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Liang Liang Sun (sun15@mail.ustc.edu.cn)
University of Science and Technology of China
Xiang Zhou
University of Science and Technology of China
Leong Chuan Kwek
National University of Singapore
Sixia Yu
University of Science and Technology of China

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No Disturbance Without Uncertainty Under Generalized Probability Theory

Liang Liang Sun[1], Xiang Zhou[1], Leong-Chuan Kwek[2,3,4], Sixia Yu [1]

[1]*Department of Modern Physics and National Laboratory for Physical Sciences at Microscale, University of Science and Technology of China, Hefei, Anhui 230026, China*

[2]*MajuLab, CNRS-UNS-NUS-NTU International Joint Research Unit, UMI 3654, Singapore*

[3]*National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616*

[4]*School of Electrical and Electronic Engineering Block S2.1, 50 Nanyang Avenue, Singapore 639798*

Quantum theory exhibits many non-classical phenomena: violating Bell's inequalities, exhibiting intrinsic uncertainty, and predicting measurement-disturbance relation. In a wider framework of generalized probability theories, these traits continue to be shared by many other post-quantum theories. In particular, Bell's inequalities are more strongly violated. Is there some undiscovered principle lying behind the generalized theory that accounts for its correlation strength and other properties? We find that there is an intimate connection between uncertainty and effect of disturbance in a measurement. The uncertainty in a measurement upper-bounds the disturbance. It can also lead to a uncertainty-disturbance principle. It turns out that this simple principle lies at the very heart of any correlations and their properties. More specifically, by considering the most widely studied two-level system,

it leads to a strong self-duality that constitutes the key geometrical structure of quantum state space, and it also explains why quantum correlations do not reach the maximum violation of Bell's inequalities and that it can result in the famous Tsirelson's bound for spatial Bell's scenario and Lüders' bound for the temporal Bell's scenario. We argue that the observation of intrinsic uncertainty and disturbance distinguish quantum theory from classical theory, and the relation between them reveals the insight into any theory under the framework of generalized probability theories.

Quantum theory is probably the best theory that we have till now. It has explained microscopic phenomena to an unprecedented level of accuracy. Notwithstanding our increasingly thorough knowledge regarding its formalism relying Hilbert's space, we still do not understand many aspects of the theory including a good grasp of uncertainty and disturbance of observables. Can the structure be understood based on the ground of physical principle? This profound question can trace back to the early days of the quantum theory and has a renew interest in recent tens years due to the advance of quantum information, especially due to the deepen understanding of quantum nonlocality.

The discover of nonlocality was inspired by the debate on uncertainty principle or specifically on whether quantum theory is deterministic. For classical and deterministic theory every quantity has a predefined value and one is able to measure the value without disturbing the system. These classical features place strict constraint on correlations between separated measurements in terms of Bell's inequalities [1,2]. Bell's inequalities can be violated by quantum theory that precludes a deterministic description of it. However, Bell's inequalities do not specify quantum theory from possible post-quantum theories as they can also violate Bell's inequality and even to a larger extant. Taking the best known spatial Clauser-Horne-Shimony-Holt (CHSH) inequality [3] for example, its maximum quantum violation is known as Tsirelson's bound [4] and strictly smaller than the violation for the Popescu-Rohrlich box [2,5]. This fact suggests potential physical principle bounding quantum correlation. Subject to the interpretation of Tsirelson's bound, the first line of research is from an information theoretic perspective. It is found that quantum correlations respect the principles of non-trivial communication complexity [6–8], information causality [9], and local orthogonality [10,11], showing restricted information theoretic power in communication and computation. From a different perspective and regarding that any bipartite systems must compose of individual sub-system, it is found that the local properties of sub-systems, such as local quantum mechanics [12] and uncertainty principle [13], can also account for the correlation strength between them, i.e., Tsirelson's bound.

Aside from the Tsirelson's bound, we still do not know much about other quantum characteristic properties, such as the quantum correlation strength for the temporal Bell scenario and the geometrical structure of quantum state space. The temporal Bell's inequalities consider correlation arising from sequential measurements on one single party at different times. The simplest temporal Bell's inequality is Leggett-Garg (LG) inequality [14], for which the maximum quantum violation for two-level system is known as Lüders' bound. This bound is smaller than its maximum possible algebraic violation. As a key geometrical structure of quantum state space and a main ingredient in the Born rule [15], the strong self-duality roughly translates into the fact that the two elementary concepts of a projective measurement and a pure state are represented as rank-one vectors in Hilbert's space. In the wider context of generalized probability theories which includes the classical and quantum theory as subsets, this structure (and the like) offers interesting paradigms that have implications on the operational tasks of steering [16], teleportation [16,17], nonlocality [15], and reversible interchangeability between pure states [18].

In this paper, we shall provide a unified understanding on the strong self-duality, Tsirelson's bound, and Lüders' bound based on a proposed uncertainty-disturbance relation. While the intrinsic *uncertainty* and *invasiveness* in measurements are necessary for any violation of Bell's inequalities in GPTs [13,19,20], here, we find an intimate connection between them holding for a quantum measurement, where the uncertainty in the measurement upper bounds the disturbance it can lead in a following measurement. Applying this relation to the most widely-studied two-level system, we find that it immediately leads to the strong self-duality, Tsirelson's bound, and Lüders' bound. Thus, we show that the uncertainty and invasiveness in a measurement, the geometrical structure of state space, and correlation strength in Bell's scenario are fundamentally related to each other, which may act as seeds for exploring two-level systems to a full understanding of other generic platforms.

