Numerical Method for Fractional Fuzzy Integral Equations

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Numerical method for fractional fuzzy integral equations

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Abstract In the present work we construct an iterative method for the numerical solution of fuzzy fractional Volterra integral equations, by using the technique of fuzzy product integration. The existence and uniqueness of the solution and the uniform boundedness of the terms of the Picard iterations are proved. The convergence of the iterative algorithm is obtained and the apriori error estimate is given in terms of the Lipschitz constants. A numerical example illustrates the accuracy of the method.

Keywords Riemann-Liouville fractional fuzzy integral equation; Iterative numerical method; Fuzzy product integration technique; Convergence analysis.

1 Introduction

Apart from the interest in Pure Mathematics, the study of fractional calculus is motivated by its applications in fractal theory, fluid flow, control theory, biology, electrochemistry and viscoelasticity (see [14], [22] and [27]). The existence and uniqueness of the solutions in the case of fractional integral equations is usually studied by using fixed point theorems (see [1], [8], [11] and [28]). The properties of the solutions were obtained in [25] and [42] and some numerical methods using quadrature schemes, product integration, variational iteration, and fuzzy transforms can be found in [11], [13], [34], [40] and [4], respectively. When such models involves uncertainty, the use of fuzzy sets is suitable, motivating the attention for fuzzy fractional differential and integral equations. The fuzzy fractional differential calculus was initiated in the work of Agarwal, Lakshmikantham and

Nieto [2], while Arshad and Lupulescu had obtained existence and uniqueness results for fuzzy fractional differential equations in [10]. Some existence results for fuzzy fractional integral equations were obtained in [3]. The concept of fractional Caputo H-differentiability based on generalized Hukuhara derivative (see [36]) was used for investigating fuzzy fractional differential (see [7]) and integro-differential (see [6]) equations. The existence of a unique solution for Volterra fuzzy fractional integral equations was obtained in [38] without Lipschitz conditions and using weakly contractive mappings, including a study of the continuous dependence by the fractional order and by the initial condition. The existence and uniqueness of the solution for fuzzy fractional Volterra-Fredholm integro-differential equations in the Caputo's sense was obtained in [5], including the approximation of the solution by the use of Adomian decomposition. Stability properties, including the Ulam-Hyers stability, were investigated for fuzzy fractional Volterra integral equations in [23], [37], and [39]. The numerical solutions for fuzzy fractional differential equations were obtained by using the variational iteration method in [26], while the solution of fractional order fuzzy initial value problem with the Caputo type fuzzy fractional derivative were studied with the modified fractional Euler method in [35]. An overview of the notions and results concerning the theory of fuzzy fractional integral and differential equations is realized in [31].

The fractional calculus for interval-valued functions involving the concepts of Riemann-Liouville fractional integral and fractional derivative was developed in [29]. After this, a deep study of nonlinear fuzzy fractional integral and differential equations, involving global existence and uniqueness and continuous dependence by the initial condition in the case of initial value problems

and integral equations with delay, was realized in [31]. The work [31] contains the study of solutions for fractional differential initial value problems with delay under Caputo's generalized Hukuhara derivative and the approximation of these solutions is made by using the modified fractional Euler method. The study from [31] is continued in [32] for interval-valued functions, where the modified fractional Euler and Adams-Bashforth-Moulton methods are presented. Since in general the initial value problem for fuzzy fractional differential equation and the corresponding fuzzy Volterra fractional integral equation are not equivalent, in [33] is given an appropriate condition for this equivalence to be valid.

Recently (see [9]), the fuzzy integral mean value theorem for fuzzy fractional Riemann-Liouville integral was obtained, and under the Caputo's gH-derivative the fuzzy generalized Taylor expansion and the fuzzy generalized Taylor formula were given with applications to the fuzzy fractional Euler method.

In this paper we construct an iterative numerical method for the following Riemann-Liouville fuzzy fractional integral equation for $t \in [0, T]$

$$x(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t H(t, s) (t-s)^{\alpha-1} \odot x(s) ds, \quad (1)$$

of fractional order $\alpha \in (0, 1)$, where $f : [0, T] \rightarrow \mathbb{R}_{\mathcal{F}}$ and $H : [0, T] \times [0, T] \rightarrow \mathbb{R}_+$ are continuous. Fractional order Volterra integral equations and their fuzzy variant in the case of input data with uncertainties (such as equation (1)) are relevant as models in various areas of science including biology, chemistry, physics, mechanics and kinetic theory of gases (see [24] and [30]).

For solving fuzzy fractional initial value problems and interval-valued fractional initial value problems, modified versions of the Euler's method and of the Adams-Bashforth-Moulton method were proposed (see [31], [32], [9]). Moreover, the variational iteration method was proposed for fuzzy fractional differential equations, but there are few results concerning the numerical solution of fuzzy fractional integral equations. In this paper, the iterative numerical method for solving the integral equation (1) is based on the fuzzy product integration method. The paper is organized as follows: in Section 2 we present some basic results about fuzzy-number-valued functions, while Section 3 is devoted to existence and uniqueness results including some properties of the Picard iterations such as uniform boundedness and uniform $(1, \alpha)$ -Lipschitz property. In Section 4 we present the iterative numerical method and the convergence of this method is investigated in Section 5. The convergence theoretical result and the accuracy of the

proposed method are tested on a numerical example in Section 6.

2 Preliminaries

Firstly, we point out some basic results concerning fuzzy calculus.

Definition 1 (see [21]). By a fuzzy number we understand a function $v : \mathbb{R} \rightarrow [0, 1]$ with the properties:

- (i) v is upper semi-continuous
- (ii) $v(t) = 0$ outside of a closed interval $[a, z]$,
- (iii) there exist the numbers x and y with $a \leq x \leq y \leq z$, such that v is non-decreasing on $[a, x]$, non-increasing on $[y, z]$, and $v(t) = 1$ for all $t \in [x, y]$.

We denote by $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers (see [15]).

The set $\mathbb{R}_{\mathcal{F}}$ is an extension of $\mathbb{R}$ since any real number $a \in \mathbb{R}$ can be considered as the fuzzy number $\tilde{a}$ by considering $\tilde{a} = \chi_{\{a\}}$. The r -level sets of $u \in \mathbb{R}_{\mathcal{F}}$ are $[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}$ and $[u]^r = [u_-^r, u_+^r]$ is a closed interval for each $r \in [0, 1]$ (see [15]). These r -level sets offer the opportunity to consider the LU-representation of a fuzzy number which has significant advantages in computation. Under this representation, for a fuzzy number described as $[u]^r = [u_-^r, u_+^r]$, $r \in [0, 1]$, are involved the functions, $u_-, u_+ : [0, 1] \rightarrow \mathbb{R}$, with u_- increasing and u_+ decreasing (see [21]).

The linear operations on $\mathbb{R}_{\mathcal{F}}$ are defined by considering the r -level sets. Taking $u, v \in \mathbb{R}_{\mathcal{F}}$, $k \in \mathbb{R}$, the addition $u + v$ and the scalar multiplication $k \odot u$ are defined by using the usual operations with intervals, having $[u + v]^r = [u]^r + [v]^r = [u_-^r + v_-^r, u_+^r + v_+^r]$, $\forall r \in [0, 1]$, and

$$[k \odot u]^r = \begin{cases} [ku_-^r, ku_+^r], & \text{if } k \geq 0 \\ [ku_+^r, ku_-^r], & \text{if } k < 0 \end{cases}, \quad r \in [0, 1].$$

It is easy to see that $(\mathbb{R}_{\mathcal{F}}, +, \tilde{0})$ is commutative monoid and for the scalar multiplication following properties hold:

- (i) $(x + y) \odot v = x \odot v + y \odot v$, for all $v \in \mathbb{R}_{\mathcal{F}}$ and $x, y \in \mathbb{R}$ with $x \cdot y \geq 0$,
- (ii) $x \odot (v + w) = x \odot v + x \odot w$, for all $v, w \in \mathbb{R}_{\mathcal{F}}$ and $x \in \mathbb{R}$,
- (iii) $x \odot (y \odot v) = (xy) \odot v$, for all $v \in \mathbb{R}_{\mathcal{F}}$ with $x, y \in \mathbb{R}$, but $(\mathbb{R}_{\mathcal{F}}, +, \odot)$ is not a linear space because $(x + y) \odot v \neq x \odot v + y \odot v$, in the case $x \cdot y < 0$.

