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Research Article

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Posted Date: February 23rd, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-1087623/v2>

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Stability analysis and soliton solutions to the new Hamiltonian amplitude equation in mathematical physics

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Abstract

The new Hamiltonian amplitude (nHA) equation deals with some of the disabilities of the modulation wave-train. The main task of this paper is to extract the analytical wave solutions of the nHA equation. Based on the unified scheme, analytical wave solutions are attained in terms of hyperbolic and trigonometric function solutions. In order to prompt the underlying wave propagation characteristics, three-dimensional (3D), two-dimensional (2D) are illustrated from the solutions obtained with the help of computational packages Mathematica and also made comparisons between wave profiles for various values. The proposed method can also be used for many other nonlinear evolution equations.

Keywords: new HA equation; nonlinear science; stability analysis; soliton solutions; unified scheme.

MSC: 35B35, 35C07, 35C08.

1. Introduction

Nonlinear science is the study of those mathematical systems and nonlinear phenomena. Nonlinear phenomena play an important role in applied mathematics, physics, engineering and other numerous areas. Scheming exact and numerical solutions, especially in mathematical physics, the traveling wave solutions of NLEEs play an important role in soliton theory. Recently, many new schemes have recently been proposed to find the exact solution of nonlinear equations such as the multiple exp-function method [1], the Hirota bilinear method [2, 3], the extended tanh-function method [4], the Sardar-sub equation method [5], the enhanced (G'/G) -expansion method [6-8], the He's semi-inverse method [9],

the Hirota's method [10], the tanh-sech method [11], the modified homotopy perturbation method [12], the improve F-expansion method [13], the $(G'/G, 1/G)$ -expansion method [14], the improved fractional sub-equation method [15], the new auxiliary equation method [16], the extended sine-Gordon equation expansion method [17], the new extended direct algebraic method [18] and so on.

In the past few decades, many researchers have developed and simplified the new equation, analyzed the closed-form soliton solutions from the nonlinear evolution equations (NLEEs). The standard NLS equation is one of the most important equations in NLEEs. Ma and Chen [19] obtained the results on traveling wave type solutions could be achieved by using the same approaches to the standard NLS equation. Ma et al [20, 21, 22] have been explored the N -soliton solution and analyzed the Hirota N -soliton conditions. In 1992, Wadati et al. [23] developed the new Hamiltonian amplitude (HA) equation from the NLS equation and is given below:

$$iu_x + u_{tt} + 2\lambda|u|^2u - \epsilon u_{xt} = 0, \quad (1.1)$$

where $\lambda = \pm 1$, $\epsilon \ll 1$.

This is an equation that deals with some instabilities of modulation wave-train, with the supplementary stretch $-u_{xt}$ get over the ill-posedness of the unstable nonlinear Schrodinger equation. It is a Hamiltonian simulation of the Kuramoto-Shivashinsky equation which is grew up in a dissipative system and is not integrable. In Ref. [24], Yomba uses the general projection Riccati equations method to obtain the exact solutions of the HA equation. Peng [25] also used the modified mapping method to acquire the exact soliton solutions of the HA equation. Kumar et al. [26], Eslami and Mirzazadeh [27] established the exact traveling wave solutions of the HA equation. Mirzazadeh [28] applied the He's semi-inverse scheme to build up the topological and non-topological soliton solutions and Demiray [29] established the exact solutions of the HA equation by using the extended trail equation method. Zafar et al. [30] construct the optical soliton solutions of the HA equation using the Jacobi elliptic functions scheme. Manafian [31] obtained the periodic and singular kink solutions of the HA equation by using the two different techniques.

The purpose of this article is to apply the unified method [32, 33] to HA equation and found optical soliton solutions. As a result, optical soliton solutions in more comprehensive and different form are attained. Hamiltonian system is used to discuss the stability of exact solutions. The obtained solutions are mainly applicable to the optics, nonlinear optic and quantum optics and other areas. The design of the paper is organized as follows. In section 2,

deals with the overview of the unified method. Applications of the methods to the HA equation are presented in section 3. The nature of the obtained solutions has been discussed in Section 4. Stability analysis is also discussed in section 5. Finally in section 6, outcomes of the present study are presented.

