective Equations (3), (4) and (10).

The gap height h at each discrete point is computed according to Equations (5) and (6). It is prevented from becoming lower than 1 nm by using truncation at this instant. Afterwards, the non-dimensional gap height h^* is determined through Equation (10). If the surfaces are chosen to be elastic, Equation (6) is discretized by assuming a constant pressure over the rectangular discretization cell [29, Ch. 3.3], [30, Ch. 3.1], the discretized equation is also provided in Appendix A.2. Since the resulting equation is a linear convolution of a kernel function with the hydrodynamic pressure field, it is computed in Fourier space by means of FFT to speed up the computation. Attention is paid to double the size of the kernel in each direction and to zero pad the hydrodynamic pressure field such that a linear instead of a circular convolution is obtained. After the convolution, the deformation and pressure fields are resized to their original size [23, 31, 32].

After computing $\vec{G}$ and $\vec{F}$, the Newton-Raphson method is used to determine the updates of non-dimensional relative pressure $\vec{\delta}_{p^*}$ and cavity fraction $\vec{\delta}_{\theta}$ [10]:

$$J\vec{\delta} = \begin{bmatrix} J_{F,p^*} & J_{F,\theta} \\ J_{G,p^*} & J_{G,\theta} \end{bmatrix} \begin{bmatrix} \vec{\delta}_{p^*} \\ \vec{\delta}_{\theta} \end{bmatrix} = - \begin{bmatrix} \vec{F} \\ \vec{G} \end{bmatrix} \quad (14)$$

The most important extension of the EHL-FBNS algorithm compared to the original FBNS algorithm is the approximation of the pressure Jacobian J_{G,p^*} of the dimensionless Reynolds equation when elastic deformation is taken into account. The idea is to consider the dependence $h^*(p^*)$ by inserting it into $\vec{c}$, thus creating the matrix A_h . Due to the kernel function, this would result in A_h being a full matrix which is prone to lose its diagonal dominance and therefore being unfeasible to invert and likely to cause unstable behaviour in the iteration process. This is rectified by approximating A_h only by some of its diagonals as already done in the literature for other EHL algorithms: for example Venner and Lubrecht [12, Ch. C.1, C.3.2] who combine it with distributive relaxation and multigrid methods or Wang *et al.* [15] who employ the semi-system method. In case of the EHL-FBNS algorithm, A_h is reduced to the 5 diagonals that correspond to the South, West, Center, East and North cells. Eventually, the Jacobians of $\vec{G}$ read $J_{G,p^*} = A_{Po} + A_h$ and $J_{G,\theta} = B$. The boundary conditions of $\vec{p}^*$ are considered in A_{Po} and $\vec{c}$ and the boundary conditions of $\vec{\theta}$ in $\vec{F}$ and $J_{F,\theta}$. If Neumann boundary conditions are used for $\vec{\theta}$, the

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Jacobian $J_{F,\theta}$ would contain several diagonals. In this case, it is approximated only by its main diagonal. It is worthwhile to note that this approximation of the Jacobians eventually only affects the updates of non-dimensional relative pressure $\vec{\delta}_p^*$ and cavity fraction $\vec{\delta}_\theta$, but never the computation of $\vec{G}$ and $\vec{F}$. The discrete formulations of the Jacobians $J_{G,p^*} = A_{Po} + A_h$ and $J_{G,\theta} = B$ are provided in Appendix A.1 and A.3. The center entries of the Jacobians of the dimensionless Fischer-Burmeister equation $J_{F,p^*,C}$ and $J_{F,\theta,C}$ for each discrete point are [10]:

$$J_{F,p^*,C} = 1 - \frac{p_{C,aux}^*}{\sqrt{p_{C,aux}^{*2} + \theta_{C,aux}^2}}, \quad (15)$$

$$J_{F,\theta,C} = 1 - \frac{\theta_{C,aux}}{\sqrt{p_{C,aux}^{*2} + \theta_{C,aux}^2}} \quad (16)$$

Here, $p_{C,aux}^*$ and $\theta_{C,aux}^*$ are the auxiliary dimensionless pressure and cavity fraction which are adjusted such that J_{F,p^*} and $J_{F,\theta}$ do not become singular [10]. To prevent them from having center entries close to zero within the range $(-\varepsilon, \varepsilon)$, they are adjusted as:

$$p_{C,aux}^* = \begin{cases} p_C^* & \text{if } p_C^* \geq \varepsilon \text{ or } p_C^* \leq -\varepsilon, \\ \varepsilon & \text{if } 0 \leq p_C^* < \varepsilon, \\ -\varepsilon & \text{if } -\varepsilon < p_C^* < 0, \end{cases} \quad (17)$$

$$\theta_{C,aux} = \begin{cases} \theta_C & \text{if } \theta_C \geq \varepsilon \text{ or } \theta_C \leq -\varepsilon, \\ \varepsilon & \text{if } 0 \leq \theta_C < \varepsilon, \\ -\varepsilon & \text{if } -\varepsilon < \theta_C < 0, \end{cases} \quad (18)$$

where machine epsilon is given by $\varepsilon \approx 2.2204 \cdot 10^{-16}$. As already done in the original FBNS algorithm, the corresponding columns of the Jacobian J and rows of the updates $\vec{\delta}$ are swapped if $J_{F,p^*,C} < J_{F,\theta,C}$ to obtain a reordered system [10]:

$$\begin{bmatrix} A_F & B_F \\ A_G & B_G \end{bmatrix} \begin{bmatrix} \vec{\delta}_a \\ \vec{\delta}_b \end{bmatrix} = - \begin{bmatrix} \vec{F} \\ \vec{G} \end{bmatrix}. \quad (19)$$

Due to the swapping, A_F is better conditioned than J_{F,p^*} which is exploited when Equation system (19) is solved [10]:

$$\vec{\delta}_b = (B_G - A_G (A_F^{-1} B_F))^{-1} \left(-\vec{G} + A_G (A_F^{-1} \vec{F}) \right), \quad (20)$$

$$\vec{\delta}_a = A_F^{-1} \left(-\vec{F} - B_F \vec{\delta}_b \right). \quad (21)$$

After obtaining $\vec{\delta}_a$ and $\vec{\delta}_b$, the earlier performed row swapping is reversed to get the updates of non-dimensional pressure $\vec{\delta}_p^n$ and cavity fraction $\vec{\delta}_\theta^n$ [10].

The new values $\bar{p}^{*,n}$ and $\bar{\theta}^n$ at iteration n are obtained by means of relaxation:

$$\bar{p}^{*,n} = \bar{p}^{*,n-1} + \alpha_{p^*} \bar{\delta}_{p^*}^n, \quad (22)$$

$$\bar{\theta}^n = \bar{\theta}^{n-1} + \alpha_{\theta} \bar{\delta}_{\theta}^n, \quad (23)$$

where α_{p^*} , $\bar{p}^{*,n-1}$, α_{θ} and $\bar{\theta}^{n-1}$ are the relaxation factors and previous solutions of non-dimensional relative pressure and cavity fraction. Depending on the simulated case, relaxation coefficients between 0.05 and 1 resulted in good tradeoffs between convergence speed and stability. Preventing $\bar{p}^{*,n}$ from having values below 0 and $\bar{\theta}^n$ from having values below 0 or above 1 through truncation furthermore enhances favourable convergence properties.