GPT provides a general framework for describing operational features of arbitrary physical theories. A GPT must specify the physical system and its state, transformations and the type of measurements. Once these specifications are set, one should be able to describe the physical theories from an operational perspective. Generally, a physical state $\mathcal{S}$ is a complete description of a system, a measurement A is a set of events $\{A^a\}$ being specified by the outcomes a. It is operationally legitimate to suppose that the state space Ω and measurement space $\mathcal{M}$ are convex, in the sense that the preparation of a state $\mathcal{S}_1$ with probability p and a different state $\mathcal{S}_2$ with probability $1-p$ gives rise to a valid mixed state $p \cdot \mathcal{S}_1 + (1-p) \cdot \mathcal{S}_2$. The same goes for measurements. The probability assigned on an event A^a happen on state $\mathcal{S}$ is denoted by $p(A^a|\mathcal{S})$, and it should also respect probabilistic mixtures of states and measurements. For example, $p(A^a|p \cdot \mathcal{S}_1 + (1-p) \cdot \mathcal{S}_2) = p \cdot p(A^a|\mathcal{S}_1) + (1-p) \cdot p(A^a|\mathcal{S}_2)$. A state that cannot be decomposed as a convex combination of different states is called a pure state. An undecomposable event is called an extremal event. Essentially, the pure states and extremal events are building blocks of the space of Ω and $\mathcal{M}$, the latter forming fundamental concepts in GPTs. An undecomposable event assumes that an ath event in measurement A would ensure a definite value in the post-measurement state $\mathcal{S}_A^a$, called an "eigenstate" and uniquely specified by A, a. A measurement with all the events being extremal is called an extremal measurement. Such generic assumptions clearly holds in quantum theory, where both of pure state and extremal measurement are represented as rank-one projectors and a post-measurement state is an eigenstate uniquely specified by setting and outcome. In the rest of the paper, we consider principally the basic measurement dealing with these undecomposable events.

All the classical observables have well-defined values, and one is able to observe the values with negligible disturbance to the state, corresponding to properties known as *realism* and *noninvasiveness*. The quantum theory employs a drastically different conceptual framework that is incompatible with the *realism* and *noninvasiveness*. This radical departure from realism and non-

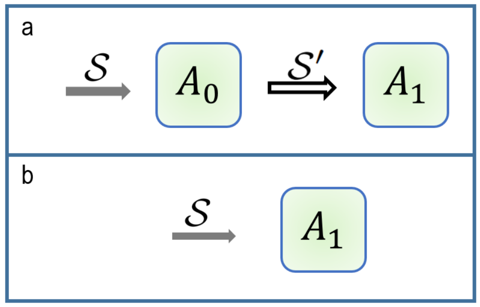

Figure 1: Sequential measurements scheme and direct measurement: (a) a first extremal measurement A_0 is performed on a physical state $\mathcal{S}$, then the post-measurement state is submitted to a following measurement A_1. (b) Had not be disturbed, a direct measurement A_1 is performed on state $\mathcal{S}$. By the distributions from scheme (a) we can characterize the uncertainty of measurement A_0, by the distance between statistics from measurement A_1 in two scenarios we define the disturbance leaded by A_0 to A_1.

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invasiveness provides the structure for uncertainty principle and gives birth to quantum theory [21]. When dealing with incompatible measurements, the uncertainty principle has two statements as: (i) Preparation uncertainty, stating that the values of incompatible observables are indeterminate in a quantum state, the intrinsic uncertainties of the observables exhibit a trade-off, (ii) Measurement uncertainty relates to the fact that a measurement on a quantum state would alter the state, resulting in a disturbance to a following incompatible measurement [22,23]. In the following, we shall show a dedicate relation between them that would be formulated in terms of operational probabilities. This enables us to impose the relation as physical principle in a GPT and resulting theories include quantum theory as a subset. We consider a simple measurement scheme as shown in Fig.1, where two sequential extremal measurements $A_0 \to A_1$ are performed on a system whose initial state is $\mathcal{S}$. In Fig.1.a, after the measurement A_0 on sufficient state copies, we obtain a distribution $p(A_0^a|\mathcal{S})$ and an average state $\mathcal{S}' = p(A_0^a|\mathcal{S}) \cdot \mathcal{S}_{A_0}^a$, where $\mathcal{S}_{A_0}^a$ denote the ath "eigenstate" of A. The average state then is submitted to a following measurement A_1, yielding a disturbed distribution as $p(A_1^{a'}|\mathcal{S}') = \sum_a p(A_0^a|\mathcal{S}) \cdot p(A_1^{a'}|\mathcal{S}_{A_0}^a)$, where $p(A_1^{a'}|\mathcal{S}_{A_0}^a)$ relates to transfer probability of obtaining a' when measuring A_1 on the "eigenstate" $\mathcal{S}_{A_0}^a$. By $p(A_1^{a'}|\mathcal{S})$ we denote the probability of directly measuring A_1 on the original state.

A classical measurement is noninvasive and may exhibit some uncertainty (due to ignorance of state preparation), in some post-quantum theory a measurement may exhibit no uncertainty but leads to a non-trivial disturbance [24,25]. We raise the question regarding the nature of quantum case. Is there some prerequisites for a given magnitude of $D_{A\to A'}$, which can be quantified in terms of distance between $p(A_1^{a'}|\mathcal{S})$ and $p(A_1^{a'}|\mathcal{S}')$: $\sum_{a'} |p(A_1^{a'}|\mathcal{S}) - p(A_1^{a'}|\mathcal{S}')|$. In quantum theory, the disturbance should relate to the δ_{A_0} and $p(A_1^{a'}|\mathcal{S}_{A_0}^a)$, here the uncertainty is defined via $\delta_{A_0}^2 = 1 - \sum_a p(A_0^0|\mathcal{S})^2$ that is convex and assumes maximum for a uniform distribution. On one hand, if the input state is "eigenstate" of A_0, then δ_{A_0} is zero, the input state is left unchanged and no disturbance is incurred. On the other hand, $D_{A_0\to A_1}$ should relate to $p(A_1^{a'}|\mathcal{S}_{A_0}^a)$, if each of them is either zero or one, then A_0 is equivalent to A_1 up to a relabel of outcomes, then no disturbance would be leaded. To quantify this aspect, we introduce a sum uncertainty

$$\delta_{A_0|A_1} = \sum_{a'} \delta_{A_0|\mathcal{S}_{A_1}^{a'}} := \sum_{a'} \sqrt{1 - \sum_a p^2(A_0^a|\mathcal{S}_{A_1}^{a'})},$$

which is zero if and only if $p(A_1^{a'}|\mathcal{S}_{A_0}^a) \in \{0, 1\}$. We find that

