New Optical Solitons And Modulation Instability Analysis of Generalized Coupled Nonlinear Schrodinger-KdV System

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Research Article

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New optical solitons and modulation instability analysis of generalized coupled nonlinear Schrödinger-KdV system

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Abstract

In this study, the generalized coupled nonlinear Schrödinger-KdV (NLS-KdV) system is investigated to obtain new optical soliton solutions. This system appears as a model for reciprocity between long and short waves in various of physical settings. Different kind of new soliton solutions including dark, bright, combined dark-bright, singular and combined singular soliton solutions are obtained using two effective methods namely, the extended sinh-Gordon equation expansion method and the solitary wave ansatz method. In addition, the modulation instability analysis of the system is presented based on the standard linear-stability analysis. The behaviours of obtained solutions are expressed by 3D graphs.

Keywords: Generalized coupled NLS-KdV system; Soliton solutions; Ansatz method; Modulation instability analysis.

1 Introduction

During the past several decades, the coupled nonlinear Schrödinge-Korteweg-de Vries (NLS-KdV) system has received extensive attention because of its important physical background [1–3]. Considers the generalized version of coupled NLS-KdV system of the form [4]:

$$
\begin{aligned}
iu_t + \lambda_1 u_{\text{xx}} + \lambda_2 |u|^2 u + \lambda_3 uv = 0,\\
v_t + \beta_1 v v_x + \beta_2 v_{\text{xxx}} + \beta_3 \left(|u|^2\right)_x = 0,
\end{aligned} \tag{1}
$$

where $u = u(x,t) \in \mathbb{C}$, $v = v(x,t) \in \mathbb{R}$ and λ_i, β_i $(i = 1,2,3)$ are constants. The NLS-KdV system occurs in phenomena of interactions between short and long dispersive waves arising in fluid mechanics, for instance the interactions of capillary-gravity water waves.

The important issues of great concern for this model are the existence and stability of the solitary wave solutions. It is known that, due to the effect of nonlinearity and dispersion, the coupled NLS-KdV system usually possesses such kind of solutions. The existence of solutions for the coupled system of NLS-KdV has been studied in [5, 6]. Furthermore, several stability theories have been used to prove the stability of solitary wave solutions of this system [7–9].

Recently, the existence and bifurcation of nontrivial solutions for the coupled NLS-KdV system is studied in [10]. The two techniques, specifically, the sub-equation method and the Kudryashov method have been utilized in [4] to fund optical soliton solutions of the above generalized coupled NLS-KdV equations. This present work investigates to obtain the soliton and combined soliton solutions to the Eq. (1) based on the extended sinh Gordon expansion method (EshGEM) [11,12] and the solitary wave ansatz method [13, 14]. In addition, the modulation instability analysis of the stationary solution of this system is studied by using the standard linear-stability analysis. Except the EshGEM and solitary wave ansatz method, analytic solutions are found to the variety of integer and fractal order models with the execution of other methods [15–25]. However, the proposed methods are powerful tools for constructing the exact solutions of nonlinear differential equations and gained considerable attention in recent years. To our best knowledge, the application of the proposed methods to the model, and the received combined soliton solutions are novel.

The paper's structure is proposed as follows. The mathematical analysis of the proposed model is presented in Section 2. The implementations of the above proposed two methods are devoted to Section 3 and 4. In last section, the conclusion is revealed.

2 Mathematical analysis for the model

In order to find the solitary wave solution of Eq. (1), consider the wave transformation

$$\begin{aligned} u =& \phi(\xi) e^{i(kx+\omega t)} \\ v =& \psi(\xi), \quad \xi = x - ct, \end{aligned} \tag{2}$$

here k, ω and c are constants. By substituting (2) into equation (1) and spliting the real and imaginary parts, we get

$$-\omega\phi(\xi) - k^2\lambda_1\phi(\xi) + \lambda_2\phi(\xi)^3 + \lambda_3\phi(\xi)\psi(\xi) + \lambda_1\phi''(\xi) = 0 \tag{3}$$

$$(-c + 2k\lambda_1)\,\phi'(\xi) = 0 \tag{4}$$

and

$$2\beta_3\phi(\xi)\phi'(\xi) - c\psi'(\xi) + \beta_1\psi(\xi)\psi'(\xi) + \beta_2\psi^{(3)}(\xi) = 0. \tag{5}$$

From Eq. (4), we get

$$c = 2k\lambda_1 \tag{6}$$

Inserting Eq. (6) into Eq. (5), and integrate once with zero constant of integration, we get

$$\frac{1}{2}\beta_1\psi(\xi)^2 + \beta_3\phi(\xi)^2 - 2k\lambda_1\psi(\xi) + \beta_2\psi''(\xi) = 0 \tag{7}$$

The Eqs. (3) and (7) will be discussed in the following two sections to obtain the solitary wave solutions of Eq. (1).

3 Implementation of the extended ShGEEM

In this section, we implement the extended ShGEEM to solve the Eq. (1). Consider the homogeneous balance $\phi''(\xi)$ with $\phi(\xi)^3$ in Eq. (3) yields $n = 1$ and $\psi''(\xi)$ with $\psi(\xi)^2$ in Eq. (7) yields $m = 2$. According to ShGEEM [11, 12], we assume that the solution structure of the form

$$\phi(\omega) = A_0 + B_1\sinh(\omega) + A_1\cosh(\omega),$$

$$\psi(\omega) = a_0 + b_1\sinh(\omega) + a_1\cosh(\omega) + \cosh(\omega)[b_2\sinh(\omega) + a_2\cosh(\omega)], \tag{8}$$

with

$$\phi(\xi) = A_0 \pm iB_1\text{sech}(\xi) - A_1\tanh(\xi),$$
$$\phi(\xi) = A_0 \pm B_1\text{csch}(\xi) - A_1\coth(\xi), \tag{9}$$

and

$$\psi(\xi) = a_0 \pm ib_1\text{sech}(\xi) - a_1\tanh(\xi) - \tanh(\xi)[\pm ib_2\text{sech}(\xi) - a_2\tanh(\xi)],$$
$$\psi(\xi) = a_0 \pm b_1\text{csch}(\xi) - a_1\coth(\xi) - \coth(\xi)[\pm b_2\text{csch}(\xi) - a_2\coth(\xi)], \tag{10}$$

Substituting Eq. (8) and its second derivative along with $\omega' = \sinh(\xi)$ into Eqs. (3) and (7), we obtain the set of parameter values by solving the algebra equations. For each set, the following solution of Eq. (1) can be found by Inserting the values of parameters into equations Eq. (9) and Eq. (10) and then, into equation Eq. (2).

