

An efficient analytical scheme for fuzzy conformable fractional differential equations arising in physical sciences

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Research Article

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Posted Date: May 12th, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-1100676/v1>

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An efficient analytical scheme for fuzzy conformable fractional differential equations arising in physical sciences

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Abstract

This article describes the fuzzy conformable fractional derivative which is based on generalized Hukuhara differentiability. On these topics, we prove a number of properties concerning this type of differentiability. In addition, fuzzy conformable Laplace transforms are used to obtain analytical solutions to the fractional differential equation. Through the use of several practical examples, such as the fuzzy conformable fractional growth equation, the fuzzy conformable fractional one-compartment model, and the fuzzy conformable fractional Newton's law of cooling, we demonstrate the effectiveness and efficiency of the approaches.

Keywords: Generalized Hukuhara conformable fractional derivative, fuzzy conformable Laplace transform, fuzzy conformable fractional growth equation, one-compartment fuzzy conformable fractional model, fuzzy conformable fractional Newton's law of cooling.

2010 AMS Subject Classification: Primary: 26A33; 34A07 Secondary: 34A08.

1 Introduction

As fuzzy set theory has developed over the past decades, it has proven to be an effective tool for modeling systems with uncertainty, providing the models with a realistic glimpse of reality and enabling them to present a more comprehensive view. Zadeh introduced fuzzy sets in 1965 [30]. Afterward, the use of fuzzy sets in modeling increased considerably.

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A large number of mathematical, physics, and engineering phenomena are explained by differential equations. Fuzzy differential equations were first constructed by Kaleva [20], and Seikkala [28]. There are several approaches to studying fuzzy differential equations [3, 10, 14, 23, 26]. The first approach [20] used the Hukuhara derivative (H-derivative) of a fuzzy function. It should be noted that, for some fuzzy differential equations in this framework, the diameter of the solution is unbounded as the time τ increases [13], which is quite different from crisp differential equations. Consequently, Bede and Gal introduced weakly generalized differentiability and strongly generalized differentiability for fuzzy functions in [9]. However, the fuzzy differential equations expressed by the strongly generalized derivative had no unique solution. As a result, Stefanini and Bede defined generalized Hukuhara difference and derivative for interval-valued functions and examined all conditions for the existence of this kind of difference [11].

Fractional calculus has developed significantly in the past few decades. Researchers have been using fractional calculus extensively for decades. A fractional-order differential equation is applied to biological population models, predator-prey models, infectious disease models, etc. Many of these models have some uncertainty or ambiguity associated with their measurements at the beginning. Considering these uncertainties and vagueness, the fuzzy problem is more closely aligned with current reality and may be able to express issues with a broader understanding. Researchers combined fuzzy-yielding algorithms with fractional notions, resulting in a hybrid operator called fuzzy fractional operator.

Fuzzy fractional calculus and fuzzy fractional differential equations have emerged as a significant topic; see [2, 17, 18]. The authors in [6] utilized the results reported in [2] and proved the existence and uniqueness of fractional differential equations with uncertainty. The generalized Hukuhara fractional Riemann-Liouville and Caputo gH-differentiability of fuzzy-valued functions are discussed in [4, 5]. In [25] the authors considered the solution to the fuzzy fractional initial value problem under Caputo gH-differentiability using a modified fractional Euler method. A weak version of the Pontryagin maximum principle for fuzzy fractional optimal control problems depending on generalized Hukuhara fractional Caputo derivatives is established in [15].

In recent years, there has been an increase in interest in the use of fuzzy conformable fractional derivatives. According to our knowledge, all papers that have used this method have rewritten the fuzzy conformable fractional differential equation as two crisp fractional differential equations and solved them using the usual methods. Moreover, the fuzzy conformable fractional derivative was defined under strongly generalized differentiability [7]. But the fuzzy conformable fractional differential equation expressed by the strongly generalized derivative does not have a unique solution. In comparison, this paper is devoted to developing a new fuzzy conformable Laplace transform through fuzzy arithmetic. We obtain the fuzzy conformable Laplace transform of fuzzy derivatives by considering the type of gH-differentiability. The fuzzy analytical solution of the fuzzy conformable fractional differential equations can be got without implicitly embedding them into crisp equations through our method. Our approaches are demonstrated by solving practical problems, including the fuzzy conformable fractional growth equation, the fuzzy conformable fractional one-compartment model, and the fuzzy conformable fractional Newton's law of cooling. In particular, this paper deals with the triangular fuzzy solutions

of the following fuzzy conformable fractional differential equation

$$\begin{cases} \mathfrak{w} : \mathbb{R}^+ \rightarrow \Omega, \\ \mathfrak{F} : \mathbb{R}^+ \times \Omega \rightarrow \Omega \\ {}_{gH}\mathbf{T}_\alpha^{\tau_0} \mathfrak{w}(\tau) = \mathfrak{F}(\tau, \mathfrak{w}(\tau)) \\ \mathfrak{w}(0) = \mathfrak{w}_0 \end{cases} \quad (1)$$

The following will be symbolized in this context: $\tau \in \mathbb{R}^+$, $\mathfrak{w}_0 \in \Omega$, $\alpha \in (0, 1]$, and Ω denotes the set of all triangular fuzzy numbers. Here, ${}_{gH}\mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau))$ is the fuzzy conformable fractional generalized Hukuhara derivative for α on the domain of \mathbb{R}^+ .

Now, let's take a quick look at the contents. Section 2 deals with aspects of background knowledge in fuzzy mathematics with emphasis on the generalized Hukuhara differentiability. Fuzzy fractional conformable derivative based on the concept of generalized Hukuhara difference is defined in Section 3 and some important properties for this kind of differentiability are proved. The fuzzy conformable Laplace method is presented in Section 4. Section 5 studies the fuzzy fractional initial value problem using the concept of conformable gH-differentiability and the fuzzy analytical triangular solutions for the fuzzy growth equation, one-compartment fuzzy fractional model and fuzzy Newton's law of cooling by considering the type of conformable differentiability obtained using the conformable Laplace method in Section 5. The paper ends with Section 6 that presents the conclusions.

2 Fundamental Awareness

Fuzzy numbers are a generalization of classical real numbers in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1. A fuzzy set $\mathcal{P} : \mathbb{R} \rightarrow [0, 1]$ represents a fuzzy number if it is normal, fuzzy convex, compactly supported and upper semi-continuous on \mathbb{R} . Then $(\mathcal{F}, \mathcal{P})$ is called the space of fuzzy numbers.

Initially, for all $r \in (0, 1]$, put $[\mathcal{P}]^r = \{s \in \mathbb{R} : \mathcal{P}(s) \geq r\}$ and $[\mathcal{P}]^0 = \overline{\{s \in \mathbb{R} : \mathcal{P}(s) > 0\}}$. Then $\mathcal{P} \in \mathcal{F}$ iff $[\mathcal{P}]^r$ is convex compact in \mathbb{R} and $[\mathcal{P}]^1 \neq \emptyset$ [12]. Indeed, if $\mathcal{P} \in \mathcal{F}$ then $[\mathcal{P}]^r = [\mathcal{P}_1(r), \mathcal{P}_2(r)]$ where, $\mathcal{P}_1(r) = \min \{s \in [\mathcal{P}]^r\}$ and $\mathcal{P}_2(r) = \max \{s \in [\mathcal{P}]^r\}$.

