Impulse Controllability Problem for the Equation of a String with an External Load on a Rectangle $QT = [0 \leq x \leq \pi] \times [0 \leq t \leq T]$.

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Research Article

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Impulse Controllability Problem for the Equation of a String with an External Load on a Rectangle $Q_T = [0 \leq x \leq \pi] \times [0 \leq t \leq T]$.

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Abstract. In this paper necessary and sufficient conditions for impulse null-controllability and approximate null-controllability are obtained for the controllability problem for the equation of the string on a rectangle. This article discusses control that is carried out using an external load. Controls solving these problems are found explicitly.

Key words. wave equation, rectangular, Dirac function, controllability problem, control by the external load, Fourier transform, function with bounded variation

0 Introduction

Controllability problems for partial differential equations are being investigated nowadays by a number of mathematicians (to illustrate the examples, given in [1–5], and references [3–4]).

Foundations of systematic control optimal problems for partial differential equations were put in the work by J.-L. Lions [1].

V.A. Il'in [2, 3] has considered bounded control of oscillations on two ends and single end, if the other end were fixed, in the terms of general solution of wave equation with fixed power.

In [4], the conditions ϵ –controllability and null-controllability in the space L^2 were considered and investigated for a series of control problems in partial differential equations.

But in some cases the controls introduced in the boundary conditions can not be realized from a practical point of view. In the article we build the controls introduced in the function of an external load, which solve the problem of approximately null-controllability and null-controllability.

The study of the controllability problem is carried out for the wave equation:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + g(x, t), \quad x \in (0, \pi), \quad t \in (0, T), \quad T > \pi, \quad (0.1)$$

$$g(x, t) = \sum_{j=1}^{n-1} \delta(x - x_j) \sum_k \mathcal{T}_{2Tk} g_j^n(t). \quad (0.2)$$

Here $g : [0, \pi] \times [0, T] \rightarrow \mathbb{R}$ is a control. Where $\delta(x)$ is the Dirac function, $\mathcal{T}_h$ is the translation operator: $(\mathcal{T}_h \varphi)(x) = \varphi(x + h, t)$. When $x_j = \frac{\pi j}{n}$, $j = \overline{1, n}$, $n \in \mathbb{N}$, $t \in (0, T)$, $T > \pi$ and $g_j^n(t)$ is a control, $j = \overline{1, n}$, $n \in \mathbb{N}$.

The initial conditions:

$$w(x, 0) = \partial w(x, 0)/\partial t = 0, \quad x \in (0, \pi), \quad t \in (0, T), \quad (0.3)$$

and Dirichlet boundary conditions:

$$w(0, t) = w(\pi, t) = 0, \quad t \in (0, T). \quad (0.4)$$

Current state of the system (0.1)–(0.4)

$$\begin{cases} w(x, t); \\ \partial w(x, t)/\partial t; \end{cases} \quad x \in (0, \pi), \quad t \in (0, T), \quad T > \pi. \quad (0.5)$$

Let us consider the spaces used in the given work.

Let us give the bijection $f : D \mapsto f(D)$, where $D = [a_1, b_1] \times [a_2, b_2]$ – rectangle, $x_1 \in [a_1, b_1]$, $x_2 \in [a_2, b_2]$. Π is the arbitrary partition of the rectangle D by straight lines $x_1 = x_{1_i}$, $x_2 = x_{2_j}$, as that $x_{1_i} < x_{1_{i+1}}$, $x_{2_j} < x_{2_{j+1}}$.

Let us introduce the space

$$V(D) = \{\phi(x_1, x_2) \mid x_1 \in [a_1, b_1], \quad x_2 \in [a_2, b_2] \quad \exists C > 0, \forall n, K \in \mathbb{N} :$$

$$\sum_{i=1}^{K-1} \sum_{j=1}^{n-1} |\phi(x_{1_{i+1}}, x_{2_{j+1}}) - \phi(x_{1_i}, x_{2_{j+1}}) - \phi(x_{1_{i+1}}, x_{2_j}) + \phi(x_{1_i}, x_{2_j})| \leq C\},$$

where $\phi : (x_1, x_2) \mapsto \phi(x_1, x_2)$ is bijection.

Then the function $f(x_1, x_2)$, defined on the rectangle D and belonging to the space $V(D)$, according to [12], is called a function of bounded variation. The exact upper bound of the sums

$\sum_{i=1}^{K-1} \sum_{j=1}^{n-1} |f(x_{1_{i+1}}, x_{2_{j+1}}) - f(x_{1_i}, x_{2_{j+1}}) - f(x_{1_{i+1}}, x_{2_j}) + f(x_{1_i}, x_{2_j})|$ over all possible finite partitions Π is called the complete variation of the function f on rectangle D and denote

$$V[f, D] = \sup_D \sum_{i=1}^{K-1} \sum_{j=1}^{n-1} |f(x_{1_{i+1}}, x_{2_{j+1}}) - f(x_{1_i}, x_{2_{j+1}}) - f(x_{1_{i+1}}, x_{2_j}) + f(x_{1_i}, x_{2_j})|.$$

Denote by Q_T the rectangular: $Q_T = [0 \leq x \leq \pi] \times [0 \leq t \leq T]$.

Remark 1. Solutions of the wave equation are functions of two variables, each of which having a kind of wave, that is, it has the form $f(x \pm t)$. Note, if $f(t) \in V(0, T)$ then $f(x \pm t) \in V(Q_T)$. And vice versa.

We assume that the control function in (0.2) is chosen from the space $V(0, T)$, that is

$$g_j^n(t) \in V(0, T). \quad (0.6)$$

Denote by $\mathcal{F}$ the Fourier transform operator

$$(\mathcal{F}\varphi)(\sigma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\sigma x} \varphi(x) dx,$$

$$(\mathcal{F}f, \varphi) = (f, \mathcal{F}^{-1}\varphi).$$

In section 1 the criterion of approximately null-controllability for system (0.1)–(0.3) with the restriction of control (0.6) is obtained. The controls solving these problems are found explicitly. In section 2 the criterion of null-controllability was obtained.