Theorem 1 *On a quantum system with finite levels, if two extremal measurements are* *performed in order $A_0 \to A_1$ then it holds the following uncertainty disturbance relation*

$$\delta_{A_0}\delta_{A_0|A_1} \geq D_{A_0\to A_1}. \tag{1}$$

The proof is presented in the supplemental materials (SM). From relation Eq.(1), a nonzero disturbance (right hand side) caused by measurement A_0 requires a nonzero uncertainty in measuring A_0 and a non-zero sum uncertainty. Thus we shall refer this figure as the principle of *no disturbance without uncertainty* (NDWU) and Eq.(1) as NDWU relation. Applied to the basic two-level systems, it turns out that Eq.(1) provides us a unified understanding on local structure of state space, temporal correlation, and spatial correlation in quantum theory.

The first essential difference between the theories is how they describe a single system. Classical states allow a realistic description, while the quantum states are represented by positive defined operators in Hilbert's space. The geometrical structure of state space lies at very heart of physical properties. Taking quantum mechanics for example, the local state space of subsystems completely determines the correlation in a bipartite quantum systems [12]. In GPTs, the geometrical structures of state space of strong and weak self-duality are intimately related to steering, teleportation, and nonlocality.

In the consideration of self-dualities, it is important to include unnormalized states, that is, defining a space Ω_+ whose elements $\omega = \lambda\mathcal{S}$ for $\mathcal{S} \in \Omega$ and $\lambda \geq 0$ [15,18]. The strong self-duality defined as

Definition 1 (strong self-duality) *A system is strongly self-dual iff there exists an isomor-*

phism $\mathcal{T}$: $\mathcal{M} \to \Omega_+$ which is symmetric and positive, i.e., $p[e|\mathcal{T}(f)] = p[f|\mathcal{T}(e)] \geq 0$ for all $e, f \in \mathcal{M}$.

Strong self-duality is an elementary structure for a quantum system, where states and measurements are equivalent to each other up to a normalization, and $\mathcal{T}$ can be defined as identity map, namely, $\mathcal{T}(e) = e$. Then the strong self-duality follows from the Born rule: $p[e|\mathcal{T}(f)] = \mathrm{tr}(e \cdot f) = p[f|\mathcal{T}(e)]$, $\forall e, f \in \mathcal{M}$ and Ω_+. Noting that A_i^a and $\mathcal{S}_{A_i}^a$ should be taken as different objects in GPTs, we can generalize the identity map as $\mathcal{T}$: $\sum p_a^i A_i^a \to \sum p_a^i \mathcal{S}_{A_i}^a$, where the sum is taken over setting i and value of observable a and the extremal measurements A_i^a in the decomposition is mapped to the prepared eigenstate $\mathcal{S}_{A_i}^a$. A sufficient and necessary condition for the strong self-duality is a symmetry condition: $p(A_i^{a'}|\mathcal{S}_{A_j}^a) = p(A_j^a|\mathcal{S}_{A_i}^{a'})$, for all extremal event A_i, A_j and outcome a, a'. The condition is necessary: $p[A_i^{a'}|\mathcal{T}(A_j^a)] = p(A_i^{a'}|\mathcal{S}_{A_j}^a) = p[A_j^a|\mathcal{T}(A_i^{a'})] = p(A_j^a|\mathcal{S}_{A_i}^{a'})$, the condition is sufficient, for two arbitrary measurement $\sum p_a^i \cdot A_i^a$ and $\sum q_{a'}^i \cdot A_i^{a'}$ we have $p[\sum p_a^j A_j^a|\mathcal{T}(q_{a'}^i A_i^{a'})] = \sum p_a^j q_{a'}^i p[A_j^a|\mathcal{T}(A_i^{a'})] = \sum p_a^j q_{a'}^i p[A_i^{a'}|\mathcal{T}(A_j^a)] = p[\sum q_{a'}^i A_i^{a'}|\mathcal{T}(\sum p_a^j A_j^a)]$. In the following, we show that Eq.(1) can lead to the symmetry condition for two-level measurements, where $a, a' \in 0, 1$.

Theorem 2 *For any two-level extremal measurement, Eq.(1) leads to a local structure of*

$$p(A_1^{a'}|\mathcal{S}_{A_0}^a) = p(A_0^a|\mathcal{S}_{A_1}^{a'}). \tag{2}$$

We note that, by proving the Theorems 2 the number of independent transfer probabilities can be reduced from four to one (see SM). For the sake of simplicity, we remove all the non-independent transfer probabilities and adhere to the notations: $c = p(A_1^0|\mathcal{S}_{A_0}^0) - p(A_1^1|\mathcal{S}_{A_0}^0) = 2p(A_1^0|\mathcal{S}_{A_0}^0) - 1$

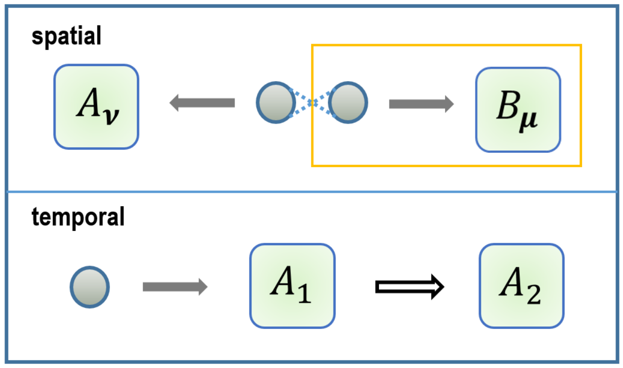

Figure 2: Spatial correlation and temporal correlation: In the spatial scenario, the separation of measurements is in space where observers can performs local measurements on the shared particles. This scenario can be also viewed as: an event happening, say, on Bob's side give rise to a conditional state is prepared on Alice side, then she performs measurement on the conditional state. For the temporal scenario, the separation is in the time domain - a first measurement is performed on original state and a second one is performed on the output state from the previous measurement.