Let $\mathcal{P}, \mathcal{Q} \in \mathcal{F}$. The generalized Hukuhara difference between in two fuzzy numbers is the fuzzy number \mathcal{R} , (if it exists), such that

$$\mathcal{P} \ominus_{gH} \mathcal{Q} = \mathcal{R} \iff \mathcal{P} = \mathcal{Q} \oplus \mathcal{R}, \quad \text{or} \quad \mathcal{Q} = \mathcal{P} \oplus (-1)\mathcal{R}.$$

Throughout of this paper we assume that $\mathcal{P} \ominus_{gH} \mathcal{Q} \in \Omega$

Definition 2.1. (See [8]). Let $\mathfrak{w} : \mathbb{J} \rightarrow \mathcal{F}$ be a fuzzy function and $\tau_0 \in \mathbb{J}$. If

$$\forall \epsilon > 0 \exists \delta > 0 \forall \tau (0 < |\tau - \tau_0| < \delta \Rightarrow D(\mathfrak{w}(\tau), L) < \epsilon),$$

where, D is the Hausdorff distance and \mathbb{J} is a bonded domain in \mathbb{R} . Then, we say that $L \in \mathcal{F}$ is limit of \mathfrak{w} in τ_0 , which is denoted by $\lim_{\tau \rightarrow \tau_0} \mathfrak{w}(\tau) = L$. Also the fuzzy function \mathfrak{w} is said to be fuzzy continuous

if

$$\lim_{\tau \rightarrow \tau_0} \mathfrak{w}(\tau) = \mathfrak{w}(\tau_0).$$

A function $\mathfrak{w} : \mathbb{J} \rightarrow \Omega$ is called fuzzy triangular function and $\mathfrak{w}(\tau) = (\mathfrak{w}_1(\tau), \mathfrak{w}_2(\tau), \mathfrak{w}_3(\tau))$, for all $\tau \in \mathbb{J}$. Suppose that the notation $C^{\mathbb{F}}(\mathbb{J}, \Omega)$ denotes the set of all fuzzy triangular functions which are fuzzy continuous on all of \mathbb{J} .

Theorem 2.1. (See [8]) Let $\nu, \mathfrak{w} : \mathbb{J} \rightarrow \mathcal{F}$ be two fuzzy functions. If $\lim_{\tau \rightarrow c} \nu(\tau) = L_1$ and $\lim_{\tau \rightarrow c} \mathfrak{w}(\tau) = L_2$, such that $L_1, L_2 \in \mathcal{F}$ then

$$\lim_{\tau \rightarrow c} [\nu(\tau) \ominus_{gH} \mathfrak{w}(\tau)] = L_1 \ominus_{gH} L_2.$$

Definition 2.2. (See [12]) The generalized Hukuhara derivative (*gH-derivative*) of a fuzzy-valued function $\mathfrak{w} : (a, b) \rightarrow \Omega$ at $\tau_0 \in (a, b)$ is defined as

$$\mathfrak{w}'_{gH}(\tau_0) = \lim_{h \rightarrow 0} \frac{\mathfrak{w}(\tau_0 + h) \ominus_{gH} \mathfrak{w}(\tau_0)}{h},$$

provided that $\mathfrak{w}'_{gH}(\tau_0) \in \mathcal{F}$. Considering the definition of *gH-difference*, this derivative has the following two cases

- Case 1. ([i] – *gH*)–differentiability)

$$\mathfrak{w}'_{i.gH}(\tau) = (\mathfrak{w}'_1(\tau), \mathfrak{w}'_2(\tau), \mathfrak{w}'_3(\tau)),$$

- Case 2. ([ii] – *gH*)–differentiability)

$$\mathfrak{w}'_{ii.gH}(\tau) = (\mathfrak{w}'_3(\tau), \mathfrak{w}'_2(\tau), \mathfrak{w}'_1(\tau)),$$

Definition 2.3. (See [27]) A fuzzy function \mathfrak{w} is piecewise continuous on the interval $[0, \infty)$ if

1. $\lim_{\tau \rightarrow 0^+} \mathfrak{w}(\tau) = \mathfrak{w}(0^+)$.
2. \mathfrak{w} is continuous on every finite interval $(0, b)$ except possibly at a finite number of points $\tau_1, \tau_2, \dots, \tau_n$ in $(0, b)$ at which \mathfrak{w} has jump discontinuity.

Definition 2.4. (See [12]) Let $\mathfrak{w} : \mathbb{J} \rightarrow \mathcal{F}$ be a triangular fuzzy function, then

$$\int_a^b \mathfrak{w}(\tau) d\tau = \left(\int_a^b \mathfrak{w}_1(\tau) d\tau, \int_a^b \mathfrak{w}_2(\tau) d\tau, \int_a^b \mathfrak{w}_3(\tau) d\tau \right).$$

Theorem 2.2. (See [16])(Integration by part) Consider $\mathfrak{w} : \mathbb{J} \rightarrow \mathcal{F}$ and $f(\tau)$ be *gH*–differentiable such that the type of differentiability does not change in \mathbb{J} . If $q(\tau)$ is a differentiable real-valued function such that $q(\tau) > 0$ and $q'(\tau) < 0$, then

1. If $\mathfrak{w}(\tau)$ is a [i] – *gH*]–differentiable function and

$$\int_a^b q(\tau) \odot \mathfrak{w}'_{gH}(\tau) d\tau = q(b) \odot \mathfrak{w}(b) \ominus q(a) \odot \mathfrak{w}(a) \oplus \int_a^b (-q'(\tau)) \odot \mathfrak{w}(\tau) d\tau.$$

2. If $\mathfrak{w}(\tau)$ is a [ii] – *gH*]–differentiable function then

$$\int_a^b q(\tau) \odot \mathfrak{w}'_{gH}(\tau) d\tau = (-q(a)) \odot \mathfrak{w}(a) \ominus (-q(b)) \odot \mathfrak{w}(b) \ominus_{gH} \int_a^b q'(\tau) \odot \mathfrak{w}(\tau) d\tau.$$

3 Fractional Conformable Calculus on Fuzzy Functions

This section introduces a new definition of a conformable derivative based on the generalized Hukuhara differentiability, and several important properties for this type of differentiability will be expressed and proven.

Definition 3.1. Let $\mathfrak{w} : [a, \infty) \rightarrow \mathcal{F}$ be a fuzzy function, $a \geq 0$ and $\alpha \in (0, 1)$. The generalized Hukuhara conformable fractional derivative of \mathfrak{w} of order α is defined by

$${}_{gH}\mathbf{T}_\alpha^a(\mathfrak{w}(\tau)) = \lim_{\epsilon \rightarrow 0} \frac{\mathfrak{w}(\tau + \epsilon(\tau - a)^{1-\alpha}) \ominus_{gH} \mathfrak{w}(\tau)}{\epsilon}, \quad \forall \tau \in [a, \infty),$$

provided that ${}_{gH}\mathbf{T}_\alpha^a(\mathfrak{w}(\tau)) \in \mathcal{F}$. If the generalized Hukuhara conformable fractional derivative of \mathfrak{w} of order α exists, then we simply say \mathfrak{w} is α_{gH} -differentiable.

Theorem 3.1. If a fuzzy function $\mathfrak{w} : [0, \infty) \rightarrow \mathcal{F}$ is α_{gH} -differentiable at $\tau_0 > 0$, $\alpha \in (0, 1)$, then \mathfrak{w} is a fuzzy continuous function at τ_0 .