1 The controllability criteria definition.

Denote

$$g_{j,r}^{n,K}(t) = g_j^n(t) \left(H\left(t - \frac{T(r-1)}{K}\right) - H\left(t - \frac{Tr}{K}\right) \right), \quad r = \overline{1, K}, \quad K \in \mathbb{N}, \quad (1.1)$$

where H is the Heaviside function: $H(t) = 1$, if $t > 0$ and $H(t) = 0$, if $t \leq 0$.

Then

$$\text{supp } g_{j,r}^{n,K}(t) \in \left[\frac{T(r-1)}{K}, \frac{Tr}{K} \right], \quad r = \overline{1, K}, \quad K \in \mathbb{N}.$$

Let $T > \pi$. The problem (0.1)–(0.4) has a single solution $w(x, t)$ in the space $V(Q_T)$. Denote by $\mathcal{R}_n^K(w)$ the set of such states $w \in V(Q_T)$, for which there are controls $g_j^n = \sum_r^K g_{j,r}^{n,K}(t) \in V(0, T)$, $j = \overline{1, n}$, $n \in \mathbb{N}$, $r = \overline{1, K}$, $K \in \mathbb{N}$.

DEFINITION 1.1. A system (0.1)–(0.4) is called approximately null-controllable at the time $T > \pi$, if 0 belongs to the closure of $\mathcal{R}(w) = \bigcup_{n, K \in \mathbb{N}} \mathcal{R}_n^K(w)$ and null-controllable if 0 belongs to $\mathcal{R}(w)$ in $V(Q_T)$.

Denote by OE odd-extension operator with respect to x , that is, to say, $(OEf)(x) = \frac{1}{2}(f(x) - f(-x)) + \operatorname{sgn}(x)\frac{1}{2}(f(x) + f(-x))$, and EE is even-extension operator, $(EEf)(x) = \frac{1}{2}(f(x) + f(-x)) + \operatorname{sgn}(x)\frac{1}{2}(f(x) - f(-x))$.

Let us, that $\operatorname{supp} \left(\begin{array}{c} w(x, t) \\ \partial w(x, t)/\partial t \end{array} \right) \in [0, \pi]$. Introduce odd 2π -periodic extension with respect to x function $w(x, t) \in V(Q_T)$:

$$w(x, t) = \sum_{k \in \mathbb{Z}} \mathcal{T}_{2\pi k} OE \left(\begin{array}{c} w(x, t) \\ \partial w(x, t)/\partial t \end{array} \right) \in V(Q_T),$$

where in this we have the property of $(\mathcal{T}_h f, \varphi) = (f, \mathcal{T}_{-h} \varphi)$.

It is easy to see that the controllable system (0.1)–(0.6) equivalent to the following problem:

$$\begin{aligned} \frac{\partial w}{\partial t} &= \begin{pmatrix} 0 & 1 \\ \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} w - \sum_{k \in \mathbb{Z}} \mathcal{T}_{2Tk} \mathcal{T}_{2\pi k} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sum_{j=1}^{n-1} \delta(x - x_j) \sum_{r=1}^K g_{j,r}^{n,K}(t), \quad n \in \mathbb{N}, \quad t \in (0, T), \\ w(x, 0) &= 0. \end{aligned} \quad (1.2)$$

Applying the Fourier transform with respect to x to the system (1.2), we obtain the following Cauchy problem in space $V(Q_T)$:

$$\begin{aligned} \frac{\partial v(\sigma, t)}{\partial t} &= \begin{bmatrix} 0 & 1 \\ -\sigma^2 & 0 \end{bmatrix} v(\sigma, t) + \\ + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \exp(2\pi k \sigma i) \sum_{j=1}^n \left(\exp\left(-\frac{\pi}{n} j \sigma i\right) - \exp\left(\frac{\pi}{n} j \sigma i\right) \right) \mathcal{T}_{2Tk} \sum_{r=1}^K g_{j,r}^{n,K}(t), \quad t \in (0, T), \end{aligned} \quad (1.3)$$

$$v(\sigma, 0) = 0, \quad (1.4)$$

where $v(\sigma, t) = \mathcal{F}w(\sigma, t) \in V(0 \leq t \leq T)$, $\sigma \in \mathbb{R}$.

Thus the function

$$v(\sigma, t) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \exp(2\pi k \sigma i) \int_0^t \Omega(\sigma, t - \tau) \left[\sum_{j=1}^n \left(\exp\left(-\frac{\pi}{n} j \sigma i\right) - \exp\left(\frac{\pi}{n} j \sigma i\right) \right) \mathcal{T}_{2Tk} \sum_{r=1}^K g_{j,r}^{n,K}(\tau) \right] d\tau, \quad (1.5)$$

where

$$\Omega(\sigma, t) = \frac{1}{2} \begin{bmatrix} \exp(i\sigma t) + \exp(-i\sigma t) & (\exp(i\sigma t) - \exp(-i\sigma t)) / (i\sigma) \\ i\sigma (\exp(i\sigma t) - \exp(-i\sigma t)) & \exp(i\sigma t) + \exp(-i\sigma t) \end{bmatrix}$$

is a solution of (1.3), (1.4).

Then

$$w(x, t) = \sum_{k \in \mathbb{Z}} \mathcal{T}_{2\pi k} \int_0^t S(t - \tau) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sum_{j=1}^{n-1} \delta(x - x_j) \mathcal{T}_{2Tk} \sum_{r=1}^K g_{j,r}^{n,K}(\tau) d\tau \quad (1.6)$$

is a unique solution of problem (1.2) in $V(Q_T)$, where

$$S(t) = \frac{1}{2} \begin{bmatrix} \mathcal{T}_t + \mathcal{T}_{-t} & (\partial/\partial x)^{-1}(\mathcal{T}_t - \mathcal{T}_{-t}) \\ (\partial/\partial x)(\mathcal{T}_t - \mathcal{T}_{-t}) & \mathcal{T}_t + \mathcal{T}_{-t} \end{bmatrix},$$

$(\partial/\partial x)^{-1}$ – operator reverse of $\partial/\partial x$.