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and $\langle A_a \rangle = p(A_a^0|\mathcal{S}) - p(A_a^1|\mathcal{S})$, Eq.(1) becomes

$$\frac{(\langle A_0 \rangle + \langle A_1 \rangle)^2}{2(1+c)} + \frac{(\langle A_0 \rangle - \langle A_1 \rangle)^2}{2(1-c)} \leq 1. \tag{3}$$

We note this relation is indeed an uncertainty relation, where c characterizes the incompatibility between measurements and the distributions in terms of expectation are constrained.

While uncertainty and disturbance are needed for any violation of Bell's inequalities, their relations captured by Eq.(1) or Eq.(3) place some constraint on their violations. We consider the simplest temporal and spatial Bell's inequalities, *i.e.*, the famous CHSH inequality and LG inequality [26] that deal with two-level measurements. The maximum quantum violation for them are known as Tsirelson's bound ($2\sqrt{2}$) and Lüders' bound ($\frac{3}{2}$) which are obtained when employing the kinds of measurement considered in Eq.(1). In contrast, the maximum violation for GPTs are 4 [5] and 3 respectively. We now show that the local relation Eq.(3) regarding a subsystem can explain not only Tsirelson's bound but also Lüders' bound.

The CHSH inequality considers a spatial Bell's scenario as shown in Fig.2, where a bipartite system is distributed to space-like separated observablers, Alice and Bob can randomly perform one of two dichotomic measurements $\{A_\nu\}$ and $\{B_\mu\}$ $(\nu, \mu = 0, 1)$ with outcomes $a, b = 0, 1$, on the particles, the joint probability is denoted by $\{p(ab|\nu\mu)\}$ where ν and μ are the settings. CHSH inequality holds for any local hidden value theory and reads,

$$\sum_{a,b,\mu,\nu} (-1)^{a+b+\mu\nu} p(ab|\nu\mu) \leq 2.$$

Since the measurements are space-like separated, without loss of generality and placed in some reference frame, one may assume that Bob first performs a measurement B_μ, obtains b with probability $p(b|\mu)$, then simultaneously a conditional state is prepared on Alice's side described by $\omega_{b|\mu}$. Alice then measures A_ν and obtains a in probability $p(\nu_a|\omega_{b|\mu})$ which is written as $p(\nu_a|\mu_b)$ for short. Thus a joint distribution $p(ab|\nu\mu)$ is decomposed into $p(b|\mu) \cdot p(\nu_a|\mu_b)$, where $p(b|\mu) = \sum_a p(ab|\nu\mu)$, $p(\nu_a|\mu_b) = \frac{p(ab|\nu\mu)}{p(b|\mu)}$. Following this analysis, we rewrite lhs of CHSH inequality as $\sum_{b,\mu,\nu} (-1)^{b+\mu\nu} p(b|\mu) \cdot (\langle A_0 \rangle_{b|\mu} + (-1)^b \langle A_1 \rangle_{b|\mu})$, where $\langle A_i \rangle_{b|\mu}$ is the expectation value of A_i in conditional state $\omega_{b|\mu}$. We note that the Eq.(3) bounds the expectations in the lhs of Eq.(4) as

$\langle A_0\rangle \pm \langle A_1\rangle \le \sqrt{2\pm 2c}$. Clearly, Tsirelson's bound follows since

$$\begin{aligned}
&\sum_{b,\mu,\nu}(-1)^{b+\mu\nu}p(b|\nu)\cdot(\langle A_0\rangle_{b|\nu}+(-1)^{\nu}\langle A_1\rangle_{b|\nu})\\
&\quad\le \quad \sum_{b,\mu,\nu}p(b|\nu)\cdot|(\langle A_0\rangle_{b|\nu}+(-1)^{\nu}\langle A_1\rangle_{b|\nu})|\\
&\quad\le \quad \sum_{b,\nu}p(b|\nu)\sqrt{2+(-1)^{\nu}2c}\\
&\le \sqrt{2+2c}+\sqrt{2-2c}\le 2\sqrt{2}
\end{aligned}$$

In the last inequality, the lhs is maximized over all c, and Tsirelson's bound is obtained when $c=0$, namely, $p(A_1^{a'}|\mathcal{S}_{A_0}^{a})=\frac{1}{2}$. Note that this is the case for the maximum violation in quantum theory where the two measurements on either side are multi-unbiased and maximum incompatible.

We now shift our attention to temporal correlations. We consider measurements on a single party at different times as shown in Fig.2. In analogy to the spatial correlation, temporal correlation also reveals rich nonclassical correlation structure. However, little is yet known about why its violation is limited. We now show that Eq.(3) can recover the Lüders' bound for the LG inequality that reads:

$$\langle A_1A_2\rangle+\langle A_2A_3\rangle-\langle A_1A_3\rangle\le 1,$$

where $\langle A_1A_2\rangle := \sum_{aa'}(-1)^{a+a'}p(A_1^a|\mathcal{S})p(A_2^{a'}|\omega_{a|A_1})$, and $\omega_{a|A_1}$ denote the post-measurement s-tate conditioning on event A_1^a. By the considering extremal measurements, $\omega_{a|A_1}=\mathcal{S}_{A_1}^a$. We can expand the correlation the $\langle A_1A_2\rangle$ and remove the independent transfer probabilities

$$\begin{aligned}
\langle A_1A_2\rangle &= \sum_{a=a'}p(A_1^a|\mathcal{S})p(A_2^{a'}|\mathcal{S}_{A_1}^a)-\sum_{a\neq a'}p(A_1^a|\mathcal{S})p(A_2^{a'}|\mathcal{S}_{A_1}^a)\\
&= 2p(A_2^0|\mathcal{S}_{A_1}^0)-1=c_{12}
\end{aligned}$$

Eq.(4) depends only on the independent transfer probabilities between A_1, A_2, and A_3 but not on the input state. Thus one can take the input state arbitrarily. Here we take it as an eigenstate as $\mathcal{S}^0_{A_1}$, so that the uncertainty of measurement A_1 is zero and introducing no disturbance to any following measurement at all, and the Eq.(4) can be casted into $\langle A_2\rangle + \langle A_2A_3\rangle - \langle A_3\rangle \leq 1$, which formula sometimes is taken as an equivalent version of LG inequality[27]. Denoting $\langle A_2A_3\rangle = c_{23}$, then we have

$$\langle A_2\rangle + c_{23} - \langle A_3\rangle \leq \sqrt{2-2c_{23}} + c_{23} \leq \frac{3}{2},$$

where we use the constraint $\langle A_0\rangle \pm \langle A_1\rangle \leq \sqrt{2\pm 2c}$ again. Then we have explained the CHSH and the LG inequalities, where our NDWU relation manifests its power.

