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Global Analysis of a time fractional order spatio-temporal SIR model

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Abstract

We deal in this paper with a diffusive *SIR* epidemic model described by reaction-diffusion equations involving a fractional derivative. The existence and uniqueness of the solution are shown, next to the boundedness of the solution. Further, it has been shown that the global behavior of the solution is governed by the value of R_0, which is known in epidemiology by the basic reproduction number. Indeed, using the Lyapunov direct method it has been proved that the disease will extinct for $R_0 < 1$ for any value of the diffusion constants. For $R_0 > 1$, the disease will persist and the unique positive equilibrium is globally stable. Some numerical illustrations have been used to confirm our theoretical results.

Keywords: Diffusive SIR model; epidemiology; fractional calculus; Caputo derivative.

Subject classification: 26A33; 34A08; 92D30; 35K57.

1. Introduction

In many sciences, experiments can be done to collect information and test hypotheses. Doing experiments for testing the outbreak of infection in different populations is generally not possible, immoral, and costly [1]. In most cases, the data are often unprecise due to underreporting. This deficit in the data collections makes a reliable estimation of the parameters impossible, which is noted hugely in our recent fight against the pandemic of COVID-19 disease. Then, it is only possible to approximate certain parameters. Based on the fact that repeatable experiments are not available in epidemiology, mathematical modeling and numerical simulations can be used to perform the theoretical experiments needed for a variety of parameter values. [2]. They also help to understand, analyzing and limiting the outbreak of infectious diseases [6, 3, 4, 5, 8, 10, 11, 12]. So we should be able to give a response to important issues as:

- The possibility of having an epidemic.
- The knowledge of the duration of this epidemic is important, for determining the proper public health intervention.
- The density of the individuals that have been touched by this disease.
- The type of control that allows authorities to make decisions about strategies as isolation, quarantine, vaccination, and treatment.

In this context, most *SIR* models have been traditionally investigated in an uniform distribution of populations which are generally formulated only by ordinary differential equations. This fact shows the possibility that the disease can outbreak over a spatial region. In reality the infected individuals have the greatest effect on spatially nearest susceptible persons. The outbreak of infectious diseases is influenced by the spatial movement of populations. The great development in transportation networks is among the main contributing factors in the growth of people's movement around the world. For these reasons, many recent researches have been devoted to the

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study of reaction-diffusion models (particularly the existence, uniqueness, positivity and stability of the equilibria). They have as goal predicting the evolution of diseases in relation to time and space simultaneously [13, 14, 15].

Recently, fractional derivatives have several applications in different fields as mechanics, control theory bioengineering and viscoelasticity [16, 17, 18, 19, 20]. We point out that derivative order (fractional) can be any positive real choice in order to best correspond to the available data [21]. Consequently, the systems of non-integer order differential equations or partial differential equations give a more realistic behavior [20, 22]. Fractional-order-derivatives are used widely in epidemiology to describe disease evolution and, in most cases, are considered to be more precise than the classical derivative [23, 24, 25]. For example, the spread of the virus is generally discontinuous, so that they are not well described by systems of ordinary differential equations. Then fractional systems naturally deal with such a property of discontinuity [26, 27]. In addition, different models have used fractional derivatives to better predict the outbreak of diseases with sufficient data, among these models we find *SIR* [28, 29, 30], *SIRC* [31], *SEIR* [32], *SIRS* and *SEIRS* [25]. In [32] the authors used the statistical data from the Florida Department of Health in the period from September 2011 to July 2014, they concluded that the absolute error between the solutions obtained statistically and that of the fractional model decreases more than those obtained by the model of integer derivative.

In the literature, several definitions of fractional derivatives have been used in different works [33, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58]. Among the most popular non-integer derivatives is that of Riemann-Liouville. It is not often adequate for modeling physical systems because it does not keep the nullity of the derivative constant and the initial conditions of the Cauchy problem are provided by fractional derivatives. Caputo presents another alternative preserving the derivative of the constant is null and the initial conditions remain expressed as in the classical case by derivatives of integer order [33, 34, 35].

Through this research, we investigate the global behavior of more real extension's of a basic *SIR* model with memory effects measured by Caputo's fractional order derivative in time. Memory function determined by the nonlocal operator of the fractional derivative highlights the other possibilities not included in the model formulation as fear from infection and the movement in the space, and closing stores, reducing the mobility of persons, and others, which makes fractional systems more realistic to describe the real life situations. Furthermore, we take into consideration the spatial behavior of populations. We show that our system is well posed in the sense that we prove the global existence, uniqueness and boundedness of the solution. By constructing suitable Lyapunov functions, the disease will be eradicated for $R_0 \leq 1$, and persist for $R_0 > 1$.

This research is structured as follows. In Section 2, we remember some basic results for fractional calculus. The proposed model formulation is given in Section 3. Next, we show in Section 4 the existence and uniqueness of a bounded solution. Section 5 is devoted for calculating all possible equilibrium states. The global behavior of the solution is the subject of interest in Section 6. Numerical simulations of the considered model in agreement with theoretical results are illustrated in Section 7.

2. Preliminaries

First, the Mittag–Leffler function, $E_\alpha(z)$, is defined as the family of entire functions of z given as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad \alpha > 0, \quad z \in \mathbb{C},$$

where $\Gamma(\cdot)$ is Gamma function . Observe that the Mittag–Leffler function generalizes the exponential function: $E_1(z) = exp(z)$.