Set 1

$$A_0 = 0,\ A_1 = 0,\ B_1 = \pm\sqrt{\frac{2\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2}},$$

$$a_0 = \frac{12\beta_2}{\beta_1},\ a_1 = 0,\ b_1 = 0,\ a_2 = -\frac{12\beta_2}{\beta_1},\ b_2 = 0,\ k = \frac{\beta_1\beta_3\lambda_1 + 24\beta_2^2\lambda_2 - 6\beta_2\beta_3\lambda_3}{12\beta_2\lambda_1\lambda_2},$$

$$\omega = -\frac{4\beta_2^2}{\lambda_1} + \lambda_1 - \frac{\beta_1^2\beta_3^2\lambda_1}{144\beta_2^2\lambda_2^2} - \frac{\beta_1\beta_3}{3\lambda_2} + \frac{\beta_1\beta_3^2\lambda_3}{12\beta_2\lambda_2^2} + \frac{2\beta_2\beta_3\lambda_3}{\lambda_1\lambda_2} - \frac{\beta_3^2\lambda_3^2}{4\lambda_1\lambda_2^2}. \tag{11}$$

Replacing the values of set 1 into Eqs. (9) and (10), we obtain the bright soliton solutions for the above model as follows:

$$u(x,t) = \pm i\sqrt{\frac{2\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2}}\ \text{sech}(x - ct)e^{i(kx+\omega t)},$$

$$v(x,t) = \frac{12\beta_2}{\beta_1}\ \text{sech}^2(x - ct), \tag{12}$$

and singular soliton solutions for the above model as follows:

$$u(x,t) = \pm\sqrt{\frac{2\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2}}\ \text{csch}(x - ct)e^{i(kx+\omega t)},$$

$$v(x,t) = -\frac{12\beta_2}{\beta_1}\ \text{csch}^2(x - ct), \tag{13}$$

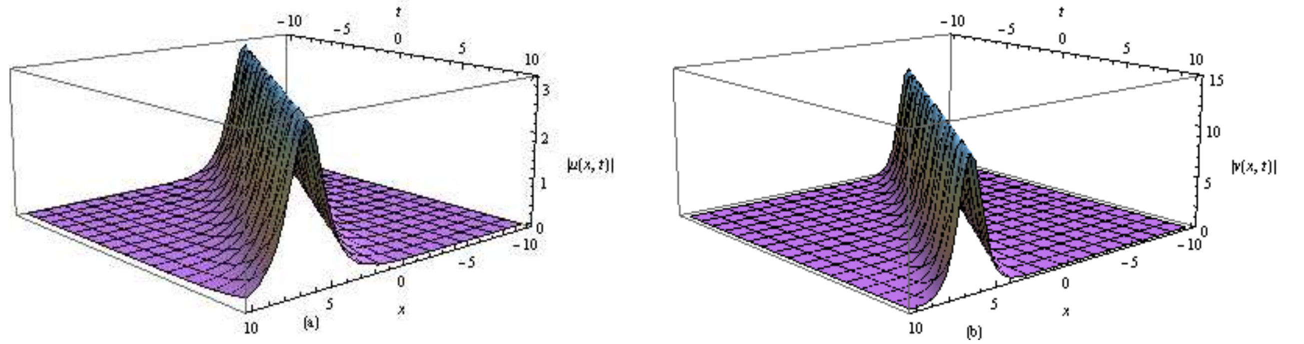

Figure 1: (a) 3D graphs for bright soliton solutions given in Eq. (12) of (a) $|u(x,t)|$ and (b) $|v(x,t)|$ with $\lambda_1 = \lambda_2 = \lambda_3 = 0.1,\ \beta_1 = \beta_2 = \beta_3 = 0.2$.

provided that $\dfrac{(-\beta_1\lambda_1 + 6\beta_2\lambda_3)}{\beta_1\lambda_2} > 0$. The behaviours of soliton solutions of Eq. (12) and (13) are presented in Fig. (1) and Fig. (2) respectively.

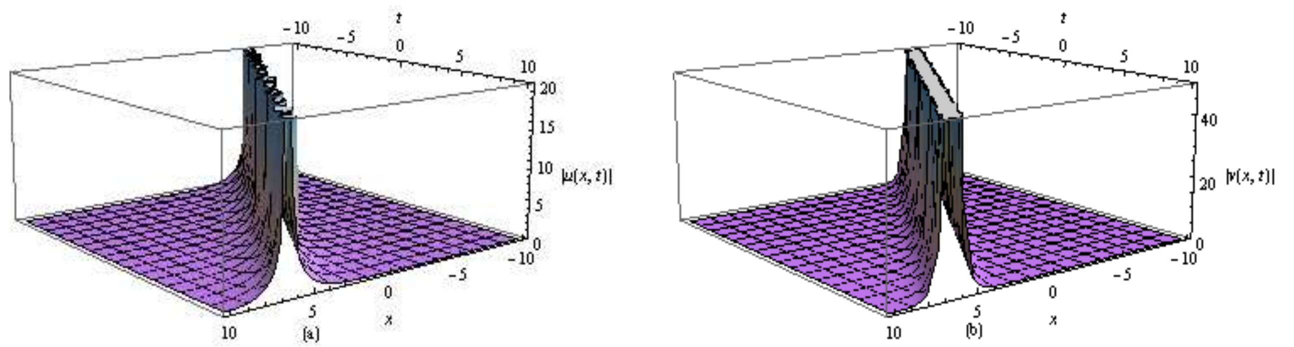

Figure 2: (a) 3D graphs for singular soliton solutions given in Eq. (13) of (a) $|u(x,t)|$ and (b) $|v(x,t)|$ with $\lambda_1 = \lambda_2 = \lambda_3 = 0.1,\ \beta_1 = \beta_2 = \beta_3 = 0.2$.