Historically, the study on uncertainty principle and the study on quantum nonclassical correlations often reinforces each other. The interpretation of uncertainty principle calls for an indeterministic description of quantum theory. Bell's theorem put this indeterministic assumption on a solid ground, while the following study triggers a question as why quantum correlation is nonlocal but not sufficient nonlocal to be maximum. It is also found that post-quantum and nonlocal theories must incorporate the properties of *indeterminism* and *invasiveness*, which are the key concepts respectively relating to preparation uncertainty and measurement uncertainty. Here, we provide new evidences for the effect of this reinforcement. We find there is an intricate connection between uncertainty and disturbance for quantum theory that reveals an previous unnoticed facet of uncertainty principle. Our connection is rephased in an operational manner, when applied to the basic two-level systems, it provides a universal understanding on the key structure of state space, and on why quantum correlation are limited in both the spatial and the temporal Bell's scenario for the basic two-level systems, bridging the global properties to the local properties. As the connection lies at the very heart of quantum characteristic properties, we claim it as a principle.

Correspondence Correspondence should be addressed to Sixia Yu (email:yusixia@ustc.edu.cn) and Liang-Liang Sun (email:sun15@mail.ustc.edu.cn).

References

1. Bell, J. S. On the einstein podolsky rosen paradox. *Physics Physique Fizika* **1**, 195–200 (1964).

2. Brunner, N., Cavalcanti, D., Pironio, S., Scarani, V. & Wehner, S. Bell nonlocality. *Rev. Mod. Phys.* **86**, 419–478 (2014).

3. Clauser, J. F., Horne, M. A., Shimony, A. & Holt, R. A. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.* **23**, 880–884 (1969).

4. Cirel'Son, B. S. Quantum generalizations of bell's inequality. *Letters in Mathematical Physics* **4**, 93–100 (1980).

5. Popescu, S. & Rohrlich, D. Quantum nonlocality as an axiom. *Foundations of Physics* **24**, 379–385 (1994).

6. Dam, W. V. Implausible consequences of superstrong nonlocality. *Natural Computing* **12**, 9–12 (2013).

7. Brassard, G. *et al.* Limit on nonlocality in any world in which communication complexity is not trivial. *Phys. Rev. Lett.* **96**, 250401 (2006).

8. Linden, N., Popescu, S., Short, A. J. & Winter, A. Quantum nonlocality and beyond: Limits from nonlocal computation. *Phys. Rev. Lett.* **99**, 180502 (2007).

9. Pawłowski, M. *et al.* Information causality as a physical principle. *Nature* **461**, 1101 (2009).

10. Cabello, A. Simple explanation of the quantum violation of a fundamental inequality. *Phys. Rev. Lett.* **110**, 060402 (2013).

11. Yan, B. Quantum correlations are tightly bound by the exclusivity principle. *Phys. Rev. Lett.* **110**, 260406 (2013).

12. Barnum, H., Beigi, S., Boixo, S., Elliott, M. B. & Wehner, S. Local quantum measurement and no-signaling imply quantum correlations. *Phys. Rev. Lett.* **104**, 140401 (2010).

13. Oppenheim, Jonathan, Wehner & Stephanie. The uncertainty principle determines the nonlocality of quantum mechanics. *Science* (2010).

14. Leggett, A. J. & Garg, A. Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks? *Physical Review Letters* **54**, 857 (1985).

15. Janotta, P., Gogolin, C., Barrett, J. & Brunner, N. Limits on nonlocal correlations from the structure of the local state space. *New Journal of Physics* **13**, 293–301 (2011).

16. Barnum, H., Barrett, J., Leifer, M. & Wilce, A. Teleportation in general probabilistic theories (2008). arXiv:quant-ph/0805.3553.

17. Barnum, H., Gaebler, C. P. & Wilce, A. Ensemble steering, weak self-duality, and the structure of probabilistic theories. *Foundations of Physics* **43** (2013).

18. Müller, M. P. & Ududec, C. Structure of reversible computation determines the self-duality of quantum theory. *Phys. Rev. Lett.* **108**, 130401 (2012).

19. Masanes, L., Acín, A. & Gisin, N. General properties of nonsignaling theories. *Phys. Rev. A* **73**, 012112 (2006).

20. Łodyga, J. *et al.* Measurement uncertainty from no-signaling and nonlocality. *Phys. Rev. A* **96**, 012124 (2017).

21. Bohr & N. *The Quantum Postulate and the Recent Development of Atomic Theory*, vol. 121.

22. Pauli, W. *Die allgemeinen Prinzipien der Wellenmechanik*, 83–272 (Springer Berlin Heidelberg, 1933).

23. Busch, P., Lahti, P. & Werner, R. F. Colloquium: Quantum root-mean-square error and measurement uncertainty relations. *Rev. Mod. Phys.* **86**, 1261–1281 (2014).

24. Carmi, A. & Cohen, E. Relativistic independence bounds nonlocality. *Science Advances* **5** (2019). arXiv:https://advances.sciencemag.org/content/5/4/eaav8370.full.pdf.

25. Sun, L.-L., Yu, S. & Chen, Z.-B. Uncertainty-complementarity balance as a general constraint on non-locality (2018). arXiv:quant-ph/1808.06416.

26. Leggett, A. J. & Garg, A. Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks? *Phys. Rev. Lett.* **54**, 857–860 (1985).