The natural topology on the set $\mathbb{R}_{\mathcal{F}}$ is generated by the Hausdorff type metric for all $u, v \in \mathbb{R}_{\mathcal{F}}$:

$$D(v, w) = \sup_{r \in [0, 1]} \{\max(|v_-^r - w_-^r|, |v_+^r - w_+^r|)\}. \quad (2)$$

Lemma 1 . The structure $(\mathbb{R}_{\mathcal{F}}, D)$ has the following properties (see [15] and [18]):

- (i) $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space;
- (ii) $D(x+z, y+z) = D(x, y)$, $\forall x, y, z \in \mathbb{R}_{\mathcal{F}}$ (invariance to translation);
- (iii) $D(x+y, z+w) \leq D(x, z) + D(y, w)$, $\forall x, y, z, w \in \mathbb{R}_{\mathcal{F}}$;
- (iv) $D(k \odot v, k \odot w) = |k| D(v, w)$, $\forall v, w \in \mathbb{R}_{\mathcal{F}}, \forall k \in \mathbb{R}$ (homogeneity);
- (v) $D(k_1 \odot v, k_2 \odot v) = |k_1 - k_2| D(v, \tilde{0})$, for any $v \in \mathbb{R}_{\mathcal{F}}$ and $k_1, k_2 \in \mathbb{R}$ with $k_1 \cdot k_2 \geq 0$ (see [20]).

Based on the metric (2), it is useful to consider the norm of a fuzzy number being defined by $\|u\|_F = D(u, \tilde{0})$. According to [15], the following properties are valid:

- (i) $\|v\|_F = 0$ if and only if $v = \tilde{0}$,
- (ii) $\|\lambda \cdot v\|_F = |\lambda| \cdot \|v\|_F$, $\forall v \in \mathbb{R}_F, \forall \lambda \in \mathbb{R}$,
- (iii) $\|v + w\|_F \leq \|v\|_F + \|w\|_F$, $\forall v, w \in \mathbb{R}_F$
- (iv) $|\|v\|_F - \|w\|_F| \leq D(v, w)$, $\forall v, w \in \mathbb{R}_F$.

Using the LU-representation, we can consider the left and the right r -level functions of a fuzzy-number-valued function $f : I \subset \mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$, as the functions $f(\cdot, r)$, $\overline{f}(\cdot, r) : I \subset \mathbb{R} \rightarrow \mathbb{R}$, defined by $\overline{f}(t, r) = (f(t))_+^r$, $\underline{f}(t, r) = (f(t))_-^r$, $\forall t \in I$, for $r \in [0, 1]$.

Definition 2 (see [41]). (i) A fuzzy-number-valued function $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ is continuous at $t_0 \in [a, b]$ if for any $\varepsilon > 0$ exists $\delta > 0$ with $D(f(t), f(t_0)) < \varepsilon$ whenever $|t - t_0| < \delta$. The function $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ is continuous on $[a, b]$ if it is continuous at each point $t_0 \in [a, b]$. (ii) A fuzzy-number-valued function $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ is bounded if and only if there exists $M \geq 0$ such that $D(f(t), \tilde{0}) \leq M$ for all $t \in [a, b]$.

Lemma 2 (see [41]). If $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ is continuous on $[a, b]$ then this function is bounded.

The set of continuous functions

$$C([a, b], \mathbb{R}_{\mathcal{F}}) = \{f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}} \mid f \text{ continuous on } [a, b]\}$$

is a metric space with the distance defined by $D^*(f, g) = \sup_{t \in [a, b]} D(f(t), g(t))$, $\forall f, g \in C([a, b], \mathbb{R}_{\mathcal{F}})$. The pair $(C([a, b], \mathbb{R}_{\mathcal{F}}), D^*)$ is a complete metric space. By using Lemma 1 we can derive corresponding properties for this metric D^* .

For fuzzy-number-valued functions the uniform modulus of continuity was defined in [20] as

$$\omega_{[a, b]}(f, \delta) =$$

$$= \sup\{D(f(x), f(y)) : x, y \in [a, b], |x - y| \leq \delta\}$$

and has the following properties (see [20], page 643):

- (i) $D(f(x), f(y)) \leq \omega_{[a, b]}(f, |x - y|)$ for all $x, y \in [a, b]$;
 - (ii) $\omega_{[a, b]}(f, \delta)$ is a non-decreasing mapping in δ ;
 - (iii) $\omega_{[a, b]}(f, \delta_1 + \delta_2) \leq \omega_{[a, b]}(f, \delta_1) + \omega_{[a, b]}(f, \delta_2)$ for all $\delta_1, \delta_2 \geq 0$,
- observing that $\lim_{\delta \rightarrow 0} \omega_{[a, b]}(f, \delta) = 0$.

According to [21], if $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ is continuous, then it is fuzzy-Riemann integrable having the fuzzy integral $(FR) \int_a^b f(t) dt$. Using the LU-representation we can write for all $r \in [0, 1]$

$$\left[(FR) \int_a^b f(t) dt \right]^r = \left[\int_a^b \underline{f}(t, r) dt, \int_a^b \overline{f}(t, r) dt \right].$$

Concerning the numerical integration of fuzzy-number-valued functions, the trapezoidal, midpoint and three point quadrature formulas were given for the first time in [16].

Lemma 3 (see [20]). If the functions $f, g : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ are continuous, then the following function $F : [a, b] \rightarrow \mathbb{R}_+$ given by $F(x) = D(f(x), g(x))$ is continuous too, and

$$D \left((FR) \int_a^b f(x) dx, (FR) \int_a^b g(x) dx \right) \leq \int_a^b \varphi(x) dx,$$

where $\varphi(x) = D(f(x), g(x))$.

Definition 3 For given $\alpha, \beta \in (0, 1]$, a function $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ is (α, β) -Lipschitz if there exist $L, L' \geq 0$ with

$$D(f(x), f(y)) \leq L|x - y|^\alpha + L'|x - y|^\beta, \forall x, y \in [a, b].$$

According to [16], $f : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ is Lipschitz if exists $L \geq 0$ with the property $D(f(x), f(y)) \leq L|x - y|$, $\forall x, y \in [a, b]$, and any Lipschitz function is continuous. For Lipschitz functions we can infer that $\omega_{[a, b]}(f, \delta) \leq L\delta$.

3 Existence, uniqueness and successive approximations

In the functional space $X = C([0, T], \mathbb{R}_{\mathcal{F}})$ we will consider both the usual metric D^* and the Bielecki's type metric D_τ^* for $u, v \in C([0, T], \mathbb{R}_{\mathcal{F}})$, defined by

$$D_\tau^*(u, v) = \max_{t \in [0, T]} D(u(t), v(t)) \cdot e^{-\tau t}$$

with suitable chosen constant $\tau > 0$. The existence and uniqueness of the solution for (1) will be obtained both

in (X, D_τ^*) and in (X, D^*) . Therefore for $t \in [0, T]$ we define on X the integral operator F by

$$F(x(t)) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t H(t, s) (t-s)^{\alpha-1} \odot x(s) ds \quad (3)$$

and we show that $F(X) \subset X$.

In this purpose, we consider the weakly singular kernel $V(t, s) = H(t, s) (t-s)^{\alpha-1}$ and since H is continuous on $C([0, T] \times [0, T])$ will exists $M_H > 0$ with $|H(t, s)| \leq M_H, \forall (t, s) \in [0, T] \times [0, T]$.

Theorem 1 *If $V(t, s) = H(t, s) (t-s)^{\alpha-1}$ for $\alpha \in (0, 1)$ with $H \in C([0, T] \times [0, T])$, $H(t, s) \geq 0, \forall (t, s) \in [0, T] \times [0, T]$, then the following integral operator $L : C([0, T], \mathbb{R}_{\mathcal{F}}) \rightarrow C([0, T], \mathbb{R}_{\mathcal{F}})$,*

$$Lx(t) = \int_0^t V(t, s) \odot x(s) ds$$

is well defined ($Lx \in C([0, T], \mathbb{R}_{\mathcal{F}})$).