2. Overview of the unified method

Let us consider the general form of the NLEEs as

$$\mathfrak{K}(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0, \quad (2.1)$$

where $u(x, y, t)$ is an unknown function, $\mathfrak{K}$ is a polynomial in $u = u(x, y, t)$. To search the travelling wave solutions of Eq. (2.1) taking the wave variable

$$u(x, y, t) = u(\zeta), \zeta = x - \theta t, \quad (2.2)$$

where θ is the traveling wave. Knocking Eq. (2.2) into Eq. (2.1) and yields the following ordinary differential equation (ODE):

$$\mathbf{K}(u, u', u'', \dots) = 0, \quad (2.3)$$

According to the unified method, the exact soliton solution of Eq. (2.3) is conjecture to be

$$u(\zeta) = A_0 + \sum_{j=1}^M [A_j w^j + B_j w^{-j}], \quad (2.4)$$

where $w = w(\zeta)$ satisfies the Riccati differential equation as follow:

$$w'(\zeta) = w^2(\zeta) + k, \quad (2.5)$$

where $w' = \frac{dw}{d\zeta}$ and $A_j (j = 1, 2, 3, \dots, M)$, $B_j (j = 1, 2, 3, \dots, M)$ and k are constants. Eq. (2.5)

has the following solutions:

Cluster 01: If $k < 0$, then the hyperbolic solutions are

$$w(\zeta) = \frac{\sqrt{-(X^2+Y^2)k} - X\sqrt{-k} \cosh(2\sqrt{-k}(\zeta+F))}{X \sinh(2\sqrt{-k}(\zeta+F)) + Y}, \quad (2.6)$$

$$w(\zeta) = \frac{-\sqrt{-(X^2+Y^2)k} - X\sqrt{-k} \cosh(2\sqrt{-k}(\zeta+F))}{X \sinh(2\sqrt{-k}(\zeta+F)) + Y}, \quad (2.7)$$

$$w(\zeta) = \sqrt{-k} + \frac{-2X\sqrt{-k}}{X + \cosh(2\sqrt{-k}(\zeta+F)) - \sinh(2\sqrt{-k}(\zeta+F))}, \quad (2.8)$$

$$w(\zeta) = -\sqrt{-k} + \frac{2X\sqrt{-k}}{X + \cosh(2\sqrt{-k}(\zeta+F)) + \sinh(2\sqrt{-k}(\zeta+F))}, \quad (2.9)$$

where the arbitrary constants X and Y are real, and F is an arbitrary constant.

Cluster 02: If $k > 0$, then the trigonometric solutions are

$$w(\zeta) = \frac{\sqrt{(X^2-Y^2)k} - X\sqrt{k} \cos(2\sqrt{k}(\zeta+F))}{X \sin(2\sqrt{k}(\zeta+F)) + Y}, \quad (2.10)$$

$$w(\zeta) = \frac{-\sqrt{(X^2-Y^2)k}-X\sqrt{k}\cos(2\sqrt{k}(\zeta+F))}{X\sin(2\sqrt{k}(\zeta+F))+Y}, \quad (2.11)$$

$$w(\zeta) = i\sqrt{k} + \frac{-2iX\sqrt{k}}{X+\cos(2\sqrt{k}(\zeta+F))-i\sin(2\sqrt{k}(\zeta+F))}, \quad (2.12)$$

$$w(\zeta) = -i\sqrt{k} + \frac{2iX\sqrt{k}}{X+\cos(2\sqrt{k}(\zeta+F))+i\sin(2\sqrt{k}(\zeta+F))}, \quad (2.13)$$

where the arbitrary constants X and Y are real, and F is an arbitrary constant.

Cluster 03: If $k = 0$, then the rational function solution is

$$w(\zeta) = -\frac{1}{\zeta+F}, \quad (2.14)$$

where F is an arbitrary constant.

We put Eq. (2.4) and (2.5) in Eq. (2.3) and associating all the coefficient of $m^i = (-N \leq i \leq N)$ to zero yield a set of algebraic equations for A_j, B_j, σ and k .

Putting A_j, B_j, σ and k into (2.4) and using the general solutions of Eq. (2.5), it can be obtained the solutions of Eq. (2.1) directly based on the value of k .

3. Formation of the solutions

Using wave transformation

$$u(x, t) = e^{iQ}u(\zeta), \quad Q = px + qt, \quad \zeta = \eta(x - \theta t). \quad (3.1)$$

Eq. (1.1) is converted to an ODE:

$$(\eta\theta^2 + \epsilon\eta^2\theta)u'' + i(\eta - 2q\eta\theta - \epsilon q\eta + \epsilon p\eta\theta)u' - (p + q^2 - \epsilon pq)u + 2\lambda u^3 = 0. \quad (3.2)$$

Equating real and imaginary parts on both sides, yields

$$(\eta\theta^2 + \epsilon\eta^2\theta)u'' - (p + q^2 - \epsilon pq)u + 2\lambda u^3 = 0, \quad (3.3)$$

and

$$(\eta - 2q\eta\theta - \epsilon q\eta + \epsilon p\eta\theta)u' = 0. \quad (3.4)$$

$$\text{From Eq. (3.4), we have } \theta = \frac{1-\epsilon q}{2q-p\epsilon}. \quad (3.5)$$

Applying balance applications in Eq. (3.3), we get $M = 1$. The trial solutions of Eq. (2.4) as

$$u(\zeta) = A_0 + A_1 w(\zeta) + B_1 \frac{1}{w(\zeta)}, \quad (3.6)$$

where A_0, A_1 and B_1 are constant to be determined later.