27. Goggin, M. E. *et al.* Violation of the leggett–garg inequality with weak measurements of photons. *Proceedings of the National Academy of Sciences* **108**, 1256–1261 (2011). arXiv:https://www.pnas.org/content/108/4/1256.full.pdf.

Methods

1 Proof of Theorem 1.

Suppose that a finite-level quantum system is prepared in state ρ and two projection measurements correspond to two orthonormal bases $\{|A_0^a\rangle\langle A_0^a|\}$ and $\{|A_1^{a'}\rangle\langle A_1^{a'}|\}$. By denoting $\langle A_0^a|A_1^{a'}\rangle = \Lambda_{a'a}$ we have expansion $|A_1^{a'}\rangle = \sum_a \Lambda_{a'a}|A_0^a\rangle$. As a result

$$p(A_1^{a'}|\mathcal{S}) = \langle A_1^{a'}|\rho|A_1^{a'}\rangle = \sum_{s,t}\Lambda^*_{a's}\Lambda_{a't}\langle A_0^s|\rho|A_0^t\rangle = \sum_{s\neq t}\Lambda^*_{a's}\Lambda_{a't}\langle A_0^s|\rho|A_0^t\rangle + p(a'|\mathcal{S}')$$

where $p(a'|\mathcal{S}') = \sum_a p(A_0^a|\mathcal{S})\cdot p(A_1^{a'}|\mathcal{S}^a_{A_0})$ with $p(A_0^a|\mathcal{S}) = \langle A_0^a|\rho|A_0^a\rangle$ and transfer probabilities are symmetric

$$p(A_1^{a'}|\mathcal{S}^a_{A_0}) = p(A_0^a|\mathcal{S}^{a'}_{A_1}).$$

Finally we calculate

$$\begin{aligned}
D_{A_0\to A_1} &= \sum_{a'}\left|p(A_1^{a'}|\mathcal{S}) - p(a'|\mathcal{S}')\right| \\
&= \sum_{a'}\Big|\sum_{s\neq t}\Lambda^*_{a's}\Lambda_{a't}\langle A_0^s|\rho|A_0^t\rangle\Big| \\
&\leq \textstyle\sum_{a'}\sqrt{\sum_{s\neq t}|\Lambda^*_{a's}\Lambda_{a't}|^2}\sqrt{\sum_{s\neq t}|\langle A_0^s|\rho|A_0^t\rangle|^2} \\
&\leq \textstyle\sum_{a'}\sqrt{\sum_{s\neq t}p(A_0^t|\mathcal{S}^{a'}_{A_1})p(A_0^s|\mathcal{S}^{a'}_{A_1})}\sqrt{\sum_{s\neq t}p(A_0^s|\mathcal{S})p(A_0^t|\mathcal{S})} \\
&= \textstyle\sum_{a'}\sqrt{1-\sum_s p(A_0^s|\mathcal{S}^{a'}_{A_1})^2}\sqrt{1-\sum_t p(A_0^t|\mathcal{S})^2} \\
&= \delta_{A_0|A_1}\delta_{A_0}.
\end{aligned}$$

Here the first inequality is due to Cauchy inequality, the second inequality is due to the fact that the state is positive semidefinite so that in the basis $\{|A_0^a\rangle\}$ it holds $|\langle A_0^s|\rho|A_0^t\rangle|^2 \leq \langle A_0^s|\rho|A_0^s\rangle\langle A_0^t|\rho|A_0^t\rangle$.

2 Proof of Theorem 2.

The transfer probabilities are symmetric, i.e., $p(A_0^a|\mathcal{S}_{A_1}^{a'}) = p(A_1^{a'}|\mathcal{S}_{A_0}^a)$, for two-outcome projection measurements if the NDWU relation holds for all pairs of projection measurements — Considering normalization conditions, there are four independent transfer probabilities (which is independent of the initial state)

$$p(A_1^0|\mathcal{S}_{A_0}^0) = \frac{1+c_1}{2}, \quad p(A_1^0|\mathcal{S}_{A_0}^1) = \frac{1+c_2}{2}, \quad p(A_0^0|\mathcal{S}_{A_1}^0) = \frac{1+c_1'}{2}, \quad p(A_0^0|\mathcal{S}_{A_1}^1) = \frac{1+c_2'}{2}.$$

It is symmetric if and only if $c_1 = -c_2 = c_1' = -c_2'$. By denoting the expectation values $\langle A_0\rangle_{\mathcal{S}} = p(A_0^0|\mathcal{S}) - p_{\mathcal{S}}(A_0^1|\mathcal{S})$ and $\langle A_1\rangle_{\mathcal{S}} = p(A_1^0|\mathcal{S}) - p(A_1^1|\mathcal{S})$ for two projection measurements in an arbitrary state $\mathcal{S}$, the NDWU relation Eq.(1) becomes

$$\frac{1}{2}\left(\sqrt{1-c_1'^2}+\sqrt{1-c_2'^2}\right)\sqrt{1-\langle A_0\rangle_{\mathcal{S}}^2} \geq \left|\langle A_1\rangle_{\mathcal{S}} - \frac{c_1+c_2}{2} - \frac{c_1-c_2}{2}\langle A_0\rangle_{\mathcal{S}}\right|$$

for the sequential measurement scenario $A_0 \to A_1$ while

$$\frac{1}{2}\left(\sqrt{1-c_1^2}+\sqrt{1-c_2^2}\right)\sqrt{1-\langle A_1\rangle_{\mathcal{S}}^2} \geq \left|\langle A_0\rangle_{\mathcal{S}} - \frac{c_1'+c_2'}{2} - \frac{c_1'-c_2'}{2}\langle A_1\rangle_{\mathcal{S}}\right|$$

for the sequential measurement scenario $A_1 \to A_0$. Let us take at first the state $\mathcal{S}_{A_1}^0$, i.e., the state on which measurement A_1 gives a definite value 0. In this case $\langle A_0\rangle_{\mathcal{S}_{A_1}^0} = c_1'$ and $\langle A_1\rangle_{\mathcal{S}_{A_1}^0} = 1$ and noting that transfer probabilities are state independent, we obtain

