Mathematical Analysis of the Concomitance of Illicit Drug Use and Banditry in a Population

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Mathematical analysis of the concomitance of illicit drug use and banditry in a population

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Abstract

One of the majors global health and social problem facing the world today is the use of illicit drug and the act banditry. The two problems have resulted into lost of precious lives, properties and even a devastating effects on the economy of some countries where such acts were been practiced. Of interest in this work is to study the global stability of illicit drug use spread dynamics with banditry compartment using a dynamical system theory approach. Illicit drug use and banditry reproduction number was evaluated analytically, which measures the potential spread of the illicit drug use and banditry in the population. The system exhibits supercritical bifurcation property, telling us that local stability of an illicit drug and banditry-present equilibrium exist and it is unique. In addition, the illicit drug and banditry-free and illicit drug and banditry-present equilibria were shown to be global asymptotically stable, this was achieved by construction of suitable Lyapunov functions. Sensitivity analysis was carried out to know the impact of each parameter on the dynamical spread of illicit drug use and banditry in a population. Numerical simulations were used to validate the obtained quantitative results, and examine the effects of some key parameters on the system. It was discovered that, to reduce the burden of banditry in the population, stringent control measures must be put in place to reduce the use of illicit drug in a population. Suggested control measures to use in curtail the menace of the illicit drug use and banditry were recommends.

Keywords: Illicit drug use and banditry model, Illicit drug use and banditry reproduction number, Bifurcation analysis, Sensitivity analysis.

1 Introduction

The world is facing many problem today, which can be classified as follows; "poverty, religious conflict, political polarization, government accountability (political scandals), education (access to schooling), food and water scarcity, poor health facilities in developing nations, lack of access to credit, discrimination, physical fitness" and the host others [1].

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One of the element of poor health facilities and poverty is the use of illicit drug and banditry [2]. The production, abuse of illicit drug and act of banditry has increased sporadically in Africa, in the last decades [3]. This increase has impacted the costs on the public health system and also increased in the spread of other epidemics, such as HIV, Gonorrhea and the host of other sexually transmitted diseases[4]. Illicit drug use remains a serious problem; with countless number of health risks, increase in the rate of social vice and also jack-up the expenses of government [4, 5].

The term Illicit drug use also known as substance abuse was defined by Nutt *et al.* [6] as an excessive use of a drug or the mode of usage of drug which is detrimental to users or people around him / her. Some of the substance abuse among humans today is Marijuana; which is dehydrated leaves [2, 7, 8, 9, 10]. Marijuana is a dehydrated leaf which has a chemical components that alters mind and makes the consumers behave irrationally [9, 10, 11, 12]. Alcohol is another type of substance abuse, which causes harm to health of the one that takes it [9, 13, 14]. When a pregnant woman takes alcohol, it has a great negative effect to the unborn baby (fetus), it can even cause miscarriage [15, 9] .

According to [15, 16], the rates of methamphetamine consumption in the World today, particularly Africa is on high side. The population of those that use methamphetamine abnormally are youths [9, 17]. The effect of illicit drug use are serious in African countries like South Africa, Nigeria and Ghana, the illicit drug use has a serious and great effect on their population particularly among their youth [15, 16]. SACENDU [18] report between January and June 2013; shows that 76%, 71%, 59%, 67% and 59% are of illicit drug use patients that were man, coloured people, jobless, single and between the ages of 15 and 29 years respectively. The population of illicit drug user varies from one place to another even in Africa, for example, the Western Cape Province has a high rate of illicit drug use consumption in comparison to some other provinces in South Africa [19, 20]. These province are the one causing trouble in South Africa, and the substance being abused are alcohol, methamphetamine and marijuana [21]. Also, Tik, is another substance abuse in Cape Town [9, 18].

Banditry can be defined as a type of organized crime committed by outlaws typically involving the threat or use of violence. A person who engages in banditry is known as a bandit [22]. According to [22] causes of banditry can be ecological and climate change, consistent shift in human and livestock populations, weak state capacity and the provision of security, proliferation of small arms and light weapon and the host of others. Examples of banditry are sexual assault of women and girls, theft, attacks on banks, market and school hostel, amongst others [10, 22].

Many researchers have used the ideal of mathematical modeling to provide solution to the problem of illicit drug use and banditry, either separately or together has a complex problem. Few of them are; Pang *et al.* [23] examined the effect of tobacco on a country being victimized by its consumptions, where China was the concerned country. It was noted that this country was seriously affected by smoking-associated sicknesses. The work major on the issue of curtailing smoking in China. Kalula and Nyabadza [24] aimed at the qualitative investigation of the dynamics of substance abuse and predicting drug abuse trends. The analysis of the model was presented in terms of the substance abuse epidemic threshold $\mathcal{R}_0$. Simulations were performed to fit the model to available data for methamphetamine use in the Western Cape, determine the role played by some key parameters and

methamphetamine users who entered rehabilitation using the least squares curve fitting method. It is important to note that their model exhibits a backward bifurcation, pointing that it is not sufficient to reduce $\mathcal{R}_0$ below unity to control the substance abuse epidemic.

In the works of [6, 24], one of their findings is that the illicit drug use problem can be eradicated if light drug users are targeted with appropriate controls, and treatment rates for drug addicts are also beefed up. One of the problems associated with Methamphetamine drug use is abnormal desire for sex which may likely increase sexually transmitted infections (STIs) and HIV [25]. In [10], a study on substance abuse which used a mathematical model to gain an insight to the dynamics spread of drug abuse and banditry in a population was carried out. It is important to note that the study exhibits backward bifurcation, pointing that it is not only sufficient to lower the $\mathcal{R}_0$ below unity, but other things must be put into consideration. Also, sensitivity analysis was done to determine impact of parameters of the model on the spread of the drug abuse and banditry menace in the population.

This study take into account the concomitant of illicit drug use and banditry, the presence of relapse of illicit drug user and bandit and the quitters compartment which is less or not concerned in the existing literatures, the formed the major purpose of this work. Hence, a new mathematical model was used to explore the co-problematic population of illicit drug user and bandit, the presence of relapse illicit drug user and bandit and the quitters compartment. The rest of study is sectionalized in the following order: Section 2 is for the model formation and analysis of the basic properties. In Section 3, stability and sensitivity analyses of the illicit drug use and banditry model are carried out. Concluding remarks are given in Section 4.

2 Model Formation

The dynamics of the concomitant of illicit drug use and banditry (IDUB) in the population was considered with the total population denoted by $N(t)$, at time t , which was sub-divided into seven well-defined classes, susceptible individuals $S(t)$ (those who are in the company of illicit drug user or/and bandit but who are neither illicit drug user nor bandit), Illicit drug user $I(t)$ (someone who abuse drug or those who absolutely depends on drugs wrongly), suspected bandit $B(t)$ (someone who commit banditry or involve in bandit act), detainee $D(t)$ (illicit drug user or/and suspected bandit who are in police custody), Prisoner $P(t)$ (illicit drug user or/and suspected bandit that was imprison) and rehabilitation population $R(t)$ (illicit drug user or/and bandit who are undergoing rehabilitation) and quitters $Q(t)$ (illicit drug user or/and bandit who quit this or these act). The population is not a constant population size because the recruitment (births) is not balanced by removal (natural deaths) rates due to induced death rate δ . Figure 1 represents the schematic diagram for the concomitant of illicit drug use and banditry population dynamics, which the nonlinear ordinary differential equations (1) is the governed system.