Set 2

$$A_0 = 0,\ A_1 = 0,\ B_1 = \pm\sqrt{\frac{2\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2}},\ \ k = \frac{-\beta_1\beta_3\lambda_1 - 24\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_3}{12\beta_2\lambda_1\lambda_2},$$

$$a_0 = \frac{-\beta_1\beta_3\lambda_1 + 12\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_3}{3\beta_1\beta_2\lambda_2},\ a_1 = 0,\ b_1 = 0,\ a_2 = -\frac{12\beta_2}{\beta_1},\ b_2 = 0,$$

$$\omega = -\frac{4\beta_2^2}{\lambda_1} + \lambda_1 - \frac{\beta_1^2\beta_3^2\lambda_1}{144\beta_2^2\lambda_2^2} - \frac{\beta_1\beta_3}{3\lambda_2} - \frac{8\beta_2\lambda_3}{\beta_1} + \frac{\beta_1\beta_3^2\lambda_3}{12\beta_2\lambda_2^2} + \frac{2\beta_2\beta_3\lambda_3}{\lambda_1\lambda_2} - \frac{\beta_3\lambda_1\lambda_3}{3\beta_2\lambda_2} - \frac{\beta_3^2\lambda_3^2}{4\lambda_1\lambda_2^2} + \frac{2\beta_3\lambda_3^2}{\beta_1\lambda_2}. \quad (14)$$

Replacing the values of set 2 into Eqs. (9) and (10), we obtain the bright soliton solutions for the above model as follows:

$$u(x,t) = \pm i\sqrt{\frac{2\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2}}\ \mathrm{sech}(x - ct)e^{i(kx+\omega t)},$$

$$v(x,t) = \frac{-\beta_1\beta_3\lambda_1 - 24\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_3}{3\beta_1\beta_2\lambda_2} + \frac{12\beta_2}{\beta_1}\,\mathrm{sech}^2(x-ct), \tag{15}$$

and singular soliton solutions for the above model as follows:

$$u(x,t) = \pm\sqrt{\frac{2\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2}}\,\mathrm{csch}(x-ct)e^{i(kx+\omega t)},$$

$$v(x,t) = \frac{-\beta_1\beta_3\lambda_1 - 24\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_3}{3\beta_1\beta_2\lambda_2} - \frac{12\beta_2}{\beta_1}\,\mathrm{csch}^2(x-ct), \tag{16}$$

provided that $\frac{(-\beta_1\lambda_1 + 6\beta_2\lambda_3)}{\beta_1\lambda_2} > 0$. The behaviours of soliton solutions of Eq. (15) and (16) are presented in Fig. (3) and Fig. (4) respectively.

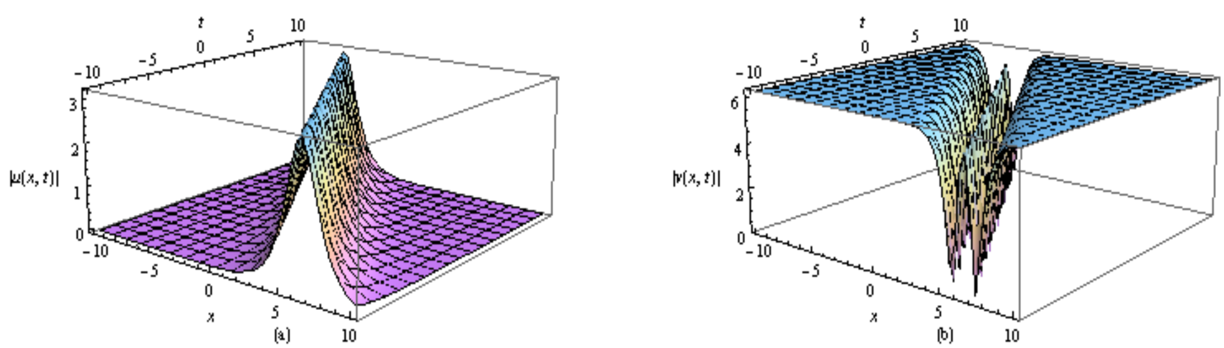

Figure 3: (a) 3D graphs for bright soliton solutions given in Eq. (15) of (a) $|u(x,t)|$ and (b) $|v(x,t)|$ with $\lambda_1 = \lambda_2 = \lambda_3 = 0.1,\ \beta_1 = \beta_2 = \beta_3 = 0.2$.

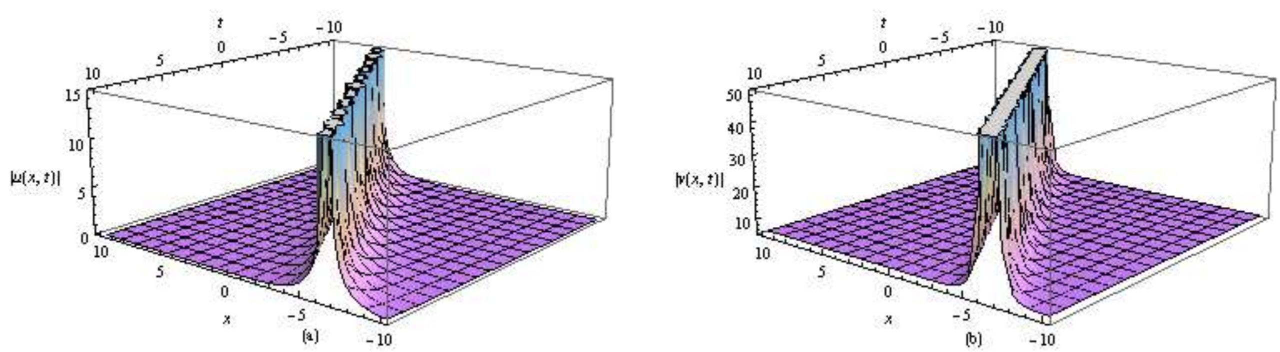

Figure 4: (a) 3D graphs for singular soliton solutions given in Eq. (16) of (a) $|u(x,t)|$ and (b) $|v(x,t)|$ with $\lambda_1 = \lambda_2 = \lambda_3 = 0.1,\ \beta_1 = \beta_2 = \beta_3 = 0.2$.