Definition 2.1. *We consider $f \in L^1(\mathbb{R}^+)$. The Riemann–Liouville fractional-integral with $\alpha > 0$ of f is*

$$I^\alpha f(t) = \int_0^t M(t,s) f(s) ds,$$

with $M(t,s) = \dfrac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}$ is a power law function.

Definition 2.2. *We consider $\alpha > 0$, and letting $n \in \mathbb{N}$ verifying $n-1 < \alpha \leq n$. The fractional derivative in sense of Caputo for α for a function $f \in C^n([0,+\infty),\mathbb{R})$ is*

$$ {}_0^C D_t^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, $$

with $D = \dfrac{d}{dt}$ for $n = 1$. In particular, for $0 < \alpha < 1$, we get

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds. $$

For more details about the definition of fractional derivative in sense Caputo , we refer to [33].

3. Model formulation

The first assumption that we will put is the disease is not fatal and there no mortality due to infection. Also, we let Ω a bounded set in $\mathbb{R}$ with smooth boundary $\partial\Omega$, and $[0,T]$ is a finite interval. The classic *SIR* epidemic model governed by following reaction–diffusion equations:

$$ \begin{aligned} \frac{\partial S(t,x)}{\partial t} - \lambda_1 \Delta S(t,x) &= \Lambda - \beta S(t,x) I(t,x) - \mu S(t,x), \\ \frac{\partial I(t,x)}{\partial t} - \lambda_2 \Delta I(t,x) &= \beta S(t,x) I(t,x) - (\mu + r) I(t,x), \\ \frac{\partial R(t,x)}{\partial t} - \lambda_3 \Delta R(t,x) &= r I(t,x) - \mu R(t,x). \end{aligned} \tag{3.1} $$

with $(t,x) \in Q_T = [0,T] \times \Omega$. The total population N is divided into three compartments of the pathological state with $S(t,x)$, $I(t,x)$ and $R(t,x)$ are respectively the densities of the susceptible population, infected population, removed population at time t and the spatial location x. The positive constants Λ, r and μ are respectively the entering flux into S-class, the recovery rate and the natural death rate. Persons in S-class acquire infection after a direct contact with person in I-class, with the rate βSI, where β is the transmission coefficient per unit of time. Positive constants λ_1, λ_2 and λ_3 denote the diffusion coefficients for the S-class, I-class, R-class, respectively. Δ represents the usual Laplacian operator.

In the actual world, and during an epidemic, there are numerous components that varies and can influence the outbreak of a disease and cannot be included in the model formulation, as the fear generated by the population from infection, weather (specific change in the weather help or reduce the spread of disease). This phenomena can be modeled by replacing the ordinary differential derivative by a fractional one, as it is used in understanding many real world phenomena, we cite for instance the researches [36, 37, 38, 9]. For the model (3.1), the state at any time t do not depend on the previous history, which is is a markovian process (memory depends on time and corresponds to a Dirac function $\delta(t,s)$). However, the evolution and control of epidemic processes in human societies cannot be envisaged without effect of memory. To study long-term memory influence on the classical *SIR* model (3.1), we reverse it to an equivalent system of integral equations. We change the function $\delta(t,s)$ by the power law function $M(t,s)$ which shows a slow decay so that the state of the system at early times also contributes to the evolution, afterward applying the fractional Caputo derivative [36, 37, 38]. One of our goals is to study the effect of vaccination on the basic reproduction number. We therefore introduced the term vaccination u into model (3.1). It is assumed that vaccination converts susceptible individuals into the removed class and confers immunity on them.Based

on the above discussion, we introduce the time fractional derivative to the diffusive *SIR* model in the following manner:

$$\begin{aligned}
{}_0^C D_t^\alpha S(t,x) &= \lambda_1 \Delta S(t,x) + \Lambda - \beta S(t,x) I(t,x) - \mu S(t,x) - uS(t,x), \\
{}_0^C D_t^\alpha I(t,x) &= \lambda_2 \Delta I(t,x) + \beta S(t,x) I(t,x) - (\mu + r) I(t,x), \quad (t,x) \in Q_T, \\
{}_0^C D_t^\alpha R(t,x) &= \lambda_3 \Delta R(t,x) + rI(t,x) - \mu R(t,x) + uS(t,x).
\end{aligned} \tag{3.2}$$

We consider that the model (3.2) is self-contained and there is a dynamic across the boundary but there is no emigration. Then the no-flux homogeneous Neumann boundary conditions are

$$\frac{\partial S(t,x)}{\partial \nu} = \frac{\partial I(t,x)}{\partial \nu} = \frac{\partial R(t,x)}{\partial \nu} = 0, \qquad (t,x) \in \Sigma_T = [0,T] \times \partial\Omega. \tag{3.3}$$

For epidemiological aspect, we consider that the following initial conditions of the three classes are positive

$$S(0,x) = S_0, I(0,x) = I_0 \text{ and } R(0,x) = R_0, \quad x \in \Omega. \tag{3.4}$$

The constant u refers to the vaccination rate. It is presumed that the vaccination transforms the persons in S-class to the removed class after acquiring immunity. We denote by ν the outward unit normal vector on the boundary $\partial\Omega$ and by $\frac{\partial}{\partial \nu} = \nu.\nabla$ the normal derivative.

