Transitional Periodic Patterns and Triple Attractors in a Predator-Prey System with Fear and Refuge

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Transitional periodic patterns and triple attractors in a predator-prey system with fear and refuge

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Abstract We report the existence of periodic structures in the transitional and chaotic regimes in bi-parameter spaces of a predator-prey system. A model is constructed taking into consideration of two important effects: namely, the prey refuge and fear of predation risk. The fixed points, their existence and stability behaviors are analyzed. The Neimark-Sacker bifurcation in the neighborhood of the interior fixed point is shown selecting refuge strength as a bifurcation parameter. The complex dynamical behaviors are explored in the biparameter space with the help of the largest Lyapunov exponent and isoperiodic diagrams. The period-bubbling transitional patterns, and triple heterogeneous attractors resulting in qualitative unpredictability are identified in the present system. The Wada basin sets for the triple coexisting attractors are found. The study reveals that the oscillations of the populations in certain control parameter regions are highly dependent upon the initial densities of the populations.

Keywords Fear effect · Refuge · Bi-parameter space · Arnold tongues · Multiple attractors · Wada basin.

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1 Introduction

In the 21st century, foraging and refuge from predation are the burning issues in many predator-prey populations. The impact of predator's fear has long-term effects on the prey population. The interactions among predator and prey are very diversified in nature. Predators exert a potent influence on the plurality and dynamics of prey species through hunting, which has a far-reaching impact on ecological communities. However, it is not the fact that predators influence the prey population through direct killing only, rather the fear of predation risk exerts a more deadly effect on the prey population than direct predation [Lima and Dill, 1990]. So, the predators' influence on the prey population can arise via direct or indirect or both ways. In the mere presence of the predators, prey species undergo various changes in their behaviors due to the fear of predation risk from the predators. Afraid preys generally go out to their pasture for a short period of time, which diminishes their growth rate and they take up some ordinary survival strategies like starvation [Creel and Christianson, 2008]. For an instance, mule deer (*Odocoileus hemionus*) forages less due to the fear of mountain lions (*Puma concolor*) [Altendorf et al., 2001]; younger backswimmers limit their foraging time due to the predation risk of cannibalistic backswimmers [Sih, 1980], etc. In addition, due to the fear of possible attack by predators, prey species often increase their surveillance rate and decrease foraging time [Lima and Dill, 1990]; sometimes frightened parents feed their heirs very little [Eggers et al., 2005], etc. Also, the physiological condition of the prey species can be affected by the fear of predators, which may result in long-lasting damage to the prey population. For example, in the Greater Yellowstone Ecosystem, Creel et al. noticed that elk (*Cervus elaphus*) make alterations in their reproductive physiology due to the fear of wolves (*Canis lupus*) [Creel et al., 2007]. Sheriff et al. observed in their experimental work on snowshoe hares that the predator-induced fear caused a decline in the reproductive output of hares, which was measured based on both the number and the quality of offsprings (offsprings from stressed dams were 37% lighter and 16% smaller than offsprings from controlled dams) [Sheriff et al., 2009]. In the field experiment on song sparrows (*Melospiza melodia*), Zanette et al. [Zanette et al., 2011] observed that only predator-induced fear could minimize the productivity of the heirs of song sparrow by 40%. In mathematical modeling, Wang et al. [Wang et al., 2016] first modeled the cost of fear in a two-species model. Subsequently, a number of researchers studied different ecological and eco-epidemiological models by incorporating the fear phenomenon, and have found interesting results [Tiwari et al., 2021, Biswas et al., 2021].

Prey species in every taxon use a range of defensive tactics to survive from predators, among which refuge is a highly effective and extensively used protective strategy. In ecology and evolution, a refuge is considered as one of the integrating concepts that provide some degree of protection to an organism from predation and other environmental difficulties [Berryman and Hawkins, 2006]. Different species of animals often look for refuges to protect themselves from predators. For instance, in a wolf-ungulate system, ungulates migrate to the wolf (*Canis lupus*) free area or outside the core region of wolves' with the aim of refuge [Mech, 1977]. Warner et al. noticed that, due to the risk of consumption by largemouth bass (*Micropterus salmoides*), bluegill sunfishes (*Lepomis macrochirus*) shift their habitat in a safer but less productive food zone [Werner et al., 1983]. To avoid the predation risk from owls, two species of gerbils (*Gerbillus allenbyi* and *Gerbillus pyramidum*) move into safer habitats and spend less time in foraging [Kotler et al., 1991]. In the presence of predators, preys make use of refuge, but not every member of a population or within social groups responds in the same magnitude [Belgrad and Griffen, 2016]. Many researchers have considered two types of refuges in their research articles [Collings, 1995], among which one type of refuge preserves a constant proportion of the prey population and the other type of refuge preserves a definite number of prey. The use of refuges makes the prey species invisible or notably less visible to aggressive predators, which prevents prey species from becoming extinct and increases the likelihood of coexistence of both prey and predator species [Hochberg and Holt, 1995]. Many researchers have explored how preys' refuge affects the dynamics of predator-prey systems and most of them concluded that refuge can produce a stabilized impact on the system [Sih, 1987]. A few researchers have observed the dynamics of predator-prey models by combining refuge with different biological phenomena like predator mutual interference [Chen et al., 2013], harvesting [Lv et al., 2014], additional food [Ghosh et al., 2017], fear effect [Zhang et al., 2019], etc.

Zhang et al. [Zhang et al., 2019] studied a continuous-time prey-predator model incorporating a prey refuge with Holling type-II response function considering the impact of fear on prey demography. Also, Wang et al. [Wang et al., 2019] investigated the influence of fear factor on the dynamics of a modified Leslie-Gower prey-predator model in a continuous-time set-up incorporating refuge for prey population. In the present paper, our aim is to explore the combined effects of fear and refuge on a prey-predator system in a discrete-time setup. The essence of this work is to study two-parameter dynamics and the existence of organized structures in transitional and chaotic regimes. The rest of the paper is arranged as follows: In Section 2, the mathematical model is outlined in short with the existence and stability of the fixed points. In Subsection 2.2, we discuss the Neimark-Sacker bifurcation in the region of the coexisting fixed point with respect to the refuge parameter. In Section 3, we

investigate the system's affluent dynamical behaviors in the parameter plane by constructing isoperiodic and Lyapunov exponent diagrams. Finally, the paper is ended with a conclusion in Section 4.

2 Model formulation and mathematical preliminaries

The interaction between prey and predator population satisfying Holling type-II response function is described as follows:

$$\begin{cases} x_{n+1} = x_n \exp \left(R - D_1 - Px_n - \frac{Ay_n}{B + x_n} \right), \\ y_{n+1} = y_n \exp \left(\frac{cAx_n}{B + x_n} - D_2 \right), \end{cases} \quad (2.1)$$

where x_n and y_n denote the prey and predator population densities at the n -th generation, R is the birth rate of prey, D_1 represents the prey's natural death rate, and the parameter P is related to the intra-species competition among prey. A represents the predator's maximum food intake (consumption) rate, B is the half-saturation constant of the predator, c ($0 < c < 1$) represents the rate of conversion of the biomass from prey to predator, and D_2 is the predator's mortality rate.