Or

$$w(x, t) = S(t) \left[\sum_{k \in \mathbb{Z}} \mathcal{T}_{2\pi k} \int_0^t S(-\tau) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sum_{j=1}^{n-1} \delta(x - x_j) \mathcal{T}_{2Tk} \sum_{r=1}^K g_{j,r}^{n,K}(\tau) d\tau \right]. \quad (1.7)$$

The system determined by the state w , defined by the set $\mathcal{R}(w)$ of the space $V(Q_T)$:

$$\mathcal{R}(w) = \bigcup_{n, K \in \mathbb{N}} \mathcal{R}_K^n(w),$$

where

$$\mathcal{R}_K^n(w) = \{R(w) \mid T > \pi \wedge g_j^n \in V(0, T)\},$$

$$R(w) = \sum_{k \in \mathbb{Z}} \mathcal{T}_{2\pi k} \int_0^t S(-\tau) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sum_{j=1}^{n-1} \delta(x - x_j) \mathcal{T}_{2Tk} \sum_{r=1}^K g_{j,r}^{n,K}(\tau) d\tau - S^{-1}(t)w(x, t),$$

$$t \in (0, T).$$

Hence, the definition 1.1 is equivalent to the following definition.

DEFINITION 1.2. The state $w = OEw$ of system (0.1)–(0.4) is called approximately null-controllable at the time $T > \pi$, if 0 belongs to the closure of $\mathcal{R}(w)$, and null-controllable if 0 belongs to $\mathcal{R}(w)$ in $V(Q_T)$.

The expression $S^{-1}(t)w(x, t)$, characterizing the system (0.1)–(0.4), is represented as

$$S^{-1}(t)w(x, t) = \frac{1}{2} \begin{bmatrix} (\mathcal{T}_{-t} + \mathcal{T}_t)w(x, t) + (\partial/\partial x)^{-1}(\mathcal{T}_{-t} - \mathcal{T}_t)(\partial/\partial t)w(x, t) \\ (\partial/\partial x)(\mathcal{T}_{-t} - \mathcal{T}_t)w(x, t) + (\mathcal{T}_{-t} + \mathcal{T}_t)(\partial/\partial t)w(x, t) \end{bmatrix}.$$

Consider the function

$$M(x, t) = \frac{1}{2} ((\partial/\partial x)(\mathcal{T}_{-t} + \mathcal{T}_t)w(x, t) + (\mathcal{T}_{-t} - \mathcal{T}_t)(\partial/\partial t)w(x, t)).$$

THEOREM 1.1. A system (0.1)–(0.4) is approximately null-controllable at the time $T > \pi$ (and state w is approximately null-controllable), if function

$$w(x, t) \in V(Q_T). \quad (1.8)$$

Besides if

$$\begin{aligned} m_{j,r}^{n,K}(x, t) &= M(x, t) \left[H \left(x - \frac{\pi(j-1)}{n} \right) - H \left(x - \frac{\pi j}{n} \right) \right] \left[H \left(t - \frac{T(r-1)}{K} \right) - H \left(t - \frac{T r}{K} \right) \right], \\ j &= \overline{1, n-1}, \quad r = \overline{1, K-1}, \\ g_{2j,2r}^{n,K}(t) &= -g_{2j,1}^{n,K} \left[-t - (2r-2)\frac{T}{K} \right] + \sum_{l=1}^r g_{1,2r-2l+1}^{n,K} \left[-t - (2j-2)\frac{\pi}{n} - (2l-1)\frac{T}{K} \right] + \\ &+ \sum_{l=1}^r \sum_{r=1}^j m_{2j-2r+1,2r-2l+1}^{n,K} \left[x - (2r-1)\frac{\pi}{n}, \quad t - (2l-1)\frac{T}{K} \right], \quad j = \overline{1, \left[\frac{n}{2} \right]}, \quad r = \overline{1, \left[\frac{K}{2} \right]}, \\ g_{2j+1,2r}^{n,K}(t) &= -g_{2j+1,1}^{n,K} \left[-t - (2r-2)\frac{T}{K} \right] - \sum_{l=1}^r g_{1,2r-2l+1}^{n,K} \left[t - (2j-1)\frac{\pi}{n} - (2l-1)\frac{T}{K} \right] + \\ &+ \sum_{l=1}^r \sum_{r=1}^j m_{2j-2r+2,2r-2l+1}^{n,K} \left[x - (2r-1)\frac{\pi}{n}, \quad t - (2l-1)\frac{T}{K} \right], \quad j = \overline{1, \left[\frac{n-1}{2} \right]}, \quad r = \overline{1, \left[\frac{K-1}{2} \right]}, \\ g_{2j,2r+1}^{n,K}(t) &= g_{2j,1}^{n,K} \left[t - (2r-1)\frac{T}{K} \right] + \sum_{l=1}^r g_{1,2r-2l+2}^{n,K} \left[-t - (2j-2)\frac{\pi}{n} - (2l-1)\frac{T}{K} \right] + \\ &+ \sum_{l=1}^r \sum_{r=1}^j m_{2j-2r+1,2r-2l+2}^{n,K} \left[x - (2r-1)\frac{\pi}{n}, \quad t - (2l-1)\frac{T}{K} \right], \quad j = \overline{1, \left[\frac{n}{2} \right]}, \quad r = \overline{1, \left[\frac{K-1}{2} \right]}, \end{aligned}$$

$$\begin{aligned}
g_{2j+1,2r+1}^{n,K}(t) &= g_{2j+1,1}^{n,K} \left[t - (2r-1) \frac{T}{K} \right] - \sum_{l=1}^r g_{1,2r-2l+2}^{n,K} \left[t - (2j-1) \frac{\pi}{n} - (2l-1) \frac{T}{K} \right] + \\
&+ \sum_{l=1}^i \sum_{r=1}^j m_{2j-2r+2,2r-2l+2}^{n,K} \left[x - (2r-1) \frac{\pi}{n}, t - (2l-1) \frac{T}{K} \right], \\
j &= 1, \left\lceil \frac{n-1}{2} \right\rceil, \quad r = 1, \left\lceil \frac{K-1}{2} \right\rceil,
\end{aligned} \tag{1.9}$$