4. Existence and uniqueness of solutions

Letting $\mathbb{X} = C(\bar{\Omega}, \mathbb{R})$ and X^3 be Banach spaces endowed by the uniform norms

$$\|h\|_{\mathbb{X}} = \sup_{x \in \Omega} |h(x)|, \forall h \in \mathbb{X},$$

and

$$\|H\|_{\mathbb{X}^3} = \sup_{x \in \Omega} \|H(x)\|_1, \forall H \in \mathbb{X}^3,$$

with $\|H(x)\|_1 = \sum_{i=1}^3 |H_i(x)|$ is the Manhattan norm [7].

We set $y = (y_1, y_2, y_3)$, $y^0 = (y_1^0, y_2^0, y_3^0)$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and we assume that A is the linear diffusion operator where

$$A : D(A) \subset \mathbb{X}^3 \to \mathbb{X}^3$$

$$Ay = \lambda \Delta J = (\lambda_1 \Delta J_1, \lambda_2 \Delta J_2, \lambda_3 \Delta J_3), \qquad \forall J \in D(A),$$

where

$$D(A) = \left\{ J \in \mathbb{X}^3 : \Delta J \in \mathbb{X}^3, \frac{\partial J}{\partial \nu} = 0_{\mathbb{R}^3} \text{ for } x \in \partial\Omega \right\}.$$

Let the function f defined by $f : [0,T] \times \mathbb{X}^3 \mapsto \mathbb{X}^3$, where

$$f(t, J(t)) := f(J(t)) = (f_1(J(t)), f_2(J(t)), f_3(J(t))),$$

with

$$\begin{cases}
f_1(J(t)) = & \Lambda - \beta J_1 J_2 - \mu J_1 - uJ_1, \\
f_2(J(t)) = & \beta J_1 J_2 - (\mu + r) J_2, \qquad\qquad t \in [0,T], \\
f_3(J(t)) = & rJ_2 - \mu J_3 + uJ_1.
\end{cases}$$

The model can expressed as

$$\begin{cases} {}_0^C D_t^\alpha J = & AJ + f(J(t)), \\ J(0) = & J^0, \end{cases} \tag{4.1}$$

where $J = (S, I, R)$ and $J^0 = (S_0, I_0, R_0)$.

Proposition 4.1. *Let $0 < \alpha \leq 1$, for any $J^0 \in D(A)$, problem* (4.1) *has a unique non-negative solution $J \in C([0,T]; X^3)$ which is*

$$J(t) = \int_0^\infty \Phi_\alpha(\theta) Q(t^\alpha \theta) J^0 d\theta + F(t),$$

where

$$F(t) = \alpha \int_0^t \int_0^\infty \theta (t-\tau)^{\alpha-1} \Phi_\alpha(\theta) Q((t-\tau)^\alpha \theta) f(\tau) d\theta d\tau \tag{4.2}$$

and $\Phi_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$.

Proof. Since A is a linear operator defined on a dense set $D(A)$ in $\mathbb{X}^3$ into itself, then it generates a C_0-semigroup $\{Q(t), t \geq 0\}$ of contractions on $\mathbb{X}^3$. It is known that function f is Lipshitz continuous in y for $t \in [0,T]$ if $y_i \geq 0$ for $i = 1, 2$ and 3. Using [39, Theorem 3.1], we have the existence and uniqueness results. ■

It remains to show that the solution is bounded. By summing the three equations of (3.2), we get

$$\begin{aligned} {}_0^C D_t^\alpha S(t,x) + {}_0^C D_t^\alpha I(t,x) + {}_0^C D_t^\alpha R(t,x) = \lambda_1 \Delta S(t,x) + \lambda_2 \Delta I(t,x) + \lambda_3 \Delta R(t,x) \\ + \Lambda - \mu(S(t,x) + I(t,x) + R(t,x)). \end{aligned}$$

Integrating in Ω the two sides of the above equality, we have

$$\begin{aligned} \int_\Omega \Big({}_0^C D_t^\alpha S(t,x) + {}_0^C D_t^\alpha I(t,x) + {}_0^C D_t^\alpha R(t,x) \Big) dx = \int_\Omega \Big(\lambda_1 \Delta S(t,x) + \lambda_2 \Delta I(t,x) \\ + \lambda_3 \Delta R(t,x) \Big) dx + \int_\Omega \Big(\Lambda - \mu(S(t,x) + I(t,x) + R(t,x)) \Big) dx. \end{aligned}$$

Applying Green's formula and using the homogeneous Neumann boundary conditions (3.3) we get

$$\begin{aligned} &\int_\Omega \Big({}_0^C D_t^\alpha S(t,x) + {}_0^C D_t^\alpha I(t,x) + {}_0^C D_t^\alpha R(t,x) \Big) dx \\ &= \int_\Omega \Big(\Lambda - \mu(S(t,x) + I(t,x) + R(t,x)) \Big) dx \\ &= \Lambda |\Omega| - \mu \int_\Omega \Big(S(t,x) + I(t,x) + R(t,x) \Big) dx. \end{aligned}$$

Note that

$$\int_\Omega \Big(S(t,x) + I(t,x) + R(t,x) \Big) dx = N(t).$$

Due to the property of the linearity of the Caputo's operator and Fubini's Theorem, one have

$${}_0^C D_t^\alpha N(t) = \Lambda |\Omega| - \mu N(t).$$

Solving this equality by using Laplace's transform, we obtain

$$N(t) = N(0) E_\alpha(-\mu t^\alpha) + \frac{\Lambda}{\mu}(1 - E_\alpha(-\mu t^\alpha)).$$

Due to $0 \leq E_\alpha(-\mu t^\alpha) \leq 1$, we conclude that $N(t) \leq N(0) + \frac{\Lambda}{\mu}$ hence the solution is bounded.