If a constant load force is prescribed, the dimensionless rigid body displacement $h_d^* = h_d/h_{ref}$ is adjusted through a PID controller to meet the load balance Equation (8) as already done by Wang *et al.* [24]. This is done by first determining the resulting normal load force F_N^n through the discretized load balance equation:

$$F_N^n = \sum_{x_2}^{N_{x_2}} \sum_{x_1}^{N_{x_1}} (p_{hd,C}^n - p_{amb}) \Delta x_1 \Delta x_2, \quad (24)$$

where N_{x_1} , Δx_1 , N_{x_2} and Δx_2 are the amount and spacing of the discretization cells in x_1 - and x_2 -direction and $p_{hd,C}^n$ is the hydrodynamic pressure at the center of each discrete cell. The residual of the load balance equation is defined as:

$$r_{F_N}^n = \frac{F_N^n - F_{N,imp}}{F_{N,ref}}, \quad (25)$$

where $F_{N,ref}$ is a reference normal force that is usually just set equal to the imposed normal force $F_{N,imp}$. Note that $r_{F_N}^n$ can be either positive or negative, depending on whether F_N^n is larger or smaller than $F_{N,imp}$. This is required for the PID controller to work properly. Finally, $r_{F_N}^n$ is fed into the PID controller to determine $h_d^{*,n+1}$ of the next iteration step [24]:

$$h_d^{*,n+1} = K_P r_{F_N}^n + K_I \sum_i^{n-1} r_{F_N}^i + K_D (r_{F_N}^n - r_{F_N}^{n-1}). \quad (26)$$

10 *An EHL extension of the unsteady FBNS algorithm* **Fig. 2** Flowchart of the most relevant steps of the EHL-FBNS algorithm.

Note that K_I is only multiplied with the sum up until $r_{F_N}^{n-1}$, since $r_{F_N}^n$ is already considered by K_P . For all of the later considered simulations with imposed normal load force, $K_P = 0.001$, $K_I = 0.02$ and $K_D = 0.001$ worked well. At last, the following residuals are computed:

$$\begin{aligned}
 r_{max,\delta p^*}^n &= \max \left(\text{abs} \left(\vec{\delta}_{p^*}^n \right) \right), \\
 r_{max,\delta \theta}^n &= \max \left(\text{abs} \left(\vec{\delta}_{\theta}^n \right) \right), \\
 r_{max,\delta G}^n &= \max \left(\text{abs} \left(\vec{G}^n - \vec{G}^{n-1} \right) \right), \\
 r_{max,G}^n &= \max \left(\text{abs} \left(\vec{G}^n \right) \right), \\
 r_{max,\delta F}^n &= \max \left(\text{abs} \left(\vec{F}^n - \vec{F}^{n-1} \right) \right), \\
 r_{max,F}^n &= \max \left(\text{abs} \left(\vec{F}^n \right) \right),
 \end{aligned} \tag{27}$$

$$r_{EHL-FBNS}^n = \max \left(r_{max,\delta p^*}^n, r_{max,\delta \theta}^n, r_{max,\delta G}^n, r_{max,G}^n, r_{max,\delta F}^n, r_{max,F}^n \right). \tag{28}$$

Note that the residuals $r_{max,\delta G}^n$ and $r_{max,\delta F}^n$ are directly affected by the relaxation factors and $\vec{G}^n$ and $\vec{F}^n$ are computed through the solutions $\vec{p}^{*,n-1}$ and $\vec{\theta}^{n-1}$. The EHL-FBNS algorithm is repeated as long as $r_{EHL-FBNS}^n$ and in case of an imposed normal load force $\text{abs}(r_{F_N}^n)$ are above the tolerance $tol = 10^{-6}$. The most important steps of the algorithm structure are also visualized in Figure 2. The initial guess is always a zero cavity fraction field and a pressure field at ambient pressure. If an unsteady simulation is performed, the solution at $t = 0$ is obtained through the steady problem caused by the geometry at $t = 0$. Furthermore, the PID controller also takes the residuals of the load balance equation of the previous time steps into account if $t > 0$.

3 Results and Discussion

In order to assess the performance of the presented EHL-FBNS algorithm, it is firstly employed in a numerical literature test case of a textured parallel slider. Afterwards, another experimental-numerical literature case is simulated with the EHL-FBNS algorithm to validate the code for textured ball-on-disc investigations, evaluate the code's stability in severe unsteady EHL conditions and compare different spatial discretization schemes. For all considered cases, the second order midpoint rule is used for the evaluation of integrals arising from the FVM and the Poiseuille term is always discretized with a second order central scheme. Consequently, the resulting order of the dimensionless Reynolds equation discretized with the FVM in the steady case is first order with the UI and second order with the QUICK scheme. In the unsteady case, only first order is achievable for both UI and QUICK since the first order Euler implicit scheme is employed. All of the EHL-FBNS simulations are performed with MATLAB[®] R2020a.

3.1 Parallel slider with a varying amount of trapezoidal pockets

The parallel slider with a varying amount of trapezoidal pockets used by Woloszynski *et al.* [10] serves as first test case. This set-up is chosen because it can cause a generic amount of distinctive cavitation regions. This is to demonstrate the good performance and stability properties of the EHL-FBNS algorithm since the simulation of such cases showed to be difficult or inefficient with other codes from the literature. Furthermore, the solid properties are chosen such that noticeable effects due to elastic deformations on the pressure profile occur. Thereby, it is shown that the consideration of elasticity does not alter the performance scaling. While being of little physical interest, this numerical set-up allows a comparison of the performance scaling to the original FBNS algorithm of Woloszynski *et al.*.

The variation of the gap height due to the rigid geometry of the surfaces h_g is constructed by assembling several unit geometries. Each unit geometry is composed of 20×20 cells with $h_g = h_{min} + h_p$. This square is surrounded by a one cell thick layer with $h_g = h_{min} + h_p/2$, which is in turn surrounded by a layer of four cells with $h_g = h_{min}$, resulting in a square of 30×30 cells. Depending on the desired size of the computational domain, a certain amount $K_{x_1} \times K_{x_2}$ of these unit cells is attached to each other and finally surrounded by a layer of cells for the Dirichlet boundary conditions of hydrodynamic pressure at p_{amb} and zero cavity fraction. At last, the coordinates of each cell center are set such they have are in the range of $[0 \ L_{x_1}] \times [0 \ L_{x_2}]$. The resulting array of cells in the exemplary case of $K_{x_1} = K_{x_2} = 2$ is visualized in Figure 3. The rigid body displacement h_d is set to 0 since it is already considered in h_g .

Fig. 3 Exemplary cell array resulting in the case of $K_{x_1} = K_{x_2} = 2$

The values of the parameters used in the EHL-FBNS simulations are summarized in Table 1. Piezoviscosity and compressibility of the liquid phase are not considered. The amount of unit geometries in x_1 -direction K_{x_1} is varied while $K_{x_2} = K_{x_1}$ is always enforced such that the resulting mesh is always quadratic. By increasing the amount of unit geometries, the total amount of discretization cells N is increased. Each resulting geometry is simulated with UI and QUICK for the rigid and elastic case. In the elastic case, the solid bodies' Young's modulus E and Poisson ratio ν are set such that the elastic deformation shows notable effects even though the overall pressures are low. At the same time, this set-up produces many distinctive cavitation regions. The rigid simulations use relaxation factors of $\alpha = 1$. Since the elastic simulations are generally more unstable, the relaxation factors have to be reduced to 0.5.