Further, we incorporate a constant proportion of prey refuge, that protects the mx_n portion of prey, where $m \in [0, 1)$ is constant, measures the holding capacity of the refuge. The remaining amount of prey, $(1 - m)x_n$ portion exists in open habitat, and those preys are readily available to the predators. Then, the model (2.1) becomes

$$\begin{cases} x_{n+1} = x_n \exp \left(Rm + R(1 - m) - D_1 - Px_n - \frac{A(1 - m)y_n}{B + (1 - m)x_n} \right), \\ y_{n+1} = y_n \exp \left(\frac{cA(1 - m)x_n}{B + (1 - m)x_n} - D_2 \right). \end{cases} \quad (2.2)$$

For the inclusion of the fear of predation in the above model, we multiply the reproduction term of the prey available to the predators by a decreasing function $f(k, y_n) = \frac{1}{1 + ky_n}$ as suggested by Wang et al. [Wang et al., 2016], where k indicates the potency of predator-induced fear on the prey. It is worth mentioning here that, depending on the predators' characteristics and other environmental constituents the potency of fear may differ.

Taking into account of the fear effect, the above model (2.2) takes the following form:

$$\begin{cases} x_{n+1} = x_n \exp \left(Rm + \frac{R(1 - m)}{1 + ky_n} - Px_n - D_1 - \frac{A(1 - m)y_n}{B + (1 - m)x_n} \right), \\ y_{n+1} = y_n \exp \left(\frac{cA(1 - m)x_n}{B + (1 - m)x_n} - D_2 \right), \end{cases} \quad n = 0, 1, 2, \dots \quad (2.3)$$

Here, the values of all the parameters included in the above system are non-negative.

2.1 Fixed points and their stabilities

From the biological standpoint, the system has three feasible fixed points. Without prolonging the work unnecessarily, the feasibility and stability conditions of the fixed points are briefly mentioned in Table 1.

Table 1: Feasibility and local stability criteria of the fixed points of system (2.3)

Fixed Point	Feasibility Criteria	Stability Criteria
$E_0(0, 0)$	always feasible	locally asymptotically stable (LAS) if $R < D_1$
$E_1 = \left(\frac{R-D_1}{P}, 0\right)$	$R > D_1$	LAS if $D_1 < R < D_1 + 2$ and $(R - D_1)(1 - m)(cA - D_2) < PBD_2$
$E^*(x^*, y^*)$, where $x^* = \frac{BD_2}{(1 - m)(cA - D_2)}$, $y^* = \frac{-L_2 + \sqrt{L_2^2 - 4L_1L_3}}{2L_1}$ with $L_1 = k(1 - m)^2(cA - D_2)^2$, $L_2 = (1 - m)^2(cA - D_2)^2 + ckB(PBD_2 + (1 - m)(cA - D_2)(D_1 - Rm))$, and $L_3 = cB(PBD_2 + (1 - m)(cA - D_2)(D_1 - R))$	$cA > D_2$ and $R > D_1 + \frac{PBD_2}{(1 - m)(cA - D_2)}$ or $Rm > D_1 + \frac{PBD_2}{(1 - m)(cA - D_2)} + \frac{(1 - m)(cA - D_2)}{kcB}$	$1 + T + D > 0$ and $D < 1$ where $T = 2 - Px^* + \frac{A(1 - m)^2x^*y^*}{(B + (1 - m)x^*)^2}$, and $D = 1 - Px^* + \frac{A(1 - m)^2x^*y^*}{(B + (1 - m)x^*)^2} + \frac{cA^2B(1 - m)^2x^*y^*}{(B + (1 - m)x^*)^3} + \frac{ckABR(1 - m)^2x^*y^*}{(B + (1 - m)x^*)^2(1 + ky^*)^2}$

2.2 Existence of Local Bifurcation

Bifurcation is the heart of dynamics resulting in structural change in the neighborhood of fixed points when the system's parameters reached the critical value [Layek, 2015]. Here, we will show that the system (2.3) undergoes Neimark-Sacker (N-S) bifurcation at the positive fixed point $E^*(x^*, y^*)$ by choosing the refuge parameter m as a bifurcation parameter. This gives a closed invariant curve that emanates from the fixed point. To determine the necessary conditions for N-S bifurcation, we go through the following steps:

Step I:

The Jacobian matrix associated with the linearization of the map (2.3) at $E^*(x^*, y^*)$ is given by

$$J^* = \begin{pmatrix} 1 - Px^* + \frac{A(1 - m)^2x^*y^*}{(B + (1 - m)x^*)^2} & -\frac{Rk(1 - m)x^*}{(1 + ky^*)^2} - \frac{A(1 - m)x^*}{B + (1 - m)x^*} \\ \frac{cAB(1 - m)y^*}{(B + (1 - m)x^*)^2} & 1 \end{pmatrix}.$$

The characteristic values of the matrix J^* are the roots of the following equation:

$$\lambda^2 - Tr(J^*)\lambda + Det(J^*) = 0,$$

where

$$\begin{cases} Tr(J^*) = 2 - Px^* + \frac{A(1-m)^2 x^* y^*}{(B + (1-m)x^*)^2}, \\ Det(J^*) = 1 - Px^* + \frac{A(1-m)^2 x^* y^*}{(B + (1-m)x^*)^2} + \frac{cA^2 B(1-m)^2 x^* y^*}{(B + (1-m)x^*)^3} + \frac{ckABR(1-m)^2 x^* y^*}{(B + (1-m)x^*)^2(1 + ky^*)^2}. \end{cases} \quad (2.4)$$

For ensuing N-S bifurcation, the necessary conditions are

$$(Tr(J^*))^2 - 4Det(J^*) < 0 \quad \text{and} \quad Det(J^*) = 1. \quad (2.5)$$

After applying the above two conditions to our proposed model, the following expressions are obtained:

$$F_1(m) \equiv \left(Px^* - \frac{A(1-m)^2 x^* y^*}{(B + (1-m)x^*)^2} \right)^2 - \frac{4cA^2 B(1-m)^2 x^* y^*}{(B + (1-m)x^*)^3} - \frac{4ckABR(1-m)^2 x^* y^*}{(B + (1-m)x^*)^2(1 + ky^*)^2} < 0, \quad \text{and}$$

$$F_2(m) \equiv \frac{ckABR(1-m)^2 x^* y^*}{(B + (1-m)x^*)^2(1 + ky^*)^2} + \frac{cA^2 B(1-m)^2 x^* y^*}{(B + (1-m)x^*)^3} + \frac{A(1-m)^2 x^* y^*}{(B + (1-m)x^*)^2} - Px^* = 0.$$