Remark 4.2. *Owing to [40, Theorem 3.1], then* (3.2)-(3.4) *has a unique solution which is non-negative and bounded.*

Noting that the two first equations does not depend on the class $R(t,x)$ and then are uncoupled with the last equation of the system (3.2). Hence our attention will concentrated on the analysis of the following reduced system: for $(t,x) \in [0,\infty) \times \Omega$.

$$\begin{aligned} {}_0^C D_t^\alpha S(t,x) &= \lambda_1 \Delta S(t,x) + \Lambda - \beta S(t,x) I(t,x) - \mu S(t,x) - u S(t,x), \\ {}_0^C D_t^\alpha I(t,x) &= \lambda_2 \Delta I(t,x) + \beta S(t,x) I(t,x) - (\mu + r) I(t,x). \end{aligned} \tag{4.3}$$

5. Existence of equilibria

The principal goal of this section is to determine the equilibria for (4.3). A crucial idea in epidemiology is the existence of significant threshold values quantifying and measuring an outbreak spread in a population. The given value

$$R_0 = \frac{\beta \Lambda}{(\mu + u)(\mu + r)}$$

is the basic reproduction number [41]. From the definition of R_0, we conclude the following results.

Theorem 5.1. *(i) There is always a disease-free equilibrium state denoted $E_f = (S_f, 0)$, with $S_f = \dfrac{\Lambda}{\mu + u}$.*

(ii) If $R_0 > 1$, there exists one endemic equilibrium state denoted $E^* = (S^*, I^*)$, with

$$S^* = \frac{\mu + r}{\beta} \text{ and } I^* = \frac{\mu + u}{\beta}(R_0 - 1).$$

Proof. (i) By a straightforward computation, we get E_f is a steady state of (4.3) which always exists.
(ii) To get the other equilibrium, we need to solve (4.3) for $(S,I) = (S^*, I^*)$. We then obtain $S^* = \frac{\mu+r}{\beta}$ *and* $I^* = \frac{\mu+u}{\beta}(R_0 - 1)$. Hence, if $R_0 > 1$, there exist a unique positive solution which is E^*. ■

6. Global Stability

Our goal now is to study the global behavior for E_f and E^* using Lyapunov's functions.At first we put the following lemma

Lemma 6.1 ([42]). *We put $y(t) \in \mathbb{R}_+^*$ be a continuous and derivable function. For all $\alpha \in (0,1)$ and for $t \geq t_0$*

$${}_{t_0}^C D_t^\alpha \left[y^* \Psi\left(\frac{y(t)}{y^*} \right) \right] \leq \left(1 - \frac{y^*}{y(t)} \right) {}_{t_0}^C D_t^\alpha y(t), \, y^* \in \mathbb{R}_+^*,$$

where Ψ is a positive function defined by $\Psi(y) = -\ln(y) + y - 1$, $y > 0$.

Theorem 6.2. *E_f is globally asymptotically stable for $R_0 \leq 1$.*

Proof. Putting:

$$V(t) = \int_\Omega \left(S_f \Psi\left(\frac{S(t,x)}{S_f} \right) + I(t,x) \right) dx.$$

Calculating the fractional derivative of V in Caputo's sense, we have

$$\begin{aligned} {}_0^C D_t^\alpha V(t) &\leq \int_\Omega \left(\left(1 - \frac{S_f}{S(t,x)} \right) {}_0^C D_t^\alpha S(t,x) + {}_0^C D_t^\alpha I(t,x) \right) dx \\ &\leq \int_\Omega \left(\left(1 - \frac{S_f}{S(t,x)} \right) \left(\Lambda - \beta S(t,x) I(t,x) - (\mu + u) S(t,x) \right) \right. \\ &\quad \left. + \beta S(t,x) I(t,x) - (\mu + r) I(t,x) \right) dx \\ &\quad + \int_\Omega \left(\lambda_1 \Delta S(t,x) - \lambda_1 \frac{S_f}{S(t,x)} \Delta S(t,x) + \lambda_2 \Delta I(t,x) \right) dx. \end{aligned}$$

Since $\Lambda = (\mu+u)S_f$, then

$$
\begin{aligned}
{}_0^C D_t^\alpha V(t) \leq \int_\Omega \Bigg(& \big(1-\frac{S_f}{S(t,x)}\big)\big((\mu+u)S_f-(\mu+u)S(t,x)\big) \\
& + \frac{\beta\Lambda}{\mu+u} I(t,x) - (\mu+r)I(t,x)\Bigg) dx \\
& + \int_\Omega \left(\lambda_1 \Delta S(t,x) - \lambda_1 \frac{S_f}{S(t,x)}\Delta S(t,x) + \lambda_2 \Delta I(t,x)\right) dx.
\end{aligned}
$$