To quantify the algorithm's performance independently of the hardware's computational power, the non-dimensional code execution time is defined as:

$$t_{ex}^* = \frac{t_{ex}(N)}{t_{ex}(N = 1024)}, \quad (29)$$

where t_{ex} is the code execution time for a certain total amount of discretization cells N . The non-dimensional code execution time t_{ex}^* of the EHL-FBNS algorithm in the rigid UI case is compared with the one of the original FBNS algorithm of Wołoszynski *et al.* [10] in Figure 4. Their code was also implemented in MATLAB[®], considered rigid geometries and was discretized with the FVM, where the Poiseuille term was discretized with second order

14 *An EHL extension of the unsteady FBNS algorithm***Table 1** Summary of the parameters and values used in the EHL-FBNS simulations of the parallel slider with a various amount of trapezoidal pockets.

Param.	Value		Param.	Value		Param.	Value
U_{up}	5 ms^{-1}		μ_0	$3 \cdot 10^{-2} \text{ Pas}$		N_{x_1}	$2 + 30 \cdot K_{x_1}$
U_{low}	0		α_R	-		N_{x_2}	$2 + 30 \cdot K_{x_2}$
u_m	2.5 ms^{-1}		$p_{0,R}$	-		L_{x_1}	$8 \cdot 10^{-2} \text{ m}$
p_{amb}	10^5 Pa		ρ_0	850 kgm^{-3}		L_{x_2}	L_{x_1}
p_{cav}	$3 \cdot 10^4 \text{ Pa}$		C_1	-		h_d	0
$p_{hd,SB}$	p_{amb}		C_2	-		h_p	$12 \cdot 10^{-6} \text{ m}$
$p_{hd,WB}$	p_{amb}		E_{up}	$-, 5 \cdot 10^9 \text{ Pa}$		h_{min}	$15 \cdot 10^{-6} \text{ m}$
$p_{hd,EB}$	p_{amb}		E_{low}	$-, E_{up}$		K_{x_2}	K_{x_1}
$p_{hd,NB}$	p_{amb}		ν_{up}	$-, 0.3$		Δx_1	$L_{x_1}/(N_{x_1} - 1)$
θ_{SB}	0		ν_{low}	$-, \nu_{up}$		Δx_2	$L_{x_2}/(N_{x_2} - 1)$
θ_{WB}	0		E'	$-, 5.49 \cdot 10^9 \text{ Pa}$			
θ_{EB}	0		$F_{N,imp}$	-			
θ_{NB}	0						
h_{ref}	h_{min}		α_{p*}	1, 0.5			
$x_{1,ref}$	L_{x_1}		α_θ	1, 0.5			
$x_{2,ref}$	L_{x_2}		ρ_{ref}	ρ_0			
p_{ref}	10^6 Pa		μ_{ref}	μ_0			
t_{ref}	-		u_{ref}	u_m			
$F_{N,ref}$	-						
Param.	Value						
N	1024, 3844, 14884, 58564, 91204, 362404, 1444804, 5769604						
K_{x_1}	1, 2, 4, 8, 10, 20, 40, 80						

central differences and a first order upwind scheme was employed for the cavity fraction. Furthermore, three reference curves for the scaling of the non-dimensional code execution time are displayed: linear $t_{ex,lin}^* = M_1 N$, logarithmic $t_{ex,log}^* = M_2 N \log(N)$ and quadratic $t_{ex,quad}^* = M_3 N^2$. The coefficients $M_1 = 9.7656 \cdot 10^{-4}$, $M_2 = 1.4089 \cdot 10^{-4}$ and $M_3 = 9.5367 \cdot 10^{-7}$ are chosen such that $t_{ex}^*(N = 1024) = 1$ in all cases. It can be seen that the original FBNS algorithm has a performance scaling close to the linear reference while the EHL-FBNS algorithm performs a little bit slower than the $N \log(N)$ reference for large N but is always much faster than the quadratic reference. The difference between the FBNS and EHL-FBNS performances might be due to the fact that the EHL-FBNS algorithm constructs the matrices A_{Po} and B and vector $\vec{c}$ at each iteration step, while the FBNS algorithm might exploit that they are constant in a rigid and isoviscous simulation with an incompressible liquid phase. However, the exact details of the implementation of the original FBNS algorithm are unknown and the difference in time scaling cannot be pinned down rigorously.

Next, it is investigated how combinations of UI or QUICK discretization and rigid or elastic geometry affect the performance. The code execution times t_{ex} are provided in Table 2. The corresponding non-dimensional code execution times t_{ex}^* are displayed in Figure 5. Because all curves have almost the

Fig. 4 Performance of EHL-FBNS and FBNS algorithm along with reference scalings. The performance of the FBNS algorithm was taken from Woloszynski *et al.* [10].

same inclinations in the double logarithmic diagram, it can be deduced that the performance scaling of the EHL-FBNS algorithm stays similar in all cases. To prove that the operating conditions were chosen such that discretization scheme and elastic model have a noticeable impact on the results while the performance scaling remains unchanged, exemplary results are considered in the following for the geometry with $K_{x_1} = 8$. The pressure profiles of the UI simulations are visualized in Figure 6 for the rigid (a) and elastic (b) case. The contour of the regions where the hydrodynamic pressure p_{hd} reaches the cavitation pressure p_{cav} is visualized by orange lines. The elastic deformation drastically reduces the resulting pressure profile in comparison to the rigid simulation. The results of the same cases but with the QUICK scheme are shown in Figure 7 for the rigid (a) and elastic (b) case for comparison. Differences between the UI and the QUICK scheme are noticeable.

Fig. 5 Performance of combinations of rigid, elastic, UI and QUICK EHL-FBNS simulations.

Fig. 6 Hydrodynamic pressure profiles p_{hd} for a UI simulation with $K_{x_1} = 8$: (a) rigid and (b) elastic case.

Fig. 7 Hydrodynamic pressure profiles p_{hd} for a QUICK simulation with $K_{x_1} = 8$: (a) rigid and (b) elastic case

Table 2 Summary of the code execution times of the FBNS algorithm given by Wołoszynski *et al.* [10] and the ones of the EHL-FBNS algorithm with UI or QUICK discretization and rigid or elastic geometry to simulate the parallel slider with a various amount of trapezoidal pockets. The FBNS simulations of Wołoszynski *et al.* were performed on a workstation with 32 GB RAM and an Intel Xeon 3.3 GHz processor while the computations of the EHL-FBNS algorithm were conducted on a workstation with a 64 GB RAM and an AMD Ryzen 9 3900X 12-Core 3.8 GHz processor.

Resolution $N[-]$	Execution time $t_{ex}[s]$				
	FBNS [10]	Ri, UI	El, UI	Ri, QUICK	El, QUICK
1024	0.1	0.0285	0.1133	0.0456	0.1958
3844	0.3	0.0828	0.2591	0.3294	0.2884
14884	1.3	0.3665	1.0631	0.9767	1.3695
58564	5.7	2.0455	5.3531	4.1428	6.0378
91204	11.3	3.5748	16.1899	6.7058	10.1669
362404	47.1	17.6511	41.6014	31.8607	54.4950
1444804	-	103.5338	225.9930	179.1673	287.9650
5769604	-	632.0061	1776.8957	1042.3062	2096.1574

3.2 Ball-on-disc tribometer with a single texture

The second set-up is the ball-on-disc tribometer with a single texture used in the simulations and experiments of Mourier *et al.* [25]. These were performed to investigate the unsteady effect of isolated dimples passing through an EHL contact at pure rolling and rolling-sliding conditions. Mourier *et al.* state that they used shallower dimples in the simulations than in the experiments because the deep dimples compromised convergence. In the following, both geometries are simulated with the EHL-FBNS algorithm to show that the presented algorithm can provide converged results in either case. Furthermore, the gap height measurements of Mourier *et al.* are used to validate the EHL-FBNS algorithm for ball-on-disc tribometers. Lastly, differences in the simulated results of Mourier *et al.* and the EHL-FBNS algorithm are discussed and deductions about the most favourable discretization scheme are drawn.