where for even n

$$\begin{aligned}
g_{1,2r}^{n,K}(t) &= -g_{n,2r}^{n,K} \left[-t - (n-2) \frac{\pi}{n} \right] + \\
&+ \sum_{l=1}^{\left\lceil \frac{n}{2} \right\rceil} g_{n-2l+1,1}^{n,K} \left[x + (2r-2) \frac{T}{K} - (n-2l-1) \frac{\pi}{n} \right] + \\
&+ \sum_{l=1}^{\left\lceil \frac{n}{2} \right\rceil} \sum_{p=1}^r m_{n-2l+1,2r-2p+1}^{n,K} \left[-x - (n-2l-1) \frac{\pi}{n}, t - (2p-1) \frac{T}{K} \right]; \\
g_{1,2r+1}^{n,K}(t) &= -g_{n,2r+1}^{n,K} \left[-t - (n-2) \frac{\pi}{n} \right] - \\
&- \sum_{l=1}^{\left\lceil \frac{n}{2} \right\rceil} g_{n-2l+1,1}^{n,K} \left[x + (2r-1) \frac{T}{K} - (n-2l-1) \frac{\pi}{n} \right] + \\
&+ \sum_{l=1}^{\left\lceil \frac{n}{2} \right\rceil} \sum_{p=1}^r m_{n-2l+1,2r-2p+2}^{n,K} \left[-x - (n-2l-1) \frac{\pi}{n}, t - (2p-1) \frac{T}{K} \right],
\end{aligned}$$

for odd n

$$\begin{aligned}
g_{1,2r}^{n,K}(t) &= g_{n,2r}^{n,K} \left[t + (n-1) \frac{\pi}{n} \right] + \\
&+ \sum_{l=1}^{\left\lceil \frac{n}{2} \right\rceil} g_{n-2l+2,1}^{n,K} \left[-x - (2r-2) \frac{T}{K} + (n+2l) \frac{\pi}{n} \right] + \\
&+ \sum_{l=1}^{\left\lceil \frac{n}{2} \right\rceil} \sum_{p=1}^r m_{n-2l+2,2r-2p+1}^{n,K} \left[x + (n+2l) \frac{\pi}{n}, t - (2p-1) \frac{T}{K} \right]; \\
g_{1,2r+1}^{n,K}(t) &= g_{n,2r+1}^{n,K} \left[t + (n-1) \frac{\pi}{n} \right] - \\
&- \sum_{l=1}^{\left\lceil \frac{n}{2} \right\rceil} g_{n-2l+2,1}^{n,K} \left[-x - (2r-1) \frac{T}{K} + (n+2l) \frac{\pi}{n} \right] + \\
&+ \sum_{l=1}^{\left\lceil \frac{n}{2} \right\rceil} \sum_{p=1}^r m_{n-2l+2,2r-2p+2}^{n,K} \left[x + (n+2l) \frac{\pi}{n}, t - (2p-1) \frac{T}{K} \right], \quad r = 1, \left\lceil \frac{K-1}{2} \right\rceil,
\end{aligned}$$

for even K

$$\begin{aligned}
g_{2j,1}^{n,K}(x) &= -g_{2j,K}^{n,K} \left[-x - (K-2) \frac{T}{K} \right] + \\
&+ \sum_{l=1}^{\left\lceil \frac{K}{2} \right\rceil} g_{1,K-2l+1}^{n,K} \left[t + (2j-2) \frac{\pi}{n} - (K-2l-1) \frac{T}{K} \right] + \\
&+ \sum_{l=1}^{\left\lceil \frac{K}{2} \right\rceil} \sum_{r=1}^j m_{2j-2r+1,K-2l+1}^{n,K} \left[x - (2r-1) \frac{\pi}{n}, -t - (K-2l-1) \frac{T}{K} \right]; \\
g_{2j+1,1}^{n,K}(x) &= -g_{2j+1,K}^{n,K} \left[-x - (K-2) \frac{T}{K} \right] - \\
&- \sum_{l=1}^{\left\lceil \frac{K}{2} \right\rceil} g_{1,K-2l+1}^{n,K} \left[t + (2j-1) \frac{\pi}{n} - (K-2l-1) \frac{T}{K} \right] + \\
&+ \sum_{l=1}^{\left\lceil \frac{K}{2} \right\rceil} \sum_{r=1}^j m_{2j-2r+2,K-2l+1}^{n,K} \left[x - (2r-1) \frac{\pi}{n}, -t - (K-2l-1) \frac{T}{K} \right],
\end{aligned}$$

for odd K

$$\begin{aligned}
g_{2j,1}^{n,K}(x) &= g_{2j,K}^{n,K} \left[x + (K-1) \frac{T}{K} \right] + \\
&+ \sum_{l=1}^{\lfloor \frac{K}{2} \rfloor} g_{1,K-2l+2}^{n,K} \left[-t - (2j-2) \frac{\pi}{n} + (K+2l) \frac{T}{K} \right] + \\
&+ \sum_{l=1}^{\lfloor \frac{K}{2} \rfloor} \sum_{r=1}^j m_{2j-2r+1,K-2l+2}^{n,K} \left[x - (2r-1) \frac{\pi}{n}, t + (K+2l) \frac{T}{K} \right]; \\
g_{2j+1,1}^{n,K}(x) &= g_{2j+1,K}^{n,K} \left[x + (K-1) \frac{T}{K} \right] - \\
&- \sum_{l=1}^{\lfloor \frac{K}{2} \rfloor} g_{1,K-2l+2}^{n,K} \left[-t - (2j-1) \frac{\pi}{n} + (K+2l) \frac{T}{K} \right] + \\
&+ \sum_{l=1}^{\lfloor \frac{K}{2} \rfloor} \sum_{r=1}^j m_{2j-2r+2,K-2l+2}^{n,K} \left[x - (2r-1) \frac{\pi}{n}, t + (K+2l) \frac{T}{K} \right], \quad j = \overline{1, \lfloor \frac{n-1}{2} \rfloor},
\end{aligned}$$

then controls $g_j^n(t) = \sum_{r=1}^K g_{j,r}^{n,K}(t)$, $j = \overline{1, n}$, solve the problem of approximately null-controllable at the time $T > \pi$.