Applying Green's formula, we get

$$
\begin{aligned}
{}_0^C D_t^\alpha V(t) \leq & -(\mu+u)\int_\Omega \frac{(S(t,x)-S_f)^2}{S(t,x)}dx + (\mu+r)\int_\Omega (R_0-1)I(t,x)dx \\
& - \lambda_1 S_f \int_\Omega \frac{|\nabla S(t,x)|^2}{S^2(t,x)}dx.
\end{aligned}
$$

For $R_0 \leq 1$, we deduce that ${}_0^C D_t^\alpha V(t) \leq 0$. In addition ${}_0^C D_t^\alpha V(t) = 0$ is equivalent to $S = S_f$ and $(R_0-1)I = 0$. Then the following two cases arise:

- If $R_0 < 1$, then $I = 0$.
- If $R_0 = 1$, using the first eq. of (4.3) together with $S = S_f$, we get

$$\Lambda - (\mu+u)S_f - \beta S_f I(t,x) = 0\,,$$

  then $\beta S_f I(t,x) = 0$. Thus, we obtain $I = 0$.

Hence, the largest invariant set of $\big\{(S,I) \in \mathbb{R}_+^2 : {}_0^C D_t^\alpha V(t) = 0\big\}$ is the singleton $\{E_f\}$. Using LaSalle's invariance principle [43], we conclude that E_f is globally asymptotically stable. ■

Similarly, we shall show global stability of E^* which is resumed in the following theorem

Theorem 6.3. *E^* is globally asymptotically stable whenever exists.*

Proof. We consider:

$$V(t) = \int_\Omega \left(S^*\Psi\left(\frac{S(t,x)}{S^*}\right) + I^*\Psi\left(\frac{I(t,x)}{I^*}\right)\right)dx.$$

We have

$$
\begin{aligned}
{}_0^C D_t^\alpha V(t) & \leq \left(1-\frac{S^*}{S(t,x)}\right){}_0^C D_t^\alpha S(t,x) + \left(1-\frac{I^*}{I(t,x)}\right){}_0^C D_t^\alpha I(t,x) \\
& \leq \int_\Omega \Bigg(\big(1-\frac{S^*}{S(t,x)}\big)\big(\Lambda - \beta S(t,x)I(t,x) - (\mu+u)S(t,x)\big) \\
& \quad + \big(1-\frac{I^*}{I(t,x)}\big)\big(\beta S(t,x)I(t,x) - (\mu+r)I(t,x)\big)\Bigg)dx + \int_\Omega \Bigg(\lambda_1 \Delta S(t,x) \\
& \quad - \lambda_1 \frac{S^*}{S(t,x)}\Delta S(t,x) + \lambda_2 \Delta I(t,x) - \lambda_2 \frac{I^*}{I(t,x)}\Delta I(t,x)\Bigg)dx.
\end{aligned}
$$

Note that $\mu+r=\beta S^*$, $\Lambda=(\mu+u)S^*+(\mu+r)I^*$. Applying then Green's formula, we obtain

$$
\begin{aligned}
{}_0^C D_t^\alpha V(t) &\leq -(\mu+u)\int_\Omega \frac{(S(t,x)-S^*)^2}{S(t,x)}dx+\int_\Omega\Bigg(2(\mu+r)I^*-(\mu+r)I^*\frac{S^*}{S(t,x)}\\
&-(\mu+r)I^*\frac{S(t,x)}{S^*}\Bigg)dx-\lambda_1 S^*\int_\Omega\frac{|\nabla S(t,x)|^2}{S^2(t,x)}dx-\lambda_2 I^*\int_\Omega\frac{|\nabla I(t,x)|^2}{I^2(t,x)}dx\\
&\leq -(\mu+u)\int_\Omega \frac{(S(t,x)-S^*)^2}{S(t,x)}dx+\int_\Omega(\mu+r)I^*\Bigg(2-\frac{S^*}{S(t,x)}\\
&-\frac{S(t,x)}{S^*}\Bigg)dx-\lambda_1 S^*\int_\Omega\frac{|\nabla S(t,x)|^2}{S^2(t,x)}dx-\lambda_2 I^*\int_\Omega\frac{|\nabla I(t,x)|^2}{I^2(t,x)}dx\\
&\leq -(\mu+u)\int_\Omega \frac{(S(t,x)-S^*)^2}{S(t,x)}dx-(\mu+r)I^*\int_\Omega\Psi\left(\frac{S^*}{S(t,x)}\right)dx\\
&-(\mu+r)I^*\int_\Omega\Psi\left(\frac{S(t,x)}{S^*}\right)dx-\lambda_1 S^*\int_\Omega\frac{|\nabla S(t,x)|^2}{S^2(t,x)}dx\\
&-\lambda_2 I^*\int_\Omega\frac{|\nabla I(t,x)|^2}{I^2(t,x)}dx.
\end{aligned}
$$

Since $\Psi(y)\geq 0$, then $D^\alpha V(t)\leq 0$ for $R_0>0$. Furthermore, the largest invariant set that verifies $\{(S,I)\in\mathbb{R}_+^2 : D^\alpha V(t)=0\}$ is $\{E^*\}$. Using LaSalle's we achieve the desired result. ■

7. Graphical representation

In this section, we present some graphical illustrations confirming our theoretical findings. The system (3.2)-(3.4) is numerically integrated by using the forward finite difference approximations to discretize the time-fractional derivative [44] and the centered finite difference schemes to approach the Laplacian's operator in one-dimensional space. This method gives an accurate of order $2-\alpha$ in time and order 2 in space.