In accordance with the above discussion, the system (2.3) exhibits N-S bifurcation at $m = m_h$, where m varies in a small neighbourhood of m_h , which is denoted by Δ and defined as

$$\Delta = \left\{ (R, m, k, P, D_1, A, B, c, D_2) : m = m_h, F_1(m_h) < 0, F_2(m_h) = 0, cA > D_2, R > D_1 + \frac{PBD_2}{(1-m)(cA - D_2)} \text{ or } \right. \\ \left. Rm > D_1 + \frac{PBD_2}{(1-m)(cA - D_2)} + \frac{(1-m)(cA - D_2)}{ckB}, R > 0, m > 0, k > 0, P > 0, \right. \\ \left. D_1 > 0, A > 0, B > 0, c > 0, D_2 > 0 \right\}.$$

Now replacing m by m_h , we get the model (2.3) as follows:

$$\begin{cases} x_{n+1} = x_n \exp \left(Rm_h + \frac{R(1-m_h)}{1 + ky_n} - Px_n - D_1 - \frac{A(1-m_h)y_n}{B + (1-m_h)x_n} \right), \\ y_{n+1} = y_n \exp \left(\frac{cA(1-m_h)x_n}{B + (1-m_h)x_n} - D_2 \right). \end{cases} \quad (2.6)$$

Next, we take into account a slight perturbation m' to the system (2.6) at the bifurcation point $m = m_h$, then the perturbed equations of the model (2.6) take the following forms:

$$\begin{cases} x_{n+1} = x_n \exp \left(R(m_h + m') + \frac{R(1-(m_h + m'))}{1 + ky_n} - Px_n - D_1 - \frac{A(1-(m_h + m'))y_n}{B + (1-(m_h + m'))x_n} \right), \\ y_{n+1} = y_n \exp \left(\frac{cA(1-(m_h + m'))x_n}{B + (1-(m_h + m'))x_n} - D_2 \right). \end{cases} \quad (2.7)$$

Step II:

For the purpose of transferring the positive equilibrium point $E^*(x^*, y^*)$ to the origin $(0, 0)$, we take the transformations $w_n = x_n - x^*$ and $z_n = y_n - y^*$ respectively. Then the map (2.7) becomes

$$\begin{cases} w_{n+1} = (w_n + x^*) \exp \left(R(m_h + m') + \frac{R(1-(m_h + m'))}{1 + k(z_n + y^*)} - P(w_n + x^*) - D_1 - \frac{A(1-(m_h + m'))(z_n + y^*)}{B + (1-(m_h + m'))(w_n + x^*)} \right) - x^*, \\ z_{n+1} = (z_n + y^*) \exp \left(\frac{cA(1-(m_h + m'))(w_n + x^*)}{B + (1-(m_h + m'))(w_n + x^*)} - D_2 \right) - y^*. \end{cases} \quad (2.8)$$

The Jacobian matrix corresponding to the linearized form of the model (2.8) at $(w_n, z_n) \equiv (0, 0)$ is

$$J_0 = \begin{pmatrix} \left\{ 1 - Px^* + \frac{A(1-(m_h + m'))^2 x^* y^*}{\{B + (1-(m_h + m'))x^*\}^2} \right\} L_1 & \left\{ -\frac{Rk(1-(m_h + m'))x^*}{(1 + ky^*)^2} - \frac{A(1-(m_h + m'))x^*}{B + (1-(m_h + m'))x^*} \right\} L_1 \\ \frac{cAB(1-(m_h + m'))y^*}{\{B + (1-(m_h + m'))x^*\}^2} L_2 & L_2 \end{pmatrix}$$

where

$$L_1 = \exp\left(R(m_h + m') + \frac{R(1 - (m_h + m'))}{1 + ky^*} - Px^* - D_1 - \frac{A(1 - (m_h + m'))x^*}{B + (1 - (m_h + m'))y^*}\right), \quad \text{and}$$

$$L_2 = \exp\left(\frac{cA(1 - (m_h + m'))x^*}{B + (1 - (m_h + m'))x^*} - D_2\right).$$

The characteristic equation of the matrix J_0 is

$$\mu^2 + \Gamma_1(m')\mu + \Gamma_2(m') = 0 \quad (2.9)$$

where $\Gamma_1(m') = -\left\{1 - Px^* + \frac{A(1 - (m_h + m'))^2 x^* y^*}{\{B + (1 - (m_h + m'))x^*\}^2}\right\} L_1 - L_2,$ and

$$\begin{aligned} \Gamma_2(m') = & \left[1 - Px^* + \frac{A(1 - (m_h + m'))^2 x^* y^*}{\{B + (1 - (m_h + m'))x^*\}^2} + \frac{cA^2 B(1 - (m_h + m'))^2 x^* y^*}{\{B + (1 - (m_h + m'))x^*\}^3} \right. \\ & \left. + \frac{ckABR(1 - (m_h + m'))^2 x^* y^*}{\{B + (1 - (m_h + m'))x^*\}^2 (1 + ky^*)^2}\right] L_1 L_2. \end{aligned}$$

The matrix J_0 has two complex conjugate characteristic values, namely $\mu_1 = \mu$ and $\mu_2 = \bar{\mu}$, which are given below:

$$\mu_{1,2} = \frac{1}{2}[-\Gamma_1(m') \pm i\sqrt{4\Gamma_2(m') - (\Gamma_1(m'))^2}]$$

with

$$|\mu| = |\mu_i| = \sqrt{\Gamma_2(m')}, \quad \text{where } i = 1, 2.$$

Now for the purpose of satisfying the transversality condition, we assume that

$$\begin{aligned} \frac{d|\mu|}{dm'} \Big|_{m'=0} = & \frac{1}{2} \left[\frac{Rky^*}{1 + ky^*} + \frac{AB(y^* - cx^*)}{(B + (1 - m_h)x^*)^2} + \frac{cA^2 B(1 - m_h)((1 - m_h)x^* - 2B)x^* y^*}{(B + (1 - m_h)x^*)^4} \right. \\ & \left. - \frac{2AB(1 - m_h)x^* y^*}{(B + (1 - m_h)x^*)^3} - \frac{2ckAB^2 R(1 - m_h)x^* y^*}{(B + (1 - m_h)x^*)^3 (1 + ky^*)^2} \right] \neq 0. \end{aligned} \quad (2.10)$$

Furthermore, we require the nondegeneracy condition, $\mu^q \neq 1$ for $q = 1, 2, 3, 4$, (here, it is better to say that, if μ complies with this condition, $\bar{\mu}$ will also comply and vice-versa) when $m' = 0$. This condition implies $\Gamma_1(0) \neq -2, 0, 1, 2$.