Set 3

$$A_0 = 0,\ A_1 = \pm\sqrt{\frac{2\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2}},\ B_1 = 0,\ k = \frac{\lambda_0}{12\beta_2\lambda_1\lambda_2},$$

$$a_0 = \frac{-\beta_1\beta_3\lambda_1 + 48\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_3 + \lambda_0}{6\beta_1\beta_2\lambda_2},\ a_1 = 0,\ b_1 = 0,\ a_2 = -\frac{12\beta_2}{\beta_1},\ b_2 = 0,$$

$$\omega = -2\lambda_1 - \frac{4\beta_2^2}{\lambda_1} - \frac{\beta_1^2\beta_3^2\lambda_1}{144\beta_2^2\lambda_2^2} + \frac{2\beta_1\beta_3}{3\lambda_2} + \frac{8\beta_2\lambda_3}{\beta_1} + \frac{\beta_1\beta_3^2\lambda_3}{12\beta_2\lambda_2^2} + \frac{\lambda_0\lambda_3}{6\beta_1\beta_2\lambda_2} - \frac{4\beta_2\beta_3\lambda_3}{\lambda_1\lambda_2} - \frac{\beta_3\lambda_1\lambda_3}{6\beta_2\lambda_2} - \frac{\beta_3^2\lambda_3^2}{4\lambda_1\lambda_2^2} + \frac{\beta_3\lambda_3^2}{\beta_1\lambda_2}, \tag{17}$$

where $\lambda_0 = \pm\sqrt{\beta_1^2\beta_3^2\lambda_1^2 - 12\beta_1\beta_2\beta_3\lambda_1\left(8\beta_2\lambda_2 + \beta_3\lambda_3\right) + 36\beta_2^2\left(16\beta_2^2\lambda_2^2 + 16\beta_2\beta_3\lambda_2\lambda_3 + \beta_3^2\lambda_3^2\right)}$.

Replacing the values of Set 3 into Eqs. (9) and (10), we obtain the dark soliton solutions for the above model as follows:

$$u(x,t) = \pm\sqrt{\frac{2\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2}}\ \tanh[x - ct]e^{i(kx+\omega t)},$$

$$v(x,t) = \frac{\lambda_0 - \beta_1\beta_3\lambda_1 + 48\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_3}{6\beta_1\beta_2\lambda_2} - \frac{12\beta_2}{\beta_1}\ \tanh^2(x - ct), \tag{18}$$

and singular soliton solutions for the above model as follows:

$$u(x,t) = \pm\sqrt{\frac{2\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2}}\ \coth(x - ct)e^{i(kx+\omega t)},$$

$$v(x,t) = \frac{\lambda_0 - \beta_1\beta_3\lambda_1 + 48\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_3}{6\beta_1\beta_2\lambda_2} - \frac{12\beta_2}{\beta_1}\ \coth^2(x - ct), \tag{19}$$

provided that $\frac{\left(-\beta_1\lambda_1 + 6\beta_2\lambda_3\right)}{\beta_1\lambda_2} > 0$. The behaviours of soliton solutions of Eq. (18) and (19) are shown in Fig. (5) and Fig. (6) respectively.

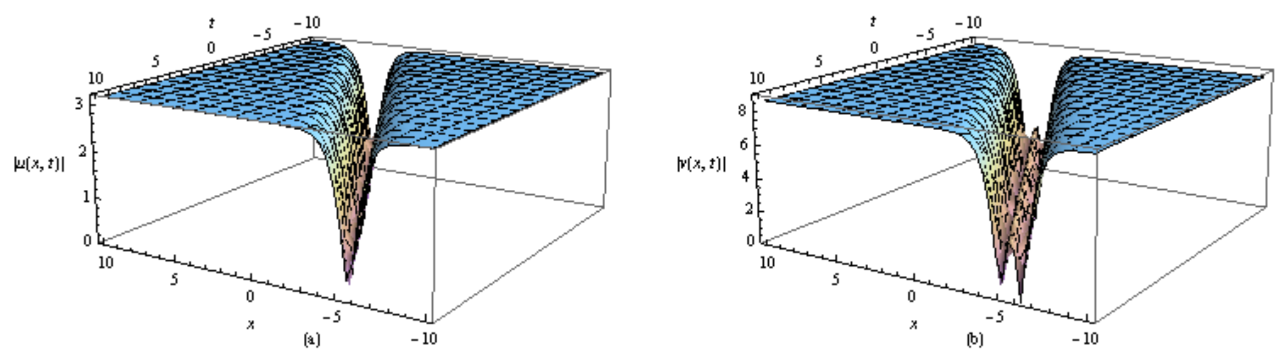

Figure 5: (a) 3D graphs for dark soliton solutions given in Eq. (18) of (a) $|u(x,t)|$ and (b) $|v(x,t)|$ with $\lambda_1 = \lambda_2 = \lambda_3 = 0.1,\ \beta_1 = \beta_2 = \beta_3 = 0.2$.

Set 4

$$A_0 = 0,\ A_1 = B_1 = \pm\sqrt{\frac{-\beta_1\lambda_1 + 6\beta_2\lambda_3}{\beta_1\lambda_2}},\ k = \frac{\lambda_0}{12\beta_2\lambda_1\lambda_2},$$

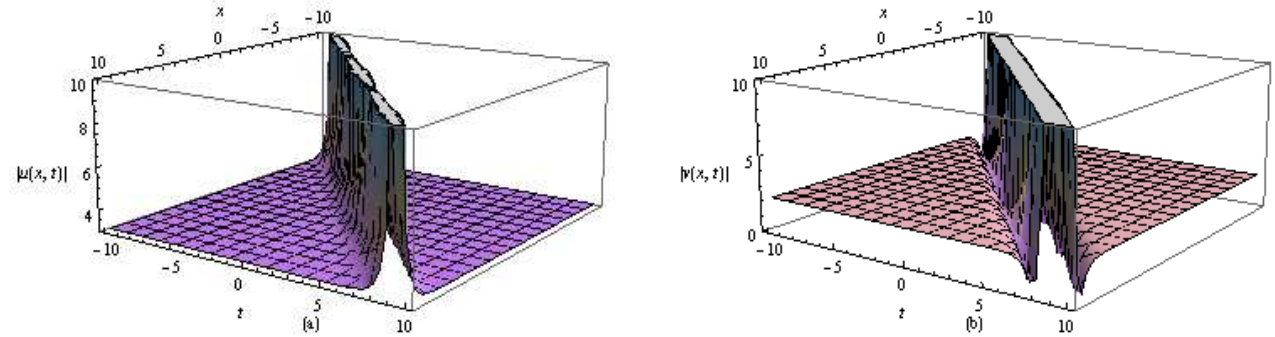

Figure 6: (a) 3D graphs for singular soliton solutions given in Eq. (19) of (a) $|u(x,t)|$ and (b) $|v(x,t)|$ with $\lambda_1 = \lambda_2 = \lambda_3 = 0.1,\ \beta_1 = \beta_2 = \beta_3 = 0.2$.