Inserting Eq. (3.6) into Eq. (3.3) along with Eq. (2.5) and then equating the coefficients of powers w^i to zero. We obtain the following algebraic equations:

$$2A_1(A_1^2\lambda + \eta\theta(\epsilon\eta + \theta)) = 0,$$

$$6\lambda A_0 A_1^2 = 0,$$

$$2A_1 \left(k\eta^2\theta\epsilon + k\eta^2\theta + 3A_1B_1\lambda + \frac{1}{2}pq\epsilon + 3\lambda A_0^2 - \frac{1}{2}q^2 - \frac{1}{2}p \right) = 0,$$

$$12A_0 \left(A_1B_1\lambda + \frac{1}{12}pq\epsilon + \frac{1}{6}\lambda A_0^2 - \frac{1}{12}q^2 - \frac{1}{12}p \right) = 0,$$

$$2B_1 \left(k\eta^2\theta\epsilon + k\theta^2\eta + 3A_1B_1\lambda + \frac{1}{2}pq\epsilon + 3\lambda A_0^2 - \frac{1}{2}q^2 - \frac{1}{2}p \right) = 0,$$

$$6\lambda A_0B_1^2 = 0,$$

$$2B_1(B_1^2\lambda + k^2\eta\theta(\epsilon\eta + \theta)) = 0.$$

To use the maple software and solve the above systems of equations, we obtain the following solutions set.

$$\text{Set 1: } \theta = \frac{-k\eta^2\epsilon \pm \sqrt{\epsilon^2\eta^4k^2 - 2\epsilon\eta k p q + 2\eta k q^2 + 2\eta k p}}{2k\eta}, A_0 = 0, A_1 = 0, B_1 = \pm \frac{\sqrt{2\lambda k(pq\epsilon - q^2 - p)}}{2\lambda}, \quad (3.7)$$

$$\text{Set 2: } \theta = \frac{-k\eta^2\epsilon \pm \sqrt{\epsilon^2\eta^4k^2 - 2\epsilon\eta k p q + 2\eta k q^2 + 2\eta k p}}{2k\eta}, A_0 = 0, A_1 = \pm \frac{\sqrt{2\lambda k(pq\epsilon - q^2 - p)}}{2\lambda}, B_1 = 0, \quad (3.8)$$

$$\text{Set 3: } \theta = \frac{-k\eta^2\epsilon \pm \sqrt{\epsilon^2\eta^4k^2 + \epsilon\eta k p q - \eta k q^2 - \eta k p}}{2k\eta}, A_0 = 0, A_1 = \pm \frac{\sqrt{-\lambda k(pq\epsilon - q^2 - p)}}{2k\lambda}, B_1 = \pm \frac{k(pq\epsilon - q^2 - p)}{2\sqrt{-\lambda k(pq\epsilon - q^2 - p)}}, \quad (3.9)$$

$$\text{Set 4: } \theta = \frac{-2k\eta^2\epsilon \pm \sqrt{4\epsilon^2\eta^4k^2 - 2\epsilon\eta k p q + 2\eta k q^2 + 2\eta k p}}{4k\eta}, A_0 = 0, A_1 = \pm \frac{\sqrt{2\lambda k(pq\epsilon - q^2 - p)}}{4k\lambda}, B_1 = \pm \frac{\sqrt{2}k(pq\epsilon - q^2 - p)}{4\sqrt{\lambda k(pq\epsilon - q^2 - p)}}. \quad (3.10)$$

Inserting Eq. (3.7) into Eq. (3.6) along with the Eqs. (2.6) -(2.13), one can attain the hyperbolic and trigonometric function solutions.

Cluster one:

The solutions are presented below for the case of $k < 0$

$$u_{1,2}(x, t) = \pm e^{iQ} \sqrt{\frac{q^2 + p - pq\epsilon}{2\lambda}} \times \frac{X \sinh(2\sqrt{-k}(\zeta + F)) + Y}{\sqrt{(X^2 + Y^2) - X \cosh(2\sqrt{-k}(\zeta + F))}}, \quad (3.11)$$

$$u_{3,4}(x, t) = \pm e^{iQ} \sqrt{\frac{q^2 + p - pq\epsilon}{2\lambda}} \times \frac{X \sinh(2\sqrt{-k}(\zeta + F)) + Y}{-\sqrt{X^2 + Y^2} - X \cosh(2\sqrt{-k}(\zeta + F))}, \quad (3.12)$$