First, we study the case without vaccination. We simulate (3.2)-(3.4) with the values: $\mu=0.8$, $\Lambda=0.9$, $\beta=0.1$, $r=0.02$, $u=0.0$, $\lambda_1=\lambda_2=\lambda_3=0.2$, and the initial conditions $S(0,x)=1.0$, $I(0,x)=2.0$ and $R(0,x)=3.0$. As a result we approximate the solutions of (3.2)-(3.4) for $\alpha=1$, $\alpha=0.8$ and $\alpha=0.6$ that displayed respectively in Figures 1, 2 and 3. We also calculate $R_0=0.1372$. Hence, (3.2)-(3.4) $E_f=(1.12,0,0)$ is unique. Using Theorem 6.2, E_f is globally stable. Finally, we notice that all the solutions are globally asymptotically stable for different values of α not just for $\alpha=1$. We also notice that the solution for $\alpha=1$ quickly converges to the equilibrium point E_f, and since fractional derivatives describe reality well, we can say that the epidemic may take longer to be stable, and this is very important in terms of economics and the study of control strategies.

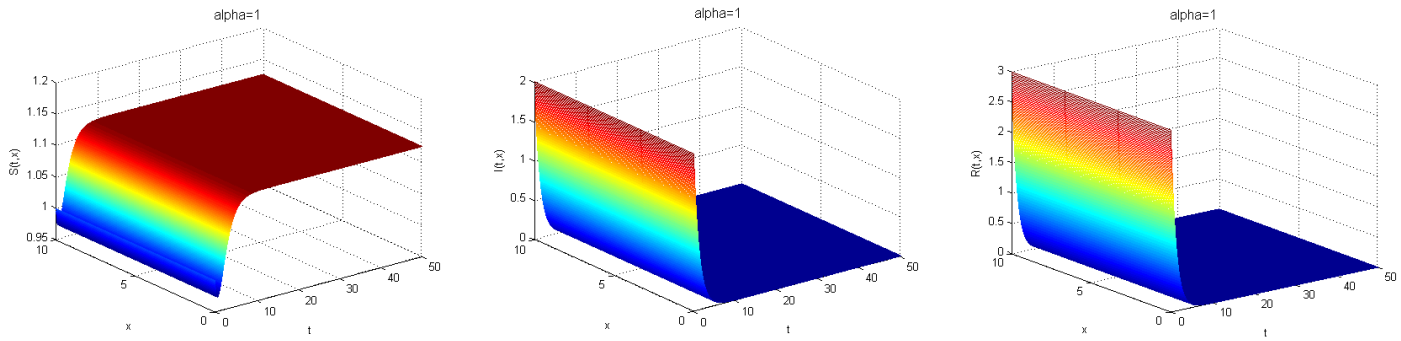

Figure 1: Dynamics of the system (3.2) for $\alpha=1$

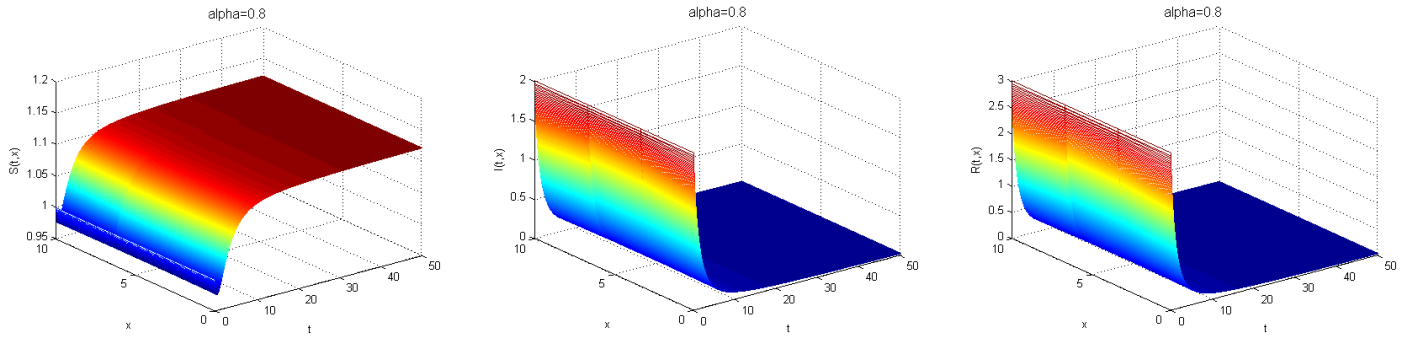

Figure 2: Global proprieties of (3.2) for $\alpha = 0.8$.