$$a_0 = \frac{-\beta_1\beta_3\lambda_1 + 30\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_3\lambda_0}{6\beta_1\beta_2\lambda_2},\ a_1 = 0,\ b_1 = 0,\ a_2 = -\frac{6\beta_2}{\beta_1},\ b_2 = -\frac{6\beta_2}{\beta_1},$$

$$\omega = -\frac{\beta_2^2}{4\lambda_1} - \frac{\lambda_1}{2} - \frac{\beta_1^2\beta_3^2\lambda_1}{144\beta_2^2\lambda_2^2} + \frac{\beta_1\beta_3}{6\lambda_2} + \frac{2\beta_2\lambda_3}{\beta_1} + \frac{\beta_1\beta_3^2\lambda_3}{12\beta_2\lambda_2^2} + \frac{\lambda_0\lambda_3}{6\beta_1\beta_2\lambda_2} - \frac{\beta_2\beta_3\lambda_3}{\lambda_1\lambda_2} - \frac{\beta_3\lambda_1\lambda_3}{6\beta_2\lambda_2} - \frac{\beta_3^2\lambda_3^2}{4\lambda_1\lambda_2^2} + \frac{\beta_3\lambda_3^2}{\beta_1\lambda_2}, \tag{20}$$

where $\lambda_0 = \pm\sqrt{\beta_1^2\beta_3^2\lambda_1^2 - 12\beta_1\beta_2\beta_3\lambda_1\left(2\beta_2\lambda_2 + \beta_3\lambda_3\right) + 36\beta_2^2\left(\beta_2^2\lambda_2^2 + 4\beta_2\beta_3\lambda_2\lambda_3 + \beta_3^2\lambda_3^2\right)}$.

Replacing the values of Set 4 into Eqs. (9) and (10), we obtain the combined dark-bright soliton solutions for the above model as follows:

$$u(x,t) = \pm\sqrt{\frac{-\beta_1\lambda_1 + 6\beta_2\lambda_3}{\beta_1\lambda_2}}[i\text{sech}(x-ct) - \tanh(x-ct)]e^{i(kx+\omega t)},$$

$$v(x,t) = \frac{-\beta_1\beta_3\lambda_1 + 30\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_0\lambda_3}{6\beta_1\beta_2\lambda_2} + \frac{6i\beta_2}{\beta_1}\ \text{sech}(x-ct)\ \tanh(x-ct) - \frac{6\beta_2}{\beta_1}\ \tanh^2(x-ct), \tag{21}$$

and combined singular soliton solutions for the above model as follows:

$$u(x,t) = \pm\sqrt{\frac{-\beta_1\lambda_1 + 6\beta_2\lambda_3}{2\beta_1\lambda_2}}[-\coth(x-ct) + \text{csch}(x-ct)]e^{i(kx+\omega t)},$$

$$v(x,t) = \frac{-\beta_1\beta_3\lambda_1 + 30\beta_2^2\lambda_2 + 6\beta_2\beta_3\lambda_0\lambda_3}{6\beta_1\beta_2\lambda_2} + \frac{6\beta_2}{\beta_1}\ \text{csch}(x-ct)\coth(x-ct) - \frac{6\beta_2}{\beta_1}\ \coth^2(x-ct), \tag{22}$$

provided that $\dfrac{(-\beta_1\lambda_1 + 6\beta_2\lambda_3)}{\beta_1\lambda_2} > 0$. The behaviour of soliton solutions of Eq. (22) is shown in Fig. (7).

4 Implementation of solitary wave ansatz method

In this section, we utilize the solitary wave ansatz method to solve the generalized coupled NLS-KdV equations. This is an effective and more powerful mathematical tool for constructing exact solutions of nonlinear differential equations, and gained considerable attention in recent years [13, 14].

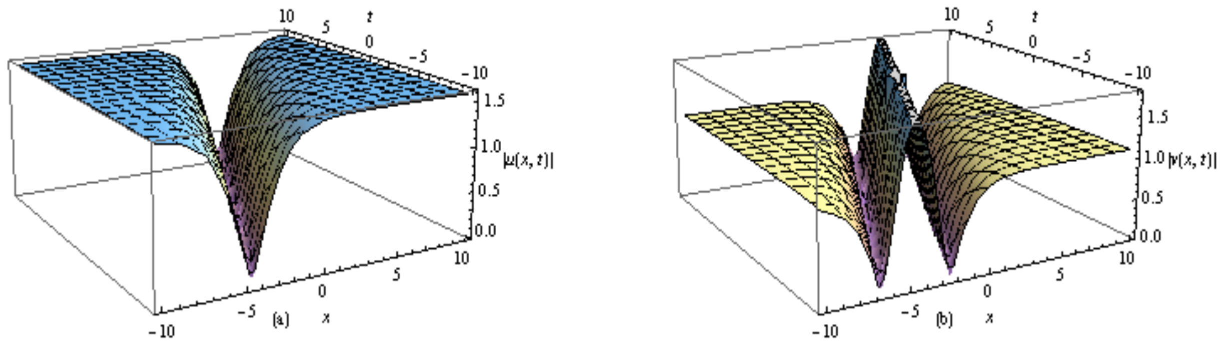

Figure 7: (a) 3D graphs for combined singular soliton solutions given in Eq. (22) of (a) $|u(x,t)|$ and (b) $|v(x,t)|$ with $\lambda_1=\lambda_2=\lambda_3=0.1,\ \beta_1=\beta_2=\beta_3=0.2$.