Proof. Sufficiency of (1.8). Let the condition (1.8) be satisfied. For all $g_j^n(t) \in V(0, T)$ we have

$$\begin{aligned}
&\int_0^t S(-\tau) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sum_{j=1}^{n-1} \delta(x - x_j) \sum_{r=1}^K g_{j,r}^{n,K}(\tau) d\tau = \\
&= \begin{bmatrix} \sum_{j=1}^n \left(\mathcal{T}_{\frac{\pi j}{n}} - \mathcal{T}_{-\frac{\pi j}{n}} \right) \partial^{-1} O E \left(\sum_{r=1}^K g_{j,r}^{n,K} \right) (x) \\ \sum_{j=1}^n \left(\mathcal{T}_{\frac{\pi j}{n}} - \mathcal{T}_{-\frac{\pi j}{n}} \right) O E \left(\sum_{r=1}^K \left(g_{j,r}^{n,K} \right)' \right) (x) \end{bmatrix}, \quad (1.10)
\end{aligned}$$

где $\partial^{-1} F(x)$ — the antiderivative of the function $F(x)$, $'$ — derivative by t .

By remark 1 from (1.8) follows

$$M \in V(Q_T). \quad (1.11)$$

With regard (1.10) the function $w(x, t)$ will take the form

$$w = S(t) \sum_{k \in Z} \mathcal{T}_{2Tk} \mathcal{T}_{2\pi k} \begin{bmatrix} \sum_{j=1}^n \left(\mathcal{T}_{\frac{\pi j}{n}} - \mathcal{T}_{-\frac{\pi j}{n}} \right) \partial^{-1} O E \left(\sum_{r=1}^K g_{j,r}^{n,K} \right) (x) \\ \sum_{j=1}^n \left(\mathcal{T}_{\frac{\pi j}{n}} - \mathcal{T}_{-\frac{\pi j}{n}} \right) O E \left(\sum_{r=1}^K \left(g_{j,r}^{n,K} \right)' \right) (x) \end{bmatrix}. \quad (1.12)$$

In coordinate form, the expression (1.12) can be represented as follows:

$$\begin{aligned}
&\frac{1}{2} ((\mathcal{T}_{-t} + \mathcal{T}_t)w + (\partial/\partial x)^{-1} (\mathcal{T}_{-t} - \mathcal{T}_t)(\partial/\partial t)w) = \\
&= \sum_{k \in Z} \mathcal{T}_{2Tk} \mathcal{T}_{2\pi k} \left(\sum_{j=1}^n \left(\mathcal{T}_{\frac{\pi j}{n}} - \mathcal{T}_{-\frac{\pi j}{n}} \right) \partial^{-1} O E \left(\sum_{r=1}^K g_{j,r}^{n,K} \right) (x) \right); \quad (1.13) \\
&\frac{1}{2} (\partial/\partial x) ((\mathcal{T}_{-t} - \mathcal{T}_t)w + (\mathcal{T}_{-t} + \mathcal{T}_t)(\partial/\partial t)w) =
\end{aligned}$$

$$= \sum_{k \in Z} \mathcal{T}_{2Tk} \mathcal{T}_{2\pi k} \left(\sum_{j=1}^n \left(\mathcal{T}_{\frac{\pi j}{n}} - \mathcal{T}_{-\frac{\pi j}{n}} \right) OE \left(\sum_{r=1}^K \left(g_{j,r}^{n,K} \right)' \right) (x) \right).$$

To find the controls g_j^n , $j = \overline{1, n}$, $n \in \mathbb{N}$, is sufficient to differentiate the (1.13) by the x

$$\begin{aligned} & \sum_{k \in Z} \mathcal{T}_{2Tk} \mathcal{T}_{2\pi k} \left(\sum_{j=1}^n \left(\mathcal{T}_{\frac{\pi j}{n}} - \mathcal{T}_{-\frac{\pi j}{n}} \right) OE \left(\sum_{r=1}^K g_{j,r}^{n,K} \right) (x) \right) = \\ & = \frac{1}{2} \left((\partial/\partial x) (\mathcal{T}_{-t} + \mathcal{T}_t) w - \frac{1}{2} (\mathcal{T}_{-t} - \mathcal{T}_t) (\partial/\partial t) w \right). \end{aligned}$$

To find the controls we divide the interval $[0, \pi]$ on n equal spaces, and the interval $[0, T]$ on K equal spaces. Then, the function $M(x, t)$ is naturally to write as a sum of nK finite functions of two variables $m_{j,r}^{n,K}(x, t)$, $\text{supp } m_{j,r}^{n,K}(x, t) \in \left[\frac{\pi(j-1)}{n}, \frac{\pi j}{n} \right] \times \left[\frac{T(r-1)}{K}, \frac{T r}{K} \right]$, $j = \overline{1, n}$, $n \in \mathbb{N}$, $r = \overline{1, K}$, $K \in \mathbb{N}$, that is

$$M(x, t) = \sum_{j,r=1}^{n,K} m_{j,r}^{n,K}(x, t), \quad n \in \mathbb{N}, \quad K \in \mathbb{N}. \quad (1.14)$$

In this the condition holds

$$\begin{cases} m_{1,r}^{n,K}(x, t) = m_{j,r}^{n,K} \left(\frac{\pi j}{n} - x, t \right); \\ m_{j,1}^{n,K}(x, t) = m_{j,r}^{n,K} \left(x, \frac{T r}{K} - t \right); \end{cases} \quad j = \overline{1, n}, \quad r = \overline{1, K}, \quad n, K \in \mathbb{N}. \quad (1.15)$$