The rolling-sliding condition of the EHL contact is characterised through the slide-to-roll ratio SSR as provided by Mourier *et al.* [25]:

$$SSR = \frac{U_{low} - U_{up}}{U_{low} + U_{up}}. \quad (30)$$

The time dependent variation of the gap height due to the rigid geometry of the surfaces is expressed as [25]:

$$h_g(x_1, x_2, t) = \frac{x_1^2}{2R_{x_1}} + \frac{x_2^2}{2R_{x_1}} + d \cos\left(\frac{\pi}{2} \cdot \frac{D}{1.2r}\right) \exp\left(-2\left(\frac{D}{1.2r}\right)^2\right), \quad (31)$$

where d is the dimple depth, r is the dimple radius and D is the distance of any position to the moving dimple center [25]:

$$D = \sqrt{(x_1 - x_{1,d})^2 + (x_2 - x_{2,d})^2}. \quad (32)$$

The dimple center has the coordinates $x_{1,d}$ and $x_{2,d}$ with [25]:

$$x_{1,d} = x_{1,0} + tu_{tex}, \quad x_{2,d} = x_{2,0}. \quad (33)$$

The initial position of the dimple center is described by $x_{1,0}$ and $x_{2,0}$ at time $t = 0$ s. The dimple moves with velocity $u_{tex} = U_{low}$. The parameters used in the EHL-FBNS simulations are summarized in Table 3. All simulations are performed with the same mean velocity u_m , but different SSR and therefore different U_{up} , U_{low} and number of time steps N_t . In the case of $SSR = 0$, $N_t = 257$ time steps and a dimple radius of $r = 15.5 \mu\text{m}$ are used. To replicate the experiment, a dimple depth of $d = 7 \mu\text{m}$ is employed, while $d = 0.175 \mu\text{m}$ is used to be consistent with the numerical set-up of Mourier *et al.* [25]. For $SSR = -0.5$, the dimple radius $r = 21.5 \mu\text{m}$ is considered. Since the dimple moves more slowly in this case, $N_t = 513$ time steps are required while the

Param.	Value	Param.	Value	Param.	Value
		μ_0	$2.5 \cdot 10^{-1}$ Pas	N	66049
		α_B	-	N_{x_1}	257
u_m	0.09 ms^{-1}	α_R	$2.2 \cdot 10^{-8} \text{ Pa}^{-1}$	N_{x_2}	257
		$p_{0,R}$	$1.96 \cdot 10^8 \text{ Pa}$	L_{x_1}	$6a$
p_{amb}	0 Pa	τ_E	-	L_{x_2}	$6a$
p_{cav}	p_{amb}	ρ_0	850 kgm^{-3}	$h_{d,ini}$	-10^{-6} m
$p_{hd,SB}$	p_{amb}	C_1	$5.9 \cdot 10^8 \text{ Pa}$	a	$136.5 \cdot 10^{-6} \text{ m}$
$p_{hd,WB}$	p_{amb}	C_2	1.34	R_{x_1}	$12.5 \cdot 10^{-3} \text{ m}$
$p_{hd,EB}$	p_{amb}	E_{up}	-	Δx_1	$L_{x_1}/(N_{x_1} - 1)$
$p_{hd,NB}$	p_{amb}	E_{low}	-	Δx_2	$L_{x_2}/(N_{x_2} - 1)$
θ_{SB}	Neumann	ν_{up}	-	$x_{1,0}$	$-3a$
θ_{WB}	0	ν_{low}	-	$x_{2,0}$	0
θ_{EB}	Neumann	E'	$110 \cdot 10^9 \text{ Pa}$	u_{tex}	U_{low}
θ_{NB}	Neumann	$F_{N,imp}$	15 N	Δt^*	$\Delta x_1/a$
h_{ref}	a^2/R_{x_1}			Δt	$\Delta t^* \cdot t_{ref}$
$x_{1,ref}$	a	α_{p^*}	0.05		
$x_{2,ref}$	a	α_θ	0.05		
p_{ref}	$385 \cdot 10^6 \text{ Pa}$	ρ_{ref}	ρ_0		
t_{ref}	a/u_m	μ_{ref}	μ_0		
$F_{N,ref}$	$F_{N,imp}$	u_{ref}	u_m		
Param.	Value				
U_{up}	$0.09 \text{ ms}^{-1}, 0.135 \text{ ms}^{-1}$				
U_{low}	$0.09 \text{ ms}^{-1}, 0.045 \text{ ms}^{-1}$				
SSR	0, -0.5				
N_t	257, 513				
r	$15.5 \cdot 10^{-6} \text{ m}, 21.5 \cdot 10^{-6} \text{ m}$				
d	$7000 \cdot 10^{-9} \text{ m}, 175 \cdot 10^{-9} \text{ m}, 1300 \cdot 10^{-9} \text{ m}, 160 \cdot 10^{-9} \text{ m}$				

The results of the EHL-FBNS simulations are provided as videos in the supplements. Exemplary results of gap height h , hydrodynamic pressure p_{hd} and cavity fraction θ at $t = 0$ s, $SSR = 0$ and $d = 7$ μm are shown for UI and QUICK simulations in Figure 8. The contour line of $p_{hd} = p_{cav}$ is marked in orange while the center line is displayed in green. Apart from more pronounced spikes in the hydrodynamic pressure at the downstream end of the EHL contact zone with the more accurate QUICK scheme, both simulations produce almost the same results at first glance.

In the following, the EHL-FBNS results are compared to the experimental and simulated counterparts of Mourier *et al.* [25]. The data of Mourier *et al.* was read in with the software Engauge Digitizer (<https://markummittchell.github.io/engauge-digitizer/>) and can therefore be subject to minor deviation from the original data. The data of Mourier *et al.* consists of gap height measurements in the experimental case and of gap height and pressure distributions in the simulated case at five distinctive dimple center positions.

Figure 9 (a) shows the UI and QUICK results of the EHL-FBNS algorithm along with the experimental results of Mourier *et al.* in case of the deep dimple with $SSR = 0$. Due to the large depth of the dimple in comparison to the remaining gap height within the EHL contact, the gap height distribution shows a strong discontinuity at the rim of the deep dimple. At the first and last dimple position, the dimple is just entering or leaving the visualized domain. Most of the domain is still unaffected by the dimple and basically corresponds to the steady solution without a dimple. Small differences in the gap height h between UI and QUICK can be observed while the more accurate QUICK scheme closely fits the experimental results in the center of the domain. The systematic difference in the gap height between UI and QUICK is mostly due to a different value of the rigid body displacement h_d which is set such that the load balance equation is eventually met. For the dimple positions 2-4, the QUICK scheme produces large deviations in the gap height compared to the experimental results at the discontinuity at the downstream rim of the dimple. There, the UI scheme manages to fit the experimentally measured gap height better. These results are consistent with the finding of LeVeque [33, Ch. 11] that for discontinuous problems, first order methods give smoother results while second order methods cause oscillations. Both schemes deviate from the experimental results at the downstream rim of the dimple when it is leaving the EHL contact at position 5. At all five positions, the hydrodynamic pressure distributions of UI and QUICK are mostly in close agreement, but the UI scheme produces higher pressure spikes at the downstream rim of the dimple which in turn cause larger elastic deformations. This explains why the UI scheme shows better agreement in the gap height since this more diffusive scheme eventually tends to smooth the discontinuity at the rim of the dimple by adjusting the hydrodynamic pressure accordingly. Consequently, the lower order UI scheme is recommendable close to discontinuities in the gap height while the QUICK scheme is more advantageous at smoother parts of the geometry.