Proof. We have $\mathfrak{w}(\tau_0 + \varepsilon\tau_0^{1-\alpha}) \ominus_{gH} \mathfrak{w}(\tau_0) = \frac{\mathfrak{w}(\tau_0 + \varepsilon\tau_0^{1-\alpha}) \ominus_{gH} \mathfrak{w}(\tau_0)}{\varepsilon} \odot \varepsilon$. By using Theorem 2.1 we conclude that

$$\lim_{\varepsilon \rightarrow 0} [\mathfrak{w}(\tau_0 + \varepsilon\tau_0^{1-\alpha}) \ominus_{gH} \mathfrak{w}(\tau_0)] = \lim_{\varepsilon \rightarrow 0} \frac{\mathfrak{w}(\tau_0 + \varepsilon\tau_0^{1-\alpha}) \ominus_{gH} \mathfrak{w}(\tau_0)}{\varepsilon} \odot \lim_{\varepsilon \rightarrow 0} \varepsilon$$

then

$$\lim_{\varepsilon \rightarrow 0} [\mathfrak{w}(\tau_0 + \varepsilon\tau_0^{1-\alpha}) \ominus_{gH} \mathfrak{w}(\tau_0)] = {}_{gH}\mathbf{T}_\alpha^0(\mathfrak{w}(\tau_0)) \odot 0$$

Now, let $\hbar = \varepsilon\tau_0^{1-\alpha}$, therefore

$$\lim_{\hbar \rightarrow 0} [\mathfrak{w}(\tau_0 + \hbar) \ominus_{gH} \mathfrak{w}(\tau_0)] = 0$$

Therefore, according to Definition 2.1, it can be concluded that the function \mathfrak{w} is a fuzzy continuous function. \square

Theorem 3.2. Let $\alpha \in (0, 1)$ and $\mathfrak{w} : [a, \infty) \rightarrow \mathcal{F}$ be α_{gH} -differentiable fuzzy function at a point $\tau > 0$. Then

$${}_{gH}\mathbf{T}_\alpha^a(\mathfrak{w}(\tau)) = (\tau - a)^{1-\alpha} \mathfrak{w}'_{gH}(\tau)$$

Proof. In Definition 3.1, let $\hbar = \varepsilon(\tau - a)^{1-\alpha}$ and then $\varepsilon = (\tau - a)^{\alpha-1} \hbar$. Hence

$$\begin{aligned} {}_{gH}\mathbf{T}_\alpha^a(\mathfrak{w}(\tau)) &= \lim_{\varepsilon \rightarrow 0} \frac{\mathfrak{w}(\tau + \varepsilon(\tau - a)^{1-\alpha}) \ominus_{gH} \mathfrak{w}(\tau)}{\varepsilon} \\ &= \lim_{\hbar \rightarrow 0} \frac{\mathfrak{w}(\tau + \hbar) \ominus_{gH} \mathfrak{w}(\tau)}{\hbar(\tau - a)^{\alpha-1}} \\ &= (\tau - a)^{1-\alpha} \lim_{\hbar \rightarrow 0} \frac{\mathfrak{w}(\tau + \hbar) \ominus_{gH} \mathfrak{w}(\tau)}{\hbar} \\ &= (\tau - a)^{1-\alpha} \mathfrak{w}'_{gH}(\tau) \end{aligned}$$

So the desired result was obtained. \square

Definition 3.2. Let $\alpha \in (0, 1)$ and $\mathbf{w} : [0, \infty) \rightarrow \Omega$ is α_{gH} -differentiable at a point $\tau_0 > 0$. We can say that $\mathbf{w}(\tau)$ is

(a) $\alpha_{i.gH}$ -differentiable function at τ_0 if and only if

$${}_{i.gH}\mathbf{T}_\alpha^0(\mathbf{w}(\tau_0)) = \left(T_\alpha^0(\mathbf{w}_1(\tau_0)), T_\alpha^0(\mathbf{w}_2(\tau_0)), T_\alpha^0(\mathbf{w}_3(\tau_0)) \right), \quad (2)$$

(b) $\alpha_{ii.gH}$ -differentiable function at τ_0 if and only if

$${}_{ii.gH}\mathbf{T}_\alpha^0(\mathbf{w}(\tau_0)) = \left(T_\alpha^0(\mathbf{w}_3(\tau_0)), T_\alpha^0(\mathbf{w}_2(\tau_0)), T_\alpha^0(\mathbf{w}_1(\tau_0)) \right). \quad (3)$$

Where $T_\alpha^0(\mathbf{w}_i(\tau_0))$ for $i = 1, 2, 3$, are the conformable fractional derivatives for the real-valued functions $\mathbf{w}_i(\tau_0)$, respectively [24].

Definition 3.3. We say that a point $\xi_0 \in (0, \infty)$, is a switching point for the differentiability of \mathbf{w} , if in any neighborhood \mathcal{V} of ξ_0 there exist points $\xi_1 < \xi_0 < \xi_2$ such that

Type I. at ξ_1 (2) holds while (3) does not hold and at ξ_2 (3) holds and (2) does not hold, or

Type II. at ξ_1 (3) holds while (2) does not hold and at ξ_2 (2) holds and (3) does not hold.

Example 3.1. Consider the fuzzy function $\mathbf{w} : [0, \pi] \rightarrow \Omega$ defined by

$$\mathbf{w}(\tau) = \left(2.3 \sin(\tau), 5.6 \sin(\tau), 9.7 \sin(\tau) \right)$$

We have the following α_{gH} -derivatives of $\mathbf{w}(\tau)$

$$\begin{cases} {}_{i.gH}\mathbf{T}_\alpha^0(\mathbf{w}(\tau)) = \left(2.3\tau^{\frac{1}{2}} \cos(\tau), 5.6\tau^{\frac{1}{2}} \cos(\tau), 9.7\tau^{\frac{1}{2}} \cos(\tau) \right) & \forall \tau \in [0, \frac{\pi}{2}] \\ {}_{ii.gH}\mathbf{T}_\alpha^0(\mathbf{w}(\tau)) = \left(9.7\tau^{\frac{1}{2}} \cos(\tau), 5.6\tau^{\frac{1}{2}} \cos(\tau), 2.3\tau^{\frac{1}{2}} \cos(\tau) \right) & \forall \tau \in [\frac{\pi}{2}, \pi] \end{cases}$$

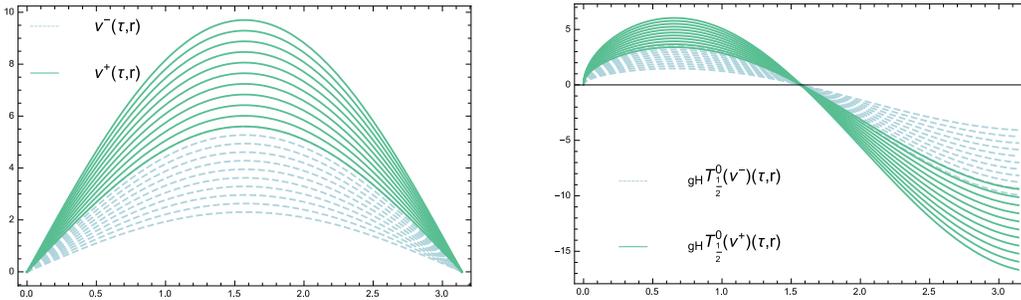


Figure 1: $\mathbf{w}(\tau)$ (left) and its ${}_{gH}\mathbf{T}_\alpha^0(\mathbf{w}(\tau))$ -derivative for different level of r -cut $r \in [0, 1]$ (right) of Example 3.1

Therefore, the fuzzy function $\mathfrak{w}(\tau)$ is $\alpha_{i.gH}$ -differentiable function on $\tau \in [0, \frac{\pi}{2}]$. This function is switched to $\alpha_{ii.gH}$ -differentiable at $\tau = \frac{\pi}{2}$. Hence, the point $\tau = \frac{\pi}{2}$ is a switching point of **Type I** for the the differentiability of \mathfrak{w} .

Theorem 3.3. Let $\mathfrak{w} : [0, \infty) \rightarrow \Omega$ be a fuzzy function and $\tau_0 \in [0, \infty)$.

(a) \mathfrak{w} is [(i) – gH]– differentiable at τ_0 if and only if \mathfrak{w} is $\alpha_{i.gH}$ -differentiable at τ_0 .

(b) \mathfrak{w} is [(ii) – gH]– differentiable at τ_0 if and only if \mathfrak{w} is $\alpha_{ii.gH}$ -differentiable at τ_0 .