$$\begin{aligned} 1 - \frac{c_1+c_2}{2} - \frac{c_1-c_2}{2}c_1' &\leq \frac{1}{2}\left(\sqrt{1-c_1'^2}+\sqrt{1-c_2'^2}\right)\sqrt{1-c_1'^2} \\ &\leq \sqrt{1-\left(\frac{|c_1'|+|c_2'|}{2}\right)^2}\sqrt{1-c_1'^2} \\ &\leq 1 - \frac{|c_1'|+|c_2'|}{2}|c_1'| \leq 1 - \frac{c_1'-c_2'}{2}c_1' \end{aligned}$$

where the first inequality is due to NDWU relation, the second inequality is due to the concavity of function $\sqrt{1-x^2}$, and the third inequality is due to inequality $\sqrt{1-x^2}\sqrt{1-y^2} \leq 1-|xy|$. By taking the state to be $\mathcal{S} = \mathcal{S}^1_{A_1}$, i.e., the state on which measurement A_1 gives a definite value 1, we have $\langle A_0\rangle_{\mathcal{S}^1_{A_1}} = c'_2$ and $\langle A_1\rangle_{\mathcal{S}^1_{A_1}} = -1$ so that

$$\begin{aligned}
1+\frac{c_1+c_2}{2}+\frac{c_1-c_2}{2}c'_2 &\leq \frac{1}{2}\left(\sqrt{1-c'^2_1}+\sqrt{1-c'^2_2}\right)\sqrt{1-c'^2_2} \\
&\leq \sqrt{1-\left(\frac{|c'_1|+|c'_2|}{2}\right)^2}\sqrt{1-c'^2_2} \\
&\leq 1-\frac{|c'_1|+|c'_2|}{2}|c'_2| \leq 1+\frac{c'_1-c'_2}{2}c'_2.
\end{aligned}$$

As a result we obtain

$$(c_1-c_2)(c'_1-c'_2) \geq (|c'_1|+|c'_2|)^2 \geq |c'_1-c'_2|^2 \geq 0.$$

Similarly from the $A_1 \to A_0$ scenario we obtain

$$(c_1-c_2)(c'_1-c'_2) \geq (|c_1|+|c_2|)^2 \geq |c_1-c_2|^2.$$

If $c'_1 = c'_2$ or $c_1 = c_2$ we have $c'_1 = c'_1 = c_1 = c_2 = 0$ then we already have symmetric transfer probabilities. If $c_1 \neq c_2$ and $c'_1 \neq c'_2$ we obtain both $c_1-c_2 \geq c'_1-c'_2$ and $c'_1-c'_2 \geq c_1-c_2$, by noting they have the same sign, so that $c_1-c_2 = c'_1-c'_2$. As a result we have both $c_1+c_2 \geq 0$ and $c_1+c_2 \leq 0$ giving the desired symmetry property of transfer probabilities $c_1 = -c_2 = c'_1 = -c'_2 = c$. The NDWU relation becomes

$$\sqrt{1-c^2}\sqrt{1-\langle A_0\rangle^2} \geq |\langle A_1\rangle - c\langle A_0\rangle|.$$

This inequality can be reformulated into

$$\frac{(\langle A_0\rangle+\langle A_1\rangle)^2}{2(1+c)}+\frac{(\langle A_0\rangle-\langle A_1\rangle)^2}{2(1-c)} \leq 1$$

which is exactly NDWU relation Eq.(3) for two-outcome projection measurements.