Fig. 8 Distribution of gap height h (a,b), hydrodynamic pressure p_{hd} (c,d) and cavity fraction θ (e,f) in the ball-on-disc tribometer with single texture at $t = 0$ s for UI (left) and QUICK (right) discretization.

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The UI and QUICK results along with experimental values of Mourier *et al.* in case of the deep dimple with $SSR = -0.5$ are depicted in Figure 9 (b). In this case, the downstream area of the dimple also gets deformed because the dimple moves at a lower speed than the mean velocity u_m . This means that some of the fluid that is initially within the dimple leaves the texture behind which causes a deformation since more volume is occupied outside of the dimple. This behaviour is principally also replicated by the EHL-FBNS results but in a more pronounced way than in the experiments. The higher order QUICK scheme matches the experimental results closer than the UI scheme in the vicinity of this effect as depicted at dimple position 3. The UI scheme causes higher hydrodynamic pressures downstream of the dimple which this time cause a larger overestimation of the occurring deformation than done by the QUICK scheme. However, the stronger this effects becomes, the larger the deviation between experiment and simulation becomes as shown at dimple position 4. Still, experimental and EHL-FBNS results generally show a good agreement in the gap height distribution. The EHL-FBNS algorithm is thereby validated for simulations of discontinuous textures in ball-on-disc tribometers under EHL operating conditions.

Fig. 9 At $SSR = 0$ (a) and $SSR = -0.5$ (b): distribution of gap height h (top) and hydrodynamic pressure p_{hd} (bottom) along the center line of the ball-on-disc tribometer with single texture for UI and QUICK discretization against the experimental results of Mourier *et al.* [25].

Next, the EHL-FBNS results are compared to the simulated results of Mourier *et al.* [25]. Figure 10 (a) shows the UI and QUICK results of the EHL-FBNS algorithm along with the simulated results of Mourier *et al.* in case of the shallow dimple with $SSR = 0$. Unlike the deep dimple, the rim of the shallow dimple is only weakly discontinuous. At the first dimple position, the simulated results of Mourier *et al.* agree well with the gap height and hydrodynamic pressure produced by the QUICK scheme. This is expected because both eventually correspond to second order spatial discretizations of an almost steady case since the shallow dimple does not affect the displayed domain yet. Similar to the deep dimple case, the first order UI scheme results in practically the same hydrodynamic pressure distribution as the QUICK scheme but a slightly different systematic offset in the gap height distribution. When the shallow dimple introduces unsteady effects at positions 2-5, the gap height distributions of QUICK stay closer to the results of Mourier *et al.* than in the case of UI discretization. Nonetheless, all three methods deliver different gap heights close to the dimple. While the hydrodynamic pressure distributions of UI and QUICK show similar oscillations around the dimple, it stays almost undisturbed in the simulations of Mourier *et al.*. Since differently to the deep dimple, the results of UI and QUICK are still reasonably close to each other at the rim of the shallow dimple, errors or oscillations caused by the discretization scheme of the Couette term are not believed to be of dominant role. Instead, the differences in the results of EHL-FBNS algorithm and Mourier *et al.* during the introduction of unsteady phenomena are likely caused by the first order Euler implicit discretization of the EHL-FBNS algorithm while Mourier *et al.* use a second order discretization in time.

These findings are complemented by the results in case of the shallow dimple at $SSR = -0.5$ as depicted in Figure 10 (b) along with the outcome of the simulations of Mourier *et al.*. Gap height and hydrodynamic pressure of QUICK and Mourier *et al.* are in close agreement while the lower order UI results slightly differ at some points. The reason for the closer agreement of the EHL-FBNS results with Mourier *et al.* is expected to be due the dimple moving at a lower speed, thus introducing slower unsteady effects. Since the discrete time steps are of the same length as in the case of $SSR = 0$, the time resolution is relatively higher for $SSR = -0.5$, thus enabling all schemes to deliver almost the same results.

Fig. 10 At $SSR = 0$ (a) and $SSR = -0.5$ (b): distribution of gap height h (top) and hydrodynamic pressure p_{hd} (bottom) along the center line of the ball-on-disc tribometer with single texture for UI and QUICK discretization against the simulated results of Mourier *et al.* [25].

In order to better understand the performance of the EHL-FBNS algorithm in unsteady EHL conditions, the required amount of iterations N_n at each position of the dimple center $x_{1,d}$ is displayed in Figure 11. The most iterations are needed to compute the steady solution at the initial position of the dimple and once the dimple reaches the EHL-contact zone. Since it produces more irregular results, the deep dimple generally requires more iterations than the shallow one. UI and QUICK simulations need a similar amount of iterations for each respective pair of simulation. Each of the eight simulations required between 2 and 6 h code execution time on a desktop computer.

Fig. 11 Amount of iterations N_n required by the EHL-FBNS algorithm for the simulation of the ball-on-disc tribometer with single texture for each position of the dimple center $x_{1,d}$.

Summarizing, the EHL-FBNS algorithm manages to deliver converged results even in unsteady EHL operating conditions with deep dimples with strong discontinuities at their rim. Moreover, the first order UI scheme gives closer agreement to the experimental results of Mourier *et al.* [25] in the vicinity of deep dimples than the higher order QUICK scheme. Therefore, lower order spatial discretization schemes are recommended close to strong gap height discontinuities while higher order schemes are recommended at smoother parts of the geometry due to their higher accuracy in these areas. Moreover, the extreme pressure spikes at the rim of the deep dimples raise the question

whether the elastic half-space assumption is still valid in this area. Nonetheless, the good agreement of the gap height distribution with the experimental data of Mourier *et al.* validates the EHL-FBNS algorithm for simulations of textures in ball-on-disc tribometers under unsteady EHL operating conditions.

4 Conclusion

The EHL-FBNS algorithm is presented in this paper. It allows the simulation of the unsteady hydrodynamic pressure build up under consideration of mass-conserving cavitation with the JFO model within lubrication gaps. Furthermore, its versatile implementation enables the simulation of various combinations of isoviscous or piezoviscous flows, incompressible or compressible liquid phases, rigid or elastic surfaces, first or second order spatial discretizations of the Couette term, imposed rigid body displacements or normal forces and Dirichlet or Neumann boundary conditions. The algorithm is explained in detail within this paper and the implemented MATLAB[®] code with the corresponding set-up and visualization scripts is provided in the supplements. That way, the results can be reproduced by downloading the code and repeating the simulations. Moreover, the usage and further development of the thoroughly commented code is encouraged through its public accessibility and maintenance on GitHub: <https://github.com/ErikHansenGit/EHL>.