Now, according to the conditions in (2.5) and the expressions given in (2.4) we get

$$(\Gamma_1(0))^2 = \left(-2 + Px^* - \frac{A(1 - m_h)^2 x^* y^*}{(B + (1 - m_h)x^*)^2}\right)^2 < 4.$$

Thus, $\Gamma_1(0) = -2 + Px^* - \frac{A(1 - m_h)^2 x^* y^*}{(B + (1 - m_h)x^*)^2} \neq \pm 2$. So, we just need $\Gamma_1(0) \neq 0, 1$, which imply that

$$Px^* - \frac{A(1 - m_h)^2 x^* y^*}{(B + (1 - m_h)x^*)^2} \neq 2, 3. \quad (2.11)$$

Step III:

Now we take the Taylor's expansion of the map (2.8) at $(w_n, z_n) \equiv (0, 0)$ up to cubic terms, then we obtain the following pair of equations:

$$\begin{cases} w_{n+1} = r_{10}w_n + r_{01}z_n + r_{20}w_n^2 + r_{11}w_n z_n + r_{02}z_n^2 + r_{30}w_n^3 + r_{21}w_n^2 z_n + r_{12}w_n z_n^2 + r_{03}z_n^3 + O((|w_n| + |z_n|)^4), \\ z_{n+1} = s_{10}w_n + s_{01}z_n + s_{20}w_n^2 + s_{11}w_n z_n + s_{02}z_n^2 + s_{30}w_n^3 + s_{21}w_n^2 z_n + s_{12}w_n z_n^2 + s_{03}z_n^3 + O((|w_n| + |z_n|)^4), \end{cases} \quad (2.12)$$

where

$$r_{10} = 1 - Px^* + \frac{A(1 - m_h)^2 x^* y^*}{(B + (1 - m_h)x^*)^2},$$

$$\begin{aligned}
r_{01} &= -\frac{Rk(1-m_h)x^*}{(1+ky^*)^2} - \frac{A(1-m_h)x^*}{B+(1-m_h)x^*}, \\
r_{20} &= \frac{x^*}{2} \left(\frac{A(1-m_h)^2 y^*}{(B+(1-m_h)x^*)^2} - P \right)^2 + \frac{AB(1-m_h)^2 y^*}{(B+(1-m_h)x^*)^3} - P, \\
r_{11} &= \frac{A(1-m_h)^2 x^*}{(B+(1-m_h)x^*)^2} - \left(1 - Px^* + \frac{A(1-m_h)^2 x^* y^*}{(B+(1-m_h)x^*)^2} \right) \left(\frac{Rk(1-m_h)}{(1+ky^*)^2} + \frac{A(1-m_h)}{B+(1-m_h)x^*} \right), \\
r_{02} &= \frac{x^*}{2} \left(\frac{Rk(1-m_h)}{(1+ky^*)^2} + \frac{A(1-m_h)}{B+(1-m_h)x^*} \right)^2 + \frac{Rk^2(1-m_h)x^*}{(1+ky^*)^3}, \\
r_{30} &= \frac{A(1-m_h)^4 x^* y^*}{(B+(1-m_h)x^*)^4} + \frac{1}{2} \left(\frac{A(1-m_h)^2 y^*}{(B+(1-m_h)x^*)^2} - P \right)^2 + \frac{x^*}{6} \left(\frac{A(1-m_h)^2 y^*}{(B+(1-m_h)x^*)^2} - P \right)^3 - \frac{A(1-m_h)^3 y^*}{(B+(1-m_h)x^*)^3} - \\
&\quad \frac{A(1-m_h)^3 x^* y^*}{(B+(1-m_h)x^*)^3} \left(\frac{A(1-m_h)^2 y^*}{(B+(1-m_h)x^*)^2} - P \right), \\
r_{21} &= \left(\frac{Rk(1-m_h)x^*}{(1+ky^*)^2} + \frac{A(1-m_h)x^*}{B+(1-m_h)x^*} \right) \left\{ \frac{A(1-m_h)^3 y^*}{(B+(1-m_h)x^*)^3} - \frac{1}{2} \left(\frac{A(1-m_h)^2 y^*}{(B+(1-m_h)x^*)^2} - P \right)^2 \right\} + \frac{AB(1-m_h)^2}{(B+(1-m_h)x^*)^3} + \\
&\quad \frac{A(1-m_h)^2 x^*}{(B+(1-m_h)x^*)^2} \left(\frac{A(1-m_h)^2 y^*}{(B+(1-m_h)x^*)^2} - P \right) - \left(\frac{A(1-m_h)^2 y^*}{(B+(1-m_h)x^*)^2} - P \right) \left(\frac{Rk(1-m_h)}{(1+ky^*)^2} + \frac{AB(1-m_h)}{(B+(1-m_h)x^*)^2} \right), \\
r_{12} &= \left(1 - Px^* + \frac{A(1-m_h)^2 x^* y^*}{(B+(1-m_h)x^*)^2} \right) \left\{ \frac{Rk^2(1-m_h)}{(1+ky^*)^3} + \frac{1}{2} \left(\frac{Rk(1-m_h)}{(1+ky^*)^2} + \frac{A(1-m_h)}{B+(1-m_h)x^*} \right)^2 \right\} - \frac{A(1-m_h)^2 x^*}{(B+(1-m_h)x^*)^2} \\
&\quad \left(\frac{Rk(1-m_h)}{(1+ky^*)^2} + \frac{A(1-m_h)}{B+(1-m_h)x^*} \right), \\
r_{03} &= -\frac{x^*}{6} \left(\frac{Rk(1-m_h)}{(1+ky^*)^2} + \frac{A(1-m_h)}{B+(1-m_h)x^*} \right)^3 - \frac{Rk^2(1-m_h)x^*}{(1+ky^*)^3} \left(\frac{Rk(1-m_h)}{(1+ky^*)^2} + \frac{A(1-m_h)}{B+(1-m_h)x^*} \right) - \frac{Rk^3(1-m_h)x^*}{(1+ky^*)^4}, \\
s_{10} &= \frac{cAB(1-m_h)y^*}{(B+(1-m_h)x^*)^2}, & s_{01} &= 1, \\
s_{20} &= \frac{c^2 A^2 B^2 (1-m_h)^2 y^*}{2(B+(1-m_h)x^*)^4} - \frac{cAB(1-m_h)^2 y^*}{(B+(1-m_h)x^*)^3}, & s_{11} &= \frac{cAB(1-m_h)}{(B+(1-m_h)x^*)^2}, & s_{02} &= 0, \\
s_{30} &= \frac{c^3 A^3 B^3 (1-m_h)^3 y^*}{6(B+(1-m_h)x^*)^6} - \frac{c^2 A^2 B^2 (1-m_h)^3 y^*}{(B+(1-m_h)x^*)^5} + \frac{cAB(1-m_h)^3 y^*}{(B+(1-m_h)x^*)^4}, \\
s_{21} &= \frac{c^2 A^2 B^2 (1-m_h)^2}{2(B+(1-m_h)x^*)^4} - \frac{cAB(1-m_h)^2}{(B+(1-m_h)x^*)^3}, & s_{12} &= 0, & s_{03} &= 0.
\end{aligned}$$

Let

$$M = \begin{pmatrix} r_{10} & r_{01} \\ s_{10} & s_{01} \end{pmatrix} = \begin{pmatrix} 1 - Px^* + \frac{A(1-m_h)^2 x^* y^*}{(B+(1-m_h)x^*)^2} & -\frac{Rk(1-m_h)x^*}{(1+ky^*)^2} - \frac{A(1-m_h)x^*}{B+(1-m_h)x^*} \\ \frac{cAB(1-m_h)y^*}{(B+(1-m_h)x^*)^2} & 1 \end{pmatrix}.$$

Now it is assumed that the eigenvalues of the above matrix are of the form $\mu', \bar{\mu}' = \omega_1 \pm i\omega_2$ with $|\omega_1 \pm i\omega_2| = 1$.