$$u_{5,6}(x, t) = \pm e^{iQ} \sqrt{\frac{q^2 + p - pq\epsilon}{2\lambda}} \times \frac{X + \cosh(2\sqrt{-k}(\zeta + F)) - \sinh(2\sqrt{-k}(\zeta + F))}{-X + \cosh(2\sqrt{-k}(\zeta + F)) - \sinh(2\sqrt{-k}(\zeta + F))}, \quad (3.13)$$

$$u_{7,8}(x, t) = \pm e^{iQ} \sqrt{\frac{q^2 + p - pq\epsilon}{2\lambda}} \times \frac{X + \cosh(2\sqrt{-k}(\zeta + F)) + \sinh(2\sqrt{-k}(\zeta + F))}{-X + \cosh(2\sqrt{-k}(\zeta + F)) + \sinh(2\sqrt{-k}(\zeta + F))}, \quad (3.14)$$

The solutions are presented below for the case of $k > 0$

$$u_{9,10}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{X \sin(2\sqrt{k}(\zeta + F)) + Y}{\sqrt{X^2 - Y^2} - X \cos(2\sqrt{k}(\zeta + F))}, \quad (3.15)$$

$$u_{11,12}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{X \sin(2\sqrt{k}(\zeta + F)) + Y}{-\sqrt{X^2 - Y^2} - X \cos(2\sqrt{k}(\zeta + F))}, \quad (3.16)$$

$$u_{13,14}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{iX + i \cos(2\sqrt{k}(\zeta + F)) + \sin(2\sqrt{k}(\zeta + F))}{X - \cos(2\sqrt{k}(\zeta + F)) + i \sin(2\sqrt{k}(\zeta + F))}, \quad (3.17)$$

$$u_{15,16}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{iX + i \cos(2\sqrt{k}(\zeta + F)) - \sin(2\sqrt{k}(\zeta + F))}{-X + \cos(2\sqrt{k}(\zeta + F)) + i \sin(2\sqrt{k}(\zeta + F))}, \quad (3.18)$$

149 where $Q = px + qt$ and $\zeta = \eta \left(x - \frac{-k\eta^2\epsilon \pm \sqrt{\epsilon^2\eta^4k^2 - 2\epsilon\eta k p q + 2\eta k q^2 + 2\eta k p}}{2k\eta} t \right)$ and the solutions
 150 will exist, if satisfied the condition $\lambda(pq\epsilon - q^2 - p) > 0$.

151 **Cluster two:**

152 The solutions are presented below for the case of $k < 0$

$$153 \quad u_{17,18}(x, t) = \pm e^{iQ} \sqrt{\frac{q^2 + p - pq\epsilon}{2\lambda}} \times \frac{\sqrt{X^2 + Y^2} - X \cosh(2\sqrt{-k}(\zeta + F))}{X \sinh(2\sqrt{-k}(\zeta + F)) + Y}, \quad (3.19)$$

$$154 \quad u_{19,20}(x, t) = \pm e^{iQ} \sqrt{\frac{q^2 + p - pq\epsilon}{2\lambda}} \times \frac{-\sqrt{X^2 + Y^2} - X \cosh(2\sqrt{-k}(\zeta + F))}{X \sinh(2\sqrt{-k}(\zeta + F)) + Y}, \quad (3.20)$$

$$155 \quad u_{21,22}(x, t) = \pm e^{iQ} \sqrt{\frac{q^2 + p - pq\epsilon}{2\lambda}} \times \frac{-X + \cosh(2\sqrt{-k}(\zeta + F)) - \sinh(2\sqrt{-k}(\zeta + F))}{X + \cosh(2\sqrt{-k}(\zeta + F)) - \sinh(2\sqrt{-k}(\zeta + F))}, \quad (3.21)$$

$$156 \quad u_{23,24}(x, t) = \pm e^{iQ} \sqrt{\frac{q^2 + p - pq\epsilon}{2\lambda}} \times \frac{-X + \cosh(2\sqrt{-k}(\zeta + F)) + \sinh(2\sqrt{-k}(\zeta + F))}{X + \cosh(2\sqrt{-k}(\zeta + F)) + \sinh(2\sqrt{-k}(\zeta + F))}, \quad (3.22)$$