For each segment $\left[\frac{\pi(j-1)}{n}, \frac{\pi j}{n} \right] \times \left[\frac{T(r-1)}{K}, \frac{T r}{K} \right]$, $j = \overline{1, n-1}$, $r = \overline{1, K-1}$ we draw up difference equations, those allow for a set of considered sections, for different j and r to construct the system of difference equations:

$$\begin{cases} g_{1,1}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) + g_{2,1}^{n,K} \left(-x - \frac{\pi}{n} + \frac{T}{K} \right) + g_{1,2}^{n,K} \left(-x + \frac{\pi}{n} + \frac{T}{K} \right) + \\ \quad + g_{2,2}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) = m_{1,1}^{n,K}(x, t), \\ -g_{1,p}^{n,K} \left(-x + \frac{\pi}{n} - \frac{T}{K} \right) - g_{2,p}^{n,K} \left(x + \frac{\pi}{n} - \frac{T}{K} \right) + g_{1,p+2}^{n,K} \left(-x + \frac{\pi}{n} + \frac{T}{K} \right) + \\ \quad + g_{2,p+2}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) = m_{1,p+1}^{n,K}(x, t), \\ \\ \left\{ \begin{aligned} & -g_{l,1}^{n,K} \left(-x + \frac{\pi}{n} + \frac{T}{K} \right) + g_{l+2,1}^{n,K} \left(-x - \frac{\pi}{n} + \frac{T}{K} \right) - g_{l,2}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) + \\ & \quad + g_{l+2,2}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) = m_{l+1,1}^{n,K}(x, t), \\ & g_{l,p}^{n,K} \left(x - \frac{\pi}{n} - \frac{T}{K} \right) - g_{l+2,p}^{n,K} \left(x + \frac{\pi}{n} - \frac{T}{K} \right) - g_{l,p+2}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) + \\ & \quad + g_{l+2,p+2}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) = m_{l+1,p+1}^{n,K}(x, t), \end{aligned} \right. \quad (1.16) \end{cases}$$

$l = \overline{1, j-1}$, $j = \overline{2, n-1}$, $p = \overline{1, r-1}$, $r = \overline{2, K-1}$. Solving this system we obtained the controls on each section.

From (1.11) follows $M(x, t) \in V \left(0 \leq x \leq \frac{(n-1)\pi}{n} \times 0 \leq t \leq \frac{(K-1)\pi}{K} \right)$. Then

$$\begin{aligned}
V \left(M, 0 \leq x \leq \frac{(n-1)\pi}{n} \times 0 \leq t \leq \frac{(K-1)\pi}{K} \right) &= \Delta_{\frac{j\pi}{n}, \frac{rT}{K}} = \\
&= \sup_{Q_T} \sum_{j=1}^{n-2} \sum_{r=1}^{K-2} \left| m_{j+1,r+1}^{n,K} - m_{j+1,r}^{n,K} - m_{j,r+1}^{n,K} + m_{j,r}^{n,K} \right| = \\
&= \sup_{Q_T} \left(\sum_{r=1}^{K-2} \left| m_{2,r+1}^{n,K} - m_{2,r}^{n,K} - m_{1,r+1}^{n,K} + m_{1,r}^{n,K} \right| + \sum_{j=2}^{n-2} \sum_{r=1}^{K-2} \left| m_{j+1,r+1}^{n,K} - m_{j+1,r}^{n,K} - m_{j,r+1}^{n,K} + m_{j,r}^{n,K} \right| \right) = \\
&= \sup_{Q_T} \left(\left| m_{2,2}^{n,K} - m_{2,1}^{n,K} - m_{1,2}^{n,K} + m_{1,1}^{n,K} \right| + \sum_{r=2}^{K-2} \left| m_{2,r+1}^{n,K} - m_{2,r}^{n,K} - m_{1,r+1}^{n,K} + m_{1,r}^{n,K} \right| + \right. \\
&\quad \left. + \sum_{j=2}^{n-2} \sum_{r=1}^{K-2} \left| m_{j+1,r+1}^{n,K} - m_{j+1,r}^{n,K} - m_{j,r+1}^{n,K} + m_{j,r}^{n,K} \right| \right) = \\
&= \sup_{Q_T} \left(\left| m_{2,2}^{n,K} - m_{2,1}^{n,K} - m_{1,2}^{n,K} + m_{1,1}^{n,K} \right| + \sum_{r=2}^{K-2} \left| m_{2,r+1}^{n,K} - m_{2,r}^{n,K} - m_{1,r+1}^{n,K} + m_{1,r}^{n,K} \right| + \right. \\
&\quad \left. + \sum_{j=2}^{n-2} \left| m_{j+1,2}^{n,K} - m_{j+1,1}^{n,K} - m_{j,2}^{n,K} + m_{j,1}^{n,K} \right| + \sum_{j=2}^{n-2} \sum_{r=2}^{K-2} \left| m_{j+1,r+1}^{n,K} - m_{j+1,r}^{n,K} - m_{j,r+1}^{n,K} + m_{j,r}^{n,K} \right| \right).
\end{aligned}$$

With regard to (1.16) we obtain

$$\begin{aligned}
\Delta_{\frac{j\pi}{n}, \frac{rT}{K}} &= \Delta_1 + \Delta_{2\frac{rT}{K}} + \Delta_{2\frac{j\pi}{n}} + \\
&+ \sup_{Q_T} \sum_{j=2}^{n-2} \sum_{r=2}^{K-2} \left| g_{j,r}^{n,K} \left(x - \frac{\pi}{n} - \frac{T}{K} \right) - g_{j+2,r}^{n,K} \left(x + \frac{\pi}{n} - \frac{T}{K} \right) - g_{j,r+2}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) + g_{j+2,r+2}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) - \right. \\
&\quad - g_{j,r-1}^{n,K} \left(x - \frac{\pi}{n} - \frac{T}{K} \right) + g_{j+2,r-1}^{n,K} \left(x + \frac{\pi}{n} - \frac{T}{K} \right) + g_{j,r+1}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) - g_{j+2,r+1}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) - \\
&\quad - g_{j-1,r}^{n,K} \left(x - \frac{\pi}{n} - \frac{T}{K} \right) + g_{j+1,r}^{n,K} \left(x + \frac{\pi}{n} - \frac{T}{K} \right) + g_{j-1,r+2}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) - g_{j+1,r+2}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) + \\
&\quad \left. + g_{j-1,r-1}^{n,K} \left(x - \frac{\pi}{n} - \frac{T}{K} \right) - g_{j+1,r-1}^{n,K} \left(x + \frac{\pi}{n} - \frac{T}{K} \right) - g_{j-1,r+1}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) + g_{j+1,r+1}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) \right| = \\
&= V(G, 0 \leq x \leq \pi \times 0 \leq t \leq T), \quad T > \pi, \tag{1.17}
\end{aligned}$$