Then $2\omega_1 = r_{10} + s_{01} = 2 - Px^* + \frac{A(1-m_h)^2 x^* y^*}{(B+(1-m_h)x^*)^2}$ and $\omega_1^2 + \omega_2^2 = r_{10} - s_{10}r_{01} = 1$.

Next, we consider the following invertible matrix

$$N = \begin{pmatrix} r_{01} & 0 \\ \omega_1 - r_{10} & \omega_2 \end{pmatrix}$$

whose columns are the eigenvectors of the matrix M corresponding to the eigenvalues μ' and $\bar{\mu}'$ respectively. The model (2.12) can be written in a simplified form using the translation

$$\begin{pmatrix} w_n \\ z_n \end{pmatrix} = N \begin{pmatrix} U_n \\ V_n \end{pmatrix},$$

126 and the desired form of the model (2.12) is as follows:

$$\begin{cases} U_{n+1} = \alpha U_n + \beta V_n + G(U_n, V_n), \\ V_{n+1} = -\beta U_n + \alpha V_n + H(U_n, V_n), \end{cases} \quad (2.13)$$

where

$$\begin{aligned} G(U_n, V_n) = & \frac{1}{r_{01}} [\{r_{20}r_{01}^2 + r_{11}r_{01}(\omega_1 - r_{10}) + r_{02}(\omega_1 - r_{10})^2\}U_n^2 + \{\omega_2r_{11}r_{01} + 2\omega_2r_{02}(\omega_1 - r_{10})\}U_nV_n \\ & + \omega_2^2r_{02}V_n^2 + \{r_{30}r_{01}^3 + r_{21}r_{01}^2(\omega_1 - r_{10}) + r_{12}r_{01}(\omega_1 - r_{10})^2 + r_{03}(\omega_1 - r_{10})^3\}U_n^3 \\ & + \{\omega_2r_{21}r_{01}^2 + 2\omega_2r_{12}r_{01}(\omega_1 - r_{10}) + 3\omega_2r_{03}(\omega_1 - r_{10})^2\}U_n^2V_n \\ & + \{\omega_2^2r_{12}r_{01} + 3\omega_2^2r_{03}(\omega_1 - r_{10})\}U_nV_n^2 + \omega_2^3r_{03}V_n^3] + O((|U_n| + |V_n|)^4), \end{aligned}$$

$$\begin{aligned} H(U_n, V_n) = & \frac{1}{\omega_2r_{01}} [\{r_{01}^2r_{20}(r_{10} - \omega_1) + r_{01}^3s_{20} + r_{01}^2s_{11}(\omega_1 - r_{10}) - r_{01}r_{11}(\omega_1 - r_{10})^2 - r_{02}(\omega_1 - r_{10})^3\}U_n^2 \\ & + \{\omega_2r_{01}^2s_{11} + \omega_2r_{01}r_{11}(r_{10} - \omega_1) - 2\omega_2r_{02}(\omega_1 - r_{10})^2\}U_nV_n + \omega_2^2r_{02}(r_{10} - \omega_1)V_n^2 \\ & + \{r_{30}r_{01}^3(r_{10} - \omega_1) + r_{01}^4s_{30} + r_{01}^3s_{21}(\omega_1 - r_{10}) - r_{01}^2r_{21}(\omega_1 - r_{10})^2 - r_{01}r_{12}(\omega_1 - r_{10})^3 \\ & - r_{03}(\omega_1 - r_{10})^4\}U_n^3 + \{\omega_2r_{01}^2r_{21}(r_{10} - \omega_1) + \omega_2r_{01}^3s_{21} - 2\omega_2r_{01}r_{12}(\omega_1 - r_{10})^2 \\ & - 3\omega_2r_{03}(\omega_1 - r_{10})^3\}U_n^2V_n + \{\omega_2^2r_{01}r_{12}(r_{10} - \omega_1) - 3\omega_2^2r_{03}(\omega_1 - r_{10})^2\}U_nV_n^2 \\ & + \omega_2^3r_{03}(r_{10} - \omega_1)V_n^3] + O((|U_n| + |V_n|)^4). \end{aligned}$$

Step-IV:

In order to find out the additional nondegeneracy condition, we compute the second and third orders partial

derivatives of G and H with respect to U_n and V_n at $(0, 0)$, which are given below:

$$\begin{aligned}
G_{U_n U_n} &= \frac{\partial^2 G}{\partial U_n^2} \Big|_{(0,0)} = \frac{2}{r_{01}} \left(r_{20} r_{01}^2 + r_{11} r_{01} (\omega_1 - r_{10}) + r_{02} (\omega_1 - r_{10})^2 \right), \\
G_{U_n V_n} &= \frac{\partial^2 G}{\partial U_n \partial V_n} \Big|_{(0,0)} = \frac{1}{r_{01}} \left(\omega_2 r_{01} r_{11} + 2\omega_2 r_{02} (\omega_1 - r_{10}) \right), \\
G_{V_n V_n} &= \frac{\partial^2 G}{\partial V_n^2} \Big|_{(0,0)} = \frac{2\omega_2^2 r_{02}}{r_{01}}, \\
G_{U_n U_n U_n} &= \frac{\partial^3 G}{\partial U_n^3} \Big|_{(0,0)} = \frac{6}{r_{01}} \left(r_{30} r_{01}^3 + r_{21} r_{01}^2 (\omega_1 - r_{10}) + r_{12} r_{01} (\omega_1 - r_{10})^2 + r_{03} (\omega_1 - r_{10})^3 \right), \\
G_{U_n U_n V_n} &= \frac{\partial^3 G}{\partial U_n^2 \partial V_n} \Big|_{(0,0)} = \frac{2}{r_{01}} \left(\omega_2 r_{21} r_{01}^2 + 2\omega_2 r_{12} r_{01} (\omega_1 - r_{10}) + 3\omega_2 r_{03} (\omega_1 - r_{10})^2 \right), \\
G_{U_n V_n V_n} &= \frac{\partial^3 G}{\partial U_n \partial V_n^2} \Big|_{(0,0)} = \frac{2}{r_{01}} \left(\omega_2^2 r_{01} r_{12} + 3\omega_2^2 r_{03} (\omega_1 - r_{10}) \right), \\
G_{V_n V_n V_n} &= \frac{\partial^3 G}{\partial V_n^3} \Big|_{(0,0)} = \frac{6\omega_2^3 r_{03}}{r_{01}}, \\
H_{U_n U_n} &= \frac{\partial^2 H}{\partial U_n^2} \Big|_{(0,0)} = \frac{2}{\omega_2 r_{01}} \left(r_{01}^2 r_{20} (r_{10} - \omega_1) + r_{01}^3 s_{20} + r_{01}^2 s_{11} (\omega_1 - r_{10}) - r_{01} r_{11} (\omega_1 - r_{10})^2 \right. \\
&\quad \left. - r_{02} (\omega_1 - r_{10})^3 \right), \\
H_{U_n V_n} &= \frac{\partial^2 H}{\partial U_n \partial V_n} \Big|_{(0,0)} = \frac{1}{\omega_2 r_{01}} \left(\omega_2 r_{01} r_{11} (r_{10} - \omega_1) + \omega_2 r_{01}^2 s_{11} - 2\omega_2 r_{02} (\omega_1 - r_{10})^2 \right), \\
H_{V_n V_n} &= \frac{\partial^2 H}{\partial V_n^2} \Big|_{(0,0)} = \frac{2\omega_2 r_{02} (r_{10} - \omega_1)}{r_{01}}, \\
H_{U_n U_n U_n} &= \frac{\partial^3 H}{\partial U_n^3} \Big|_{(0,0)} = \frac{6}{\omega_2 r_{01}} \left(r_{01}^3 r_{30} (r_{10} - \omega_1) + r_{01}^4 s_{30} + r_{01}^3 s_{21} (\omega_1 - r_{10}) - r_{01}^2 r_{21} (\omega_1 - r_{10})^2 \right. \\
&\quad \left. - r_{01} r_{12} (\omega_1 - r_{10})^3 - r_{03} (\omega_1 - r_{10})^4 \right), \\
H_{U_n U_n V_n} &= \frac{\partial^3 H}{\partial U_n^2 \partial V_n} \Big|_{(0,0)} = \frac{2}{\omega_2 r_{01}} \left(\omega_2 r_{01}^2 r_{21} (r_{10} - \omega_1) + \omega_2 r_{01}^3 s_{21} - 2\omega_2 r_{01} r_{12} (\omega_1 - r_{10})^2 - 3\omega_2 r_{03} (\omega_1 - r_{10})^3 \right), \\
H_{U_n V_n V_n} &= \frac{\partial^3 H}{\partial U_n \partial V_n^2} \Big|_{(0,0)} = \frac{2}{\omega_2 r_{01}} \left(\delta^2 r_{01} r_{12} (r_{10} - \omega_1) - 3\omega_2^2 r_{03} (r_{10} - \omega_1)^2 \right), \\
H_{V_n V_n V_n} &= \frac{\partial^3 H}{\partial V_n^3} \Big|_{(0,0)} = \frac{6\omega_2^2 r_{03} (r_{10} - \omega_1)}{r_{01}},
\end{aligned}$$

Next, we calculate the following quantities:

$$\begin{aligned}
\zeta_{20} &= \frac{1}{8} [(G_{U_n U_n} - G_{V_n V_n} + 2H_{U_n V_n}) + i(H_{U_n U_n} - H_{V_n V_n} - 2G_{U_n V_n})], \\
\zeta_{11} &= \frac{1}{4} [(G_{U_n U_n} + G_{V_n V_n}) + i(H_{U_n U_n} + H_{V_n V_n})], \\
\zeta_{02} &= \frac{1}{8} [(G_{U_n U_n} - G_{V_n V_n} - 2H_{U_n V_n}) + i(H_{U_n U_n} - H_{V_n V_n} + 2G_{U_n V_n})], \\
\zeta_{21} &= \frac{1}{16} [(G_{U_n U_n U_n} + G_{U_n V_n V_n} + H_{U_n U_n V_n} + H_{V_n V_n V_n}) + i(H_{U_n U_n U_n} + H_{U_n V_n V_n} - G_{U_n U_n V_n} - G_{V_n V_n V_n})].
\end{aligned}$$

Finally, we compute the following discriminatory quantity

$$\rho^* = -Re \left[\frac{(1 - 2\mu') \bar{\mu}'^2}{1 - \mu'} \zeta_{11} \zeta_{20} \right] - \frac{1}{2} |\zeta_{11}|^2 - |\zeta_{02}|^2 + Re(\bar{\mu}' \zeta_{21}),$$

which must be non-zero for the occurrence of the N-S bifurcation in the system (2.3). The nature of the N-S bifurcation and the stability of the bifurcated solutions are determined depending on the sign of ρ^* . If $\rho^* < 0$ then

the type of N-S bifurcation is supercritical or soft, in this case, the fixed point $E^*(x^*, y^*)$ loses stability and an initially stable invariant closed curve bifurcates from E^* . On the contrary, when $\rho^* > 0$ then the fixed E^* losses stability and an unstable invariant closed curve bifurcates from it, which is directed towards the occurrence of subcritical or sharp N-S bifurcation. In addition, a degenerate N-S bifurcation occurs when $\rho^* = 0$.

3 Lyapunov Exponents and Isoperiodic Diagrams

A chaotic dynamical system is random and unpredictable. However, in the midst of random behavior, there may have structures of regularity. Thus, fundamentally different behaviors are possible, e.g., periodicity, regularity, chaos, etc. and they occur in an unpredictable manner. Chaos may characterize measuring the average of the gradient of the map as a positive value. The long-time statistical average of the exponent with respect to an orbit is called the Lyapunov exponent or number. There are n Lyapunov exponents for a system with n variables. The system is chaotic if the largest Lyapunov exponent is positive. Dynamics in bi-parameter space now have a special attraction for the occurrence of well-organized patterns with regular and chaotic oscillations. The organized structures are connected with populations' behaviors for varying relevant parameter sets. Among these organized structures, Arnold tongues and shrimp-like patterns are commonly encountered in several nonlinear systems (see Refs. [Gallas, 1993, Pati et al., 2020]). The phase-locking (frequency-locking) structure that appeared on the parameter plane is known as Arnold's tongue or Arnold's horn [Arnold, 1965]. In recent years, the shrimp-like spiral structures have attracted particular attention. This structure has special importance as transition to chaos occurs through the period-bubbling route. This structural route to chaos is delaying chaotic dynamics under the constraints of varying two parameters. Both the periodic windows (Arnold tongues and shrimps) display self-similarities. In the following subsections, we discuss in detail various complex and well-organized structures.