Proof. The desired result can be obtained easily by using Theorem 3.2 and Definition 2.2. \square

Lemma 3.1. Let $\alpha \in (0, 1]$ and \mathfrak{w}, ν are α_{gH} -differentiable at a point $\tau > 0$. Then

$$1. \quad {}_{i.gH} \mathbf{T}_\alpha^0[\mathfrak{w}(\tau) \oplus \nu(\tau)] = {}_{i.gH} \mathbf{T}_\alpha^0(\mathfrak{w}(\tau)) \oplus {}_{i.gH} \mathbf{T}_\alpha^0(\nu(\tau)).$$

$$2. \quad {}_{ii.gH} \mathbf{T}_\alpha^0[\mathfrak{w}(\tau) \oplus \nu(\tau)] = {}_{ii.gH} \mathbf{T}_\alpha^0(\mathfrak{w}(\tau)) \oplus {}_{ii.gH} \mathbf{T}_\alpha^0(\nu(\tau)).$$

Proof. Let \mathfrak{w}, ν are $[\alpha_{i.gH}]$ -differentiable functions for all $\tau > 0$, then

$$\begin{aligned} & {}_{i.gH} \mathbf{T}_\alpha^0(\mathfrak{w}(\tau)) \oplus {}_{i.gH} \mathbf{T}_\alpha^0(\nu(\tau)) \\ &= \left(T_\alpha(\mathfrak{w}_1(\tau)), T_\alpha(\mathfrak{w}_2(\tau)), T_\alpha(\mathfrak{w}_3(\tau)) \right) \oplus \left(T_\alpha(\nu_1(\tau)), T_\alpha(\nu_2(\tau)), T_\alpha(\nu_3(\tau)) \right) \\ &= \left(T_\alpha(\mathfrak{w}_1(\tau)) + T_\alpha(\nu_1(\tau)), T_\alpha(\mathfrak{w}_2(\tau)) + T_\alpha(\nu_2(\tau)), T_\alpha(\mathfrak{w}_3(\tau)) + T_\alpha(\nu_3(\tau)) \right) \\ &= \left(T_\alpha(\mathfrak{w}_1(\tau) + \nu_1(\tau)), T_\alpha(\mathfrak{w}_2(\tau) + \nu_2(\tau)), T_\alpha(\mathfrak{w}_3(\tau) + \nu_3(\tau)) \right) \\ &= {}_{i.gH} \mathbf{T}_\alpha^0(\mathfrak{w}(\tau) \oplus \nu(\tau)) \end{aligned}$$

In a similar way, we can proof the other case. \square

Definition 3.4. (See [7]) Assume $\mathfrak{w} \in C^{\mathbb{F}}([a, \infty), \mathcal{F})$ and $\alpha \in (0, 1)$. Then the fuzzy conformable fractional integral is constructed as

$$\mathcal{I}_\alpha^a(\mathfrak{w}(\tau)) = \int_a^\tau (\xi - a)^{\alpha-1} \mathfrak{w}(\xi) d\xi$$

Remark 1. If $\mathfrak{w} \in C^F([a, \infty), \Omega)$ and $\mathfrak{w}(\tau) = (\mathfrak{w}_1(\tau), \mathfrak{w}_2(\tau), \mathfrak{w}_3(\tau))$ then it is clear that

$$\mathcal{I}_a^\alpha(\mathfrak{w}(\tau)) = \left(I_a^\alpha(\mathfrak{w}_1(\tau)), I_a^\alpha(\mathfrak{w}_2(\tau)), I_a^\alpha(\mathfrak{w}_3(\tau)) \right),$$

where $I_a^\alpha(\mathfrak{w}_i(\tau))$ for $i = 1, 2, 3$ are the conformable fractional integral definition for the real-valued functions $\mathfrak{w}_i(\tau)$ [1].

4 The Fuzzy Fractional Conformable Laplace Transform

This section will introduce the fuzzy conformable Laplace transform for one-variable fuzzy-valued functions and prove some important properties for this transformation.

Definition 4.1. A fuzzy function \mathfrak{w} is said to be fuzzy conformable exponentially bounded if it satisfies in the following inequality

$$D(\mathfrak{w}(\tau), 0) \leq M e^{c \frac{\tau^\alpha}{\alpha}},$$

where M and c are positive real constants and $0 < \alpha < 1$, for all sufficiently large t .

Definition 4.2. Let $\mathfrak{w} : [\tau_0, \infty) \rightarrow \mathcal{F}$ be a fuzzy function. The fuzzy fractional conformable Laplace transform of order α of fuzzy function $\mathfrak{w}(\tau)$ is defined as follows

$$\mathfrak{L}_\alpha^{\tau_0} \{\mathfrak{w}(\tau)\}(s) = \mathbf{W}_\alpha^{\tau_0}(s) = \int_{\tau_0}^{\infty} (\tau - \tau_0)^{\alpha-1} e^{-s \frac{(\tau-\tau_0)^\alpha}{\alpha}} \mathfrak{w}(\tau) d\tau,$$

whenever the limit exist.

Remark 2. Consider the fuzzy function $\mathfrak{w}(\tau) = (\mathfrak{w}_1(\tau), \mathfrak{w}_2(\tau), \mathfrak{w}_3(\tau))$, then by attention to Definition 2.4, the fuzzy fractional conformable Laplace transform of this function is denoted by

$$\mathfrak{L}_\alpha^{\tau_0} \{\mathfrak{w}(\tau)\}(s) = \left(L_\alpha^{\tau_0} \{\mathfrak{w}_1(\tau)\}, L_\alpha^{\tau_0} \{\mathfrak{w}_2(\tau)\}, L_\alpha^{\tau_0} \{\mathfrak{w}_3(\tau)\} \right),$$

where $L_\alpha^{\tau_0}$ is the definition of classical fractional conformable Laplace transform for the real-valued functions $\mathfrak{w}_i(\tau)$ [1, 24], and

$$L_\alpha^{\tau_0} \{\mathfrak{w}_i(\tau)\} = \int_{\tau_0}^{\infty} (\tau - \tau_0)^{\alpha-1} e^{-s \frac{(\tau-\tau_0)^\alpha}{\alpha}} v_i(\tau) d\tau = \lim_{\mathcal{R} \rightarrow \infty} \int_{\tau_0}^{\mathcal{R}} (\tau - \tau_0)^{\alpha-1} e^{-s \frac{(\tau-\tau_0)^\alpha}{\alpha}} v_i(\tau) d\tau, \quad i = 1, 2, 3.$$

Definition 4.3. Let $\mathfrak{w}(\tau)$ is a fuzzy function. If $\mathfrak{L}_\alpha^{\tau_0} \{\mathfrak{w}(\tau)\}(s) = \mathbf{W}_\alpha^{\tau_0}(s)$ then $\{\mathfrak{L}_\alpha^{\tau_0}\}^{-1}$ is used to denote the inverse fuzzy fractional conformable Laplace transform of $\mathbf{W}_\alpha^{\tau_0}(s)$ and we have

$$\{\mathfrak{L}_\alpha^{\tau_0}\}^{-1}[\mathbf{W}_\alpha^{\tau_0}(s)] = \mathfrak{w}(\tau), \quad \tau \geq 0,$$

which maps the fuzzy fractional conformable Laplace transform of a fuzzy function back to the original function.

Lemma 4.1. Let \mathfrak{w} be piecewise continuous on $[\tau_0, \infty)$ and fuzzy conformable exponentially bounded. Then the fuzzy conformable Laplace transform of $\mathfrak{w}(\tau)$ exists for all s provided $\text{Re}(s) > c$.