Figure 1: Schematic diagram of the model (1)

$$\begin{aligned}
\frac{dS}{dt} &= \pi - \beta \frac{I(t) + \eta B(t)}{N(t)} S(t) - \mu S(t) \\
\frac{dI}{dt} &= \beta \frac{I(t) + \eta B(t)}{N(t)} S(t) + \alpha_1 R(t) - (\sigma_1 + \gamma_1 + \delta_1 + \mu) I(t) \\
\frac{dB}{dt} &= \gamma_1 I(t) - (\sigma_2 + \gamma_2 + \delta_2 + \mu) B(t) \\
\frac{dD}{dt} &= \sigma_1 I(t) + \sigma_2 B(t) - (\sigma_3 + \gamma_3 + \tau_1 + \mu) D(t) \\
\frac{dP}{dt} &= \gamma_2 B(t) + \gamma_3 D(t) - (\sigma_4 + \tau_2 + \delta_4 + \mu) P(t) \\
\frac{dR}{dt} &= \tau_1 D(t) + \tau_2 P(t) - (\sigma_5 + \alpha_1 + \mu) R(t) \\
\frac{dQ}{dt} &= \sigma_3 D(t) + \sigma_4 P(t) + \sigma_5 R(t) - \mu Q(t),
\end{aligned} \tag{1}$$

with initial conditions when $t = 0$:

$$S(0) = S_0, B(0) = B_0, I(0) = I_0, D(0) = D_0, P(0) = P_0, R(0) = R_0, Q(0) = Q_0. \tag{2}$$

The following assumptions guide the dynamics of the model been formulated : the interaction of susceptible humans with illicit drug users and bandits first results into illicit drug users then progresses to bandit population thereafter. Also, rehabilitated individuals moves back to illicit drug users due to one reason or the other during treatment. Furthermore, the illicit drug users and bandit populations moved to detention before being

transferred to prison. Moreover, the detainees can either move to prison or rehabilitation center. In summary, the illicit drug use and banditry model is given by the system of non-linear ordinary differential equations (1) with the nonnegative initial conditions (2) and the schematic diagram represented by Figure 1. For limpidity, the definition of variables and parameters used in model (1) are given in Table 1.

Looking through the system of equation (1), one can notice that (Q) only appear in the last equation of system; on this note, for theoretical analysis, the (Q) compartment of system (1) will be silence [26].

The equation of the total population which is obtained by adding all the equations of (1) is given below:

$$\frac{dN}{dt} \leq \pi - \mu N \quad (3)$$

Table 1. Definition of variables and parameters

Variable/Parameter	Definition
$S(t)$	susceptible individuals
$I(t)$	illicit drug users
$B(t)$	suspected bandits
$D(t)$	detainees
$P(t)$	prisoners
$R(t)$	rehabilitate individual
$Q(t)$	quitters
π	recruitment rate into the susceptible population
α_1	movement of rehabilitate to illicit drug user
η	modification parameter for suspected bandits
σ_1	detention rate of illicit drug users
γ_1	progression rate of illicit drug users to suspected bandits
δ_1	induce death rate of illicit drug users
β	effective influence rate
δ_4	induce death rate of those in prisoners
μ	natural rate
σ_2	detention rate of suspected bandits
γ_2	movement of suspected bandits to prisoners
δ_2	induced death rate of suspected bandits
σ_3	quitting rate of detainee
γ_3	movement of detentes to prison
τ_1	rate of rehabilitate detnee
σ_4	quitting rate of individuals in the prison
τ_2	rehabilitating rate of those in prison
σ_5	treatment rate of those in rehabilitation center

2.1 Basic Qualitative Properties

The basic properties of the model can now be investigated.

2.1.1 Positivity and boundedness of solutions

Since model (1) monitors human population, all the parameters are non-negative. Therefore, it is important to proof that all the state variables are also non-negative for all time $t > 0$.

Theorem 1. *The state variables, $S(t)$, $I(t)$, $B(t)$, $D(t)$, $P(t)$, $R(t)$ and $Q(t)$, of the model (1), with the initial data (2), remain non-negative for all $t > 0$.*

Proof. Noting that

$$\lambda = \frac{\beta(I(t) + \eta B(t))}{N(t)},$$

one sees from the first equation of (1) that

$$\frac{dS}{dt} \geq -(\lambda + \mu) S(t), \quad (4)$$

simplifying (4) gives,

$$\frac{d}{dt} \left(S(t) \exp \left(\mu t + \int_0^t \lambda(\varpi) d\varpi \right) \right) \geq 0, \quad (5)$$

solving (5), gives

$$S(t) \geq S(0) \exp \left(- \left(\mu t + \int_0^t \lambda(\varpi) d\varpi \right) \right) > 0, \forall t > 0.$$

The remaining state variables $I(t)$, $B(t)$, $D(t)$, $P(t)$, $R(t)$ and $Q(t)$, can be proved to be positive for all $t > 0$ when the same approach been used for $S(t)$ is employed. $\square$

Next, consider the biologically feasible region, defined by $\Gamma \subset \mathbb{R}_+^7$ where

$$\Gamma = \left\{ (S, I, B, D, P, R, Q) \in \mathbb{R}_+^7 : N \leq \frac{\pi}{\mu} \right\}.$$

Γ is shown positively invariant in a certain region.

The total population is given by

$$\frac{dN}{dt} \leq \pi - \mu N, \quad (6)$$

which results into the solution $N(t) = N(0) \exp(-\mu t) + \frac{\pi}{\mu} (1 - \exp(-\mu t))$. It follows that $N(t) \rightarrow \frac{\pi}{\mu}$ as $t \rightarrow \infty$ in particular, $N(t) \leq \frac{\pi}{\mu}$ if $N(0) \leq \frac{\pi}{\mu}$ with respect to the illicit drug use and banditry model (1). Hence, it suffices to consider the dynamics of the model in Γ . In this region, the illicit drug use and banditry model can be considered as being mathematically and biologically well-posed [27].

3 Stability and Sensitivity Analysis

3.1 Illicit drug use and banditry-free Equilibrium ($\mathcal{D}_0$)

This is the equilibrium point where the illicit drug use and banditry is absent in the population, denoted by $\mathcal{D}_0$. At this point $I = B = 0$ and so, setting the vector field of (1) to zero gives

$$\mathcal{D}_0 = (S_0, I_0, B_0, D_0, P_0, R_0) = \left(\frac{\pi}{\mu}, 0, 0, 0, 0, 0 \right) \quad (7)$$

The illicit drug use and banditry reproduction number $\mathcal{R}_0$ is established next.

3.2 Illicit drug use and banditry threshold ($\mathcal{R}_0$)

The IDUB reproduction number also noted as IDUB threshold, $\mathcal{R}_0$, given by (8), is the criterion of the IDUB spread in a complete naive population. Also, $\mathcal{R}_0$ can be defined as the average number of new cases of illicit drug user and bandit influenced by a typical illicit drug users and bandits in a naive population.