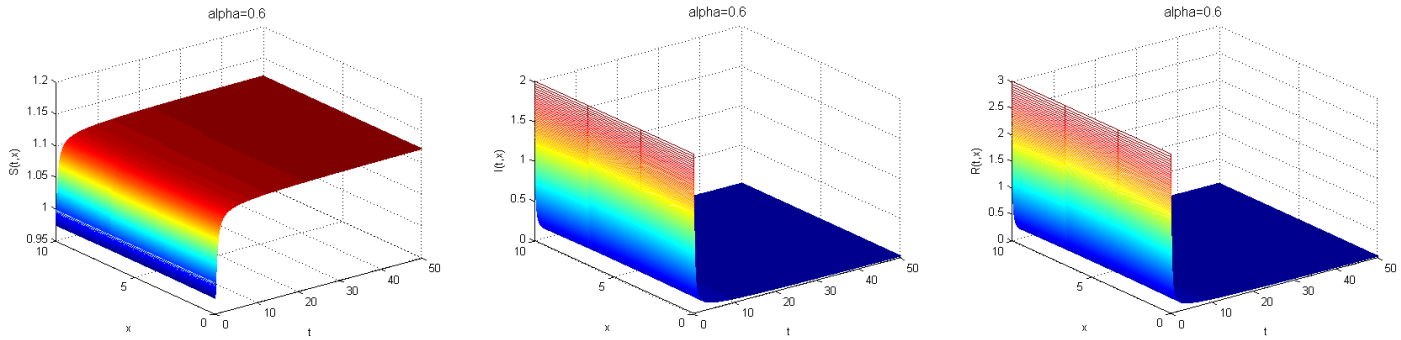

Figure 3: Global proprieties of (3.2) for $\alpha = 0.6$.

Now, we consider $\mu = 0.2$ and letting the same previous set of parameter. Then, $R_0 = 2.0455$. From Theorem 6.3, E^* is globally asymptotically stable. Figures 4, 5 and 6 illustrate this result for different values of α, which means biologically that the infection persists but it is under control.

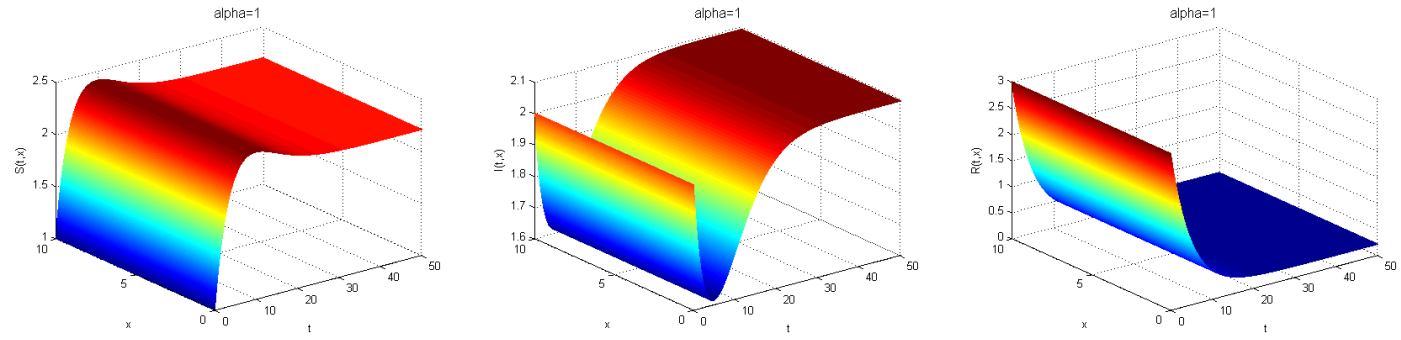

Figure 4: Global proprieties of (3.2) for $\alpha = 1$.

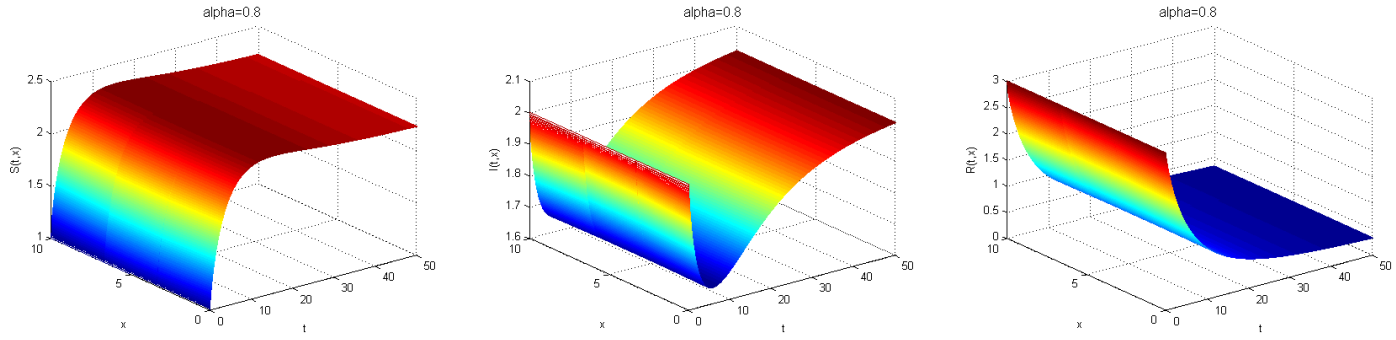

Figure 5: Global proprieties of (3.2) for $\alpha = 0.8$.