Proof. \mathfrak{w} is conformable exponentially bounded function, so there exists τ_0, M_1 and c such that

$$D(\mathfrak{w}(\tau), 0) \leq M_1 e^{c \frac{\tau^\alpha}{\alpha}},$$

for all $\tau \geq \tau_0$. Furthermore, \mathfrak{w} is piecewise continuous on $[0, \tau_0]$ and hence bounded there, so

$$D(\mathfrak{w}(\tau), 0) \leq M_2, \quad \forall \tau \in [0, \tau_0].$$

This means that, a constant \mathcal{M} can be chosen sufficiently large so that $D(\mathfrak{w}(\tau), 0) \leq \mathcal{M} e^{c \frac{\tau^\alpha}{\alpha}}$. Now, let $\mathfrak{L}_\alpha^0 \{\mathfrak{w}(\tau)\} = \mathbf{W}_\alpha^0(s)$, therefore,

$$\begin{aligned} D\left(\int_0^{\mathcal{R}} \tau^{\alpha-1} e^{-s \frac{\tau^\alpha}{\alpha}} \mathfrak{w}(\tau) d\tau, 0\right) &\leq \int_0^{\mathcal{R}} \tau^{\alpha-1} e^{-s \frac{\tau^\alpha}{\alpha}} D(\mathfrak{w}(\tau), 0) d\tau \\ &\leq \mathcal{M} \int_0^{\mathcal{R}} \tau^{\alpha-1} e^{(c-s) \frac{\tau^\alpha}{\alpha}} d\tau \\ &= \frac{\mathcal{M}}{s-c} - \frac{e^{(c-s) \frac{\mathcal{R}^\alpha}{\alpha}}}{s-c} \end{aligned}$$

Letting $\mathcal{R} \rightarrow \infty$, so when $\text{Re}(s) > c$ we have

$$D\left(\mathbf{W}_\alpha^0(s), 0\right) \leq \frac{\mathcal{M}}{s-c}.$$

The proof is complete. \square

Lemma 4.2. Let $\mathfrak{w} : [\tau_0, \infty) \rightarrow \mathbb{R}_F$ be a fuzzy function such that $\mathfrak{L}_\alpha^{\tau_0}\{\mathfrak{w}(\tau)\}(s) = \mathbf{W}_\alpha^{\tau_0}(s)$ exists. Then

$$\mathbf{W}_\alpha^{\tau_0}(s) = \mathcal{L}[\mathfrak{w}(\tau_0 + (\alpha\tau)^{\frac{1}{\alpha}})], \quad 0 < \alpha \leq 1$$

where $\mathcal{L}[\nu(\tau)] = \int_0^\infty e^{-s\tau}\nu(\tau)d\tau$ is the fuzzy Laplace transform for the fuzzy function $\nu(\tau)$ [27].

Proof. Let $u = \frac{(\tau-\tau_0)^\alpha}{\alpha}$, so the proof is clear. \square

Lemma 4.3. Consider $\mathfrak{w}(\tau)$ and $\mathfrak{v}(\tau)$ are two fuzzy functions. Let a, b are two real constant such that $a, b > 0$ (or $a, b < 0$). If the fuzzy conformable Laplace transform \mathfrak{w} and \mathfrak{v} exist hence the fuzzy conformable Laplace transform of $a\mathfrak{w}(\tau) \oplus b\mathfrak{v}(\tau)$ and $a\mathfrak{w}(\tau) \ominus_{gH} b\mathfrak{v}(\tau)$ exist, and

1. $a\mathfrak{L}_\alpha^{\tau_0}\{\mathfrak{w}(\tau)\}(s) \ominus_{gH} b\mathfrak{L}_\alpha^{\tau_0}\{\mathfrak{v}(\tau)\}(s) = \mathfrak{L}_\alpha^{\tau_0}\{a\mathfrak{w}(\tau) \ominus_{gH} b\mathfrak{v}(\tau)\}(s).$
2. $a\mathfrak{L}_\alpha^{\tau_0}\{\mathfrak{w}(\tau)\}(s) \oplus b\mathfrak{L}_\alpha^{\tau_0}\{\mathfrak{v}(\tau)\}(s) = \mathfrak{L}_\alpha^{\tau_0}\{a\mathfrak{w}(\tau) \oplus b\mathfrak{v}(\tau)\}(s)$

Proof. The proof of this Lemma is clear with respect to Lemma 4.2 and Lemma 3.1 in [22]. \square

Theorem 4.1. Let us consider $\tau_0 \in \mathbb{R}$, $0 < \alpha \leq 1$ and $\mathfrak{w} : [\tau_0, \infty) \rightarrow \mathbb{R}_F$ be α_{gH} -differentiable fuzzy function provided that the type of α_{gH} -differentiability doesn't change in interval $[\tau_0, \infty)$. Then

$$\mathfrak{L}_\alpha^{\tau_0}\left\{i.gH \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau))\right\}(s) = s \mathbf{W}_\alpha^{\tau_0}(s) \ominus \mathfrak{w}(\tau_0),$$

$$\mathfrak{L}_\alpha^{\tau_0}\left\{ii.gH \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau))\right\}(s) = (-1)\mathfrak{w}(\tau_0) \ominus_{gH} (-1)s \mathbf{W}_\alpha^{\tau_0}(s),$$

where $\mathbf{W}_\alpha^{\tau_0}(s) = \mathfrak{L}_\alpha^{\tau_0}\{\mathfrak{w}(\tau)\}(s).$

Proof. First, let $\mathfrak{w}(\tau)$ is a $\alpha_{i.gH}$ -differentiable triangular fuzzy function. According to the definition of conformable Laplace transform for a fuzzy function, we have

$$\mathfrak{L}_\alpha^{\tau_0}\{gH \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau))\}(s) = \lim_{\mathcal{R} \rightarrow \infty} \int_{\tau_0}^{\mathcal{R}} (\tau - \tau_0)^{\alpha-1} e^{-s\frac{(\tau-\tau_0)^\alpha}{\alpha}} gH \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau))d\tau,$$

provided that the limit exists. Using Theorem 3.2, we conclude that

$$\mathfrak{L}_\alpha^{\tau_0}\{gH \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau))\}(s) = \lim_{\mathcal{R} \rightarrow \infty} \int_{\tau_0}^{\mathcal{R}} e^{-s\frac{(\tau-\tau_0)^\alpha}{\alpha}} \mathfrak{w}'_{gH}(\tau)d\tau,$$

Now, let \mathfrak{w} be a $\alpha_{i.gH}$ - differentiable function. Then Theorem 3.3 and Theorem 2.2 result

$$\begin{aligned} \mathfrak{L}_\alpha^{\tau_0}\{i.gH \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau))\}(s) &= \lim_{\mathcal{R} \rightarrow \infty} \int_{\tau_0}^{\mathcal{R}} e^{-s\frac{(\tau-\tau_0)^\alpha}{\alpha}} \mathfrak{w}'_{i.gH}(\tau)d\tau \\ &= \lim_{\mathcal{R} \rightarrow \infty} \left[\left(e^{-s\frac{(\mathcal{R}-\tau_0)^\alpha}{\alpha}} \odot \mathfrak{w}(\mathcal{R}) \ominus \mathfrak{w}(\tau_0) \right) \oplus s \int_0^{\mathcal{R}} (\tau - \tau_0)^{\alpha-1} e^{-s\frac{(\tau-\tau_0)^\alpha}{\alpha}} \odot \mathfrak{w}(\tau)d\tau \right]. \end{aligned}$$

Therefore, we have

$$\mathfrak{L}_\alpha^{\tau_0} \{ {}_{i.gH} \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau)) \}(s) = s \mathbf{W}_\alpha^{\tau_0}(s) \ominus \mathfrak{w}(\tau_0).$$

Now, let \mathfrak{w} be a $\alpha_{ii.gH}$ -differentiable function, then Theorem 3.3 and Theorem 2.2 result

$$\begin{aligned} \mathfrak{L}_\alpha^{\tau_0} \{ {}_{ii.gH} \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau)) \}(s) &= \lim_{\mathcal{R} \rightarrow \infty} \int_{\tau_0}^{\mathcal{R}} e^{-s \frac{(\tau - \tau_0)^\alpha}{\alpha}} \mathfrak{w}'_{ii.gH}(\tau) d\tau \\ &= \lim_{\mathcal{R} \rightarrow \infty} \left[\left((-1) \mathfrak{w}(\tau_0) \ominus (-1) e^{-s \frac{(\mathcal{R} - \tau_0)^\alpha}{\alpha}} \odot \mathfrak{w}(\mathcal{R}) \right) \ominus_{gH} (-1) s \int_0^{\mathcal{R}} (\tau - \tau_0)^{\alpha-1} e^{-s \frac{(\tau - \tau_0)^\alpha}{\alpha}} \odot \mathfrak{w}(\tau) d\tau \right] \\ &= (-1) \mathfrak{w}(\tau_0) \ominus_{gH} (-1) \lim_{\mathcal{R} \rightarrow \infty} s \int_0^{\mathcal{R}} (\tau - \tau_0)^{\alpha-1} e^{-s \frac{(\tau - \tau_0)^\alpha}{\alpha}} \odot \mathfrak{w}(\tau) d\tau. \end{aligned}$$