The next generation matrix method [28], the IDUB reproduction number, $\mathcal{R}_0$ is obtained as follows:

$$F = \begin{pmatrix} \beta & \beta\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and let } d_1 = \sigma_1 + \gamma_1 + \delta_1 + \mu, \quad d_2 = \sigma_2 + \gamma_2 + \delta_2 + \mu, \quad d_3 = \sigma_3 + \gamma_3 + \tau_1 + \mu, \\ d_4 = \sigma_4 + \tau_2 + \delta_4 + \mu, \quad d_5 = \sigma_5 + \alpha_1 + \mu.$$

$$V = \begin{pmatrix} d_1 & 0 & 0 & 0 & -\alpha_1 \\ -\gamma_1 & d_2 & 0 & 0 & 0 \\ -\sigma_1 & -\sigma_2 & d_3 & 0 & 0 \\ 0 & -\gamma_2 & -\gamma_3 & d_4 & 0 \\ 0 & 0 & -\tau_1 & -\tau_2 & d_5 \end{pmatrix}$$

Therefore, the spectral radius of the matrix $\mathcal{FV}^{-1}$ is obtained, which is also $\mathcal{R}_0$, was determined in the sense of [9, 10, ?, ?] as

$$\mathcal{R}_0 = \frac{\beta d_3 d_4 d_5 (d_2 + \eta \gamma_1)}{d_1 d_2 d_3 d_4 d_5 - ((d_4 \tau_1 \alpha_1 + \tau_2 \alpha_1 \gamma_3)(d_2 \sigma_1 + \gamma_1 \sigma_2) + d_3 \tau_2 \alpha_1 \gamma_1 \gamma_2)}, \quad (8)$$

Algebraic simplification shows that $d_1 d_2 d_3 d_4 d_5 > ((d_4 \tau_1 \alpha_1 + \tau_2 \alpha_1 \gamma_3)(d_2 \sigma_1 + \gamma_1 \sigma_2) + d_3 \tau_2 \alpha_1 \gamma_1 \gamma_2)$.

To study the local asymptotic stability (LAS) of $\mathcal{D}_0$ given by (7), the illicit drug use and banditry reproduction number, $\mathcal{R}_0$, given by (8), is a major key needed for this analysis. The local asymptotically stability of $\mathcal{D}_0$ will be examined by the following theorem.

Theorem 2. *The illicit drug use and banditry-free equilibrium, $\mathcal{D}_0$, of the system (1) is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$.*

Proof. Since the illicit drug use and banditry reproduction number, $\mathcal{R}_0$, was obtained using next generation matrix method by [28], it is suffice to proofed that the model (1) is locally asymptotically stable when $\mathcal{R}_0 < 1$ $\square$

The significance of Theorem 2 shows that the problem of IDUB model governed by (1) will be wiped out from the population, if the initial size of the illicit drug user and bandits sub-populations are in the basin of attraction of the $\mathcal{D}_0$. However, illicit drug user and bandits obliteration is independent of the initial sizes if the $\mathcal{D}_0$ is globally asymptotically stable. The global stability result is established in the following theorem.

Theorem 3. *The IDBFE of the model (1), given by (7), is globally asymptotically stable (GAS) in the region Γ whenever $\mathcal{R}_0 \leq 1$.*

Proof. Consider the linear Lyapunov function $\mathcal{L} : \Gamma \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{L} = & S(t) + \frac{\alpha_1[\sigma_2(\gamma_3\tau_2+\tau_1d_4)+d_3\gamma_2\tau_2]}{d_2d_3d_4d_5}B(t) + \frac{\gamma_3\tau_2\alpha_1+\alpha_1\tau_1d_4}{d_3d_4d_5}D(t) \\ & + \frac{\tau_2\alpha_1}{d_4d_5}P(t) + \frac{\alpha_1}{d_5}R(t), \end{aligned} \quad (9)$$

The time derivative of (9) along the solution path of the system (1) is given by

$$\begin{aligned} \dot{\mathcal{L}} = & [\lambda S + \alpha_1 R - d_1 I] + \frac{\alpha_1[\sigma_2(\gamma_3\tau_2+\tau_1d_4)+d_3\gamma_2\tau_2]}{d_2d_3d_4d_5} [\gamma_1 I - d_2 B] \\ & + \frac{\gamma_3\tau_2\alpha_1+\alpha_1\tau_1d_4}{d_3d_4d_5} [\sigma_1 I + \sigma_2 B - d_3 D] + \frac{\tau_2\alpha_1}{d_4d_5} [\gamma_2 B + \gamma_3 D - d_4 P] \\ & + \frac{\alpha_1}{d_5} [\tau_1 D + \tau_2 P - d_5 R], \\ = & \left[\frac{\mu(d_2+\eta\gamma_1)}{\pi} \beta S(t) - d_1 + \frac{(\alpha_1\sigma_2(\gamma_3\tau_2+\tau_1d_4)+\alpha_1d_3\gamma_2\tau_2)\gamma_1}{d_2d_3d_4d_5} + \frac{(\gamma_3\alpha_1\tau_2+\alpha_1\tau_1d_4)}{d_3d_4d_5} \right] I(t), \\ \leq & \left[\frac{\beta(d_2+\eta\gamma_1)}{d_2} - \frac{d_1d_2d_3d_4d_5 - ((\alpha_1\sigma_2\gamma_1+d_2\sigma_1\alpha_1)(\gamma_3\tau_2+\tau_1d_4)+\alpha_1\gamma_1\gamma_2\tau_2d_3)}{d_2d_3d_4d_5} \right] I(t), \text{ since } S(t) \leq \frac{\pi}{\mu} \text{ in } \Gamma, \\ = & \frac{d_2d_3d_4d_5}{d_1d_2d_3d_4d_5 - ((\alpha_1\sigma_2\gamma_1+d_2\sigma_1\alpha_1)(\gamma_3\tau_2+\tau_1d_4)+\alpha_1\gamma_1\gamma_2\tau_2d_3)} [1 - \mathcal{R}_0] I(t). \end{aligned}$$

Thus, $\dot{\mathcal{L}} \leq 0$ if $\mathcal{R}_0 \leq 1$ with $\dot{\mathcal{L}} = 0$ if and only if $I(t) = 0$. This shows that as $t \rightarrow \infty$, then $(S(t), I(t), B(t), D(t), P(t), R(t)) \rightarrow (\frac{\pi}{\mu}, 0, 0, 0, 0, 0)$. It follows that the largest compact invariant set in $\{(S(t), I(t), B(t), D(t), P(t), R(t)) \in \Gamma : \dot{\mathcal{L}} = 0\}$ is the singleton $\{\mathcal{D}_0\}$. Therefore, by LaSalle's Invariance Principle [29], the illicit drug use and banditry-free equilibrium given by $\mathcal{D}_0$ is GAS in Γ if $\mathcal{R}_0 \leq 1$. $\square$

The deduction of Theorem 4 proofed that diminution or evacuation of IDUB is not dependent on the initial sizes of the illicit drug users and bandits in the population. The stability property is shown in Figure 2, where all the solutions meet to illicit drug use and banditry-free equilibrium. Hence, illicit drug use and banditry can be evacuated, if the associated illicit drug use and banditry reproduction number is less than one.