Proof Let $t, t' \in [0, T]$ be such that $0 \leq t \leq t' \leq T$ and $\delta > 0$. Since

$$\begin{aligned} Lx(t') &= \int_0^{t'} V(t', s) \odot x(s) ds = \int_0^{t-\delta} V(t', s) \odot x(s) ds + \\ &+ \int_{t-\delta}^{t'} V(t', s) \odot x(s) ds + \int_{t-\delta}^{t'} V(t', s) \odot x(s) ds \end{aligned}$$

we see that

$$\begin{aligned} D(Lx(t'), Lx(t)) &\leq \\ &\leq \int_0^{t-\delta} D(V(t', s) \odot x(s), V(t, s) \odot x(s)) ds + \\ &+ \int_{t-\delta}^t D(V(t, s) \odot x(s), \tilde{0}) ds \\ &+ \int_{t-\delta}^{t'-\delta} D(V(t', s) \odot x(s), \tilde{0}) ds + \\ &+ \int_{t'-\delta}^{t'} D(V(t', s) \odot x(s), \tilde{0}) ds = I_1 + I_2 + I_3 + I_4 \end{aligned}$$

and because $x \in C([0, T], \mathbb{R}_{\mathcal{F}})$ will be $M_x > 0$ such that $D(x(s), \tilde{0}) \leq M_x, \forall s \in [0, T]$.

For $s \in [0, T]$ such that $t-s \geq \delta > 0$ the weakly singular kernel V is continuous and thus will be uniformly continuous on the compact $[t-\frac{\delta}{2}, T] \times [0, t-\delta]$. So, $\forall \varepsilon > 0$ there exists $\delta' > 0$ such that

$$|V(t', s) - V(t, s)| < \frac{\varepsilon}{3M_x}$$

if $|t-t'| < \delta'$ and $t-s \geq \delta$. Then, we deduce

$$\begin{aligned} I_1 &\leq \int_0^{t-\delta} |V(t', s) - V(t, s)| \cdot D(x(s), \tilde{0}) ds \leq \\ &\leq \frac{\varepsilon}{3M_x} \cdot TM_x = \frac{\varepsilon}{3} \end{aligned}$$

$$I_2 \leq \int_{t-\delta}^t |H(t, s)| (t-s)^{\alpha-1} \cdot D(x(s), \tilde{0}) ds \leq$$

$$\leq M_H M_x \int_{t-\delta}^t (t-s)^{\alpha-1} ds = \frac{M_H M_x}{\alpha} \cdot \delta^\alpha$$

and

$$I_4 \leq \int_{t'-\delta}^{t'} |H(t', s)| (t'-s)^{\alpha-1} D(x(s), \tilde{0}) ds \leq$$

$$\leq M_H M_x \int_{t'-\delta}^{t'} (t'-s)^{\alpha-1} ds = \frac{M_H M_x}{\alpha} \cdot \delta^\alpha.$$

Moreover, for the third integral we get

$$I_3 \leq \int_{t-\delta}^{t'-\delta} |H(t', s)| (t'-s)^{\alpha-1} \cdot D(x(s), \tilde{0}) ds \leq$$

$$\begin{aligned} &\leq M_H M_x \int_{t-\delta}^{t'-\delta} (t'-s)^{\alpha-1} ds = \\ &= \frac{M_H M_x}{\alpha} \cdot [(t'-t+\delta)^\alpha - \delta^\alpha] \leq \frac{M_H M_x}{\alpha} \cdot (t'-t+\delta)^\alpha \\ &< \frac{M_H M_x}{\alpha} \cdot (\delta'' + \delta)^\alpha \end{aligned}$$

if $|t-t'| < \delta''$. Taking $\frac{M_H M_x}{\alpha} \cdot \delta^\alpha = \frac{\varepsilon}{6}$, that is $\delta = \left(\frac{\varepsilon \alpha}{6 M_H M_x}\right)^{\frac{1}{\alpha}}$, we get $I_2 \leq \frac{\varepsilon}{6}$ and $I_4 \leq \frac{\varepsilon}{6}$. Now, we can choose $\delta'' > 0$ such that $I_3 < \frac{\varepsilon}{5}$, for instance we can take $\delta'' < \left(\frac{\varepsilon \alpha}{M_H M_x}\right)^{\frac{1}{\alpha}} \left(\left(\frac{1}{5}\right)^\alpha - \left(\frac{1}{6}\right)^\alpha\right)$. Finally, if $|t-t'| < \min(\delta, \delta', \delta'')$ we obtain

$$D(Lx(t'), Lx(t)) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{5} = \frac{13\varepsilon}{15} < \varepsilon$$

and thus the operator L is well defined.

Now, since $\Gamma(\alpha) > 0, \forall \alpha > 0$, and $f \in C([0, T], \mathbb{R}_{\mathcal{F}})$ we infer $F(x) \in C([0, T], \mathbb{R}_{\mathcal{F}})$ for any $x \in C([0, T], \mathbb{R}_{\mathcal{F}})$ and thus $F(X) \subset X$, the integral operator being well-defined. Considering $R > 0$ be such that $D(f(t), \tilde{0}) \leq R, \forall t \in [0, T]$, the existence and uniqueness result in the framework of the space (X, D_{τ}^*) will be done in the closed ball $B_R = \{x \in X : D_{\tau}^*(x, f) \leq R\}$ without any contraction condition.

Theorem 2 *If $f \in C([0, T], \mathbb{R}_{\mathcal{F}})$ and H is continuous on $C([0, T] \times [0, T])$ with $H(t, s) \geq 0, \forall (t, s) \in [0, T] \times [0, T]$, taking $\tau \geq (2M_H)^{\frac{1}{\alpha}}$ the integral equation (1) will have solution $x^* \in B_R \subset X$. Moreover, the Picard iterations $x_{k+1} = F(x_k), k \in \mathbb{N}$, converges to x^* in (X, D_{τ}^*) not depending by the choice of the term $x_0 \in B_R$ and if $x_0 = f$ the apriori error estimate is:*

$$D_{\tau}^*(x_k, x^*) \leq \frac{RM_H T^{\alpha} \cdot \left(\frac{M_H}{\tau^{\alpha}}\right)^k}{\Gamma(\alpha + 1) \left(1 - \frac{M_H}{\tau^{\alpha}}\right)}, \quad \forall k \in \mathbb{N}^*. \quad (4)$$

Proof Firstly we prove that $F(B_R) \subset B_R$. For $x \in B_R$ we have

$$\begin{aligned} D(F(x(t)), f(t)) &\leq \\ &\leq D\left(\frac{1}{\Gamma(\alpha)} \int_0^t H(t, s) (t-s)^{\alpha-1} \odot x(s) ds, \tilde{0}\right) \leq \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t H(t, s) (t-s)^{\alpha-1} \cdot D(x(s), \tilde{0}) ds \leq \\ &\leq \frac{M_H \cdot \|x\|_{\tau}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\tau s} ds \end{aligned}$$

and after change variable $t-s = \frac{w}{\tau}$ we get

$$\begin{aligned} D(F(x(t)), f(t)) &\leq \frac{M_H \cdot \|x\|_{\tau} e^{\tau t}}{\Gamma(\alpha) \tau^{\alpha}} \int_0^{\tau t} w^{\alpha-1} e^{-w} dw \leq \\ &\leq \frac{M_H \cdot \|x\|_{\tau} e^{\tau t}}{\Gamma(\alpha) \tau^{\alpha}} \int_0^{\infty} w^{\alpha-1} e^{-w} dw. \end{aligned}$$