157 The solutions are presented below for the case of $k > 0$

$$158 \quad u_{25,26}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{\sqrt{X^2 - Y^2} - X \cos(2\sqrt{k}(\zeta + F))}{X \sin(2\sqrt{k}(\zeta + F)) + Y}, \quad (3.23)$$

$$159 \quad u_{27,28}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{-\sqrt{X^2 - Y^2} - X \cos(2\sqrt{k}(\zeta + F))}{X \sin(2\sqrt{k}(\zeta + F)) + Y}, \quad (3.24)$$

$$160 \quad u_{29,30}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{-iX + i \cos(2\sqrt{k}(\zeta + F)) + \sin(2\sqrt{k}(\zeta + F))}{-X - \cos(2\sqrt{k}(\zeta + F)) + i \sin(2\sqrt{k}(\zeta + F))}, \quad (3.25)$$

$$161 \quad u_{31,32}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{-iX + i \cos(2\sqrt{k}(\zeta + F)) - \sin(2\sqrt{k}(\zeta + F))}{X + \cos(2\sqrt{k}(\zeta + F)) + i \sin(2\sqrt{k}(\zeta + F))}, \quad (3.26)$$

162 where $Q = px + qt$ and $\zeta = \eta \left(x - \frac{-k\eta^2\epsilon \pm \sqrt{\epsilon^2\eta^4k^2 - 2\epsilon\eta k p q + 2\eta k q^2 + 2\eta k p}}{2k\eta} t \right)$ and the solutions
 163 will exist, if satisfied the condition $\lambda(pq\epsilon - q^2 - p) > 0$.

164 **Cluster three:**

165 For $k < 0$, one can obtain the following solutions:

$$166 \quad u_{33,34}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{\lambda}} \times \frac{X(\sqrt{X^2 + Y^2} \cosh(2\sqrt{-k}(\zeta + F)) + (\sinh(2\sqrt{-k}(\zeta + F))Y - X))}{(X \sinh(2\sqrt{-k}(\zeta + F)) + Y)(X \cosh(2\sqrt{-k}(\zeta + F)) - \sqrt{X^2 + Y^2})}, \quad (3.27)$$

$$167 \quad u_{35,36}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{\lambda}} \times \frac{X(\cosh(2\sqrt{-k}(\zeta + F))\sqrt{X^2 + Y^2} - (\sinh(2\sqrt{-k}(\zeta + F))Y - X))}{(X \sinh(2\sqrt{-k}(\zeta + F)) + Y)(X \cosh(2\sqrt{-k}(\zeta + F)) + \sqrt{X^2 + Y^2})}, \quad (3.28)$$

$$168 \quad u_{37,38}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{\lambda}} \times \frac{2X(\cosh(2\sqrt{-k}(\zeta + F)) - \sinh(2\sqrt{-k}(\zeta + F)))}{2\cosh^2(2\sqrt{-k}(\zeta + F)) - 2\cosh(2\sqrt{-k}(\zeta + F))\sinh(2\sqrt{-k}(\zeta + F)) - X^2 - 1}, \quad (3.29)$$

$$169 \quad u_{39,40}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{\lambda}} \times \frac{2X(\cosh(2\sqrt{-k}(\zeta + F)) - \sinh(2\sqrt{-k}(\zeta + F)))}{2\cosh^2(2\sqrt{-k}(\zeta + F)) - 2\cosh(2\sqrt{-k}(\zeta + F))\sinh(2\sqrt{-k}(\zeta + F)) - X^2 - 1}, \quad (3.30)$$

170 For $k > 0$, one can yield the following solutions:

$$171 \quad u_{41,42}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{\lambda}} \times \frac{X(\cos(2\sqrt{k}(\zeta + F))\sqrt{X^2 - Y^2} - (\sin(2\sqrt{k}(\zeta + F))Y + X))}{i(X \sin(2\sqrt{k}(\zeta + F)) + Y)(X \cos(2\sqrt{k}(\zeta + F)) - \sqrt{X^2 - Y^2})}, \quad (3.31)$$

$$u_{43,44}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{\lambda}} \times \frac{X(\cos(2\sqrt{k}(\zeta+F))\sqrt{X^2-Y^2} + (\sin(2\sqrt{k}(\zeta+F))Y+X))}{i(X\sin(2\sqrt{k}(\zeta+F))+Y)(X\sqrt{k}\cos(2\sqrt{k}(\zeta+F))+\sqrt{X^2-Y^2})}, \quad (3.32)$$

$$u_{45,46}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{\lambda}} \times \frac{2X(i\sin(2\sqrt{k}(\zeta+F)) - \cos(2\sqrt{k}(\zeta+F)))}{2i\cos(2\sqrt{k}(\zeta+F))\sin(2\sqrt{k}(\zeta+F)) - 2\cos^2(2\sqrt{k}(\zeta+F)) + X^2 + 1}, \quad (3.33)$$

$$u_{47,48}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{\lambda}} \times \frac{2X(\cos(2\sqrt{k}(\zeta+F)) + i\sin(2\sqrt{k}(\zeta+F)))}{2\cos^2(2\sqrt{k}(\zeta+F)) + 2i\cos(2\sqrt{k}(\zeta+F))\sin(2\sqrt{k}(\zeta+F)) - X^2 - 1}, \quad (3.34)$$

where $Q = px + qt$ and $\zeta = \eta \left(x - \frac{-k\eta^2\epsilon \pm \sqrt{\epsilon^2\eta^4k^2 + \epsilon\eta k p q - \eta k q^2 - \eta k p}}{2k\eta} t \right)$. The obtained solutions will exist, if satisfied the condition $\lambda(pq\epsilon - q^2 - p) > 0$.