Hence

$$\mathfrak{L}_\alpha^{\tau_0} \{ {}_{ii.gH} \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau)) \}(s) = (-1) \mathfrak{w}(\tau_0) \ominus_{gH} (-1) s \mathbf{W}_\alpha^{\tau_0}(s).$$

□

5 The Fuzzy Conformable Fractional Initial Value Problem

Consider the following fuzzy conformable fractional initial value problem

$$\begin{cases} {}_{gH} \mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau)) = \mathfrak{F}(\tau, \mathfrak{w}(\tau)), \\ \mathfrak{w}(\tau_0) = (\mathfrak{w}_1(\tau_0), \mathfrak{w}_2(\tau_0), \mathfrak{w}_3(\tau_0)), \end{cases}, \quad (4)$$

where $\mathfrak{w}(\tau_0)$ is a fuzzy triangular continuous function. There exists at most one solution for the problem (4) [7].

The fuzzy conformable fractional initial value problem (4) can be solved using the fuzzy Laplace transform method and based on the following steps

- i. Take the fuzzy Laplace transform of this fuzzy initial value problem using all theorems and properties in Section 4 as necessary.
- ii. Put initial conditions into the resulting equation.
- iii. Solve for the output variable.
- iv. Use the fuzzy inverse Laplace transform.

In the following, we will obtain an analytical triangular fuzzy solution for several practical examples such as the fuzzy conformable fractional Growth equation, the one-compartment fuzzy conformable fractional model and the fuzzy conformable fractional Newton's law of cooling by using the fuzzy conformable Laplace transform.

5.1 The Fuzzy Growth Equation

The growth of microorganisms, such as bacteria, fungi, and algae, is governed by differential equations or systems of differential equations. Despite the fact that the conditions in the laboratory may vary over time, actual conditions in the real world may differ from laboratory conditions. Thus, getting a precise figure for the colony's population is virtually impossible. If $\mathfrak{w}(\tau)$ represents the population at any given time, then τ would represent the population at that time. In addition, let \mathfrak{w}_0 be the fuzzy initial population at time τ_0 , that is, $\mathfrak{w}(\tau_0) = \mathfrak{w}_0$. Then if the population grows exponentially

$$(\text{Rate of change of population at time } \tau) = \kappa(\text{Current population at time } \tau)$$

In mathematical terms, this can be written as

$$\begin{cases} {}_{gH}\mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau)) = \kappa \odot \mathfrak{w}(\tau), & 0 < \alpha < 1, \\ \mathfrak{w}(\tau_0) = \mathfrak{w}_0, \end{cases} \quad (5)$$

The value κ is known as the relative growth rate and is a positive real constant and $\mathfrak{w}_0 = (\mathfrak{w}_{0_1}, \mathfrak{w}_{0_2}, \mathfrak{w}_{0_3})$ is a triangular fuzzy constant.

Remark 3. *It is easy to investigate that κ if in Eq.(5) is a positive real constant then the fuzzy solution of this equation is a $\alpha_{i.gH}$ -differentiable function and if κ is a negative real constant then Eq.(5) has $\alpha_{ii.gH}$ -differentiable fuzzy solution.*

In the following, we will obtain the fuzzy analytical triangular solution of problem (5) with the fuzzy initial condition $\mathfrak{w}(\tau_0)$ by the fuzzy conformable Laplace transform. The fuzzy conformable Laplace transform is applied to the problem (5). We had assumed that κ is a positive real constant, hence according to Remark 3 and Theorem 4.1 and the fuzzy initial condition $\mathfrak{w}(\tau_0)$, we have

$$\mathfrak{L}_\alpha^{\tau_0} \{ {}_{gH}\mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau)) \}(s) = \kappa \odot \mathfrak{L}_\alpha^{\tau_0} \{ \mathfrak{w}(\tau) \}(s) \Rightarrow s\mathbf{W}_\alpha^{\tau_0}(s) \ominus \mathfrak{w}_0 = \kappa \odot \mathbf{W}_\alpha^{\tau_0}(s)$$

We can rearrange the above equation by using some basic rules of fuzzy arithmetic as follows

$$(s - \kappa) \odot \mathbf{W}_\alpha^{\tau_0}(s) = \mathfrak{w}_0, \quad \Rightarrow \quad \mathbf{W}_\alpha^{\tau_0}(s) = \frac{\mathfrak{w}_0}{(s - \kappa)}$$

The fuzzy inverse Laplace transform is applied to the above equation

$$\mathbf{W}_\alpha^{\tau_0}(s) = \left(\{L_\alpha^{\tau_0}\}^{-1} \left\{ \frac{\mathfrak{w}_{0_1}}{(s - \kappa)} \right\}, \{L_\alpha^{\tau_0}\}^{-1} \left\{ \frac{\mathfrak{w}_{0_2}}{(s - \kappa)} \right\}, \{L_\alpha^{\tau_0}\}^{-1} \left\{ \frac{\mathfrak{w}_{0_3}}{(s - \kappa)} \right\} \right)$$

Therefore, the fuzzy analytical triangular solution of problem (5) with the fuzzy initial condition $\mathfrak{w}(\tau_0)$ is

$$\mathfrak{w}(\tau) = \left(\mathfrak{w}_{0_1} e^{\kappa \frac{(\tau - \tau_0)^\alpha}{\alpha}}, \mathfrak{w}_{0_2} e^{\kappa \frac{(\tau - \tau_0)^\alpha}{\alpha}}, \mathfrak{w}_{0_3} e^{\kappa \frac{(\tau - \tau_0)^\alpha}{\alpha}} \right)$$

Example 5.1. In the refrigerator, you have an old jar of yogurt that is growing bacteria. The number of bacteria in the yogurt jar will be denoted by $\mathfrak{w}(\tau)$ in this problem. Let the initial fuzzy population of bacteria at time zero equal $\mathfrak{w}(0) = (516, 540, 598)$ and $\kappa = \frac{1}{30}$.

As bacteria number changes over time τ , the fuzzy initial valued problem describes the changing number of bacteria as being

$$\begin{cases} {}_gH\mathbf{T}_\alpha^0(\mathfrak{w}(\tau)) = \kappa \odot \mathfrak{w}(\tau), \\ \mathfrak{w}(0) = (516, 540, 598), \end{cases}$$

Now, suppose that $\alpha = \frac{1}{5}$, so the number of bacteria over time τ is obtained by using the fuzzy conformable Laplace transform as follows

$$\mathfrak{w}(\tau) = \left(516e^{\frac{1}{6}\tau^{\frac{1}{5}}}, 540e^{\frac{1}{6}\tau^{\frac{1}{5}}}, 598e^{\frac{1}{6}\tau^{\frac{1}{5}}} \right)$$

This exact solution for various τ in $[0, 1]$ have been reported in Figures 2.