4.1 Bright Soliton solution

The solitary wave ansatz for the bright soliton solution, the hypothesis is [13, 14]

$$\phi(\xi) = A_1 \operatorname{sech}^{p_1}(\mu\xi),$$
$$\psi(\xi) = A_2 \operatorname{sech}^{p_2}(\mu\xi), \tag{23}$$

where $\xi = x - ct$ and A_1, A_2, p_1, p_2, μ are real constant. By inserting (23) into (3) and (7) we obtain the following two relations:

$$\begin{aligned} &- A_1\omega \operatorname{sech}^{p_1}(\mu\xi) + A_1^3\lambda_2 \operatorname{sech}^{3p_1}(\mu\xi) + A_1A_2\lambda_3 \operatorname{sech}^{p_1+p_2}(\mu\xi) - \mu^2 A_1 p_1\lambda_1 \operatorname{sech}^{2+p_1}(\mu\xi) \\ &+ \mu^2 A_1 p_1^2 \lambda_1 \operatorname{sech}^{p_1}(\mu\xi) - \mu^2 A_1 p_1^2\lambda_1 \operatorname{sech}^{2+p_1}(\mu\xi) - A_1\lambda_1 k^2 \operatorname{sech}^{p_1}(\mu\xi) = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} &\frac{1}{2}A_2^2\beta_1 \operatorname{sech}^{2p_2}(\mu\xi) + A_1^2\beta_3 \operatorname{sech}^{2p_1}(\mu\xi) - \mu^2 A_2 p_2\beta_2 \operatorname{sech}^{2+p_2}(\mu\xi) \\ &+ \mu^2 A_2 p_2^2\beta_2 \operatorname{sech}^{p_2}(\mu\xi) - \mu^2 A_2 p_2^2 \beta_2 \operatorname{sech}^{2+p_2}(\mu\xi) - 2A_2\lambda_1 k \operatorname{sech}^{p_2}(\mu\xi) = 0. \end{aligned} \tag{25}$$

By virtue of balancing principle, on equating the exponents of each pair of the sech functions, we find

$$\begin{aligned} &3p_1 = p_1 + p_2, \\ &2p_2 = 2 + p_2, \quad \text{then} \quad p_1 = 1 \ \text{ and } \ p_2 = 2. \end{aligned} \tag{26}$$

Substituting $p_1 = 1$, $p_2 = 2$ into (24) and (25) and setting the coefficients of $\operatorname{sech}^j(\mu\xi)$ $(j = 0, 1, 2, 3, 4)$ to zero, we get:

$$\begin{aligned} &A_1\left[\omega + \lambda_1\left(-\mu^2 + k^2\right)\right] = 0, \\ &A_1\left(A_1^2\lambda_2 + A_2\lambda_3 - 2\mu^2\lambda_1\right) = 0, \\ &\left[A_1^2\beta_3 + 2A_2\left(2\mu^2\beta_2 - \lambda_1 k\right)\right] = 0, \\ &\frac{1}{2}A_2\left(A_2\beta_1 - 12\mu^2\beta_2\right) = 0. \end{aligned} \tag{27}$$

Solving the above system of equations, we get:

$$A_1 = \pm\frac{\sqrt{2}\mu\sqrt{\beta_1\lambda_1 - 6\beta_2\lambda_3}}{\sqrt{\beta_1}\sqrt{\lambda_2}},\ A_2 = \frac{12\mu^2\beta_2}{\beta_1},\ k = \frac{\beta_3\left(\beta_1\lambda_1 - 6\beta_2\lambda_3\right) + 24\mu^2\beta_2^2\lambda_2}{12\beta_2\lambda_1\lambda_2},$$

$$\omega = \lambda_1 \left[\mu^2 - \frac{\left(\beta_3\left(\beta_1\lambda_1 - 6\beta_2\lambda_3\right) + 24\mu^2\beta_2^2\lambda_2\right)^2}{144\beta_2^2\lambda_1^2\lambda_2^2} \right]. \tag{28}$$

From (23) and (28), the bright soliton solutions of the generalized coupled system of NLS-KdV equations are given by the formula

$$u(x,t) = \pm \frac{\sqrt{2}\mu\sqrt{\beta_1\lambda_1 - 6\beta_2\lambda_3}}{\sqrt{\beta_1}\sqrt{\lambda_2}} \operatorname{sech}\left[\mu(x-ct)\right] \; e^{i(kx+\omega t)}. \tag{29}$$

$$v(x,t) = \frac{12\mu^2\beta_2}{\beta_1} \operatorname{sech}^2\left[\mu(x-ct)\right], \tag{30}$$

provided that $\frac{(\beta_1\lambda_1 - 6\beta_2\lambda_3)}{\beta_1\lambda_2} > 0$. Note that this result is consistent with the results derived by the extended ShGEEM.

4.2 Singular Soliton solution

The solitary wave ansatz for the singular soliton solution, the hypothesis is [13, 14]

$$\phi(\xi) = A_1 \operatorname{csch}^{p_1}(\mu\xi),$$
$$\psi(\xi) = A_2 \operatorname{csch}^{p_2}(\mu\xi), \tag{31}$$

where $\xi = x - ct$ and A_1, A_2, p_1, p_2, μ are reai constant. By Inserting (31) into (3) and (7) we obtain the following two relations:

$$\begin{aligned} &- A_1\omega\operatorname{csch}^{p_1}(\mu\xi) - A_1\lambda_1k^2\operatorname{csch}^{p_1}(\mu\xi) + A_1p_1\lambda_1\mu^2\operatorname{csch}^{2+p_1}(\mu\xi) + A_1p_1^2\lambda_1\mu^2\operatorname{csch}^{p_1}(\mu\xi) \\ &+ A_1p_1^2\lambda_1\mu^2\operatorname{csch}^{2+p_1}(\mu\xi) + A_1^3\lambda_2\operatorname{csch}^{3p_1}(\mu\xi) + A_1A_2\lambda_3\operatorname{csch}^{p_1+p_2}(\mu\xi) = 0 \end{aligned} \tag{32}$$

$$\begin{aligned} &\frac{1}{2}A_2^2\beta_1\operatorname{csch}^{2p_2}(\mu\xi) + A_2p_2\beta_2\mu^2\operatorname{csch}^{2+p_2}(\mu\xi) + A_2p_2^2\beta_2\mu^2\operatorname{csch}^{p_2}(\mu\xi) + A_2p_2^2\beta_2\mu^2\operatorname{csch}^{2+p_2}(\mu\xi) \\ &+ A_1^2\beta_3\operatorname{csch}^{2p_1}(\mu\xi) - 2A_2\lambda_1k\operatorname{csch}^{p_2}(\mu\xi) = 0. \end{aligned} \tag{33}$$

By virtue of balancing principle, on equating the exponents of each pair of the sech functions, we find

$$\begin{aligned} &3p_1 = p_1 + p_2, \\ &2p_2 = 2 + p_2, \quad \text{then} \quad p_1 = 1 \;\; \text{and} \;\; p_2 = 2. \end{aligned} \tag{34}$$