Cluster four:

For $k < 0$, one can get the following solutions:

$$u_{49,50}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{(-X\sqrt{X^2+Y^2}\cosh(2\sqrt{-k}(\zeta+F)) + (\cosh^2(2\sqrt{-k}(\zeta+F))X^2 + Y(X\sinh(2\sqrt{-k}(\zeta+F))+Y)))}{i(X\sinh(2\sqrt{-k}(\zeta+F))+Y)(X\cosh(2\sqrt{-k}(\zeta+F)) - \sqrt{X^2+Y^2})}, \quad (3.35)$$

$$u_{51,52}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{(-X\sqrt{X^2+Y^2}\cosh(2\sqrt{-k}(\zeta+F)) - (\cosh^2(2\sqrt{-k}(\zeta+F))X^2 + Y(X\sinh(2\sqrt{-k}(\zeta+F))+Y)))}{i(X\sinh(2\sqrt{-k}(\zeta+F))+Y)(X\sqrt{-k}\cosh(2\sqrt{-k}(\zeta+F)) + \sqrt{X^2+Y^2})}, \quad (3.36)$$

$$u_{53,54}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{2\cosh^2(2\sqrt{-k}(\zeta+F)) - 2\cosh(2\sqrt{-k}(\zeta+F))\sinh(2\sqrt{-k}(\zeta+F)) + X^2 - 1}{i(2\cosh^2(2\sqrt{-k}(\zeta+F)) - 2\cosh(2\sqrt{-k}(\zeta+F))\sinh(2\sqrt{-k}(\zeta+F)) - X^2 - 1)}, \quad (3.37)$$

$$u_{55,56}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{2\cosh^2(2\sqrt{-k}(\zeta+F)) - 2\cosh(2\sqrt{-k}(\zeta+F))\sinh(2\sqrt{-k}(\zeta+F)) + X^2 - 1}{i(2\cosh^2(2\sqrt{-k}(\zeta+F)) - 2\cosh(2\sqrt{-k}(\zeta+F))\sinh(2\sqrt{-k}(\zeta+F)) - X^2 - 1)}, \quad (3.38)$$

For $k > 0$, one can determine the following solutions:

$$u_{57,58}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{(X\sqrt{X^2-Y^2}\cos(2\sqrt{k}(\zeta+F)) - (\cos^2(2\sqrt{k}(\zeta+F))X^2 - \sin(2\sqrt{k}(\zeta+F))XY - Y^2))}{(X\sin(2\sqrt{k}(\zeta+F))+Y)(X\cos(2\sqrt{k}(\zeta+F)) - \sqrt{X^2-Y^2})}, \quad (3.39)$$

$$u_{59,60}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{(X\sqrt{X^2-Y^2}\cos(2\sqrt{k}(\zeta+F)) + (\cos^2(2\sqrt{k}(\zeta+F))X^2 - \sin(2\sqrt{k}(\zeta+F))XY - Y^2))}{(X\sin(2\sqrt{k}(\zeta+F))+Y)(X\cos(2\sqrt{k}(\zeta+F)) + \sqrt{X^2-Y^2})}, \quad (3.40)$$

$$u_{61,62}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{(2i\cos^2(2\sqrt{k}(\zeta+F)) + iX^2 + 2\cos(2\sqrt{k}(\zeta+F)) + 2\sin(2\sqrt{k}(\zeta+F)) - i)}{2i\cos(2\sqrt{k}(\zeta+F))\sin(2\sqrt{k}(\zeta+F)) - 2\cos^2(2\sqrt{k}(\zeta+F)) + X^2 + 1}, \quad (3.41)$$

$$u_{63,64}(x, t) = \pm e^{iQ} \sqrt{\frac{pq\epsilon - q^2 - p}{2\lambda}} \times \frac{(-2i\cos^2(2\sqrt{k}(\zeta+F)) - iX^2 + 2\cos(2\sqrt{k}(\zeta+F))\sin(2\sqrt{k}(\zeta+F)) + i)}{2\cos^2(2\sqrt{k}(\zeta+F)) + 2i\cos(2\sqrt{k}(\zeta+F))\sin(2\sqrt{k}(\zeta+F)) - X^2 - 1}, \quad (3.42)$$

where $Q = px + qt$ and $\zeta = \eta \left(x - \frac{-2k\eta^2\epsilon \pm \sqrt{4\epsilon^2\eta^4k^2 - 2\epsilon\eta k p q + 2\eta k q^2 + 2\eta k p}}{4k\eta} t \right)$ and the solutions will exist, if satisfied the condition $\lambda(pq\epsilon - q^2 - p) > 0$.