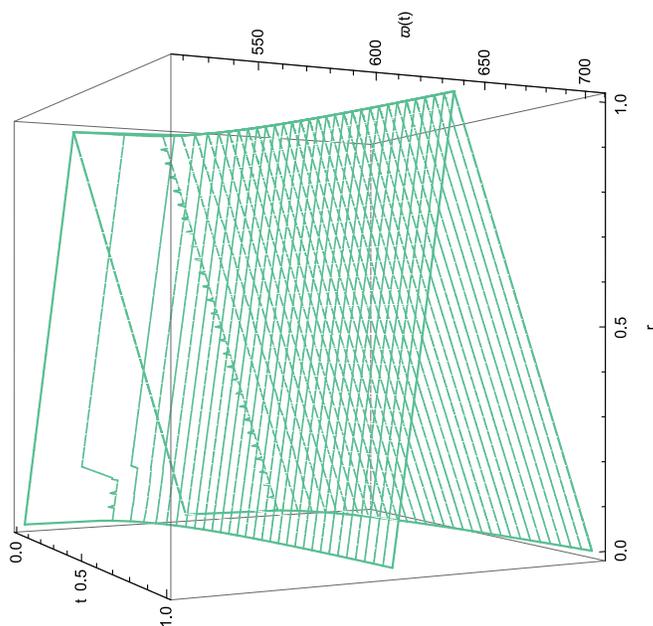
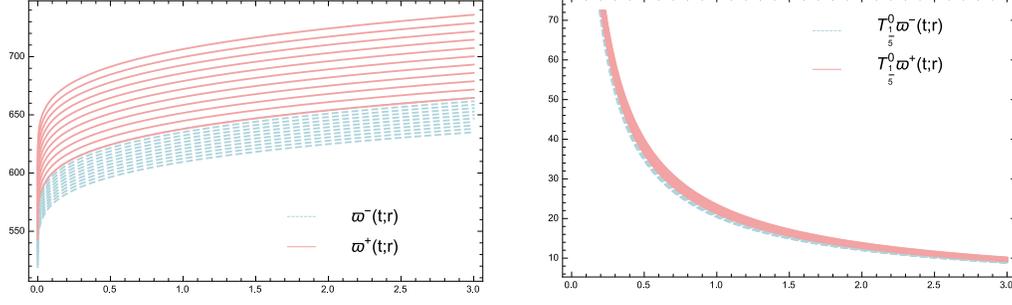


Figure 2: Graphs of fuzzy exact solution of Example 5.1

The r -cut of this solution, $\mathfrak{w}(\tau)$ and ${}_gH\mathbf{T}_{\frac{1}{5}}^0(\mathfrak{w}(\tau))$ of this solution for $0 \leq r \leq 1$ are showed in Figures 3. As you can see, the position of lower cut (blue) and upper cut (red) isn't changed. It shows that $\mathfrak{w}(\tau)$ is $\alpha_{i.gH}$ -differentiable.

(a). r -cut of $\mathfrak{w}(\tau)$ (b). r -cut of ${}_gH\mathbf{T}_{\frac{1}{5}}^0(\mathfrak{w}(\tau))$ Figure 3: r -cut representation of different functions for Example 5.1 for all $r \in [0, 1]$.

5.2 The One-compartment Fuzzy Fractional Model

The one-compartment open model refers to the rate and extent of distribution of a drug to different tissues, and the rate of elimination of the drug. The rate of drug movement between compartments is described by the first order kinetics. If $\mathfrak{w}(\tau)$ denotes the concentration of drug in the compartment at time τ , then the rate of change of $\mathfrak{w}(\tau)$ is

$${}_gH\mathbf{T}_{\alpha}^{\tau_0}(\mathfrak{w}(\tau)) = (-1)\kappa \odot \mathfrak{w}(\tau), \quad 0 < \alpha < 1,$$

where κ is the elimination rate constant [22].

Now, by the fuzzy conformable Laplace transform we have

$$\mathfrak{L}_{\alpha}^0\{ {}_gH\mathbf{T}_{\alpha}^0(\mathfrak{w}(\tau))\}(s) = (-1)\kappa \mathfrak{L}_{\alpha}^0\{\mathfrak{w}(\tau)\}(s)$$

Let \mathfrak{w}_0 represents the fuzzy initial amount of a drug. Remark 3 and Theorem 3.2 yield to

$$(-1)\mathfrak{w}_0 \ominus {}_gH(-1)s\mathbf{W}_{\alpha}^{\tau_0}(s) = (-1)\kappa \mathbf{W}_{\alpha}^{\tau_0}(s), \quad \Rightarrow \quad \mathbf{W}_{\alpha}^{\tau_0}(s) = \frac{\mathfrak{w}_0}{s + \kappa}$$

By applying the inverse conformable Laplace transform we obtain

$$\mathfrak{w}(\tau) = \left(\mathfrak{w}_{0_1} e^{-\kappa \frac{(\tau-\tau_0)^{\alpha}}{\alpha}}, \mathfrak{w}_{0_2} e^{-\kappa \frac{(\tau-\tau_0)^{\alpha}}{\alpha}}, \mathfrak{w}_{0_3} e^{-\kappa \frac{(\tau-\tau_0)^{\alpha}}{\alpha}} \right)$$

Example 5.2. Assume that $(3.5, 4.3, 5.1)$ mg of a highly hydrophilic drug (like Aminoglycosides) is injected into the body at time τ_0 . The fuzzy initial valued problem that describes the amount of this drug in the tissues is

$$\begin{cases} {}_gH\mathbf{T}_{\alpha}^{\tau_0}(\mathfrak{w}(\tau)) = (-1)\kappa \odot \mathfrak{w}(\tau) \\ \mathfrak{w}(\tau_0) = (3.97, 4.3, 5.1) \end{cases}$$

Hence, the amount of this drug in the tissues, $\mathfrak{w}(\tau)$, is

$$\mathfrak{w}(\tau) = \left(3.97e^{-\kappa \frac{(\tau-\tau_0)^{\alpha}}{\alpha}}, 4.3e^{-\kappa \frac{(\tau-\tau_0)^{\alpha}}{\alpha}}, 5.1e^{-\kappa \frac{(\tau-\tau_0)^{\alpha}}{\alpha}} \right)$$

5.3 The Fuzzy Newton Fractional Conformable Cooling Law

The temperature difference in any situation results from energy flow into a system or energy flow from a system to the surroundings. The former leads to heating, whereas the latter leads to cooling. Newton's law of cooling states that the rate of change temperature is proportional to the difference between the body's temperature and the surrounding medium's temperature.

Consider a body in the temperature \mathfrak{w} placed in a medium of temperature \mathcal{M} medium, which is considered a fuzzy constant. This law can be written in the form.

$${}_gH\mathbf{T}_\alpha^0(\mathfrak{w}(\tau)) = (-1)\kappa \odot (\mathfrak{w} \ominus \mathcal{M}) \quad 0 < \alpha < 1, \quad (6)$$

where κ is a real positive constant of the cooling coefficient. Consider that the initial temperature of the body $\mathfrak{w}(0) = \mathfrak{w}_0$ is a fuzzy constant.

By Remark 3, the Eq.(6) has a $\alpha_{ii.gH}$ - differential solution. Applying the fuzzy conformable Laplace transform to problem (6) and using Lemma 4.3 we have

$$\mathfrak{L}_\alpha^0\{ {}_gH\mathbf{T}_\alpha^0(\mathfrak{w}(\tau))\}(s) = (-1)\kappa \odot \left(\mathfrak{L}_\alpha^0\{\mathfrak{w}(\tau)\}(s) \ominus \mathfrak{L}_\alpha^0\{\mathcal{M}\}(s) \right)$$

So the body's temperature is

$$\mathfrak{w}(\tau) = (\mathfrak{w}_0 \ominus \mathcal{M})e^{-\kappa \frac{\tau^\alpha}{\alpha}} \oplus \mathcal{M}$$