3.3 Illicit drug use and banditry-present equilibrium

The steady-state solution of model (1) when all the state variables are positive is referred to as the illicit drug use and banditry-present equilibrium point (IDBPE) denoted and given by

$$\mathcal{D}^* = (S^*, I^*, B^*, D^*, P^*, R^*). \quad (10)$$

Then, setting the right hand sides of (1) to zero, the following are obtained in terms of λ :

$$\begin{aligned} S_\lambda^* &= \frac{\pi}{\lambda + \mu}, \\ I_\lambda^* &= \frac{\pi\lambda\alpha_1 A_1}{d_1[k_4 - A_8\alpha_1(\lambda + \mu)]}, \\ B_\lambda^* &= \frac{\pi\lambda(A_7 - (A_5 + A_6)\alpha_1(\lambda + \mu))}{k_9[k_4 - A_8\alpha_1(\lambda + \mu)]}, \\ D_\lambda^* &= \frac{\pi\lambda k_2(k_4 - \alpha_1(A_2 - A_3(\lambda + \mu)))}{k_7[k_4 - A_8\alpha_1(\lambda + \mu)]}, \\ P_\lambda^* &= \frac{\pi\lambda A_4(k_4 + \alpha_1 A_5(\lambda + \mu))}{k_8[k_4 - A_8\alpha_1(\lambda + \mu)]}, \\ R_\lambda^* &= \frac{\pi\lambda A_2}{k_4 - A_8\alpha_1(\lambda + \mu)}, \end{aligned} \quad (11)$$

where

$$\begin{aligned} k_1 &= d_1\tau_1 + \tau_2\gamma_2, & k_2 &= \sigma_1 d_2 + \sigma_2\gamma_1, & k_3 &= \tau_2\gamma_1\gamma_2 d_3, & k_4 &= d_1 d_2 d_3 d_4 d_5, \\ k_5 &= \tau_1 d_4 + \gamma_1\gamma_2 d_3 + \gamma_3\tau_2, & k_6 &= \gamma_1\gamma_2 d_3, & k_7 &= d_1 d_2 d_3, & k_8 &= d_1 d_2 d_3 d_4, \\ k_9 &= d_1 d_2 & A_1 &= k_4 - k_2 k_6 + k_3, & A_2 &= k_1 k_2 + k_3, & A_3 &= k_1 k_2 + k_6, \\ A_4 &= k_6 + \gamma_3 k_2, & A_5 &= k_6(\tau_2 - k_2), & A_6 &= k_1 k_2(\gamma_1 - 1), \\ A_7 &= \gamma_1 k_4, & A_8 &= k_2 k_5 \alpha_1. \end{aligned}$$

$$\lambda^* = \frac{\beta(I^* + \eta B^*)}{N^*}, \quad (12)$$

and

$$N^* = S^* + I^* + B^* + D^* + P^* + R^*. \quad (13)$$

Substituting (11) and (13) into equation (12) after algebraic manipulations, gives the following

$$\lambda = 0 \quad \text{or} \quad \lambda = \mu(\mathcal{R}_0 - 1). \quad (14)$$

When $\lambda = 0$ from system (11), it means illicit drug use and banditry-present equilibrium does not exist, but we have illicit drug use and banditry-free equilibrium, and when $\lambda = \mu(\mathcal{R}_0 - 1)$, we have a unique illicit drug use and banditry-present equilibrium if $\mathcal{R}_0 > 1$. Simplifying (11), the following unique illicit drug use and banditry-present equilibrium $\mathcal{D}^*$ is obtained in terms of $\mathcal{R}_0$ as follows

$$S^* = \frac{\pi}{\mu\mathcal{R}_0},$$

Figure 2: Simulated results of model (1) illustrating the global dynamics of illicit drug use and banditry-free with different initial sizes. Parameter values used are given in Table 3 except: $\beta = 0.04$ and $\sigma_5 = 0.7$, so that, $\mathcal{R}_0 < 1$.

$$\begin{aligned}
I^* &= \frac{\pi\mu(\mathcal{R}_0 - 1)\alpha_1 A_1}{d_1[k_4 - A_8\alpha_1\mu\mathcal{R}_0]}, \\
B^* &= \frac{\mu\pi(\mathcal{R}_0 - 1)(A_7 - (A_5 + A_6)\alpha_1\mu\mathcal{R}_0)}{k_9[k_4 - A_8\alpha_1\mu\mathcal{R}_0]}, \\
D^* &= \frac{\mu\pi(\mathcal{R}_0 - 1)k_2(k_4 - \alpha_1(A_2 - A_3\mu\mathcal{R}_0))}{k_7[k_4 - A_8\alpha_1\mu\mathcal{R}_0]}, \\
P^* &= \frac{\pi\mu(\mathcal{R}_0 - 1)A_4(k_4 + \alpha_1 A_5\mu\mathcal{R}_0)}{k_8[k_4 - A_8\alpha_1\mu\mathcal{R}_0]}, \\
R^* &= \frac{\pi\mu A_2(\mathcal{R}_0 - 1)}{k_4 - A_8\alpha_1\mu\mathcal{R}_0}.
\end{aligned} \tag{15}$$

Also, when $\mathcal{R}_0 = 1$ from system (15), it means we have illicit drug use and banditry-free equilibrium but illicit drug use and banditry-present equilibrium does not exist. Similarly, we have a unique illicit drug use and banditry-present equilibrium when $\mathcal{R}_0 > 1$.

To further examine the nature of its stability when $\mathcal{R}_0 = 1$, the bifurcation analysis will be explored next.

3.4 Forward Bifurcation

To study the existence of bifurcation of the model (1), center manifold theory being described in Castillo-Chavez and Song [30] will be explored. The illicit drug use and banditry model (1) will be written in the vector form for this purpose:

$$\frac{dX}{dt} = F(X),$$

Figure 3: Simulated results of model (1) illustrating the convergence of solution trajectories with different initial sizes to the illicit drug use and banditry-present equilibrium. Parameter values used are given in Table 3, such that $\mathcal{R}_0 > 1$.

where $\mathbf{X} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$ and $\mathbf{F} = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ with $S(t) = x_1, I(t) = x_2, B(t) = x_3, D(t) = x_4, P(t) = x_5, R(t) = x_6$. Then, model (1) becomes

$$\begin{aligned}
 f_1 &= \frac{dx_1}{dt} = \pi - \lambda x_1 - \mu x_1, \\
 f_2 &= \frac{dx_2}{dt} = \lambda x_1 + \alpha_1 x_6 - (\sigma_1 + \gamma_1 + \delta_1 + \mu) x_2, \\
 f_3 &= \frac{dx_3}{dt} = \gamma_1 x_2 - (\sigma_2 + \gamma_2 + \delta_2 + \mu) x_3, \\
 f_4 &= \frac{dx_4}{dt} = \sigma_1 x_2 + \sigma_2 x_3 - (\sigma_3 + \gamma_3 + \tau_1 + \mu) x_4, \\
 f_5 &= \frac{dx_5}{dt} = \gamma_2 x_3 + \gamma_3 x_4 - (\sigma_4 + \tau_2 + \delta_4 + \mu) x_5, \\
 f_6 &= \frac{dx_6}{dt} = \tau_1 x_4 + \tau_2 x_5 - (\sigma_5 + \alpha_1 + \mu) x_6.
 \end{aligned} \tag{16}$$

At $\mathcal{R}_0 = 1$ in (8), the bifurcation parameter β^* can be obtained as

$$\beta^* = \frac{d_1 d_2 d_3 d_4 d_5 - ((d_4 \tau_1 \alpha_1 + \tau_2 \alpha_1 \gamma_3)(d_2 \sigma_1 + \gamma_1 \sigma_2) + d_3 \tau_2 \alpha_1 \gamma_1 \gamma_2)}{d_3 d_4 d_5 (d_2 + \eta \gamma_1)}. \tag{17}$$