Remark: We have simplified all the above solutions and tested them with Maple. All solutions have satisfied the original equation.

4. Discuss the nature of the obtained solutions

In this segment, we will discuss the effect of the parameter ϵ of the HA equation through its obtained solutions. To explain the impact of the parameter ϵ , we have presented some 3D and 2D wave profile of the attained solutions under selection of the different values of ϵ ($\epsilon \ll 1$). The nature of the solution $|u_1(x, t)|$ and the effect of its free parameter (ϵ) are displayed in **Figs. 1(a)-(d)**. The 3D wave structure of the solution $|u_1(x, t)|$ are prepared

under taking the free parameters values as $p = 1, q = 1, k = -0.01, \lambda = 1, \eta = 0.01, X = 0.5, Y = 0.1, F = 0.02$, and different values of $\epsilon = -3.6, -0.5, 0.5$. It is seen from the **Figs. 1(a)-1(b)** that solution $|u_1(x, t)|$ represents the bell shape wave structure when we choose $\epsilon = -0.5$ and 0.5 , respectively. On the other hand, under selection of very small values of ϵ ($\epsilon = -3.6$), the bell shape wave structure can change into the singular bell shape wave, which is depicted in **Fig. 1(c)**. The above behaviors are displayed in comparison graph (see **Fig. 1(d)**). It is also seen from the **Fig. 1(d)** that the amplitude of the wave profiles is increasing with the decrease of the values ϵ .

Again, we have illustrated the 3D wave structure of the solution $|u_9(x, t)|$ in **Figs. 2(a)-(c)** under selection of the parameters $p = 1, q = 1, k = 31.5, \lambda = 1, \eta = 0.3, X = 0.01, Y = 0.1, F = 0.04$ and for different values of $\epsilon = -0.9, 0.02, 0.9$. The 3D wave profile of the solutions represents the periodic wave structure. Meanwhile, **Fig. 2(d)** represents the 2D cross-sectional comparison plots between the different wave profiles at $t = 5$. It is seen from its comparison graph (**Fig. 2(d)**) that the signal similarities are almost identical for $\epsilon = -0.9$, and 0.02 . But, the wave amplitude for $\epsilon = -0.9$ is lower than for $\epsilon = 0.02$. On the other hand, for $\epsilon = 0.9$, the signals are almost 90-degree phase that of $\epsilon = -0.9$, and 0.02 . Therefore, the amplitude of the wave profiles is increases when the value of parameter ϵ decreases.

Figure 2. 3D profile of the periodic wave solutions of $|u_9(x, t)|$ with the effects of ϵ for (a) $\epsilon = -0.9$, (b) $\epsilon = 0.02$, (c) $\epsilon = 0.9$, and (d) The corresponding 2D wave profile of (a)-(c) at $t = 5$.

Finally, we have displayed the 3D wave profile of the solution $|u_{23}(x, t)|$ in **Figs. 3(a)-(c)**. As seen from **Fig. 3(a)**, the wave profile represents the U shape when we select the free parameters as $p = 1.5$, $q = 1.7$, $k = -0.01$, $\lambda = 1$, $\eta = 0.02$, $X = 0.1$, $F = 0.023$. For the value of $\epsilon = 0.9$, we have found the U shape wave profile in the range $-5 \leq x, t \leq 5$. If we put $\epsilon = -15$ and -28 in the solution $|u_{23}(x, t)|$ and plotted them, then it shapes of the wave profile represents periodic, which are exhibited in **Fig. 3(b)** and **Fig. 3(c)**. Fig. 3(d) represents the comparison graphs between the wave profiles for $\epsilon = 0.9, -15$ and -28 . We can perceive that the number of oscillations and amplitude are increasing as the values of ϵ decreases. It is also seen from the wave signal of the solution $|u_{23}(x, t)|$ that wave length is decreases when the value of ϵ decreases. Therefore, it is obvious from the graphical illustrations that the parameter ϵ have an influential role to depict the wave solutions of the HA equation.

Figure 3. 3D wave profile of the solutions of $|u_{23}(x, t)|$ with the effects of ϵ for (a) $\epsilon = 0.9$, (b) $\epsilon = -15$, (c) $\epsilon = -28$, and (d): The corresponding 2D wave profile of (a)-(c) at $x = 0$.