Since $x \in B_R$ we infer that $D_{\tau}^*(x, f) \leq R$ and

$$\begin{aligned} \|x\|_{\tau} &= D_{\tau}^*(x, \tilde{0}) + D_{\tau}^*(f, \tilde{0}) \leq R + \\ &+ \max_{t \in [0, T]} D(f(t), \tilde{0}) \cdot e^{-\tau t} \leq \\ &\leq R + \max_{t \in [0, T]} D(f(t), \tilde{0}) \leq 2R \end{aligned}$$

obtaining

$$D(F(x(t)), f(t)) \cdot e^{-\tau t} \leq \frac{2RM_H}{\tau^{\alpha}}, \quad \forall t \in [0, T]$$

which means $D_{\tau}^*(F(x), f) \leq \frac{2M_H}{\tau^{\alpha}} \cdot R$. Now, the condition $\tau \geq (2M_H)^{\frac{1}{\alpha}}$ ensures the fact $F(x) \in B_R$, and thus $F(B_R) \subset B_R$. Under the same condition the integral operator F is contraction in (X, D_{τ}^*) . Indeed,

$$\begin{aligned} D(F(x(t)), F(y(t))) &\leq \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t |H(t, s)| (t-s)^{\alpha-1} D(x(s), y(s)) \cdot e^{-\tau s} e^{\tau s} ds \leq \\ &\leq \frac{M_H \cdot D_{\tau}^*(x, y)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{\tau s} ds = \\ &= \frac{M_H \cdot D_{\tau}^*(x, y) e^{\tau t}}{\Gamma(\alpha) \tau^{\alpha}} \int_0^{\tau t} e^{-w} \cdot w^{\alpha-1} dw \leq \\ &\leq \frac{M_H \cdot D_{\tau}^*(x, y) e^{\tau t}}{\Gamma(\alpha) \tau^{\alpha}} \int_0^{\infty} e^{-w} \cdot w^{\alpha-1} dw = \\ &= \frac{M_H \cdot D_{\tau}^*(x, y) e^{\tau t}}{\tau^{\alpha}}, \quad \forall t \in [0, T] \end{aligned}$$

and we get

$$\begin{aligned} D_{\tau}^*(F(x), F(y)) &= \max_{t \in [0, T]} D(F(x(t)), F(y(t))) e^{-\tau t} \leq \\ &\leq \frac{M_H}{\tau^{\alpha}} \cdot D_{\tau}^*(x, y) \leq \frac{1}{2} D_{\tau}^*(x, y). \end{aligned}$$

According to [17], (X, D_{τ}^*) is complete metric space and therefore the closed set $B_R \subset X$ will be complete. Consequently, F has unique fixed point $x^* \in B_R$ that is the unique solution of equation (1). From the Banach's fixed point principle, we have that the Picard iterations generated by F converge to x^* obtaining the estimate

$$D_{\tau}^*(x_k, x^*) \leq \frac{q^k}{1-q} \cdot D_{\tau}^*(x_1, x_0), \quad \forall k \in \mathbb{N}^*$$

where $q = \frac{M_H}{\tau^{\alpha}} \leq \frac{1}{2}$. Now, we see that

$$\begin{aligned} D(x_1(t), x_0(t)) &\leq \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t |H(t, s)| (t-s)^{\alpha-1} D(f(s), \tilde{0}) ds \leq \\ &\leq \frac{RM_H}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \leq \frac{RM_H T^{\alpha}}{\alpha \Gamma(\alpha)}, \quad \forall t \in [0, T] \end{aligned}$$

and thus,

$$\leq D_{\tau}^*(x_1, x_0) = \max_{t \in [0, T]} D(x_1(t), x_0(t)) e^{-\tau t} \leq$$

$$\max_{t \in [0, T]} D(x_1(t), x_0(t)) \leq \frac{RM_H T^{\alpha}}{\Gamma(\alpha + 1)}$$

obtaining the error estimate (4).

For solving the integral equation (1) we construct in the next section an iterative numerical method and the condition that ensures the convergence of this method is the same as the contraction condition imposed in the existence and uniqueness result for equation (1) on the metric space (X, D^*) . This is the reason for which we approach the existence and uniqueness separately in the space (X, D^*) , under the following conditions:

- (i) $f \in C([0, T], \mathbb{R}_{\mathcal{F}})$ and $H \in C([0, T] \times [0, T])$
- (ii) $H(t, s) \geq 0, \forall (t, s) \in [0, T] \times [0, T]$ and $\frac{M_H T^\alpha}{\Gamma(\alpha+1)} < 1$, where $M_H > 0$ is the upper bound of the kernel function, $|H(t, s)| \leq M_H, \forall (t, s) \in [0, T] \times [0, T]$
- (iii) there exist $\gamma, \mu \geq 0$ with the properties $D(f(t), f(t')) \leq \gamma |t - t'|$ and

$$|H(t, s) - H(t', s)| \leq \mu |t - t'|, \quad \forall s, t, t' \in [0, T].$$

Theorem 3 Suppose that the conditions (i) and (ii) are fulfilled. Then the integral equation (1) has unique solution $x^* \in X$ and the Picard iterations $x_{k+1} = F(x_k)$, $k \in \mathbb{N}$, converge to x^* in (X, D^*) not depending by the choice of the term $x_0 \in X$, being uniformly bounded. If $x_0 = f$, the apriori and "a posteriori" error estimates are:

$$D^*(x_k, x^*) \leq \frac{RM_H T^\alpha \cdot \left(\frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)^k}{\Gamma(\alpha+1) \left(1 - \frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)}, \quad \forall k \in \mathbb{N}^* \quad (5)$$

and $\forall t \in [0, T], \forall k \in \mathbb{N}^*$

$$\begin{aligned} D(x_k(t), x^*(t)) &\leq \\ &\leq \frac{M_H T^\alpha}{\Gamma(\alpha+1) - M_H T^\alpha} \cdot D(x_k(t), x_{k-1}(t)), \end{aligned} \quad (6)$$

respectively.

Proof By Theorem 1 the operator $F : X \rightarrow X$ is well defined and for testing the contraction quality, for arbitrary $x, y \in X$ we have

$$\begin{aligned} D(F(x(t)), F(y(t))) &\leq \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t |H(t, s)| (t-s)^{\alpha-1} \cdot D(x(s), y(s)) ds \leq \\ &\leq \frac{M_H \cdot D^*(x, y)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \leq \\ &\leq \frac{M_H T^\alpha \cdot D^*(x, y)}{\alpha \Gamma(\alpha)} = \frac{M_H T^\alpha}{\Gamma(\alpha+1)} \cdot D^*(x, y) \end{aligned}$$

for all $t \in [0, T]$ and by condition (ii) this operator is contraction on (X, D^*) . Then the integral equation (1)

has unique solution $x^* \in X$ and the Picard iterations converge to x^* . Moreover, the following error estimates:

$$\begin{aligned} D(x_k(t), x^*(t)) &\leq \\ &\leq \frac{\left(\frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)^k}{\Gamma(\alpha+1) \left(1 - \frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)} \cdot D(x_1(t), x_0(t)), \quad \forall t \in [0, T] \end{aligned}$$

and $\forall t \in [0, T], \forall k \in \mathbb{N}^*$

$$D(x_k(t), x^*(t)) \leq \frac{\frac{M_H T^\alpha}{\Gamma(\alpha+1)}}{1 - \frac{M_H T^\alpha}{\Gamma(\alpha+1)}} \cdot D(x_k(t), x_{k-1}(t))$$

hold. Since

$$\begin{aligned} D(x_1(t), x_0(t)) &\leq \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t |H(t, s)| (t-s)^{\alpha-1} \cdot D(f(s), \tilde{0}) ds \leq \\ &\leq \frac{RM_H}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \end{aligned}$$

the estimate (5) follows. Now, by induction we obtain

$$\begin{aligned} D(x_{k+1}(t), x_k(t)) &\leq \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t |H(t, s)| (t-s)^{\alpha-1} \cdot D(x_k(s), x_{k-1}(s)) ds \leq \\ &\leq \frac{M_H T^\alpha}{\Gamma(\alpha+1)} \cdot D^*(x_k, x_{k-1}) \leq \dots \leq \\ &\leq \left(\frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)^k \cdot D^*(x_1, x_0) \end{aligned}$$

and thus

$$\begin{aligned} D^*(x_k, x_0) &\leq \\ &\leq [1 + \frac{M_H T^\alpha}{\Gamma(\alpha+1)} + \dots + \left(\frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)^{k-1}] \cdot D^*(x_1, x_0) \leq \\ &\leq \frac{D^*(x_1, x_0)}{1 - \frac{M_H T^\alpha}{\Gamma(\alpha+1)}}. \end{aligned}$$