Within this paper, the EHL-FBNS algorithm results were compared to literature data of Woloszynski *et al.* [10] and Mourier *et al.* [25]. The key findings are:

- the performance of the EHL-FBNS code almost scales with $N \log(N)$ in simulations with N discretization cells and a constant rigid body displacement h_d .
- The EHL-FBNS code can deliver converged results even when extreme gap height discontinuities are present.
- Higher order spatial discretizations of the Couette term can cause large errors in the gap height distributions when gap height discontinuities are present in the EHL contact. Therefore, lower order spatial discretization schemes are recommended close to strong gap height discontinuities while higher order schemes are recommended at smoother parts of the geometry due to their higher accuracy in these regions.
- The EHL-FBNS algorithm is validated for the investigation of deep dimples with discontinuous rims in ball-on-disc tribometers under severe EHL operating conditions.

Appendix A Discretized equations

A.1 Discretized dimensionless Reynolds equation considering mass-conserving cavitation and discretized dimensionless Fischer-Burmeister equation

Fig. A1 Schematic sketch of a two dimensional FVM grid of size $L_{x_1}^* \cdot L_{x_2}^*$ with an exemplary stencil within the computational domain.

Applying the FVM with a two dimensional grid as schematically depicted in Figure A1 to Equation (9) eventually results in the discrete dimensionless Reynolds equation considering mass-conserving cavitation at each cell center G_C :

$$\begin{aligned}
 G_C &= \int_{\partial A^*} \left(\xi_{Po}^* \left(\left(\frac{\frac{\partial}{\partial x_1^*}}{\left(\frac{x_{1,ref}}{x_{2,ref}} \right)^2} \frac{\partial}{\partial x_2^*} \right) p^* \right) \cdot \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} dL^* \right. \\
 &\quad \left. - \int_{\partial A^*} \xi_{Co}^* (1 - \theta) n_1 dL^* - \iint_{A^*} \frac{\partial}{\partial t^*} (\xi_{Ti}^* (1 - \theta)) dx_1^* dx_2^* \right) \\
 &= A_{Po,SP_S^*} + A_{Po,WP_W^*} + A_{Po,CP_C^*} + A_{Po,EP_E^*} + A_{Po,NP_N^*} \\
 &\quad + B_{WW}\theta_{WW} + B_W\theta_W + B_C\theta_C + B_E\theta_E + c_C
 \end{aligned} \tag{A1}$$

where the two-dimensional cell is of area A^* , has the boundary ∂A^* and its outward pointing normal vector $\vec{n}$ is of length 1. Since a rectangular mesh

aligned with a cartesian coordinate system is used, the components n_1 and n_2 can take the values of ± 1 or 0. The line increment of the cell boundary is denoted by L^* . The scalar product is implied by $\cdot$. The integrals are discretized with the second order midpoint rule. Values or derivatives at the boundaries of the Poiseuille term are discretized with a second order central interpolation or differential scheme. Its coefficients A_{Po} read:

$$A_{Po,S} = -\frac{1}{\Delta x_2^*} \left(\frac{x_{1,ref}}{x_{2,ref}} \right)^2 (-\Delta x_1^* \xi_{Po,s}^*), \quad (A2)$$

$$A_{Po,W} = -\frac{1}{\Delta x_1^*} (-\Delta x_2^* \xi_{Po,w}^*), \quad (A3)$$

$$A_{Po,C} = \frac{1}{\Delta x_2^*} \left(\frac{x_{1,ref}}{x_{2,ref}} \right)^2 (-\Delta x_1^* \xi_{Po,s}^*) + \frac{1}{\Delta x_1^*} (-\Delta x_2^* \xi_{Po,w}^*) \\ - \frac{1}{\Delta x_1^*} (\Delta x_2^* \xi_{Po,e}^*) - \frac{1}{\Delta x_2^*} \left(\frac{x_{1,ref}}{x_{2,ref}} \right)^2 (\Delta x_1^* \xi_{Po,n}^*), \quad (A4)$$

$$A_{Po,E} = \frac{1}{\Delta x_1^*} (\Delta x_2^* \xi_{Po,e}^*), \quad (A5)$$

$$A_{Po,N} = \frac{1}{\Delta x_2^*} \left(\frac{x_{1,ref}}{x_{2,ref}} \right)^2 (\Delta x_1^* \xi_{Po,n}^*). \quad (A6)$$

The dimensionless cell spacings are defined as $\Delta x_1^* = \Delta x_1 / x_{1,ref}$ and $\Delta x_2^* = \Delta x_2 / x_{2,ref}$. At the cell boundaries, ξ_{Po}^* is determined as:

$$\xi_{Po,s}^* = \frac{\xi_{Po,C}^* + \xi_{Po,S}^*}{2}, \quad (A7)$$

$$\xi_{Po,w}^* = \frac{\xi_{Po,C}^* + \xi_{Po,W}^*}{2}, \quad (A8)$$

$$\xi_{Po,e}^* = \frac{\xi_{Po,E}^* + \xi_{Po,C}^*}{2}, \quad (A9)$$

$$\xi_{Po,n}^* = \frac{\xi_{Po,N}^* + \xi_{Po,C}^*}{2}. \quad (A10)$$

The components of $B = B_{Co} + B_{Ti}$ and $\vec{c} = \vec{c}_{Co} + \vec{c}_{Ti}$ contain contributions of the Couette and unsteady term, which are detailed now. Values at the boundaries of the Couette term are discretized with a generic order upwind interpolation scheme. Depending on how a , b and c are set, first order upwind interpolation (UI: $a = 0$, $b = 1$, $c = 0$), third order quadratic upwind interpolation (QUICK: $a = -1/8$, $b = 6/8$, $c = 3/8$), second order linear upwind interpolation (LUI: $a = -1/2$, $b = 3/2$, $c = 0$) or second order cubic upwind interpolation (CUI: $a = -1/6$, $b = 5/6$, $c = 2/6$) can be chosen [28, Ch. 4.4]. The coefficients of B_{Co} read:

$$B_{Co,WW} = a(-\Delta x_2^* \xi_{Co,w}^*), \quad (A11)$$

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$$B_{Co,W} = b(-\Delta x_2^* \xi_{Co,w}^*) + a(\Delta x_2^* \xi_{Co,e}^*), \quad (A12)$$

$$B_{Co,C} = c(-\Delta x_2^* \xi_{Co,w}^*) + b(\Delta x_2^* \xi_{Co,e}^*), \quad (A13)$$

$$B_{Co,E} = c(\Delta x_2^* \xi_{Co,e}^*). \quad (A14)$$

Special attention has to be paid close to the boundaries if no WestWest neighbour cell exists. In this case, the first order upwind interpolation scheme is used for the approximation of $\theta_w = \theta_W$. Then, the coefficients of B_{Co} read:

$$B_{Co,WW} = 0, \quad (A15)$$

$$B_{Co,W} = (-\Delta x_2^* \xi_{Co,w}^*) + a(\Delta x_2^* \xi_{Co,e}^*), \quad (A16)$$

$$B_{Co,C} = b(\Delta x_2^* \xi_{Co,e}^*), \quad (A17)$$

$$B_{Co,E} = c(\Delta x_2^* \xi_{Co,e}^*). \quad (A18)$$

At the cell boundaries, ξ_{Co}^* is determined as:

$$\xi_{Co,w}^* = a\xi_{Co,WW}^* + b\xi_{Co,W}^* + c\xi_{Co,C}^*, \quad (A19)$$

$$\xi_{Co,e}^* = a\xi_{Co,W}^* + b\xi_{Co,C}^* + c\xi_{Co,E}^*, \quad (A20)$$

except when there is no WestWest neighbour cell, such that $\xi_{Co,e}^*$ stays the way it is but $\xi_{Co,w}^*$ becomes:

$$\xi_{Co,w}^* = \xi_{Co,W}^*. \quad (A21)$$

The unsteady term is discretized with the first order Euler implicit scheme. The coefficient of B_{Ti} reads:

$$B_{Ti,C} = \Delta x_1^* \Delta x_2^* \frac{\xi_{Ti,C}^*}{\Delta t^*}, \quad (A22)$$

where $\Delta t^* = \Delta t / t_{ref}$ is the dimensionless time step. The components of $\vec{c}_{Co}$ and $\vec{c}_{Ti}$ read:

$$c_{Co,C} = -\Delta x_2^* \cdot (-\xi_{Co,w}^*) + \Delta x_2^* \cdot (-\xi_{Co,e}^*), \quad (A23)$$

$$c_{Ti,C} = -\Delta x_1^* \Delta x_2^* \frac{\xi_{Ti,C}^* - \xi_{Ti,C}^{*,prev} (1 - \theta_C^{prev})}{\Delta t^*}, \quad (A24)$$

where $\xi_{Ti,C}^{*,prev}$ and θ_C^{prev} correspond to the values of the previous time step. The discrete non-dimensional Fischer-Burmeister Equation (12) at each cell center reads:

$$F_C = p_C^* + \theta_C - \sqrt{p_C^{*2} + \theta_C^2}. \quad (A25)$$

A.2 Discretized elastic deformation

In its non-dimensional form, the gap height equation (5) reads:

$$h^* = \frac{h_d}{h^*} + \frac{h_g}{h^*} + \frac{h_{el}}{h^*} = h_d^* + h_g^* + h_{el}^*, \quad (\text{A26})$$

where the discretized dimensionless elastic deformation of the gap height can be expressed as a linear convolution of a non-dimensional kernel function K^* = Kp_{ref}/h_{ref} with the non-dimensional hydrodynamic pressure field p_{hd}^* :

$$h_{el,C}^* = h_{el}^*(x_{1,C}^*, x_{2,C}^*) = \sum_{x_2'^*}^{N_{x_2'^*}} \sum_{x_1'^*}^{N_{x_1'^*}} K^*(x_{1,C}^* - x_1'^*, x_{2,C}^* - x_2'^*) p_{hd}^*(x_1'^*, x_2'^*). \quad (\text{A27})$$

This convolution can be interpreted as the alignment of the non-dimensional hydrodynamic pressure field p_{hd}^* below the mirrored non-dimensional kernel function K^* as shown in Figure A2. This is explained in detail in the following. The non-dimensional kernel function is mirrored in x_1^* - and x_2^* -direction. Afterwards, the center entry $K^*(0,0)$ is aligned with $p_{hd,C}^* = p_{hd}^*(x_{1,C}^*, x_{2,C}^*)$. That way, aligned pairs of K^* and p_{hd}^* are created. The product of each respective pair is computed, for example $K^*(0,0) \cdot p_{hd,C}^*$ or $K^*(-\Delta x_1^*, 0) \cdot p_{hd,E}^*$. The sum over all of the products is equal to $h_{el,C}^*$. If instead of $h_{el,C}^* = h_{el}^*(x_{1,C}^*, x_{2,C}^*)$ the dimensionless elastic deformation at the West cell $h_{el,W}^* = h_{el}^*(x_{1,W}^*, x_{2,W}^*)$ is desired, the center entry $K^*(0,0)$ of the mirrored non-dimensional kernel function has to be aligned with $p_{hd,W}^* = p_{hd}^*(x_{1,W}^*, x_{2,W}^*)$.

Fig. A2 Schematic sketch of the alignment of the mirrored dimensionless kernel function K^* with the dimensionless pressure field p_{hd}^* to obtain the dimensionless elastic deformation $h_{el,C}^*$ via convolution.

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Assuming the hydrodynamic pressure being constant over the rectangular discretization cell of area $A = \Delta x_1 \Delta x_2$, the kernel K can be expressed as [29, Ch. 3.3], [30, Ch. 3.1]:

$$\begin{aligned}
 K(\tilde{x}_1, \tilde{x}_2) = & \frac{2}{\pi E'} \left((\tilde{x}_1 + q_1) \ln \left(\frac{(\tilde{x}_2 + q_2) + \sqrt{(\tilde{x}_2 + q_2)^2 + (\tilde{x}_1 + q_1)^2}}{(\tilde{x}_2 - q_2) + \sqrt{(\tilde{x}_2 - q_2)^2 + (\tilde{x}_1 + q_1)^2}} \right) \right. \\
 & + (\tilde{x}_2 + q_2) \ln \left(\frac{(\tilde{x}_1 + q_1) + \sqrt{(\tilde{x}_2 + q_2)^2 + (\tilde{x}_1 + q_1)^2}}{(\tilde{x}_1 - q_1) + \sqrt{(\tilde{x}_2 + q_2)^2 + (\tilde{x}_1 - q_1)^2}} \right) \\
 & + (\tilde{x}_1 - q_1) \ln \left(\frac{(\tilde{x}_2 - q_2) + \sqrt{(\tilde{x}_2 - q_2)^2 + (\tilde{x}_1 - q_1)^2}}{(\tilde{x}_2 + q_2) + \sqrt{(\tilde{x}_2 + q_2)^2 + (\tilde{x}_1 - q_1)^2}} \right) \\
 & \left. + (\tilde{x}_2 - q_2) \ln \left(\frac{(\tilde{x}_1 - q_1) + \sqrt{(\tilde{x}_2 - q_2)^2 + (\tilde{x}_1 - q_1)^2}}{(\tilde{x}_1 + q_1) + \sqrt{(\tilde{x}_2 - q_2)^2 + (\tilde{x}_1 + q_1)^2}} \right) \right), \quad (\text{A28})
 \end{aligned}$$

where the certain terms are consolidated as:

$$\tilde{x}_1 = x_{1,C} - x'_1, \quad (\text{A29})$$

$$\tilde{x}_2 = x_{2,C} - x'_2, \quad (\text{A30})$$

$$q_1 = \frac{\Delta x_1}{2}, \quad (\text{A31})$$

$$q_2 = \frac{\Delta x_2}{2}. \quad (\text{A32})$$

A.3 Discretized pressure Jacobian of the dimensionless Reynolds Equation

In the rigid case, the following procedure is obsolete and the pressure Jacobian of the dimensionless Reynolds Equation is simply $J_{G,p^*} = A_{Po}$. In order to consider the relationship between h^* and p^* in J_{G,p^*} in the elastic case, the coefficients of the Couette and unsteady term are reformulated:

$$\xi_{Co}^* = \xi_{Co,h}^* h^*, \quad \xi_{Ti}^* = \xi_{Ti,h}^* h^*, \quad (\text{A33})$$

with

$$\xi_{Co,h}^* = 12 \frac{x_{1,ref} u_{m,ref} \mu_{ref}}{h_{m,ref}^2 p_{ref}} \rho^* u^*, \quad (\text{A34})$$

$$\xi_{Ti,h}^* = 12 \frac{x_{1,ref}^2 \mu_{ref}}{t_{ref} h_{ref}^2 p_{ref}} \rho^*. \quad (\text{A35})$$