The linearized matrix of the system (16) around $\mathcal{D}_0$ and evaluated at β^* is given by

$$\mathcal{J}_{(\mathcal{D}_0, \beta^*)} = \begin{pmatrix} -\mu & -\beta^* & -\beta^*\eta & 0 & 0 & 0 \\ 0 & \beta^* - d_1 & \beta^*\eta & 0 & 0 & \alpha_1 \\ 0 & \gamma_1 & -d_2 & 0 & 0 & 0 \\ 0 & \sigma_1 & \sigma_2 & -d_3 & 0 & 0 \\ 0 & 0 & \gamma_2 & \gamma_3 & -d_4 & 0 \\ 0 & 0 & 0 & \tau_1 & \tau_2 & -d_5 \end{pmatrix}$$

The eigenvalues λ of $\mathcal{J}_{(\mathcal{D}_0, \beta^*)}$ given by the matrix above are the roots of the characteristic equation of the form

$$(\lambda + \mu)P(\lambda) = 0, \quad (18)$$

where $P(\lambda)$ is a polynomial of degree four whose roots are all negative except one zero eigenvalue. The left eigenvector, $v = (v_1, v_2, \dots, v_6)$, corresponding to the simple zero eigenvalue of (16) is obtained from $v\mathcal{J}_{(\mathcal{D}_0, \beta^*)} = 0$ as

$$\begin{aligned} v_1 = 0, \quad v_2 = \frac{d_5}{\alpha_1}v_6, \quad v_3 = \frac{\beta^*\eta d_3 d_4 d_5 + \sigma_2 \alpha_1 (\tau_2 \gamma_3 + \tau_1 d_4) + \gamma_2 \tau_2 \alpha_1 d_3}{\alpha_1 d_2 d_3 d_4}v_6, \\ v_4 = \frac{\tau_2 \gamma_3 + \tau_1 d_4}{d_3 d_4}v_6, \quad v_5 = \frac{\tau_2}{d_4}v_6. \end{aligned} \quad (19)$$

Further, the right eigenvector, $w = (w_1, w_2, \dots, w_6)^T$, associated with this simple zero eigenvalue can be obtained from $w\mathcal{J}_{(\mathcal{D}_0, \beta^*)} = 0$. As a result, we have

$$\begin{aligned} w_1 = \frac{-\beta^*(d_2 + \eta\gamma_1)}{\mu d_2}w_2, \quad w_3 = \frac{\gamma_1}{d_2}w_2, \quad w_5 = \frac{\gamma_1 \gamma_2 d_3 + \gamma_3(\sigma_1 d_2 + \sigma_2 \gamma_1)}{d_2 d_3 d_4}w_2, \\ w_4 = \frac{\sigma_1 d_2 + \sigma_2 \gamma_1}{d_2 d_3}w_2, \quad w_6 = \frac{\tau_1 d_4(\sigma_1 d_2 + \sigma_2 \gamma_1) + \tau_2(\gamma_1 \gamma_2 d_3 + \gamma_3(\sigma_1 d_2 + \sigma_2 \gamma_1))}{d_2 d_3 d_4 d_5}w_2. \end{aligned} \quad (20)$$

It should be noted that the components of w and v are obtained so that $v \cdot w = 1$ as required in [30]. All the second-order partial derivatives of $f_i, i = 1, 2, \dots, 6$, from the system (16) are zero at point $(\mathcal{D}_0, \beta^*)$ except the following :

$$\begin{aligned} \frac{\partial^2 f_1}{\partial x_1 \partial x_2} = \frac{\partial^2 f_1}{\partial x_2 \partial x_1} = \frac{-\beta^* \mu}{\pi}, \quad \frac{\partial^2 f_1}{\partial x_1 \partial x_3} = \frac{\partial^2 f_1}{\partial x_3 \partial x_1} = \frac{-\beta^* \mu \eta}{\pi}, \\ \frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \frac{\partial^2 f_2}{\partial x_2 \partial x_1} = \frac{\beta^* \mu}{\pi}, \quad \frac{\partial^2 f_2}{\partial x_1 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_1} = \frac{\beta^* \mu \eta}{\pi}, \end{aligned} \quad (21)$$

with

$$\frac{\partial^2 f_1}{\partial x_2 \partial \beta} = -1, \quad \frac{\partial^2 f_1}{\partial x_3 \partial \beta} = -\eta, \quad \frac{\partial^2 f_2}{\partial x_2 \partial \beta} = 1, \quad \frac{\partial^2 f_2}{\partial x_3 \partial \beta} = \eta. \quad (22)$$

The direction of the bifurcation at $\mathcal{R}_0 = 1$ is determined by the signs of the bifurcation coefficients **a** and **b**, define as follow:-

$$a = \sum_{k,i,j=1}^6 v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(\mathcal{D}_0, \beta^*) \quad (23)$$

and

$$b = \sum_{k,i=1}^6 v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta}(\mathcal{D}_0, \beta^*). \quad (24)$$

Since $v_1 = 0$, then (23) and (24) can be written as

$$a = v_2 w_1 w_2 \frac{\partial^2 f_2}{\partial x_1 \partial x_2} + v_2 w_1 w_3 \frac{\partial^2 f_2}{\partial x_1 \partial x_3} \quad (25)$$

and

$$b = v_2 w_2 \frac{\partial^2 f_2}{\partial x_2 \partial \beta} + v_2 w_3 \frac{\partial^2 f_2}{\partial x_3 \partial \beta}. \quad (26)$$

Substituting (17), (19), (20), (21) and (22) into (25) and (26) gives the following

$$a = \frac{-d_5 \mu^2}{\pi \alpha_1} w_2^2 v_6 \quad (27)$$

and

$$b = \frac{d_5(d_2 + \gamma_1 \eta)}{\alpha_1 d_2}. \quad (28)$$

As we can see $a < 0$ and $b > 0$, it follows that the IDUB model (1) exhibits a forward bifurcation and $\mathcal{D}^*$ is locally stable. This result is claimed as follows.

Theorem 4. *The IDUB model governed by (1) exhibits a forward bifurcation at the threshold $\mathcal{R}_0 = 1$ (or, equivalently, there exists an illicit drug use and banditry-present equilibrium, $\mathcal{D}^*$, which is locally asymptotically stable whenever $\mathcal{R}_0 > 1$, but near $\mathcal{R}_0 = 1$).*

The significance of the above result is that a small inflow of illicit drug user and bandit into a completely susceptible population will cause a raise the spread of illicit drug use and banditry the population within the community whenever $\mathcal{R}_0 > 1$. Nevertheless, the initial sizes of the illicit drug users and bandits in the population is a critical factor of this result. Hence, to show that the elimination and persistence of illicit drug use and banditry does not depend on the initial sizes of the illicit drug users and bandits, we established the global stability of IDBPE in the next Section.

3.5 Global stability of IDBPE

Theorem 5. *The unique IDBPE of the illicit drug use and banditry model (1), given by (15), is globally asymptotically stable (GAS) in $\Gamma \setminus \Gamma_0$ when $\mathcal{R}_0 > 1$, $DR^{**} \leq D^{**}R, DP^{**} \leq D^{**}P$ and $PR^{**} \leq P^{**}R$.*

Figure 4: Simulation results of model (1) with the effect of σ_1 and σ_5 on (a) Individuals in detention; (b) Bandits; (c) Illicit drug users (d) Individuals in prison and (e)-(f) Individuals in rehabilitation center.

(a)

(b)

(c)

(d)

(e)

(f)

Figure 5: Simulation results of model (1) with the effect of σ_2 and τ_1 on (a) Individuals in detention; (b) Bandits; (c) Individuals in rehabilitation center (d) Individuals in detention; (e) Individuals in prison and (f) Individuals in rehabilitation center.