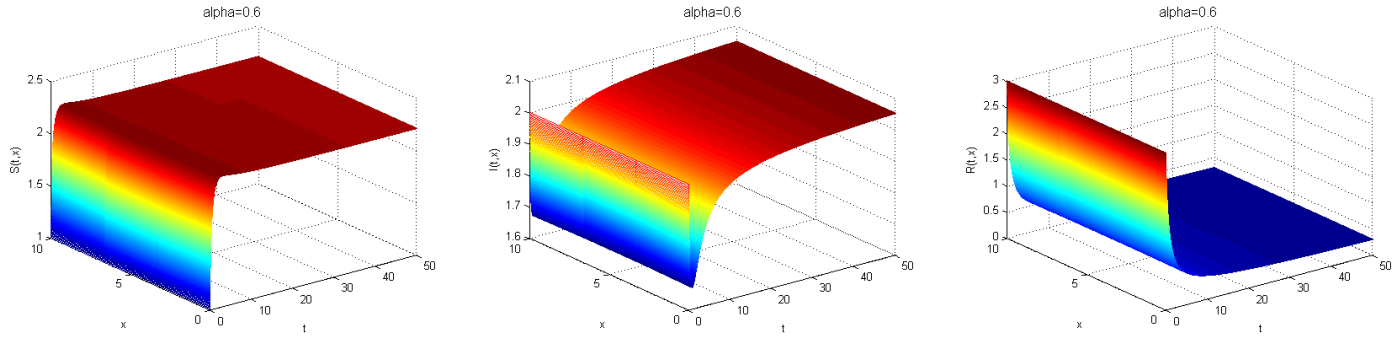

Figure 6: Global proprieties of (3.2) for $\alpha = 0.6$.

Next, we determine the influence of the perfect vaccine on R_0. We are only interested in the endemic case for $\alpha = 0.8$. We note that the basic reproduction rate decreases when the vaccine rate increases (for example $u = 0.8$ we have $R_0 = 0.4091$ (see Figure 7)). Consequently, the density of infected persons is also decreasing. Besides, the number of removed persons increases at the expense of susceptible people (see Figure 8). It reflects the importance of the vaccine to eradicate the disease.

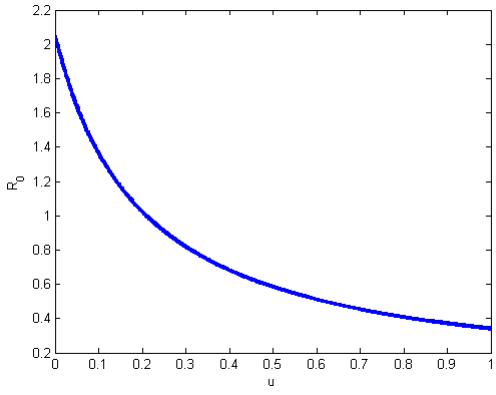

Figure 7: Variation of R_0 according to the vaccine effect.

8. Conclusion

We dealt in this research with the qualitative analysis of the solutions for a reaction-diffusion system under the influence of the fractional derivative α. Taking advantage of the Lyapunov function method we have shown that R_0 plays an crucial role in determining the global threshold of the system. We have established the global stability of the two equilibria: E_f and E^* for different values of α.

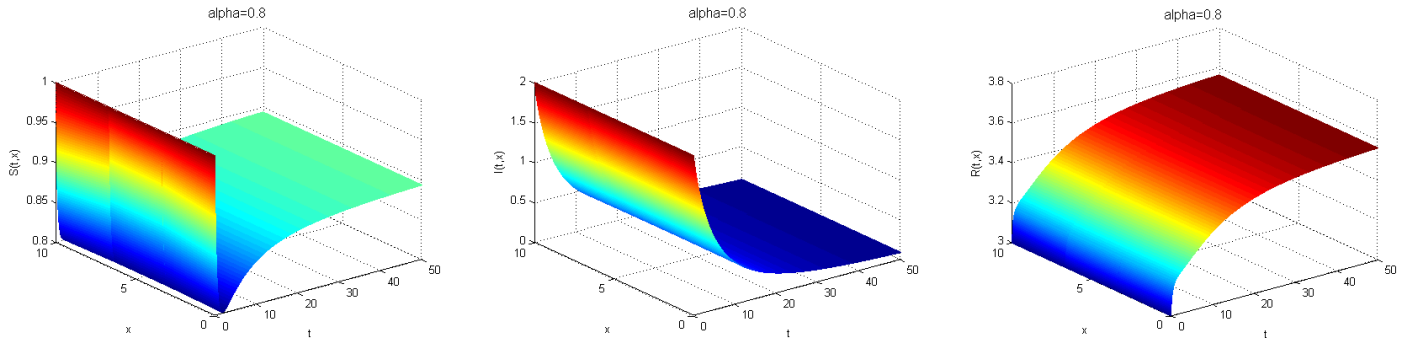

Figure 8: Behavior of the solution for $\alpha = 0.8$ under the effect of vaccine ($u = 0.8$).

From epidemiological point of view, this means that the infection will eradicated or persisted while respecting certain restrictions on the parameters. According to our theoretical analysis, we obtained the stability of the equilibria not only for the integer derivative ($\alpha = 1$) but also for all $0 < \alpha \leq 1$, which confirms the generality of our system. In addition, fractional derivatives have provided other means of predicting the progression of the disease and, in some cases, affecting the time required to reach stable states. Our future work is to control the vaccination term u to get a better optimal strategy with other fractional derivatives having a non singular kernel. Also, to mention that the main assumption is to consider that the diffusion coefficients are different which shows the possibility of having Turing instability. Our paper proves that it is not possible to have that the system undergoes Turing instability and have threshold results determined by comparing R_0 with 1, where for $R_0 < 1$ we get the stability of DFE, and for $R_0 > 1$ we get the global stability of the endemic equilibrium.

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