Since $D^*(x_1, x_0) \leq \frac{RM_H T^\alpha}{\Gamma(\alpha+1)}$ it obtains

$$\begin{aligned} D^*(x_k, \tilde{0}) &\leq D^*(x_k, x_0) + D^*(x_0, \tilde{0}) \leq R + \frac{\frac{RM_H T^\alpha}{\Gamma(\alpha+1)}}{1 - \frac{M_H T^\alpha}{\Gamma(\alpha+1)}} \\ &= \frac{R\Gamma(\alpha+1)}{\Gamma(\alpha+1) - M_H T^\alpha} \end{aligned} \quad (7)$$

for all $k \in \mathbb{N}^*$, and the sequence of Picard iterations is uniformly bounded with the following upper bound

$$M = \frac{R\Gamma(\alpha+1)}{\Gamma(\alpha+1) - M_H T^\alpha}.$$

Theorem 4 Under the conditions (i)-(iii) the sequence of Picard iterations $x_{k+1} = F(x_k)$, $k \in \mathbb{N}$, $x_0 = f$, is uniform $(1, \alpha)$ -Lipschitz.

Proof Suppose that $0 \leq t' < t \leq T$ (the case $t < t'$ being investigated analogously). Then $0 \leq t' - s < t - s$ and since $0 < \alpha < 1$ we infer that $(t - s)^{\alpha-1} < (t' - s)^{\alpha-1}$. Consequently, we can observe that

$$\begin{aligned} & H(t, s)(t - s)^{\alpha-1} - H(t', s)(t' - s)^{\alpha-1} < \\ & < H(t, s)(t' - s)^{\alpha-1} - H(t', s)(t' - s)^{\alpha-1} = \\ & = (H(t, s) - H(t', s)) \cdot (t' - s)^{\alpha-1} \end{aligned}$$

obtaining the inequality

$$\begin{aligned} & \left| H(t, s)(t - s)^{\alpha-1} - H(t', s)(t' - s)^{\alpha-1} \right| \leq \\ & |H(t, s) - H(t', s)| (t' - s)^{\alpha-1} \leq \mu |t - t'| (t' - s)^{\alpha-1}. \end{aligned}$$

We can see that in the case $t < t'$ it follows

$$\begin{aligned} & H(t', s)(t' - s)^{\alpha-1} - H(t, s)(t - s)^{\alpha-1} < \\ & < (H(t', s) - H(t, s)) \cdot (t - s)^{\alpha-1} \end{aligned}$$

and similarly we get

$$\begin{aligned} & \left| H(t, s)(t - s)^{\alpha-1} - H(t', s)(t' - s)^{\alpha-1} \right| \leq \\ & \leq \mu |t - t'| (t - s)^{\alpha-1}. \end{aligned}$$

Therefore, it is enough to take only the case $0 \leq t' < t \leq T$, and based on these inequalities, by (7) we obtain

$$\begin{aligned} D(x_k(t), x_k(t')) & \leq D(f(t), f(t')) + \frac{1}{\Gamma(\alpha)} \cdot \\ & \cdot D \left(\int_0^t H(t, s)(t - s)^{\alpha-1} \odot x_{k-1}(s) ds, \right. \\ & \left. , \int_0^{t'} H(t', s)(t' - s)^{\alpha-1} \odot x_{k-1}(s) ds \right) \leq \gamma |t - t'| + \\ & + \frac{1}{\Gamma(\alpha)} \cdot D \left(\int_0^{t'} H(t, s)(t - s)^{\alpha-1} \odot x_{k-1}(s) ds + \right. \\ & \left. + \int_{t'}^t H(t, s)(t - s)^{\alpha-1} \odot x_{k-1}(s) ds, \right. \\ & \left. , \int_0^{t'} H(t', s)(t' - s)^{\alpha-1} \odot x_{k-1}(s) ds \right) \leq \gamma |t - t'| + \\ & + \frac{1}{\Gamma(\alpha)} \int_0^{t'} \left| H(t, s)(t - s)^{\alpha-1} - H(t', s)(t' - s)^{\alpha-1} \right| \cdot \end{aligned}$$

$$\cdot D(x_{k-1}(s), \tilde{0}) ds + \frac{1}{\Gamma(\alpha)} \int_{t'}^t |H(t, s)| (t - s)^{\alpha-1} \cdot$$

$$\cdot D(x_{k-1}(s), \tilde{0}) ds \leq \gamma |t - t'| +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{t'} \mu |t - t'| M (t' - s)^{\alpha-1} ds +$$

$$+ \frac{MM_H}{\Gamma(\alpha)} \int_{t'}^t (t - s)^{\alpha-1} ds \leq \gamma |t - t'| +$$

$$+ \frac{M\mu |t - t'|}{\Gamma(\alpha)} \cdot \int_0^{t'} (t' - s)^{\alpha-1} ds + \frac{MM_H}{\Gamma(\alpha)} \cdot \frac{|t - t'|^\alpha}{\alpha} \leq$$

$$\leq \left(\gamma + \frac{M\mu T^\alpha}{\alpha \Gamma(\alpha)} \right) |t - t'| + \frac{MM_H}{\Gamma(\alpha + 1)} |t - t'|^\alpha =$$

$$= L_1 |t - t'| + L_2 |t - t'|^\alpha$$

for all $k \in \mathbb{N}^*$, where $L_1 = \gamma + \frac{M\mu T^\alpha}{\Gamma(\alpha+1)}$ and $L_2 = \frac{MM_H}{\Gamma(\alpha+1)}$. Finally, we get for all $t, t' \in [0, T]$ and for all $k \in \mathbb{N}^*$

$$D(x_k(t), x_k(t')) \leq L_1 |t - t'| + L_2 |t - t'|^\alpha$$

and consequently, $\omega_{[0, T]}(x_k, \delta) \leq L_1 \delta + L_2 \delta^\alpha$ for $\delta > 0$ with $|t - t'| \leq \delta$.

4 The iterative numerical method

The Picard iterations generated by the integral operator F , is $x_{k+1} = F(x_k)$, $k \in \mathbb{N}$, with $x_0 = f$, and can be expressed for $t \in [0, T]$ and $k \in \mathbb{N}$ in the form

$$x_{k+1}(t) = f(t) + \frac{1}{\Gamma(\alpha)} \int_0^t H(t, s) \cdot (t - s)^{\alpha-1} \odot x_k(s) ds. \quad (8)$$

In the numerical integration of the integrals (8) we can consider the fuzzy product integration. The method of product integration was presented in [12], [13], and [34] for the crisp case. For constructing the method of fuzzy product integration we take a uniform partition of $[0, T]$ with the set of nodes $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$ where $t_i = i \cdot h = i \cdot \frac{T}{n}$, $i = \overline{0, n}$, and $h = \frac{T}{n}$ is the stepsize. On these nodes, the sequence (8) is

$$x_0(s) = f(s), \quad s \in [0, T], \quad x_0(t_i) = f(t_i), \quad i = \overline{0, n}$$

$$x_{k+1}(t_i) = f(t_i) +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{t_i} H(t_i, s) \cdot (t_i - s)^{\alpha-1} \odot x_k(s) ds, \quad i = \overline{0, n} \quad (9)$$

and in the context of fuzzy trapezoidal product integration we get the approximation of the integrals as follows,

$$\begin{aligned} I(k, i) &= \int_0^{t_i} H(t_i, s) \cdot (t_i - s)^{\alpha-1} \odot x_k(s) ds = \\ &= \sum_{j=1}^i [H(t_i, t_{j-1}) \cdot h\varphi_{2,i}(j-1) \odot x_k(t_{j-1}) + \\ &+ H(t_i, t_j) \cdot h\varphi_{1,i}(j-1) \odot x_k(t_j)] + R_{k,i} = I_{k,i} + R_{k,i} \end{aligned} \quad (10)$$

for $i = \overline{1, n}$ and for all $k \in \mathbb{N}^*$, where

$$\begin{aligned} \varphi_{2,i}(j) &= h^{\alpha-1} \int_0^1 (1-u)(i-j-u)^{\alpha-1} du, \\ \varphi_{1,i}(j) &= h^{\alpha-1} \int_0^1 u(i-j-u)^{\alpha-1} du \end{aligned} \quad (11)$$

for $i = \overline{1, n}$, $j = \overline{0, i-1}$, are obtained by computing the following integrals respectively

$$\int_{t_{j-1}}^{t_j} (t_j - s)(t_i - s)^{\alpha-1} ds, \quad \int_{t_{j-1}}^{t_j} (s - t_{j-1})(t_i - s)^{\alpha-1} ds.$$