3.1 Dynamical behaviors in (m, k) biparameter plane

The two system parameters, namely, the fear parameter k and the refuge parameter m play important roles in shaping the asymptotic dynamics of the prey-predator system. For this purpose, we focus on these two control parameters, and the variation of these two parameters creates a variety of complex structures. In order to identify the complex dynamics of a deterministic model in the parameter space by forming the stability diagrams, two techniques are frequently used: (i) Lyapunov diagram in terms of the largest Lyapunov exponent, and (ii) isoperiodic diagram in terms of the periodicity of the attractor. For the Lyapunov stability diagram, we calculate the Lyapunov exponents with respect to the control parameters m and k . Moreover, we draw the isoperiodic diagrams to explore the self-similarities between the emerging periodic structures. In favor of this, at each point of the parameter plane, we calculate the period of the attractors. Here, both types of diagrams including the patterns repeated on various scales of parameters are constructed in a grid of equally spaced 3.6×10^5 values of the m - k parameter plane with initial state fixed at $(x_0, y_0) = (0.3, 0.2)$ and the rest of the parameters are fixed as $R = 3.2$, $D_1 = 0.3$, $P = 0.1$, $A = 2$, $B = 5$, $c = 0.9$, and $D_2 = 0.1$. For removing the transient, we discard a sufficiently large number (50000) of iterations for each parameter value. A spacious and clear view of the parameter plane of the system (2.3) is illustrated in Fig. 1, where the colors of Fig. 1(a) are selected according to the values of the largest Lyapunov exponents. Continuously changing white to golden color indicates the periodic attractors (the corresponding Lyapunov exponent is negative), black region specifies the quasi-periodicity of the system for which the exponent value is zero. On the other hand, the chaotic region has shown a continuous color scale ranging from blue to red, connected with positive exponent values. In Fig. 1(b), the periodic windows are characterized by different colors, and their periods have mentioned in the color bar provided on the right side of the diagram. From Fig. 1, it is observed that the parameter plane contains a number of V-shaped periodic islands (Arnold horns). The periods of the noticeable tongues are 54, 55, 56, 57, 58, 59, 60, 61,, occur in symmetrical pair from both sides. In this figure, a completely new kind of U-shaped structure has emerged, which is formed by the merging of two Arnold tongues of the same period from both directions. In this way an infinite number of U-shaped structures are formed, whose patterns are self-similar and the periods of those U-shaped structures are 64, 65, 66, 67, 68, and so on. It is very significant to mention here that the periodicity of the appeared tongues reveals the period-adding phenomenon, where the difference between the periods of the tongues appearing in succession is always the same constant 1. From the figure, one can easily see that the tips of all tongues are submerged in the quasi-periodic regime while the tails of these tongues are embedded in the chaotic region. A special part of the parameter plane has been widened to view more subtle structures, which has manifested in Fig. 2.

Fig. 1: Plot of largest Lyapunov exponent (Subfig. (a)) and isoperiodic (Subfig. (b)) diagrams of the model (2.3) in the m - k biparameter plane.

An extension of the region bounded by the cyan box in Fig. 1(a) has been disclosed in Fig. 2(a). On the other, Fig. 2(b) reveals the magnification of the area indicated by the rust color box in Fig. 1(b). Here, the presence of innumerable periodic Arnold's tongues between the two big Arnold's tongues is clearly understood. It is very significant to mention the periods of the intermediate periodic tongues. As in the familiar Fibonacci sequence, each new term in the sequence is exactly the sum of the previous two, similarly, the period of the largest Arnold tongue between two Arnold tongues is equal to the sum of the periods of these two tongues. In the present model, we investigate the small-scale structure between the periodic tongues with periodicities 55 and 56. The sequence that indicates the in-between tongues periods, consists of the numbers: $55 \rightarrow 56 \rightarrow 111 \rightarrow 167 \rightarrow 278 \rightarrow 445 \rightarrow \dots$. It is worth mentioning here that the ratio between the successive terms of the sequence of periodicity gradually tends to the constant $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618033989$ [Dixon et al., 1996], this fixed number is termed as Golden ratio or Divine proportion [de Souza et al., 2012]. This is a very interesting feature in predator-prey population dynamics. The appearance of the golden ratio signifies the saturation or optimal level of periodic patterns. From both diagrams of Fig. 2, it is observable that an infinite number of shrimp-like periodic structures with different periods are rooted in the chaotic regime of the parameter space.

Fig. 2: Zoom-in of the rectangles marked in cyan and rust colors in Figs. 1 (a) and (b) respectively, disclose the complexity of the structures more clearly.

3.2 Period-bubbling process

The period-bubbling transition to chaos has been found in a large class of non-linear dynamical systems [Vandermeer, 1997]. This phenomenon is one of the most common routes to chaos where the transition process is delayed and acts as controlling the chaos. In this process of bifurcation pattern, the creation and the destruction of periodic orbits occur via two bifurcations. The first one, that is, the forward period-doubling bifurcation gives birth of a 2-periodic cycle from a period-1 cycle. However, the other one, the period-halving (reverse period-doubling) bifurcation transforms this period-2 cycle into a period-1 cycle. This is how a closed bubble-like loop is formed, known as the periodic bubble. Following the same, ever smaller and smaller bubbles are generated within the original ones and these cascades of period-bubbling appear to continue until the chaotic attractors fill the smaller bubbles. Period-bubbling is a two parameters process, one of them causes pair of bifurcations, and the other controls the width of the bubbles.

With the aim of exploring the period-bubbling phenomena in the present model, we adopt an augmentation of a specific region of the m - k plane, which is depicted in Fig. 3. From this figure, one can easily conclude that chaos occurs via period-bubbling root. Here, the refuge parameter m generates the bifurcations, where $m \in [0.15, 0.18]$, and the width of the bubbles are controlled by the fear parameter k . So, two important parameters of the model equation are linked with the occurrence of period-bubbling, an important route to chaos. From Fig. 3(b), it is observed that the bubbling phenomena is originated through the destabilization of the period-57 attractor as the value of the parameter k decreases from $k_1 \approx 11.5$, which in turn gives rise to the period- 57×2 bubbles sandwiched between period-57 attractors. In a similar manner, the branches of these periodic bubbles give birth to period- 57×4 bubbles when the value of k diminishes from $k_2 \approx 10.3$. Identically, the periodic bubbles of higher periods are formed through the instability of lower periodic bubbles with decreasing values of k . To make this process more apparent, we draw the bifurcation diagrams with respect to m for four distinct values of k , those are manifested in Fig. 4. Here, Fig. 4(a) shows period-1 cycle for $k = 11.5$, for $k = 10.3$ a period-2 bubble is formed (see Fig. 4 (b)), and a period-4 bubble is developed from its branches when $k = 10.2$ (look Fig. 4(c)). Finally, in Fig. 4(d), it is observed that numerous small bubbles have filled the two medium bubbles for $k = 10$, that indicating the occurrence of chaos.

Fig. 3: Figure shows the period-bubbling phenomena for the model (2.3) in m - k biparameter space, where $(m, k) \in [0.15, 0.18] \times [9.5, 12]$.

Fig. 4: Bifurcation diagrams with respect to the refuge parameter m , disclose the period-bubbling scenario for some discrete values of k .

3.3 Multistability

Multistability often occurs in many physical processes [Pati et al., 2021, Hossain et al., 2021]. It plays an important role in predicting system behaviors [Pisarchik and Feudel, 2014]. Multistability is identified in terms of the coexistence of two or more heterogeneous attractors depending on the initial states for a fixed set of the control parameters. The coexistence of multiple incompatible attractors in a deterministic system is an innate characteristic, for which the system's behavior becomes incomprehensible.