Figures

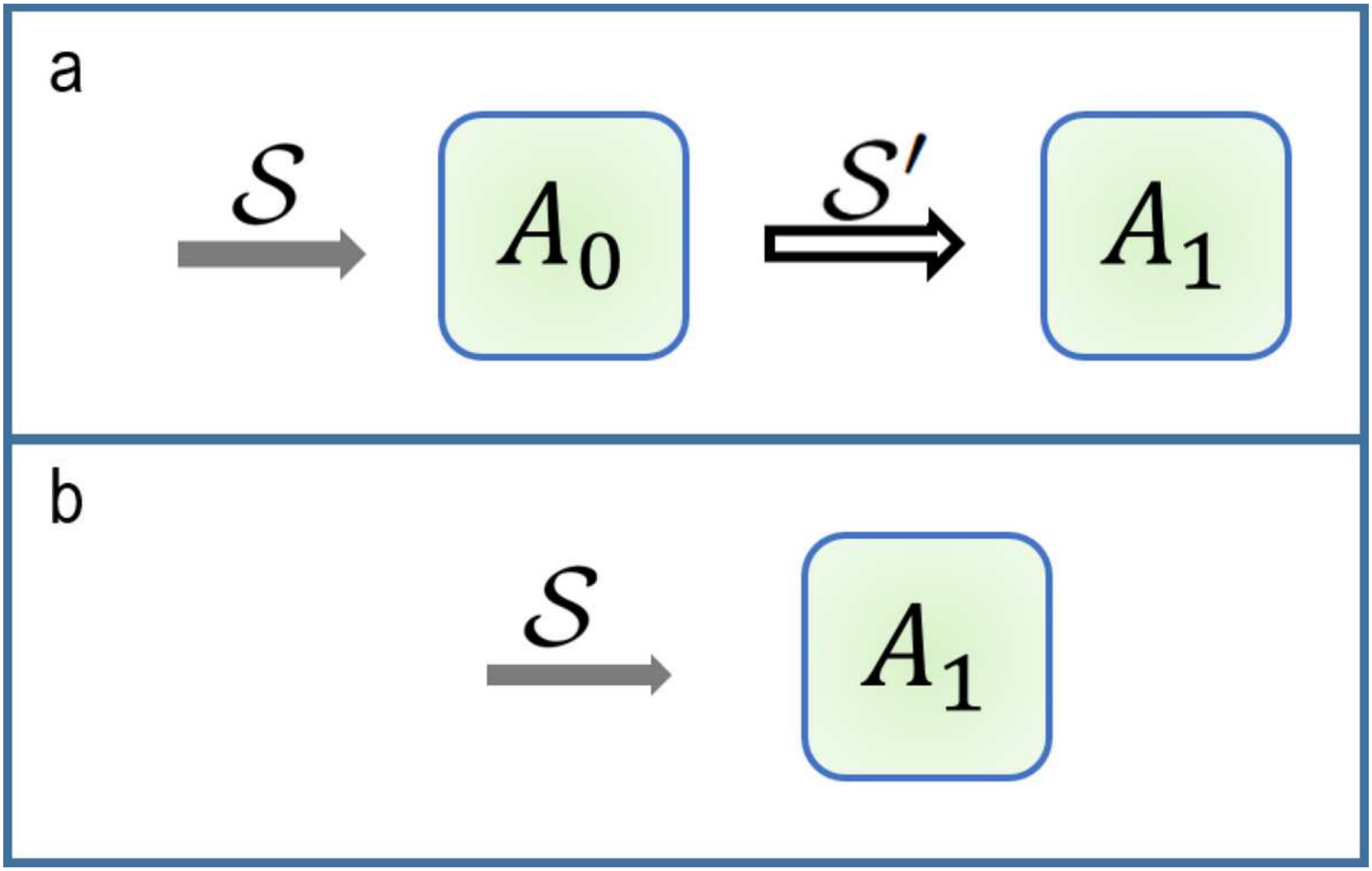

Figure 1

Sequential measurements scheme and direct measurement: (a) a first extremal measurement A0 is performed on a physical state S, then the post-measurement state is submitted to a following measurement A1. (b) Had not be disturbed, a direct measurement A1 is performed on state S. By the distributions from scheme (a) we can characterize the uncertainty of measurement A0, by the distance between statistics from measurement A1 in two scenarios we define the disturbance leaded by A0 to A1.

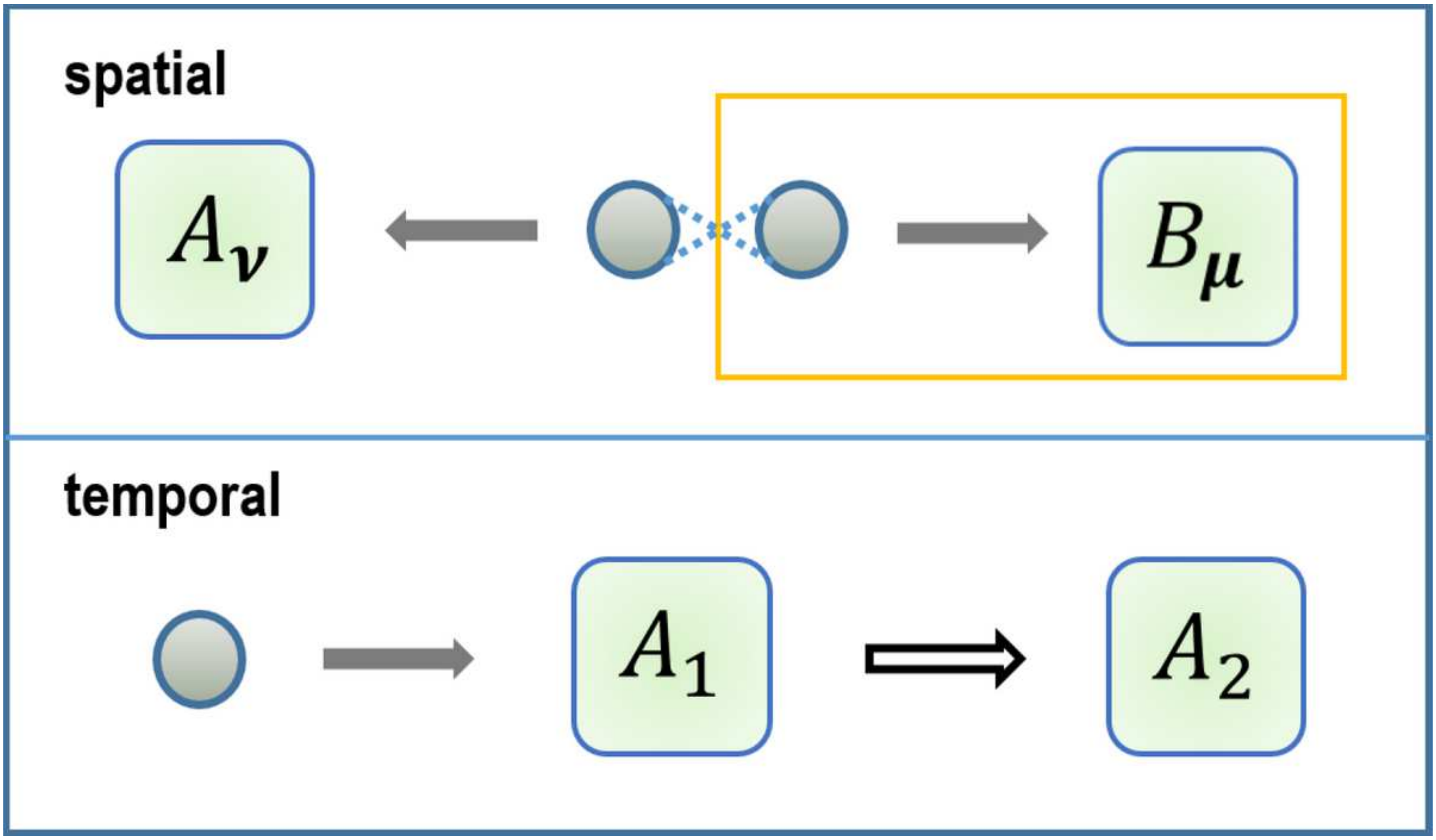

Figure 2

Spatial correlation and temporal correlation: In the spatial scenario, the separation of measurements is in space where observers can performs local measurements on the shared particles. This scenario can be also viewed as: an event happening, say, on Bob’s side give rise to a conditional state is prepared on Alice side, then she performs measurement on the conditional state. For the temporal scenario, the separation is in the time domain - a first measurement is performed on original state and a second one is performed on the output state from the previous measurement.