Since in (10) we have approximated the terms of the sequence of Picard iterations by the fuzzy polygonal line,

$$\begin{aligned} x_k(s) &\approx \frac{1}{h} [(t_j - s) \odot x_k(t_{j-1}) + (s - t_{j-1}) \odot x_k(t_j)] = \\ &= \overline{L_k}(s), \quad s \in [t_{j-1}, t_j], \quad j = \overline{1, n} \end{aligned}$$

for all $k \in \mathbb{N}^*$, and according to the proof of Theorem 4, we infer that

$$\begin{aligned} D(x_k(s), \overline{L_k}(s)) &\leq \frac{(t_j - s)}{h} \cdot D(x_k(s), x_k(t_{j-1})) + \\ &+ \frac{(s - t_{j-1})}{h} \cdot D(x_k(s), x_k(t_j)) \leq \\ &\leq \max\{D(x_k(s), x_k(t_{j-1})), D(x_k(s), x_k(t_j))\} \cdot \\ &\cdot \left(\frac{(t_j - s)}{h} + \frac{(s - t_{j-1})}{h} \right) \leq \omega_{[0,T]}(x_k, h) \leq \\ &\leq L_1 h + L_2 h^\alpha, \quad \forall s \in [t_{j-1}, t_j], \quad j = \overline{1, n}. \end{aligned}$$

Then, since for $i = \overline{1, n}$ and $k \in \mathbb{N}^*$

$$I_{k,i} = \int_0^{t_i} H(t_i, s) \cdot (t_i - s)^{\alpha-1} \odot \overline{L_k}(s) ds,$$

we obtain

$$\begin{aligned} D(I(k, i), I_{k,i}) &\leq \|R_{k,i}\|_F \leq \\ &\leq D^*(x_k, \overline{L_k}) \cdot \int_0^{t_i} |H(t_i, s)| (t_i - s)^{\alpha-1} ds \leq \\ &\leq (L_1 h + L_2 h^\alpha) \int_0^{t_i} M_H \cdot (t_i - s)^{\alpha-1} ds \leq \\ &\leq \frac{M_H T^\alpha (L_1 h + L_2 h^\alpha)}{\alpha} \end{aligned} \quad (12)$$

for all $i = \overline{1, n}$, $k \in \mathbb{N}^*$.

By (9) and (10), in inductive way we get

$$\begin{aligned} x_{k+1}(t_i) &= f(t_i) + \frac{1}{\Gamma(\alpha)} \odot I(k, i), \quad i = \overline{1, n}, \quad k \in \mathbb{N}^*, \\ x_0(t_i) &= f(t_i), \quad i = \overline{0, n}, \\ x_1(t_i) &= f(t_i) + \frac{1}{\Gamma(\alpha)} \int_0^{t_i} H(t_i, s) \cdot (t_i - s)^{\alpha-1} \odot f(s) ds \\ &= f(t_i) + \frac{1}{\Gamma(\alpha)} \cdot \sum_{j=0}^i H(t_i, t_j) \cdot w_{i,j} \odot f(t_j) + \overline{R_{1,i}} = \\ &= \overline{x_1(t_i)} + \overline{R_{1,i}}, \quad i = \overline{0, n} \end{aligned} \quad (13)$$

and

$$\begin{aligned} x_2(t_i) &= f(t_i) + \frac{1}{\Gamma(\alpha)} \int_0^{t_i} H(t_i, s) \cdot (t_i - s)^{\alpha-1} \odot x_1(s) ds \\ &= f(t_i) + \frac{1}{\Gamma(\alpha)} \left(\sum_{j=0}^i H(t_i, t_j) \cdot w_{i,j} \odot x_1(t_j) + R_{2,i} \right) \\ &= f(t_i) + \frac{1}{\Gamma(\alpha)} \cdot \left(\sum_{j=0}^i H(t_i, t_j) \cdot w_{i,j} \odot (\overline{x_1(t_j)} + \overline{R_{1,j}}) \right) \\ &+ \frac{1}{\Gamma(\alpha)} \cdot R_{2,i} = \overline{x_2(t_i)} + \overline{R_{2,i}}, \quad i = \overline{0, n}, \end{aligned} \quad (14)$$

where

$$\begin{aligned} w_{i,0} &= \frac{1}{h} \int_{t_0}^{t_1} (t_1 - s)(t_i - s)^{\alpha-1} ds, \\ w_{i,n} &= \frac{1}{h} \int_{t_{n-1}}^{t_n} (s - t_{n-1})(t_i - s)^{\alpha-1} ds, \quad i = \overline{1, n} \end{aligned} \quad (15)$$

and

$$w_{i,j} = \frac{1}{h} \left(\int_{t_{j-1}}^{t_j} (s - t_{j-1})(t_i - s)^{\alpha-1} ds + \right.$$

$$+ \int_{t_j}^{t_{j+1}} (t_j - s) (t_i - s)^{\alpha-1} ds \Big), \quad j = \overline{1, n-1}. \quad (16)$$

Generally, for $i = \overline{0, n}$, the computed values are

$$\overline{x_k(t_i)} = f(t_i) + \frac{1}{\Gamma(\alpha)} \cdot \sum_{j=0}^i H(t_i, t_j) \cdot w_{i,j} \odot \overline{x_{k-1}(t_j)} \quad (17)$$

and we have $x_k(t_i) = \overline{x_k(t_i)} + \overline{R_{k,i}}$, for $k \in \mathbb{N}^*$, $i = \overline{0, n}$.

A stopping condition for the iterative algorithm (13)-(17) can be expressed as follows: taking $\varepsilon > 0$ we get the first number $k \in \mathbb{N}^*$ with,

$$D(\overline{x_k(t_i)}, \overline{x_{k-1}(t_i)}) < \varepsilon, \quad \forall i = \overline{1, n}$$

and stop at this iterative step. The solution $x^*(t_i)$ will be approximated by the terms of this last iteration $\overline{x_k(t_i)}$ on the nodes t_i , $i = \overline{1, n}$, with $x^*(t_0) = f(t_0)$.

5 The convergence and error analysis

In this main section, we prove the convergence of the iterative numerical method (10) and the effectiveness of the algorithm (13)-(17), by establishing the error estimate when the terms $x_k(t_i)$ are approximated by the values $\overline{x_k(t_i)}$, $i = \overline{1, n}$.

Theorem 5 *Under the conditions (i)-(iii), in the approximation of the solution $x^*(t_i)$ by the computed values $\overline{x_k(t_i)}$, on the nodes t_i , $i = \overline{1, n}$, the error estimate is*

$$D(x^*(t_i), \overline{x_k(t_i)}) \leq \quad (18)$$

$$\leq \frac{RM_H T^\alpha \cdot \left(\frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)^k}{\Gamma(\alpha+1) \left(1 - \frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)} + \frac{\frac{M_H T^\alpha (L_1 h + L_2 h^\alpha)}{\Gamma(\alpha+1)}}{1 - \frac{M_H T^\alpha}{\Gamma(\alpha+1)}}$$

for all $i = \overline{1, n}$, $k \in \mathbb{N}^*$.