We now focus on the overlapping or coexistence of multiple dissimilar attractors in our model (2.3) and their basin sets. Owing to explore the multistability in our model, we take a magnification of a particular part of the m - k parameter plane, indicated in Fig. 2. The existence of multistability is evident in Fig. 5, where the diagrams are the extension of the regions bounded by the cyan and rust rectangular boxes in Figs. 2(a) and 2(b), respectively. From Fig. 5(b), the overlapping of three periodic attractors with periods 110, 111, and 222 is well understood. This figure (Fig. 5(b)) also shows the coexistence of period-110, period-111, and chaotic attractors.

Fig. 5: Enlargement of the areas inside the cyan and rust rectangular boxes marked in Figs. 2(a) and 2(b), respectively. Subfig. (b) exhibits two types of multistability: (i) among period-110, period-111 and period-222 attractors, and (ii) among period-110, period-111 and chaotic attractors.

3.4 Wada basin for coexisting attractors

We now train our focus on basin boundary. Wada basins are one of those unforeseen events that have been found in different phenomena. Over the years, several researchers have explored the occurrence of Wada basins in both dissipative and Hamiltonian systems [Poon et al., 1996]. In 1917, Kuniyo Yoneyama explained topologically the process of the separation of three or more open regions in a plane by a common boundary, which was suggested by Takeo Wada [Yoneyama, 1917]. Bearing the name of this Japanese mathematician, these complex topological structures are termed as Wada lakes or lakes of Wada [Hocking and Young, 1961]. The strange features of Wada lake were first explored from a topological standpoint [Hocking and Young, 1988]. In a dynamical system, Wada boundaries are special types of basin boundaries that separate three or more basins of attraction that may or may not be connected. In other words, the Wada characteristic is satisfied by a basin boundary if the intersection of every open neighborhood of any boundary point with at least three distinct basins is non-empty [Zhang, 2013]. In nonlinear dynamics, Wada basins are fractals but the reverse is not true [Aguirre and Sanjuán, 2002]. As for the appearance of fractal basin boundary, the coexistence of two attractors is sufficient. On the other hand, at least three coexisting attractors are needed for the Wada basin boundary [Aguirre and Sanjuán, 2002]. When the basin boundaries satisfy the Wada property, then a special kind of uncertainty occurs in the final state of the system, which is quite inconvenient in analyzing the system behavior, a fascinating property of Wada basins [McDonald et al., 1985]. Wada basin boundary was first observed in the forced damped pendulum by Kennedy and Yorke [Kennedy and Yorke, 1991]. Following this pioneering work, several researchers have investigated the Wada feature in a plethora of systems [Vandermeer, 2004].

Here, we have identified the presence of Wada basins in our model (2.3) by adopting numerical approach. To generate the basins of attraction, we select a grid of 600×600 cells, where each cell specifies an initial state. As in the case of our model, there are two types of multistability, so we draw two sets of basin diagrams. The first one is drawn for $m = 0.104$, $k = 7.935$ and the typical look is displayed in Fig. 6, where the basin set corresponding

to the period-110 attractor are plotted in green color, whereas the basin set of the period-111 are plotted in blue color and for the period-222 are in red. In order to draw the basin diagram for the coexisting period 110, 111 attractors, and chaotic attractor, we choose $m = 0.104$ and $k = 7.88$. Fig. 7 depicts the complex and beautiful structure of the basin sets for these coexisting attractors. In this figure, the initial conditions that are plotted in red and green colors correspond to the period-110 and period-111 attractors, respectively. On the other hand, for the chaotic attractor, the set of initial conditions is highlighted in black color. Although the complexity of the structure is evident from the first images of both figures (Fig. 6(a) and Fig. 7(a)), for better understanding, an extension of a certain portion is taken from each of the first three diagrams of both figures. In both cases, the amplified forms of these portions have exhibited in Subfigs. (b), (c), and (d), respectively, which unveil the Wada features more clearly.

Fig. 6: The Wada basin boundaries for the coexisting periodic attractors of periods 110, 111, and 222 is shown in Subfig. (a), whereas Subfig. (b) is the amplification of the black-boxed region indicated in Subfig. (a). Subfigs. (c) and (d) represent the magnification of the regions that are indicated by the black color rectangular boxes in Subfigs. (b) and (c), respectively.

Fig. 7: The Wada basin boundaries for the period-110, period-111, and chaotic coexisting attractors is uncovered in Subfig. (a), while Subfig. (b) is the augmentation of the blue-boxed region indicated in Subfig. (a). Subfigs. (c) and (d) present the amplification of the regions that are indicated by the blue color rectangular boxes in Subfigs. (b) and (c), respectively.

4 Conclusions

This is a work exploring the dynamics of a predator-prey system taking into account the combined effects of prey refuge and fear under the framework of a discrete nonlinear dynamical system. The occurrence of biologically feasible fixed points, their existence criteria, stability analysis, and the associated local bifurcation are determined using dynamical principle theory. We have seen that the coexisting fixed point undergoes Neimark-Sacker (N-S) bifurcation satisfying certain conditions of the system parameters. This leads to quasi-periodic oscillations in the system. The global dynamics of the system are explored by adopting Lyapunov and isoperiodic diagrams in (m, k) bi-parameter space. A stability diagram in bi-parameter space for simultaneously varying two parameters is an important tool to elucidate the complex organized structures appearing in this nonlinear model. Sometimes, it reveals new, more intriguing dynamical features that can not be obtained in single-parameter space. Using bi-

parametric stability diagrams, we see that the present system exhibits four types of topologically non-equivalent dynamics, viz. fixed point (steady-state), periodic, quasi-periodic, and chaos. The organizations of these regular and irregular oscillations in the bi-parameter space are studied elaborately. Different kinds of periodic structures, viz., Arnold tongues, shrimp structures, are obtained for the organization of periodic windows in quasi-periodic and chaotic regimes in the bi-parameter space. Furthermore, we observe the important route, period-bubbling transition to chaos, a mechanism in which chaos occurs via the creation and annihilation of period-doubling bubbles. The existence of coexisting multiple attractors is definitely a new feature of nonlinear dynamics with potential applications in population dynamics. Tri-stability with coexisting three different periodic attractors, and two heterogeneous periodic attractors and one chaotic attractor are new features in the present predator-prey model and obtained due to the crossing of a periodic strip and a shrimp-shaped structure in the chaotic regime. The basin sets of these coexisting attractors are studied, and Wada basin boundaries of three different attractors sharing a common boundary are obtained. The occurrence of multistability depends on initial population sizes. A slight deviation in the population size can lead to completely different attractors in the subsequent time. Further research will explore useful mechanisms for determining specific attractors that give interesting results in prey and predator populations under the variation of two control parameters.

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Conflict of interest

The authors declare that they have no conflict of interest.

Authors contributions

All authors contributed equally to this work.

Data availability

The data (set of parameters) that supports the findings of this study are available within the article.

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