5. Stability Analysis

Hamiltonian system is a mathematical tool to describe the evolution of physical system. We practice the general form of the given Hamiltonian system

$$\varphi(\theta) = \int_{-\infty}^{\infty} \frac{\rho^2(\zeta)}{2} d\zeta, \quad (5.1)$$

where $\varphi(\theta)$ represents the momentum and $\rho(\zeta)$ represent traveling wave solutions. The adequate condition for stability is expressed as

$$\frac{\partial \varphi(\theta)}{\partial \theta} > 0, \quad (5.2)$$

where θ is the speed of the velocity. Eq. (5.1) and Eq. (5.2) are used to control the explicit parameters and intermissions at which the traveling wave solutions for the HA is stable. Applying the sufficient conditions of Eq. (5.1) and Eq. (5.2) in selecting interval $[-5,5]$ for the traveling wave solution, we obtain

$$\begin{aligned} \varphi(\theta) = & \frac{X\sqrt{X^2+Y^2}\epsilon pq}{2\lambda\sqrt{-k}Y^2(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))+\Omega)} - \frac{X\sqrt{X^2+Y^2}q^2}{2\lambda\sqrt{-k}Y^2(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))+\Omega)} - \frac{X\sqrt{X^2+Y^2}p}{2\lambda\sqrt{-k}Y^2(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))+\Omega)} - \\ & \frac{X^2\epsilon pq}{2\lambda\sqrt{-k}Y^2(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))+\Omega)} + \frac{X^2q^2}{2\lambda\sqrt{-k}Y^2(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))+\Omega)} + \frac{X^2p}{2\lambda\sqrt{-k}Y^2(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))+\Omega)} + \\ & \frac{\ln(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))-1)\epsilon pq}{8\lambda\sqrt{-k}} - \frac{\ln(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))-1)q^2}{8\lambda\sqrt{-k}} - \frac{\ln(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))-1)p}{8\lambda\sqrt{-k}} - \\ & \frac{\ln(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))+1)\epsilon pq}{8\lambda\sqrt{-k}} + \frac{\ln(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))+1)q^2}{8\lambda\sqrt{-k}} + \frac{\ln(\tanh(\sqrt{-k}F+\sqrt{-k}\eta(x-\theta t))+1)p}{8\lambda\sqrt{-k}}. \end{aligned}$$

where $\Omega = \frac{X}{Y} - \frac{\sqrt{X^2+Y^2}}{Y}$.

$$\begin{aligned} \frac{\partial \varphi(\theta)}{\partial \theta} = & \frac{\eta t (\cosh^2(\sqrt{-k}((x-\theta t)\eta+F)) + \sinh(\sqrt{-k}((x-\theta t)\eta+F))) Y(\sqrt{X^2+Y^2}-X)}{4\lambda((\sqrt{X^2+Y^2}X-X^2-Y^2) \cosh^2(\sqrt{-k}((x-\theta t)\eta+F)) + \sinh(\sqrt{-k}((x-\theta t)\eta+F)))} \cdot \\ & \frac{\cosh(\sqrt{-k}((x-\theta t)\eta+F)) - \sqrt{X^2+Y^2}X + X^2 + \frac{Y^2}{2}(\epsilon pq - q^2 - p)}{Y(\sqrt{X^2+Y^2}-X) \cosh(\sqrt{-k}((x-\theta t)\eta+F)) + \frac{Y^2}{2}}. \end{aligned}$$

By picking the parameter values $p = 1, q = 1, k = -0.1, \lambda = 1, \eta = -0.2, \epsilon = -0.1$ and $\theta = \frac{-k\eta^2\epsilon \pm \sqrt{\epsilon^2\eta^4k^2 - 2\epsilon\eta kpq + 2\eta kq^2 + 2\eta kp}}{2k\eta}$, we have $\frac{\partial \varphi(\theta)}{\partial \theta} > 0$. Therefore, we assume that the traveling wave solutions is stable in the interval $[-5, 5]$.

6. Conclusion

In summary, we have applied the unified method applied to find the new exact solution to the nHA equation. We discussed the nature of the solution and the attained solution is expressed by the bell soliton, the periodic wave soliton, the singular bell soliton, which are is shown in Figs. 1-3. Moreover, we proved the stability of the solution. It seems that the unified scheme is powerful, suitable, direct and provides a universal wave solution for NLEEs in science, engineering and mathematical physics and other numerous areas. This technique could be applied to study many other NLEEs in future.

Conflict of interest

There is no conflict of interest.

Authors Contributions

All authors contributed equally and real and approved the final version of the manuscript.

Funding sources

We have no funding.

Ethical statement

Compliance with ethical standards.

Data Availability

My manuscript has no associated data.

Acknowledgement

We would express our sincere thanks to referee for his enthusiastic help and valuable suggestions.

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