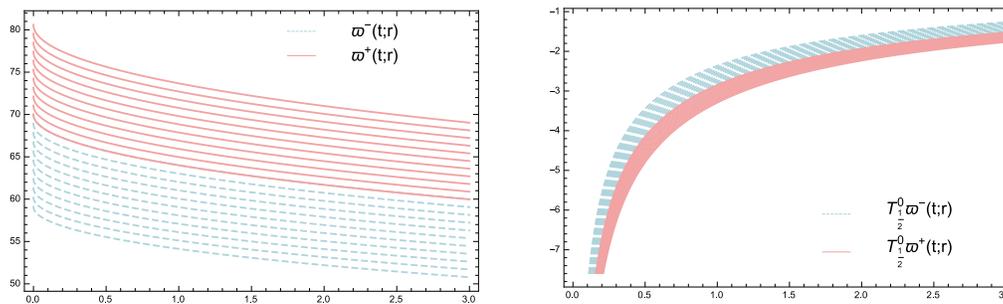
Example 5.3. *At midnight the furnace fails inside a building such that the outside temperature at a constant $(6.8, 7, 7.85)^\circ$ F and the inside temperature at $(59.1, 70, 80.6)^\circ$ F. The fuzzy initial valued problem that describes the temperature inside the building is*

$$\begin{cases} {}_gH\mathbf{T}_\alpha^{\tau_0}(\mathfrak{w}(\tau)) = (-1)\kappa \odot \left(\mathfrak{w} \ominus (6.8, 7, 7.85) \right) \\ \mathfrak{w}(0) = (59.1, 70, 80.6) \end{cases}$$

Let $\alpha = \frac{1}{2}$ and $\kappa = \frac{1}{20}$. Then by using the method described above, the temperature inside the building decrease

$$\mathfrak{w}(\tau) = (52.6, 63, 72.75)e^{-\frac{1}{10}\tau^{\frac{1}{2}}} \oplus (6.8, 7, 7.85)$$

The r -cut of this solution, $\mathfrak{w}(\tau)$ and ${}_gH\mathbf{T}_{\frac{1}{2}}^0(\mathfrak{w}(\tau))$ of this solution for $0 \leq r \leq 1$ are showed in Figures 4. As you can see, the position of lower cut (blue) and upper cut (red) is changed. It shows that $\mathfrak{w}(\tau)$ is $\alpha_{ii.gH}$ -differentiable.

(a). r -cut of $\mathfrak{w}(\tau)$ (b). r -cut of ${}_{gH}\mathbf{T}_{\frac{1}{2}}^0(\mathfrak{w}(\tau))$ Figure 4: r -cut representation of different functions for Example 5.3 for all $r \in [0, 1]$.

6 Conclusion

This study discusses the fuzzy conformable fractional order initial problem within the context of conformable generalized Hukuhara differentiability. A fuzzy conformable fractional derivative based on generalized Hukuhara differentiability was introduced, and a number of associated properties were shown on the topics. As a result, the fuzzy conformable Laplace transform was employed to determine the analytical solutions for the fractional differential equation. A number of practical applications such as fuzzy Newton's law of cooling, fuzzy growth equation, and one-compartment fuzzy fractional model, have been used to demonstrate the effectiveness and efficiency of the approaches. According to the results, the fuzzy conformable Laplace transform is an effective and convenient tool for solving fuzzy conformable fractional differential equations. We have obtained interesting results here that could be used to conduct future studies on fuzzy fractional partial differential equations under conformable gH -differentiability.

Acknowledgements

This work was partially supported by the Central Tehran Branch of Islamic Azad University.

Compliance with Ethical Standards

Conflict of interest

All authors declare that they have no conflict of interest.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

References

- [1] T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.*, 279 (2015) 57–66.
- [2] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.*, 72 (2010) 2859–2862.
- [3] T. Allahviranloo, S. Salahshour, S. Abbasbandy, Explicit solutions of fractional differential equations with uncertainty, *Soft Comput.*, 16 (2012) 297–302.
- [4] T. Allahviranloo, Z. Gouyandeh, A. Armand, Fuzzy fractional differential equations under generalized fuzzy Caputo derivative, *J. Intell. Fuzzy Syst*, 26 (2014) 1481–1490.
- [5] T. Allahviranloo, S. Salahshour, S. Abbasbandy, Explicit solutions of fractional differential equations with uncertainty, *Soft Comput.*, 16 (2012) 297–302.
- [6] S. Arshad, V. Lupulescu, On the fractional differential equations with uncertainty, *Nonlinear Anal.*, 74 (2011) 3685–3693.
- [7] O. A. Arqub, M. Al-Smadi, Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions, *Soft Comput.*, 24 (2020) 12501–12522.
- [8] A. Armand, T. Allahviranloo, Z. Gouyandeh, Some fundamental results on fuzzy calculus, *Iranian Journal of Fuzzy Systems*, 15 (2013) 27–46.
- [9] B. Bede, S.G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets and Systems*, 151 (2005) 581–599.
- [10] B. Bede, I. J. Rudas, A. L. Bencsik, First order linear fuzzy differential equations under generalized differentiability, *Inf. Sci*, 177 (2007) 1648–1662.
- [11] B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions, *Fuzzy Sets and Systems*, 230 (2013) 119–141.
- [12] B. Bede, *Mathematics of fuzzy sets and fuzzy logic*, Springer, London, (2013).
- [13] P. Diamond, Brief note on the variation of constants formula for fuzzy differential equations, *Fuzzy Sets and Systems*, 129 (2002) 65–71.
- [14] D. Dubois, H. Prade, Towards fuzzy differential calculus, *Fuzzy Sets and Systems*, 8 (1) (1982) 1–17.

- [15] O. S. Fard, J. Soolaki, D. F. M. Torres, A necessary condition of Pontryagin type for fuzzy fractional optimal control problems, *Discrete Contin. Dyn. Syst. Ser. S*, 11 (2018) 59–76.
- [16] M. Ghaffari, T. Allahviranloo, S. Abbasbandy, M. Azhini, On the fuzzy solutions of time-fractional problems, *Iranian Journal of Fuzzy Systems*, 18 (3) (2021) 51–66.
- [17] N.V. Hoa, Fuzzy fractional functional differential equations under Caputo gH-differentiability, *Communications in Nonlinear Science and Numerical Simulation*, 22 (1-3) (2015) 1134–1157.
- [18] N. V. Hoa, H. Vu, T. Minh Duc, Fuzzy fractional differential equations under Caputo Katugampola fractional derivative approach, *Fuzzy Sets and Systems*, 375 (2019) 70–99.
- [19] M. Hukuhara, Integration des applications mesurables dont l'ensemble est un compact convexe, *Funkcialaj Ekvacioj*, 10 (1967) 205–223.
- [20] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, 24 (1987) 301–317.
- [21] A. Kaufmann, M. M. Gupta, *Introduction Fuzzy Arithmetic*, Van Nostrand Reinhold, New York, (1985).
- [22] M. Keshavarz, T. Allahviranloo, S. Abbasbandy, M. H. Modarressi, A Study of Fuzzy Methods for Solving System of Fuzzy Differential Equations, *New Mathematics and Natural Computation*, 17 (2021) 1–27.
- [23] A. Khastan, J. J. Nieto, R.R. Rodríguez-Lopez, Periodic boundary value problems for first-order linear differential equations with uncertainty under generalized differentiability, *Inf. Sci*, 222 (2013) 544–558.
- [24] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, 264 (2014) 65–70.
- [25] M. Mazandarani, A. V. Kamyad, Modified fractional Euler method for solving fuzzy fractional initial value problem, *Commun. Nonlinear Sci. Numer. Simul*, 18 (2013) 12–21.
- [26] S. Rahimi Chermahini, M. S. Asgari, Analytical fuzzy triangular solutions of the wave equation, *Soft Comput.*, 25 (2021) 363–378.
- [27] S. Salahshour, T. Allahviranloo, Applications of fuzzy Laplace transforms. *Soft Comput.*, 17 (2013) 145–158.
- [28] S. Seikkala, On the fuzzy initial value problem, *Fuzzy Sets and Systems*, 24 (1987) 319–330.
- [29] L. Stefanini, A generalization of hukuhara difference and division for interval and fuzzy arithmetic, *Fuzzy Sets and Systems* 161 (11) (2010) 1564–1584.
- [30] L. A. Zadeh, Fuzzy sets, *Inf. Control* 8 (1965) 338–353.