where

$$\Delta_1 = \sup_{Q_T} \left| g_{1,1}^{n,K} \left(x - \frac{\pi}{n} - \frac{T}{K} \right) - g_{3,1}^{n,K} \left(x + \frac{\pi}{n} - \frac{T}{K} \right) - g_{1,3}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) + g_{3,3}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) + \right.$$

$$\begin{aligned}
& +g_{1,1}^{n,K}\left(-x+\frac{\pi}{n}+\frac{T}{K}\right)-g_{3,1}^{n,K}\left(-x-\frac{\pi}{n}+\frac{T}{K}\right)+g_{1,2}^{n,K}\left(x-\frac{\pi}{n}+\frac{T}{K}\right)-g_{3,2}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)+ \\
& +g_{1,1}^{n,K}\left(-x+\frac{\pi}{n}-\frac{T}{K}\right)+g_{2,1}^{n,K}\left(x+\frac{\pi}{n}-\frac{T}{K}\right)-g_{1,3}^{n,K}\left(-x+\frac{\pi}{n}+\frac{T}{K}\right)-g_{2,3}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)+ \\
& +g_{1,1}^{n,K}\left(x-\frac{\pi}{n}+\frac{T}{K}\right)+g_{2,1}^{n,K}\left(-x-\frac{\pi}{n}+\frac{T}{K}\right)+g_{1,2}^{n,K}\left(-x+\frac{\pi}{n}+\frac{T}{K}\right)+g_{2,2}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)\Big|; \\
\Delta_{2_{\frac{rT}{K}}} &= \sup_{Q_T} \sum_{r=2}^{K-2} \Big| g_{1,r}^{n,K}\left(x-\frac{\pi}{n}-\frac{T}{K}\right)-g_{3,r}^{n,K}\left(x+\frac{\pi}{n}-\frac{T}{K}\right)-g_{1,r+2}^{n,K}\left(x-\frac{\pi}{n}+\frac{T}{K}\right)+g_{3,r+2}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)- \\
& -g_{1,r-1}^{n,K}\left(x-\frac{\pi}{n}-\frac{T}{K}\right)+g_{3,r-1}^{n,K}\left(x+\frac{\pi}{n}-\frac{T}{K}\right)+g_{1,r+1}^{n,K}\left(x-\frac{\pi}{n}+\frac{T}{K}\right)-g_{3,r+1}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)- \\
& +g_{1,r}^{n,K}\left(-x+\frac{\pi}{n}-\frac{T}{K}\right)+g_{2,r}^{n,K}\left(x+\frac{\pi}{n}-\frac{T}{K}\right)-g_{1,r+2}^{n,K}\left(-x+\frac{\pi}{n}+\frac{T}{K}\right)-g_{2,r+2}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)- \\
& -g_{1,r-1}^{n,K}\left(-x+\frac{\pi}{n}-\frac{T}{K}\right)-g_{2,r-1}^{n,K}\left(x+\frac{\pi}{n}-\frac{T}{K}\right)+g_{1,r+1}^{n,K}\left(-x+\frac{\pi}{n}+\frac{T}{K}\right)+g_{2,r+1}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)\Big|; \\
\Delta_{2_{\frac{j\pi}{n}}} &= \sup_{Q_T} \sum_{j=2}^{n-2} \Big| g_{j,1}^{n,K}\left(x-\frac{\pi}{n}-\frac{T}{K}\right)-g_{j+2,1}^{n,K}\left(x+\frac{\pi}{n}-\frac{T}{K}\right)-g_{j,3}^{n,K}\left(x-\frac{\pi}{n}+\frac{T}{K}\right)+g_{j+2,3}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)- \\
& -g_{j,1}^{n,K}\left(-x+\frac{\pi}{n}+\frac{T}{K}\right)-g_{j+2,1}^{n,K}\left(-x-\frac{\pi}{n}+\frac{T}{K}\right)+g_{j,2}^{n,K}\left(x-\frac{\pi}{n}+\frac{T}{K}\right)-g_{j+2,2}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)- \\
& -g_{j-1,1}^{n,K}\left(x-\frac{\pi}{n}-\frac{T}{K}\right)+g_{j+1,1}^{n,K}\left(x+\frac{\pi}{n}-\frac{T}{K}\right)+g_{j-1,3}^{n,K}\left(x-\frac{\pi}{n}+\frac{T}{K}\right)-g_{j+1,3}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)- \\
& -g_{j-1,1}^{n,K}\left(-x+\frac{\pi}{n}+\frac{T}{K}\right)+g_{j+1,1}^{n,K}\left(-x-\frac{\pi}{n}+\frac{T}{K}\right)-g_{j-1,2}^{n,K}\left(x-\frac{\pi}{n}+\frac{T}{K}\right)+g_{j+1,2}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)\Big|.
\end{aligned}$$

G is a linear combination of functions $g_{j,r}^{n,K}$ with shifted supports. As each of g represents itself as a function of two variables and has a kind of wave, then, with respect to remark 1 and the one that each function of bounded variation can be represented as a linear combination of functions of bounded variation we come to a conclusion that $g_j^n(t) \in V(0, T)$. Hence, $w \in \mathcal{R}(w)$.

The set of states $w(x, t)$, described by a function $M(x, t) = \sum_{j,r=1}^{n,K} m_{j,r}^{n,K}(x, t)$, $\text{supp } M(x, t) \in [0 \leq x \leq \pi] \times [0 \leq t \leq T]$, is a set $\mathcal{R}(M)$.