(a)

(b)

(c)

(d)

Figure 6: Simulation results of model (1) with the effect of σ_3 on (a) Individuals in detention; (b) Illicit drug users; (c) Individuals in prison and (d) Individuals in rehabilitation center.

Proof. Consider the non-linear Lyapunov function $\mathfrak{F} : \Gamma \setminus \Gamma_0 \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathfrak{F} = & S - S^{**} - S^{**} \ln \frac{S}{S^{**}} + \left(I - I^{**} - I^{**} \ln \frac{I}{I^{**}} \right) + \frac{\alpha_1 \tau_1 (\gamma_3 + d_4)}{d_3 d_4 d_5} \left(D - D^{**} - D^{**} \ln \frac{D}{D^{**}} \right) \\ & + \frac{d_3 d_4 d_5 (d_1 + \beta S^{**}) + \sigma_1 \alpha_1 \tau_1 (\gamma_3 + d_4)}{d_3 d_4 d_5 \gamma_1} \left(B - B^{**} - B^{**} \ln \frac{B}{B^{**}} \right) \\ & + \frac{\alpha_1 \tau_1}{d_4 d_5} \left(P - P^{**} - P^{**} \ln \frac{P}{P^{**}} \right) + \frac{\alpha_1}{d_5} \left(R - R^{**} - R^{**} \ln \frac{R}{R^{**}} \right), \end{aligned} \quad (29)$$

which is of Goh-Volterra type (see, e.g., [31, 32, 33]). The Lyapunov derivative is given by

$$\begin{aligned} \frac{d\mathfrak{F}}{dt} = & \frac{dS}{dt} - \frac{S^{**}}{S} \frac{dS}{dt} + \left(\frac{dI}{dt} - \frac{I^{**}}{I} \frac{dI}{dt} \right) + \frac{d_3 d_4 d_5 (d_1 + \beta S^{**}) + \sigma_1 \alpha_1 \tau_1 (\gamma_3 + d_4)}{d_3 d_4 d_5 \gamma_1} \left(\frac{dB}{dt} - \frac{B^{**}}{B} \frac{dB}{dt} \right) \\ & + \frac{\alpha_1 \tau_1 (\gamma_3 + d_4)}{d_3 d_4 d_5} \left(\frac{dD}{dt} - \frac{D^{**}}{D} \frac{dD}{dt} \right) + \frac{\alpha_1 \tau_1}{d_4 d_5} \left(\frac{dP}{dt} - \frac{P^{**}}{P} \frac{dP}{dt} \right) + \frac{\alpha_1}{d_5} \left(\frac{dR}{dt} - \frac{R^{**}}{R} \frac{dR}{dt} \right). \end{aligned} \quad (30)$$

Let $\frac{\beta(I(t) + \eta B(t))}{N(t)} = \tilde{\beta}(I(t) + \eta B(t))$, then putting the appropriate equations of the system (1) into (30), one obtains

$$\begin{aligned} \frac{d\mathfrak{F}}{dt} = & \left(1 - \frac{S^{**}}{S} \right) (\pi - \tilde{\beta}(I(t) + \eta B(t))S(t) - \mu S(t)) \\ & + \left(1 - \frac{I^{**}}{I} \right) (\tilde{\beta}(I(t) + \eta B(t))S(t) + \alpha_1 R(t) - (\sigma_1 + \gamma_1 + \delta_1 + \mu)I(t)) \\ & + \frac{d_3 d_4 d_5 (d_1 + \beta S^{**}) + \sigma_1 \alpha_1 \tau_1 (\gamma_3 + d_4)}{d_3 d_4 d_5 \gamma_1} \left(1 - \frac{B^{**}}{B} \right) (\gamma_1 I(t) - (\sigma_2 + \gamma_2 + \delta_2 + \mu)B(t)) \\ & + \frac{\alpha_1 \tau_1 (\gamma_3 + d_4)}{d_3 d_4 d_5} \left(1 - \frac{D^{**}}{D} \right) (\sigma_1 I(t) + \sigma_2 B(t) - (\sigma_3 + \gamma_3 + \tau_1 + \mu)D(t)) \\ & + \frac{\alpha_1 \tau_1}{d_4 d_5} \left(1 - \frac{P^{**}}{P} \right) (\gamma_2 B(t) + \gamma_3 D(t) - (\sigma_4 + \tau_2 + \delta_4 + \mu)P(t)) \\ & + \frac{\alpha_1}{d_5} \left(1 - \frac{R^{**}}{R} (\tau_1 D(t) + \tau_2 P(t) - (\sigma_5 + \alpha_1 + \mu)R(t)) \right). \end{aligned} \quad (31)$$

At the illicit drug use and banditry-present equilibrium, the following relations hold

from the system (1):

$$\begin{aligned}
\pi &= \tilde{\beta}(I^{**} + \eta B^{**})S^{**} + \mu S^{**}, \\
\sigma_1 + \gamma_1 + \delta_1 + \mu &= \frac{\tilde{\beta}(I^{**} + \eta B^{**})}{I^{**}} + \frac{\alpha_1 R^{**}}{I^{**}}, \\
\sigma_2 + \gamma_2 + \delta_2 + \mu &= \frac{\gamma_1 I^{**}}{B^{**}}, \\
\sigma_3 + \gamma_3 + \tau_1 + \mu &= \frac{\sigma_1 I^{**} + \sigma_2 B^{**}}{D^{**}}, \\
\sigma_4 + \tau_2 + \delta_4 + \mu &= \frac{\gamma_2 B^{**} + \gamma_3 D^{**}}{P^{**}}, \\
\sigma_5 + \alpha_1 + \mu &= \frac{\tau_1 D^{**} + \tau_2 P^{**}}{R^{**}}.
\end{aligned} \tag{32}$$

Using the relations (32) in (31) and simplifying yields

$$\begin{aligned}
\frac{d\mathfrak{F}}{dt} &= \tilde{\beta}S^{**}(I^{**} + \eta B^{**}) \left(1 - \frac{S^{**}}{S}\right) + \mu S^{**} \left(2 - \frac{S}{S^{**}} - \frac{S^{**}}{S}\right) + \alpha_1 R^{**} \left(1 - \frac{RI^{**}}{R^{**}I}\right) \\
&+ \tilde{\beta}S^{**}(I + \eta B) \left(1 - \frac{SI^{**}}{S^{**}I}\right) + \frac{\sigma_1 \alpha_1 \tau_1 \gamma_3}{d_3 d_4 d_5} I^{**} \left(2 - \frac{IB^{**}}{I^{**}B} - \frac{ID^{**}}{I^{**}D}\right) \\
&+ (d_1 + \tilde{\beta}S^{**}) \left(1 - \frac{IB^{**}}{I^{**}B}\right) + \frac{\sigma_1 \alpha_1 \tau_1}{d_3 d_5} I^{**} \left(2 - \frac{IB^{**}}{I^{**}B} - \frac{ID^{**}}{I^{**}D}\right) \\
&+ \frac{\sigma_2 \alpha_1 \tau_1}{d_3 d_5} B^{**} \left(1 - \frac{BD^{**}}{B^{**}D}\right) + \frac{\sigma_2 \alpha_1 \tau_1 \gamma_3}{d_3 d_4 d_5} B^{**} \left(1 - \frac{BD^{**}}{B^{**}D}\right) + \frac{\gamma_2 \alpha_1 \tau_1}{d_4 d_5} B^{**} \left(1 - \frac{BP^{**}}{B^{**}P}\right) \\
&+ \frac{\gamma_3 \alpha_1 \tau_1}{d_4 d_5} D^{**} \left(1 - \frac{DP^{**}}{D^{**}P}\right) + \frac{\alpha_1 \tau_1}{d_5} D^{**} \left(1 - \frac{DR^{**}}{D^{**}R}\right) + \frac{\alpha_1 \tau_2}{d_5} P^{**} \left(1 - \frac{PR^{**}}{P^{**}R}\right).
\end{aligned} \tag{33}$$

Since $DR^{**} \leq D^{**}R$, $DP^{**} \leq D^{**}P$ and $PR^{**} \leq P^{**}R$ then $1 - \frac{D^{**}R}{DR^{**}} \leq 0$, $1 - \frac{D^{**}P}{DP^{**}} \leq 0$ and $1 - \frac{P^{**}R}{PR^{**}} \leq 0$ with equality if $D^{**}R = DR^{**}$, $D^{**}P = DP^{**}$ and $P^{**}R = PR^{**}$.