Substituting $p_1 = 1$, $p_2 = 2$ into (32) and (33) and setting the coefficients of $\operatorname{sech}^j(\mu\xi)$ $(j = 0, 1, 2, 3, 4)$ to zero, we get:

$$\begin{aligned} &A_1\left[\omega + \left(k^2 - \mu^2\right)\lambda_1\right] = 0, \\ &A_1\left(2\mu^2\lambda_1 + A_1^2\lambda_2 + A_2\lambda_3\right) = 0, \\ &A_1^2\beta_3 + A_2\left(4\mu^2\beta_2 - 2k\lambda_1\right) = 0, \\ &\frac{1}{2}A_2\left(A_2\beta_1 + 12\mu^2\beta_2\right) = 0. \end{aligned} \tag{35}$$

Solving the above system of equations, we get:

$$A_1 = \pm\frac{\sqrt{2}\mu\sqrt{-\beta_1\lambda_1 + 6\ \beta_2\lambda_3}}{\sqrt{\beta_1}\sqrt{\lambda_2}},\ A_2 = -\frac{12\mu^2\beta_2}{\beta_1},\ k = \frac{24\mu^2\beta_2^2\lambda_2 + \beta_3\left(\beta_1\lambda_1 - 6\beta_2\lambda_3\right)}{12\beta_2\lambda_1\lambda_2},$$

$$\omega = \lambda_1\left(\mu^2\ - \frac{\left(24\mu^2\beta_2^2\lambda_2 + \beta_3\left(\beta_1\lambda_1 - 6\beta_2\lambda_3\right)\right)^2}{144\beta_2^2\lambda_1^2\lambda_2^2}\right). \tag{36}$$

From (31) and (36), the soliton soliton solutions of the generalized coupled system of NLS-KdV equations are given by the formula

$$u(x,t) = \pm\frac{\sqrt{2}\mu\sqrt{-\beta_1\lambda_1 + 6\ \beta_2\lambda_3}}{\sqrt{\beta_1}\sqrt{\lambda_2}}\ \text{csch}\left[\mu(x-ct)\right]\ e^{i(kx+\omega t)}. \tag{37}$$

$$v(x,t) = -\frac{12\mu^2\beta_2}{\beta_1}\ \text{csch}^2\left[\mu(x-ct)\right], \tag{38}$$

provided that $\dfrac{(-\beta_1\lambda_1 + 6\beta_2\lambda_3)}{\beta_1\lambda_2} > 0$. Note that this result is consistent with the results derived by the extended ShGEEM.

5 Modulation instability analysis

In this section, we derive the modulation instability (MI) of the stationary solutions of the coupled NLS-KdV equations by employing the standard linear stability analysis [26].

Based on the linear stability analysis, the stationary solutions of Eq. (1) have the following form [26]

$$u = q_0\ e^{i\omega t}, \quad v = p_0, \tag{39}$$

where p_0 and q_0, are the real constant-amplitudes (initial incidence power). Substituting Eq. (39) into Eq. (1), we get

$$\omega = q_0^2\lambda_2 + p_0\lambda_3, \tag{40}$$

In order to find the linear stability analysis of Eq. (1), the perturbed stationary solutions can be written as

$$u = (q_0 + \theta\tilde{q}[x,t])\ e^{i\omega t}, \quad v = (p_0 + \theta\tilde{p}[x,t]) \tag{41}$$

where $\theta << 1$ is a perturbation parameter. Substituting Eq. (41) into Eq. (1), we obtain

$$\tilde{\text{q}}\text{c}q_0^2\lambda_2 + q_0\lambda_3\tilde{p}[x,t] + q_0^2\lambda_2\tilde{q}[x,t] + i\tilde{q}^{(0,1)}[x,t] + 2ik\lambda_1\tilde{q}^{(1,0)}[x,t] + \lambda_1\tilde{q}^{(2,0)}[x,t] = 0,$$

$$\tilde{p}^{(0,1)}[x,t] + p_0\beta_1\tilde{p}^{(1,0)}[x,t] + q_0\beta_3\tilde{q}^{(1,0)}[x,t] + \beta_2\tilde{p}^{(3,0)}[x,t] = 0, \tag{42}$$

where $'*'$ means complex conjugate. Now, we introduce $\tilde{q}$ and $\tilde{p}$ in the following form:

$$\tilde{q}[x,t] = q_1\ e^{i\left(\hat{k}x - \hat{\omega}t\right)} + q_2\ e^{-i\left(\hat{k}x - \hat{\omega}t\right)},$$

$$\tilde{p}[x,t] = p_1\ e^{i\left(\hat{k}x - \hat{\omega}t\right)} + p_2\ e^{-i\left(\hat{k}x - \hat{\omega}t\right)}, \tag{43}$$

Then, we substitute Eq. (43) into Eq. (42), yields the following homogeneous equations for p_1, q_1, p_2, q_2:

$$\begin{aligned}
p_1\left(\hat{\omega}-\hat{k}p_0\beta_1+\hat{k}^3\beta_2\right)-\hat{k}q_0q_1\beta_3=0,\\
p_2\left(\hat{\omega}-\hat{k}p_0\beta_1+\hat{k}^3\beta_2\right)-\hat{k}q_0q_2\beta_3=0,\\
q_0^2q_1\lambda_2-q_2\left(\hat{\omega}+\hat{k}\left(-2k+\hat{k}\right)\lambda_1-q_0^2\lambda_2\right)+p_2q_0\lambda_3=0,\\
\hat{\omega}q_1-2k\hat{k}q_1\lambda_1-\hat{k}^2q_1\lambda_1+q_0\left(q_0\left(q_1+q_2\right)\lambda_2+p_1\lambda_3\right)=0,
\end{aligned} \tag{44}$$