Proof Since

$$D(x^*(t_i), \overline{x_k(t_i)}) \leq$$

$$\leq D(x^*(t_i), x_k(t_i)) + D(x_k(t_i), \overline{x_k(t_i)})$$

by inequality (5) we have to estimate $D(x_k(t_i), \overline{x_k(t_i)})$, $k \in \mathbb{N}^*$, $i = \overline{1, n}$. In this purpose by (13) and (12) we see that

$$D(x_1(t_i), \overline{x_1(t_i)}) \leq \|\overline{R_{1,i}}\|_F \leq \frac{1}{\Gamma(\alpha)} \|R_{1,i}\|_F \leq$$

$$\leq \frac{M_H T^\alpha (L_1 h + L_2 h^\alpha)}{\alpha \Gamma(\alpha)} = \frac{M_H T^\alpha (L_1 h + L_2 h^\alpha)}{\Gamma(\alpha+1)} \quad (19)$$

for $i = \overline{0, n}$ and according to (14) it follows,

$$D(x_2(t_i), \overline{x_2(t_i)}) \leq \|\overline{R_{2,i}}\|_F \leq \frac{1}{\Gamma(\alpha)} \cdot \|R_{2,i}\|_F +$$

$$+ \frac{1}{\Gamma(\alpha)} \cdot \sum_{j=0}^i |H(t_i, t_j)| \cdot |w_{i,j}| \odot \|\overline{R_{1,j}}\|_F \leq$$

$$\leq \frac{M_H T^\alpha (L_1 h + L_2 h^\alpha)}{\Gamma(\alpha+1)} \cdot \left(1 + \frac{M_H}{\Gamma(\alpha)} \cdot \sum_{j=0}^i w_{i,j}\right)$$

because $w_{i,j} \geq 0$, $\forall i = \overline{1, n}$, $j = \overline{0, i-1}$. Now, according to (11), (15) and (16), since $\varphi_{1,i}(j) + \varphi_{2,i}(j) = \frac{h^{\alpha-1}}{\alpha} [(i-j)^\alpha - (i-j-1)^\alpha]$ we infer that

$$\sum_{j=0}^i w_{i,j} \leq h \sum_{j=0}^i (\varphi_{1,i}(j) + \varphi_{2,i}(j)) =$$

$$= \frac{h^\alpha}{\alpha} \sum_{j=0}^i [(i-j)^\alpha - (i-j-1)^\alpha] = \frac{(ih)^\alpha}{\alpha} \leq \frac{T^\alpha}{\alpha}$$

and thus,

$$D(x_2(t_i), \overline{x_2(t_i)}) \leq$$

$$\leq \frac{M_H T^\alpha (L_1 h + L_2 h^\alpha)}{\Gamma(\alpha+1)} \cdot \left(1 + \frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right), \quad i = \overline{0, n}. \quad (20)$$

By (14), (17) and using the estimate (20), in iterative manner, for all $k \in \mathbb{N}^*$ and for all $i = \overline{1, n}$, we obtain,

$$D(x_k(t_i), \overline{x_k(t_i)}) \leq \|\overline{R_{k,i}}\|_F$$

$$\leq \frac{M_H T^\alpha (L_1 h + L_2 h^\alpha)}{\Gamma(\alpha+1)} \cdot [1 + \frac{M_H T^\alpha}{\Gamma(\alpha+1)} + \dots +$$

$$+ \left(\frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)^{k-1}] \leq \frac{M_H T^\alpha (L_1 h + L_2 h^\alpha)}{\Gamma(\alpha+1) \left(1 - \frac{M_H T^\alpha}{\Gamma(\alpha+1)}\right)}$$

and the estimate (18) follows.

By (18) we get

$$\lim_{k \rightarrow \infty, h \rightarrow 0} D(x^*(t_i), \overline{x_k(t_i)}) = 0$$

and then the iterative method (10) is convergent.

6 Numerical experiment

In this section, the theoretical results stated in Theorem 5 are tested and the performances of the iterative numerical method (10) are illustrated, by presenting the following numerical example.

Example 1 The linear fuzzy Volterra integral equation

$$x(t) = f(t) + \frac{1}{\sqrt{\pi}} \int_0^t \frac{s(t-s)^{-\frac{1}{2}}}{8} \odot x(s) ds, \quad t \in [0, 1]$$

with $\alpha = \frac{1}{2}$ and

$$f(t, r) = [f(t, r)_-, f(t, r)_+] =$$

$$\left[\frac{(3+2r-r^2)}{4} \cdot g(t), \frac{(5-2r+r^2)}{4} \cdot g(t) \right], \quad r \in [0, 1]$$

where $g(t) = \left(\sqrt{\pi}t - \frac{2}{15}t^{\frac{5}{2}} \right)$, has the exact solution

$$x^*(t, r) = [x^*(t, r)_-, x^*(t, r)_+] =$$

$$= \left[\frac{(3+2r-r^2)}{4} \cdot \sqrt{\pi}t, \frac{(5-2r+r^2)}{4} \cdot \sqrt{\pi}t \right].$$

By applying the algorithm (13)-(17) for $r \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ with $\varepsilon = 10^{-15}$ and $n = 10$ the number of iterations is $k = 10$. For the test of convergence we apply again the algorithm with $n = 100$. The results expressed by the errors $e_{i,-}^r = |x^*(t_i, r)_- - x_k(t_i)_-^r|$, $e_{i,+}^r = |x^*(t_i, r)_+ - x_k(t_i)_+^r|$, $i = \overline{0, n}$, for $r \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ are presented for $n = 10$ in Table 1 and Table 3, and in Tables 2 and 4 when $n = 100$. We observe that the convergence stated in Theorem 5 is confirmed.

$t_i \backslash r$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0	0	0	0	0	0
0.2	1.2E-4	1.3E-4	1.5E-4	1.5E-4	1.6E-4
0.4	1.8E-4	2.0E-4	2.2E-4	2.3E-4	2.4E-4
0.6	2.3E-4	2.6E-4	2.8E-4	3.0E-4	3.0E-4
0.8	2.7E-4	3.1E-4	3.4E-4	3.6E-4	3.6E-4
1	3.1E-4	3.6E-4	3.9E-4	4.1E-4	4.2E-4

Table 1. Numerical results for $e_{i,-}^r$ and $n = 10$

$t_i \backslash r$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0	0	0	0	0	0
0.2	1.3E-6	1.5E-6	1.7E-6	1.7E-6	1.8E-6
0.4	1.9E-6	2.2E-6	2.4E-6	2.5E-6	2.6E-6
0.6	2.4E-6	2.8E-6	3.1E-6	3.2E-6	3.3E-6
0.8	2.9E-6	3.3E-6	3.6E-6	3.8E-6	3.9E-6
1	3.3E-6	3.8E-6	4.2E-6	4.4E-6	4.4E-6

Table 2. Numerical results for $e_{i,-}^r$ and $n = 100$

$t_i \backslash r$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0	0	0	0	0	0
0.2	2.1E-4	1.8E-4	1.7E-4	1.6E-4	1.6E-4
0.4	3.1E-4	2.8E-4	2.6E-4	2.5E-4	2.4E-4
0.6	3.8E-4	3.5E-4	3.3E-4	3.1E-4	3.1E-4
0.8	4.6E-4	4.2E-4	3.9E-4	3.7E-4	3.7E-4
1	5.3E-4	4.8E-4	4.5E-4	4.3E-4	4.2E-4

Table 3. Numerical results for $e_{i,+}^r$ and $n = 10$

$t_i \backslash r$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0	0	0	0	0	0
0.2	2.2E-6	2.1E-6	1.9E-6	1.8E-6	1.8E-6
0.4	3.3E-6	2.9E-6	2.8E-6	2.6E-6	2.6E-6
0.6	4.1E-6	3.7E-6	3.5E-6	3.3E-6	3.3E-6
0.8	4.8E-6	4.4E-6	4.1E-6	3.9E-6	3.9E-6
1	5.5E-6	5.1E-6	4.7E-6	4.5E-6	4.4E-6

Table 4. Numerical results for $e_{i,+}^r$ and $n = 100$

As can be viewed in Tables 1-4, the accuracy of the numerical results for $n = 10$ (stepsize $h = 10^{-1}$) is $O(10^{-4})$, and the accuracy is improved when $n = 100$ (stepsize $h = 10^{-2}$), being $O(10^{-6})$ in this case. Consequently, the convergence of the proposed method proved in Theorem 5 is numerically confirmed.