Using the previously mentioned generic order upwind interpolation scheme for the Couette term, this results in:

$$\xi_{Co,w}^* = a \xi_{Co,h,WW}^* h_{WW}^* + b \xi_{Co,h,W}^* h_W^* + c \xi_{Co,h,C}^* h_C^*, \quad (\text{A36})$$

$$\xi_{Co,e}^* = a\xi_{Co,h,W}^* h_W^* + b\xi_{Co,h,C}^* h_C^* + c\xi_{Co,h,E}^* h_E^*. \quad (A37)$$

Except if no WestWest neighbour cell exists and $\xi_{Co,w}^*$ has to be replaced by:

$$\xi_{Co,w}^* = \xi_{Co,h,W}^* h_W^*. \quad (A38)$$

Afterwards, these expressions and Equations (A26) and (A27) are inserted into c_C of Equation (A1) (thus equations (A23) and (A24)) and only the coefficients of p_S^* , p_W^* , p_C^* , p_E^* and p_N^* are considered to obtain $J_{G,p^*} = A_{Po} + A_h = A_{Po} + A_{Co} + A_{Ti}$. Eventually, a generic scheme can be found to find the coefficients of an arbitrary diagonal D of matrix A_{Co} :

$$\begin{aligned} A_{Co,D}(i_d, j_d) = \Delta x_2^* (& a\xi_{Co,h,WW}^* K^*(\Delta x_{1,WW}^*, \Delta x_{2,WW}^*) + b\xi_{Co,h,W}^* K^*(\Delta x_{1,W}^*, \Delta x_{2,W}^*) \\ & + c\xi_{Co,h,C}^* K^*(\Delta x_{1,C}^*, \Delta x_{2,C}^*) \\ & - (a\xi_{Co,h,W}^* K^*(\Delta x_{1,W}^*, \Delta x_{2,W}^*) + b\xi_{Co,h,C}^* K^*(\Delta x_{1,C}^*, \Delta x_{2,C}^*) \\ & + c\xi_{Co,h,E}^* K^*(\Delta x_{1,E}^*, \Delta x_{2,E}^*))). \end{aligned} \quad (A39)$$

Note that i_d and j_d are not the indices of matrix A_{Co} , but counters that correspond to its diagonals $A_{Co,D}$. For cells close to the boundary that do not have a WestWest neighbour cell the following expression is used instead:

$$\begin{aligned} A_{Co,D}(i_d, j_d) = \Delta x_2^* (& \xi_{Co,h,W}^* K^*(\Delta x_{1,W}^*, \Delta x_{2,W}^*) \\ & - (a\xi_{Co,h,W}^* K^*(\Delta x_{1,W}^*, \Delta x_{2,W}^*) + b\xi_{Co,h,C}^* K^*(\Delta x_{1,C}^*, \Delta x_{2,C}^*) \\ & + c\xi_{Co,h,E}^* K^*(\Delta x_{1,E}^*, \Delta x_{2,E}^*))). \end{aligned} \quad (A40)$$

Some of the above equations are consolidated as:

$$\begin{aligned} \Delta x_{1,WW}^* &= (-i_d - 2)\Delta x_1^*, & \Delta x_{2,WW}^* &= -j_d\Delta x_2^* \\ \Delta x_{1,W}^* &= (-i_d - 1)\Delta x_1^*, & \Delta x_{2,W}^* &= -j_d\Delta x_2^* \\ \Delta x_{1,C}^* &= -i_d\Delta x_1^*, & \Delta x_{2,C}^* &= -j_d\Delta x_2^* \\ \Delta x_{1,E}^* &= (-i_d + 1)\Delta x_1^*, & \Delta x_{2,E}^* &= -j_d\Delta x_2^* \end{aligned} \quad (A41)$$

In order to construct the South, West, Center, East and North diagonals of A_{Co} , the counters i_d and j_d have to be set as shown in Table A1.

Table A1 Description on how to set i_d and j_d to obtain the desired diagonal entry of A_{Co} .

$A_{Co,D}$	i_d	j_d
$A_{Co,S}$	0	-1
$A_{Co,W}$	-1	0
$A_{Co,C}$	0	0
$A_{Co,E}$	1	0
$A_{Co,N}$	0	1

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The unsteady term is discretized with the first order Euler implicit scheme.
The coefficients of A_{Ti} read:

$$\begin{aligned}
 A_{Ti,S} &= -\Delta x_1^* \Delta x_2^* \frac{\xi_{Ti,h,C}^*}{\Delta t^*} K^*(0, \Delta x_2^*) \\
 A_{Ti,W} &= -\Delta x_1^* \Delta x_2^* \frac{\xi_{Ti,h,C}^*}{\Delta t^*} K^*(\Delta x_1^*, 0) \\
 A_{Ti,C} &= -\Delta x_1^* \Delta x_2^* \frac{\xi_{Ti,h,C}^*}{\Delta t^*} K^*(0, 0) \\
 A_{Ti,E} &= -\Delta x_1^* \Delta x_2^* \frac{\xi_{Ti,h,C}^*}{\Delta t^*} K^*(-\Delta x_1^*, 0) \\
 A_{Ti,N} &= -\Delta x_1^* \Delta x_2^* \frac{\xi_{Ti,h,C}^*}{\Delta t^*} K^*(0, -\Delta x_2^*)
 \end{aligned} \tag{A42}$$

Supplementary information. The supplements within the folder `code` consist of set-up (`EHL_01_run_setup.m`), computation (`EHL_02_run_mainprocessing.m`) and visualization scripts (`EHL_03_run_visualization.m`). Executing the master script `EHL_00_run_Study_C4_gen.m` will automatically set-up and run all simulations of section 3.1 while `EHL_00_run_Study_A_gen.m` will generate the results of section 3.2. The data will be saved in a folder named `data`. The figures containing simulation data can then be reproduced by executing the corresponding visualization scripts. The supplemented videos of section 3.2 can be recreated in .avi format by executing the script `EHL_04_animation_Study_A.m`.

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The Scientific colour map `batlow` [34] is used in this study to prevent visual distortion of the data and exclusion of readers with colour-vision deficiencies [35].

Declarations

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Conflicts of Interest

The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

Availability of data and materials

The used MATLAB[®] code, set-up and visualization scripts are provided in the supplements. All of the results can be reproduced at will by downloading the code and repeating the simulations. Further or original data can be supplied from the corresponding author upon request.

Code availability

The supplied MATLAB[®] scripts are thoroughly commented to encourage their usage and further development. A maintained and publicly available version of the code can also be found on GitHub: <https://github.com/ErikHansenGit/EHL>.

Authors' contributions

Conceptualization: Erik Hansen, Bettina Frohnapfel, Andrea Codrignani; Methodology: Erik Hansen; Software: Erik Hansen, Altay Kaçan; Formal analysis and investigation: Erik Hansen; Validation: Erik Hansen; Data curation: Erik Hansen, Altay Kaçan; Visualization: Erik Hansen, Altay Kaçan; Writing - original draft preparation: Erik Hansen; Writing - review and editing: Erik Hansen, Altay Kaçan, Andrea Codrignani, Bettina Frohnapfel; Project administration: Erik Hansen, Bettina Frohnapfel; Funding acquisition: Bettina Frohnapfel; Resources: Bettina Frohnapfel; Supervision: Bettina Frohnapfel, Andrea Codrignani. All authors have read and agreed to the published version of the manuscript.

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