From Eq. (44), we obtain the following coefficient matrix of p_1, q_1, p_2, q_2

$$\begin{pmatrix} \hat{\omega}-\hat{k}p_0\beta_1+\hat{k}^3\beta_2 & -\hat{k}q_0\beta_3 & 0 & 0 \\ 0 & 0 & \hat{\omega}-\hat{k}p_0\beta_1+\hat{k}^3\beta_2 & -\hat{k}q_0\beta_3 \\ 0 & q_0^2\lambda_2 & q_0\lambda_3 & -\hat{\omega}-\hat{k}^2\lambda_1+q_0^2\lambda_2 \\ q_0\lambda_3 & \hat{\omega}-\hat{k}^2\lambda_1+q_0^2\lambda_2 & 0 & q_0^2\lambda_2 \end{pmatrix}\begin{pmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The above coefficient matrix has a nontrivial solution when the determinant vanishes. By expanding the determinant, we derive the following dispersion relation:

$$-\hat{\omega}^4+m_3\hat{\omega}^3+m_2\hat{\omega}^2+m_1\hat{\omega}+m_0=0, \tag{45}$$

where

$$\begin{aligned}
m_0&=\hat{k}^2\left(\hat{k}^2\left(-p_0\beta_1+\hat{k}^2\beta_2\right)\lambda_1-q_0^2\beta_3\lambda_3\right)\left(\left(-p_0\beta_1+\hat{k}^2\beta_2\right)\left(\hat{k}^2\lambda_1-2q_0^2\lambda_2\right)-q_0^2\beta_3\lambda_3\right),\\
m_1&=2\hat{k}^3\left(-p_0\beta_1+\hat{k}^2\beta_2\right)\lambda_1\left(\hat{k}^2\lambda_1-2q_0^2\lambda_2\right)+2\hat{k}q_0^2\beta_3\left(-\hat{k}^2\lambda_1+q_0^2\lambda_2\right)\lambda_3,\\
m_2&=-\hat{k}^2\left(p_0^2\beta_1^2-2\hat{k}^2p_0\beta_1\beta_2+\hat{k}^4\beta_2^2-\hat{k}^2\lambda_1^2+2q_0^2\lambda_1\lambda_2\right),\\
m_3&=2\hat{k}p_0\beta_1-2\hat{k}^3\beta_2.
\end{aligned}$$

It is noted that the coupled NLS-KdV equations are modulational stable for any wavenumber $\hat{k}$ if and only if four roots $\hat{\omega}$ of Eq. (45) are all positive real numbers. However, it is not so easy to find the roots of Eq. (45), since we have to employ the existing complicated analytical formulae and the associated criteria for the roots of a fourth-order polynomial. Therefore, we consider a special case of initial incidence powers, namely, $p_0=\dfrac{\hat{k}^2\beta_2}{\beta_1}, q_0=\dfrac{\hat{k}\sqrt{\lambda_1}}{\sqrt{\lambda_2}}$, which simplifies Eq. (45) to

$$-\hat{\omega}^4-\hat{k}^4\lambda_1^2\hat{\omega}^2+\frac{\hat{k}^6\beta_3^2\lambda_1^2\lambda_3^2}{\lambda_2^2}=0. \tag{46}$$

Now, the solution of dispersion relation of Eq. (46) is,

$$\hat{\omega}=\pm\frac{1}{\sqrt{2}}\sqrt{\hat{k}^4\lambda_1^2\pm\frac{\sqrt{\hat{k}^6\lambda_1^2\left(\hat{k}^2\lambda_1^2\lambda_2^2-4\beta_3^2\lambda_3^2\right)}}{\lambda_2}}. \tag{47}$$

Thus, we observe that the modulation instability of the Eq. (1) occurs when either

$$\hat{k}^6\lambda_1^2\left(\hat{k}^2\lambda_1^2\lambda_2^2-4\beta_3^2\lambda_3^2\right)<0 \tag{48}$$

or

$$\hat{k}^4\lambda_1^2 \pm \frac{\sqrt{\hat{k}^6\lambda_1^2\left(\hat{k}^2\lambda_1^2\lambda_2^2 - 4\beta_3^2\lambda_3^2\right)}}{\lambda_2} < 0. \tag{49}$$

Moreover, we investigate the modulation Instability gain spectrum $G(\hat{k})$, which is determined by the maximum absolute value for the imaginary part of the wavenumber and defined as

$$G(\hat{k}) = 2\text{Im}(\hat{\omega}) = \sqrt{2}\text{Im}\left[\sqrt{\hat{k}^4\lambda_1^2 \pm \frac{\sqrt{\hat{k}^6\lambda_1^2\left(\hat{k}^2\lambda_1^2\lambda_2^2 - 4\beta_3^2\lambda_3^2\right)}}{\lambda_2}}\right], \tag{50}$$

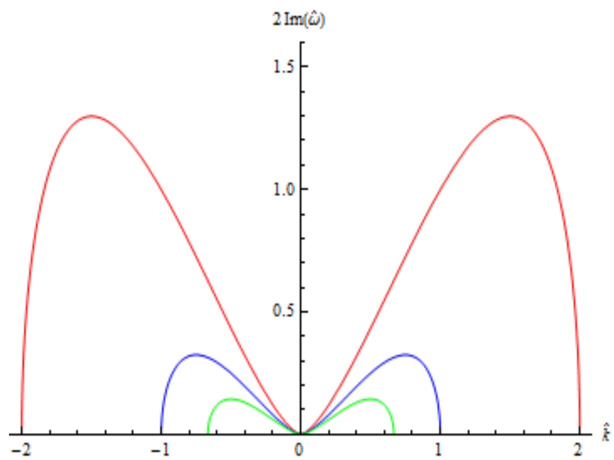

Figure 8: Gain spectrum of modulation instability for Eq. (1) when $\lambda_1 = \lambda_3 = \beta_3 = 1$ and $\lambda_2 = 3,\ 2,\ 1$ (From bottom to up).

In Fig. 8, we plot the gain spectrum of modulation instability for three different values of $\lambda_1, \lambda_2, \lambda_3$ and β_3.

6 Conclusions

In this study, two effective methods, namely, the extended sinh-Gordon equation expansion method and the solitary wave ansatz method have been successfully applied to obtain dark, bright, combined dark-bright, singular and combined singular soliton solutions of the generalized coupled NLS-KdV equations. To our best knowledge, the application of proposed methods to the model, and the received combined soliton solutions are new, which have not been reported earlier. Moreover, by applying the concept of linear stability analysis, the modulation instability analysis of the stationary solution is studied and the MI gain spectrum is reported for the proper choice of initial incidence powers. The dynamical behaviour of the obtained solutions are demonstrated in Figs 1-8.

Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interest with regard to the publication of this manuscript. .

Data Availability Statements
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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