Consequently, (33) becomes

$$\begin{aligned}
\frac{d\mathfrak{F}}{dt} = & \tilde{\beta}S^{**}(I^{**} + \eta B^{**}) \left(1 - \frac{S^{**}}{S}\right) + \mu S^{**} \left(2 - \frac{S}{S^{**}} - \frac{S^{**}}{S}\right) + \alpha_1 R^{**} \left(1 - \frac{RI^{**}}{R^{**}I}\right) \\
& + \tilde{\beta}S^{**}(I + \eta B) \left(1 - \frac{SI^{**}}{S^{**}I}\right) + \frac{\sigma_1 \alpha_1 \tau_1 \gamma_3}{d_3 d_4 d_5} I^{**} \left(2 - \frac{IB^{**}}{I^{**}B} - \frac{ID^{**}}{I^{**}D}\right) \left(1 + \frac{\gamma_3}{d_4}\right) \\
& + (d_1 + \tilde{\beta}S^{**}) \left(1 - \frac{IB^{**}}{I^{**}B}\right) + \frac{\sigma_2 \alpha_1 \tau_1}{d_3 d_5} B^{**} \left(1 - \frac{BD^{**}}{B^{**}D}\right) \left(1 + \frac{\gamma_3}{d_4}\right) \\
& + \frac{\gamma_2 \alpha_1 \tau_1}{d_4 d_5} B^{**} \left(1 - \frac{BP^{**}}{B^{**}P}\right).
\end{aligned} \tag{34}$$

Using Arithmetic Mean -Geometric Mean inequality [34], the following inequalities hold:

$$\begin{aligned}
\left(1 - \frac{S}{S^{**}}\right) \leq 0, \quad \left(2 - \frac{S}{S^{**}} - \frac{S^{**}}{S}\right) \leq 0, \quad \left(2 - \frac{IB^{**}}{I^{**}B} - \frac{ID^{**}}{I^{**}D}\right) \leq 0, \quad \left(1 - \frac{BP^{**}}{B^{**}P}\right) \leq 0, \\
\left(1 - \frac{RI^{**}}{R^{**}I}\right) \leq 0, \quad \left(1 - \frac{SI^{**}}{S^{**}I}\right) \leq 0, \quad \left(1 - \frac{IB^{**}}{I^{**}B}\right) \leq 0, \quad \left(1 - \frac{BD^{**}}{B^{**}D}\right) \leq 0.
\end{aligned}$$

Moreover, since all the model parameters are non-negative, it follows from (34) that $\frac{d\mathfrak{F}}{dt} \leq 0$ with equality if and only if $S = S^{**}$, $I = I^{**}$, $B = B^{**}$, $D = D^{**}$, $P = P^{**}$, $R = R^{**}$. Hence, by LaSalle's invariance principle [29], $(S, I, B, D, P, R) \rightarrow \mathcal{D}_0$ as $t \rightarrow \infty$. $\square$

Literally Theorem 5 means that IDUB will remain, regardless of the initial sizes of illicit drug users and bandits in the population, whenever $\mathcal{R}_0 > 1$. This stability property is shown in Figure 3, where all the solutions tend to illicit drug use and banditry-present equilibrium (IDBPE). This now leads us to sensitivity analysis.

3.6 Sensitivity Analysis

Following the idea in [9, 10, 31, 35, 36], we perform a sensitivity analysis of the model (1) in order to determine the contributory effects of the model parameters on the transmission and spread of the illicit drug and banditry menace in a population.

The normalized forward-sensitivity index of a variable, v , that depends differentially on a parameter, p , is defined as

$$\Upsilon_p^v = \frac{\partial v}{\partial p} \times \frac{p}{v}. \tag{35}$$

In particular, sensitivity indices of the basic reproduction number, $\mathcal{R}_0$, with respect to the model parameters are computed and the summary is given below in Table 2.

Table 2. Sensitivity sign of each parameters of the model

Parameter	Sign
β	Positive
α_1	Positive
τ_1	Positive
η	Positive
δ_4	Positive
σ_1	Negative
δ_1	Negative
μ	Negative
σ_2	Negative
γ_2	Negative
δ_2	Negative
σ_3	Negative
γ_3	Negative
γ_1	Negative
σ_5	Negative

The sign of the sensitivity index plays a key role in determining how the parameters of the model relate to the basic reproduction number, $\mathcal{R}_0$, of the model.

In Table 2, the parameters with positive sign have a direct relation on $\mathcal{R}_0$ while the ones with negative sensitivity sign as an inverse relation on $\mathcal{R}_0$. This means that increasing any parameter with positive sign will in turn increase the value of $\mathcal{R}_0$ (i.e the menace will persist) and the vice versa, while increasing any parameter with negative sign will in turn decrease the value of $\mathcal{R}_0$ (i.e the menace will fade out) and the vice versa. With sensitivity analysis, one can get insight on the appropriate intervention strategies to prevent and control the effect and spread of the IDUB in the population. It means that effect should be made to reduce the value of parameters with positive sign, in like manner, the value of parameters with negative sign must be increase at all cost.

In addition, the graphs of the IDUB model (1) are plotted, using the value of parameters in Table 3, to study the behaviour effects of detention rate of illicit drug users σ_1 , treatment rate of individuals in rehabilitation center σ_5 , detention rate of bandits σ_2 , rehabilitation rate of detainee τ_1 , quitting rate of detainee σ_3 , loss of determination of individuals in rehabilitation center α_1 , effective influence rate β , progression rates of illicit drug user to bandits γ_1 and rates at which bandits are imprisoned γ_2 , as well as rates at which detainee are imprisoned γ_3 on the dynamics of the population. From Figure 4(a-e), it can be observed in that as σ_1 increases (decreases), the population of bandits decreases (increases), illicit drug user decreases (increases) with time while as σ_1 increases (decreases), the population of detainee and individuals in prison increases (decreases) with time. It shows that σ_1 has a positive effect on population of bandits and illicit drug users. The physical meaning of this is that as the detention rate of illicit drug users increases, the number of the illicit drug users and bandits in the population decreases. Thus, σ_1 , is a good corrective measure that can reduce the spread of IDUB in the population. Figure 4(f) shows the result of the rehabilitation rate σ_5 on the individuals in rehabilitation center. As σ_5 increases, the population of individuals in rehabilitation center reduce accordingly.