7 Concluding remarks

In this work we proposed an iterative numerical method that uses the fuzzy trapezoidal product integration technique for solving Riemann-Liouville type fractional fuzzy Volterra linear integral equations. The existence and uniqueness of the solution was approached both under the Bielecki's type metric and Hausdorff type metric. The use of the Bielecki's type metric does not requires any contraction condition for obtaining the existence and uniqueness of the solution, but since the condition which ensures the convergence of the method is the same as the contraction condition imposed under the Hausdorff type metric D^* we investigate the properties of the Picard iterations using this metric D^* . This sequence of iterations (8) is uniformly bounded and has uniform $(1, \alpha)$ -Lipschitz type property, this last notion being introduced in Definition 3 for this purpose. The convergence of the iterative method was proved obtaining the error estimate and was tested on a numerical example.

8 Compliance with Ethical Standards

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References

1. S. Abbas, M. Benchohra, Fractional order Riemann-Liouville integral equations with multiple time delays, *Appl. Math. E-Notes* 12 (2012) 79-87
2. R.P. Agarwal, V. Lakshmikantham, J.J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.* 72 (2010) 2859-2862
3. R.P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, Fuzzy fractional integral equations under compactness type condition, *Fract. Calc. Appl. Anal.* 15 (2012) 572-590
4. B. Agheli, M. Adabatabar Firozja, A Fuzzy Transform Method for numerical solution of fractional Volterra integral equations, *Int. J. Appl. Comput. Math.* (2020) 6:5.
5. N. Ahmad, A. Ullah, A. Ullah, S. Ahmad, K. Shah, I. Ahmad, On analysis of the fuzzy fractional order Volterra-Fredholm integro-differential equation, *Alexandria Engineering J.* 60 (2021) 1827-1838
6. T. Allahviranloo, A. Armand, Z. Gouyandeh, H. Ghadiri, Existence and uniqueness of solutions for fuzzy fractional Volterra-Fredholm integro-differential equations, *J. Fuzzy Set Valued Anal.* 2013 (2013), 1-9
7. T. Allahviranloo, A. Armand, Z. Gouyandeh, Fuzzy fractional differential equations under generalized fuzzy Caputo derivative, *J. Intelligent & Fuzzy Syst.* 26 (2014) 1481-1490
8. S. András, Weakly singular Volterra and Fredholm-Volterra integral equations, *Studia Univ. Babeş-Bolyai Math.* 48 (3) (2003) 147-155
9. A. Armand, T. Allahviranloo, S. Abbasbandy, Z. Gouyandeh, The fuzzy generalized Taylor's expansion with application in fractional differential equations, *Iranian J. Fuzzy Syst.* 16, no. 2 (2019) 57-72
10. S. Arshad, V. Lupulescu, On the fractional differential equations with uncertainty, *Nonlinear Anal.* 74 (2011) 85-93
11. A. Atangana, N. Bildik, Existence and numerical solution of the Volterra fractional integral equations of the second kind, *Math. Probl. Eng.* 2013 (2013), 1-12.
12. K.E. Atkinson, *An Introduction to Numerical Analysis*, second ed. John Wiley & Sons, New York 1989
13. K.E. Atkinson, The numerical solution of an Abel integral equation by a product trapezoidal method, *SIAM J. Numer. Anal.* 11 (1974) 97-101
14. D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, in: *Series on Complexity, Nonlinearity and Chaos*, vol. 3, World Scientific Publishers, Co., N. Jersey, London, Singapore, 2012
15. B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer-Verlag, Berlin Heidelberg, 2013.
16. B. Bede, S.G. Gal, Quadrature rules for integrals of fuzzy-number-valued functions, *Fuzzy Sets Syst.* 145 (2004) 359-380.
17. A.M. Bica, C. Popescu, Numerical solutions of the nonlinear fuzzy Hammerstein-Volterra delay integral equations, *Information Sciences* 233 (2013) 236-255
18. P. Diamond, P. Kloeden, Metric topology of fuzzy numbers and fuzzy analysis. In: Dubois, D., Prade, H., et al. (eds.) *Handbook Fuzzy Sets Ser.*, vol. 7, pp. 583-641, Kluwer Academic Publishers, Dordrecht (2000)
19. P. Diamond, Theory and applications of fuzzy Volterra integral equations, *IEEE Transactions on Fuzzy Systems* 10 (2002) 97-102.
20. S.G. Gal, Approximation theory in fuzzy setting, in: G.A. Anastassiou (ed.), *Handbook of analytic-computational methods in applied mathematics*, pp. 617-666, Chapman & Hall/CRC Press, Boca Raton, (Chapter 13) 2000.
21. R. Goetschel, W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets Syst.* 18 (1986) 31-43.
22. R. Gorenflo, *Fractals and Fractional Calculus in Continuum Mechanics*, in: A. Carpinteri, F. Mainardi (Eds.), Springer Verlag, Wien, 1997
23. N.V. Hoa, On the stability for implicit uncertain fractional integral equations with fuzzy concept, *Iranian J. Fuzzy Syst.* 18, no. 1 (2021) 185-201
24. S. Hu, M. Khavanin, W.A.N. Zhuang, Integral equations arising in the kinetic theory of gases. *Appl. Anal.* 34 no. 3-4 (1989) 261-266
25. R.W. Ibrahim, S. Momani, Upper and lower bounds of solutions for fractional integral equations, *Surv. Math. Appl.* 2 (2007) 145-156
26. E. Khodadadi, E. Celik, The variational iteration method for fuzzy fractional differential equations with uncertainty, *Fixed Point Theory Appl.* 2013 (2013), 1-13
27. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006
28. V. Lakshmikantham, Theory of fractional functional differential equations, *Nonlinear Anal.* 69 (2008) 3337-3343
29. V. Lupulescu, Fractional calculus for interval-valued functions, *Fuzzy Sets Syst.* 265 (2015) 63-85
30. N.I. Muskhelishvili, J.R.M. Radok, *Singular Integral Equations: Boundary Problems of Function Theory and their Application to Mathematical Physics*, Courier Corporation, Chelmsford (2008)
31. H.V. Ngo, Fuzzy fractional functional integral and differential equations, *Fuzzy Sets Syst.* 280 (2015) 58-90
32. N.V. Hoa, V. Lupulescu, D. O'Regan, Solving interval-valued fractional initial value problems under Caputo gH-fractional differentiability, *Fuzzy Sets Syst.* 309 (2017) 1-34
33. H.V. Ngo, V. Lupulescu, D. O'Regan, A note on initial value problems for fractional fuzzy differential equations, *Fuzzy Sets Syst.* 347 (2018) 54-69
34. S. Micula, An iterative numerical method for fractional integral equations of the second kind, *J. Comput. Appl. Math.* 339 (2018) 124-133
35. M. Mazandarani, A. Vahidian Kamyad, Modified fractional Euler method for solving fuzzy fractional initial value problem, *Commun. Nonlinear Sci. Numer. Simul.* 18 (2013) 12-21
36. L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval valued functions and interval differential equations, *Nonlinear Anal.* 71 (2009) 1311-1328
37. H. Vu, J.M. Rassias, N. Van Hoa, Ulam-Hyers-Rassias stability for fuzzy fractional integral equations, *Iranian J. Fuzzy Syst.* 17, no. 2 (2020) 17-27
38. H. Vu, N. Van Hoa, Applications of contractive-like mapping principles to fuzzy fractional integral equations with the kernel ψ -functions, *Soft Computing* 24, no. 24 (2020) 18841-18855
39. H. Vu, N.V. Hoa, Hyers-Ulam stability of fuzzy fractional Volterra integral equations with the kernel ψ -function via successive approximation method, *Fuzzy Sets Syst.* 419 (2021) 67-98

- 40. G.C. Wu, D. Băleanu, Variational iteration method for fractional calculus - a universal approach by Laplace transform, *Adv. Differential Equations* 2013 (18) (2013) 1-9
- 41. Congxin Wu, Cong Wu, The supremum and infimum of the set of fuzzy numbers and its applications, *J. Math. Anal. Appl.* 210 (1997) 499-511.
- 42. B. Zheng, Explicit bounds derived by some new inequalities and applications in fractional integral equations, *J. Inequalities Appl.* 2014, 2014: 4