(a)

(b)

(c)

(d)

(e)

Figure 7: Simulation results of model (1) with the effect of α_1 and τ_1 on (a) Individuals in detention; (b) Bandits; (c) Illicit drug users (d) Individuals in prison and (e) Individuals in rehabilitation center.

(a)

(b)

(c)

(d)

(e)

Figure 8: Simulation results of model (1) with the effect of β on (a) Bandits; (b) Individuals in detention; (c) Individuals in prison (d) Illicit drug users and (e) Individuals in rehabilitation center.

(a)

(b)

(c)

(d)

(e)

(f)

Figure 9: Simulation results of model (1) with the effect of γ_1 and γ_2 on (a) Bandits; (b) Illicit drug users (c) Individuals in prison (d) Bandits (e) Individuals in prison and (f) Individuals in detention.

Figure 10: Simulation results of model (1) with the effect of γ_3 (a) Individuals in detention and (b) Individuals in prison.

Table 3. Values of parameters of the model

Parameter	Range	Baseline value	Source
π	40 - 60	50	[19, 20]
β	0.2 - 0.5	0.64	[24, 17]
η	0.6 - 0.9	0.7	[37]
μ	0.009 - 0.04	0.02	[37]
α_1	0.6 - 0.9	0.75	[19, 38]
σ_1	0.5 - 0.8	0.65	[20, 37]
γ_1	0.5 - 0.8	0.6	Assumed
δ_1	0.09 - 0.2	0.14	[19]
σ_2	0.5 - 0.8	0.6	[19, 37]
γ_2	0.5 - 0.8	0.65	[17]
δ_2	0.09 - 0.2	0.14	[19]
σ_3	0.6 - 0.9	0.7	[17]
γ_3	0.6 - 0.9	0.7	[17]
τ_1	0.7 - 0.95	0.85	[19]
σ_4	0.1 - 0.3	0.2	[17]
τ_2	0.05 - 0.2	0.1	[19, 4]
δ_4	0.08 - 0.2	0.14	[19]
σ_5	0.4 - 0.9	0.55	Assumed

In a similar manner, Figure 5(a-c) reveals the behaviour of detention rate of bandits σ_2 on detainees, bandits and individuals in rehabilitation center. It is observed that the populations of detainee and individuals in rehabilitation center increase (decrease) as σ_2 increase (decrease) but the population of bandits decreases. It shows that σ_2 has a positive

effect on population of bandits. Literally, the higher the detention rate of bandits, the lesser the number of bandits in the population. This shows that σ_2 , is also a good corrective measure that can reduce the spread of banditry in the population. Also, Figure 5(d-f) shows the effects of τ_1 on the population of detainees, prisoners and individuals in rehabilitation center. More so, as τ_1 rises, the populations of detainees and prisoners falls while the population of individuals in rehabilitation center increases. It shows that τ_1 has a positive effect on populations of detainees and prisoners. Figuratively, the higher the rehabilitation rate of detainees, the lesser the number of detainees in the population which means illicit drug use and banditry will reduce. This also shows that τ_1 is also a good corrective measure that can reduce the spread of illicit drug use and banditry in the population.

In addition, the populations of detainees, illicit drug users, prisoners and individuals in rehabilitation center increase (decrease) as quitting rate of detainees decreases (increases) (Figure 6a-d). It shows that all effort must be put in place to encourage detainee to quit the act of illicit drug use or banditry as the case may be. Literally, this means that decreasing the quitting rate of detainees, increasing the number of detainees in the population, which means illicit drug use and banditry will reduce. This also shows that τ_1 is also a good corrective measure that can reduce the spread of IDUB in the population. In another development, Figure 7(a-e) shows the effects of relapsed rate of individuals in rehabilitation center, α_1 , on the detainees, bandits, illicit drug users, prisoners and individuals in rehabilitation center. It is observed that detainees, bandits, illicit drug users and prisoners increase (decrease) as α_1 increases (decreases), while the population of individuals in rehabilitation center decreases (increases). It shows that α_1 has a negative effect on population of illicit drug users and bandits. The physical meaning of this is that the higher the relapsed rate of individuals in rehabilitation center, the number of illicit drug users and bandits population increases and that increasing α_1 will have a positive effect on the dynamics of IDUB. This shows that rehabilitated individuals should be monitored closely to avoid relapsed or re-occurrence of IDUB in the population.

It can be observed in Figure 8(a-e) that as β increases (decreases), the populations of bandits, detainees, prisoners, illicit drug users and individuals in rehabilitation center increase (decrease) with time. The physical meaning of this is that the bandits, detainee, prisoner, illicit drug user and individual in rehabilitation center increases as the effective influence rate of IDUB increases. This connotes that a control measures must be targeted at effective influence rate to inhibit the spread of IDUB in the population. Similarly, it is seen that the populations of bandit and prisoner increase (decrease) as the movement rate of illicit drug user to bandit increases (decreases) while as the population of illicit drug user decreases (increases) as the movement rate of illicit drug user to bandit increases (decreases) as shown in Figure 9(a-c). It shows that γ_1 has a positive effect on the act of illicit drug use. In physical sense, the higher the movement rate of illicit drug user to bandit, the lesser the movement of illicit drug user. Also, It is observed in Figure 9(d-f) that the population of bandit decreases (increases) as γ_2 increases (decreases) while the population of prisoner and detainee increase (decrease) with time. It shows that γ_2 has a positive effect on the population of bandit. The meaning of this is that the higher the number of movement of bandit to prison, the lesser the number of bandits in the population, but good corrective measure must be put in place in the prison so as to compliment the work of γ_2 . This shows that increasing γ_2 will increase the spread of banditry in the population.

Finally, in Figure 10(a-b), it is observed that γ_3 increases (decreases) as the population of detainees decreases (increases) while the population of prisoners increases (decreases). It shows that γ_3 has a positive effect on the population of detainees. The physical interpretation of this is that the higher the progression rate of detainees to prison, the lesser the number of detainees in the population, which means illicit drug use and banditry will reduce, but good corrective measure must be put in place in the prison so as to compliment the work of γ_3 .

4 Conclusion

This work presented and analysed a suitable compartmental deterministic model for IDUB population dynamics. The illicit drug and banditry model was formulated, and analyzed to study the effect of illicit drug use and banditry on the population. The existence of illicit drug and banditry-free and illicit drug and banditry-present equilibria was determined. The illicit drug and banditry reproduction number was evaluated at illicit drug use and banditry-free equilibrium and the local stability of it was shown. The formulated model was proved to have global asymptotic stability when the illicit drug use and banditry-free equilibrium is less than unity by constructing a suitable linear Lyapunov functional. Also, the model is shown to have unique illicit drug use and banditry-present equilibrium whenever the associated illicit drug use and banditry reproduction number exceeds unity.

Further, by a suitably constructed nonlinear Lyapunov functional of Goh-Volterra type, the unique illicit drug use and banditry-present equilibrium was shown to be globally asymptotically stable. Bifurcation analysis of the mode was further carried out on the illicit drug and banditry-present equilibrium and proved the same to be forward bifurcation. The sensitivity analysis of the model was done to know the contributory effects of each parameter on the dynamic spread of the illicit drug use and banditry in the population. Finally, corrective measures like detention rate, rehabilitation and quitting rate should be beefed up so as to have illicit drug use and banditry free population. This points that, if the law against illicit drug use and banditry is strengthened and good corrective measure is put in place in the prison, it will reduce illicit drug users and bandits in the population.

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