Let us $M_1(x, t) = \sum_{j,r=1}^{n-1,K-1} m_{j,r}^{n,K}(x, t)$, $\text{supp } M_1(x, t) \in [0 \leq x \leq \frac{(n-1)\pi}{\pi}] \times [0 \leq t \leq \frac{(K-1)T}{K}]$.

Then $\overline{\mathcal{R}(M)} = \mathcal{R}(M_1) \cup \mathcal{R}(m_{n,K}^{n,K}(x, t))$.

Then the condition of approximate controllability for the system (0.1)–(0.4), will be

$$g_{1,1}^{n,K}\left(x-\frac{\pi}{n}+\frac{T}{K}\right)+g_{2,1}^{n,K}\left(-x-\frac{\pi}{n}+\frac{T}{K}\right)+g_{1,2}^{n,K}\left(-x+\frac{\pi}{n}+\frac{T}{K}\right)+g_{2,2}^{n,K}\left(x+\frac{\pi}{n}+\frac{T}{K}\right)+$$

$$\begin{aligned}
& + \sum_{p=3}^{K-2} \left(g_{1,p+2}^{n,K} \left(-x + \frac{\pi}{n} + \frac{T}{K} \right) + g_{2,p+2}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) - g_{1,p}^{n,K} \left(-x + \frac{\pi}{n} - \frac{T}{K} \right) - g_{2,p}^{n,K} \left(x + \frac{\pi}{n} - \frac{T}{K} \right) \right) + \\
& + \sum_{l=3}^{n-2} \left(g_{l+2,1}^{n,K} \left(-x - \frac{\pi}{n} + \frac{T}{K} \right) + g_{l+2,2}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) - g_{l,1}^{n,K} \left(-x + \frac{\pi}{n} + \frac{T}{K} \right) - g_{l,2}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) \right) + \\
& + \sum_{p=3}^{K-2} \sum_{l=3}^{n-2} \left(g_{l,p}^{n,K} \left(x - \frac{\pi}{n} - \frac{T}{K} \right) - g_{l+2,p}^{n,K} \left(x + \frac{\pi}{n} - \frac{T}{K} \right) \right) - \tag{1.18}
\end{aligned}$$

$$- \sum_{p=3}^{K-2} \sum_{l=3}^{n-2} \left(g_{l,p+2}^{n,K} \left(x - \frac{\pi}{n} + \frac{T}{K} \right) - g_{l+2,p+2}^{n,K} \left(x + \frac{\pi}{n} + \frac{T}{K} \right) \right) = \sum_{p=1}^{K-1} \sum_{l=1}^{n-1} m_{l,p}^{n,K} (x, t)$$

in $V(Q_T)$. Hence, 0 belongs to the $\overline{\mathcal{R}(M)}$ in $V(Q_T)$. Thus, it has been established that the system (0.1)–(0.4) is approximately null-controllable, and the state $w(x, t)$ is approximately controllable.

Necessity of (1.8). Let the system (0.1)–(0.4) is approximate null-controllable. Then, by definition 1.2 0 belongs to the $\overline{\mathcal{R}(M)}$ in $V(Q_T)$. Therefore, for any $w \in \mathcal{R}(w)$ there are such controls $g_j^n(t) \in V(0, T)$, $j = \overline{1, n}$, that the condition (1.16) is satisfied. Using (1.16), based on the fact that a linear combination of bounded variation functions is also a bounded variation function, we obtain the condition (1.8). The theorem is proved. $\square$

Let us illustrate the theorem 1.1 with examples.

EXAMPLE 1.1. Let $w(x, t) = t \sin \frac{x}{2}$. It can be seen that $w(x, t) \in V(Q_T)$; hence $w(x, t) = t \sin \frac{x}{2}$. According to Remark 1, the function $M(x, t) = \cos \frac{x}{2} \left(\sin \frac{t}{2} - \frac{t}{2} \cos \frac{t}{2} \right)$ also belongs to $V(Q_T)$. Then, according to the theorem 1.1, the state $w(x, t) = t \sin \frac{x}{2}$ of the system (0.1)–(0.4) is approximately null-controllable. Suppose that $n = K = 3$. Then controls $g_{1,1}^{3,3} = g_{3,1}^{3,3} \left[x + \frac{2\pi}{3} \right] - m_{3,1}^{3,3} \left[x + \frac{5\pi}{3}, t \right]$,

$$g_{1,2}^{3,3} = g_{3,2}^{3,3} \left[x + \frac{2\pi}{3} \right] - m_{3,2}^{3,3} \left[x + \frac{5\pi}{3}, t \right],$$

$$g_{2,1}^{3,3} = g_{2,3}^{3,3} \left[t + \frac{2T}{3} \right] + g_{1,3}^{3,3} \left[-t + \frac{5T}{3} \right] + m_{1,3}^{3,3} \left[x - \frac{\pi}{3}, t + \frac{5T}{3} \right],$$

$$g_{2,2}^{3,3} = -g_{2,1}^{3,3} \left[-t \right] + g_{1,1}^{3,3} \left[-t - \frac{T}{3} \right] + m_{1,1}^{3,3} \left[x - \frac{\pi}{3}, t - \frac{T}{3} \right]$$

solve the problem of approximately null-controllability of the system (0.1)–(0.3).

EXAMPLE 1.2. Let $w(x, t) = tx \sin \frac{1}{t}$. It is known that $w(x, t) \notin V(0 \leq t \leq T)$, $T > \pi$ for all $x \in (0, \pi)$; and $w(x, t) \in V(0 \leq x \leq \pi)$ for all $t \in (0, T)$. Hence, function $w(x, t) \notin V(Q_T)$. Then the condition (1.8) of the theorem 1.1 is not satisfied. Hence state $w(x, t) = tx \sin \frac{1}{t}$ of the system (0.1)–(0.4), like the system itself, is not approximately null-controllability in $V(Q_T)$.

2 The criterion of null-controllability

An important concept in the investigation of controllable systems is a null-controllability.

Actually it makes sense to consider the case of null-controllable system: by definition of null-controllability we have $0 \in \mathcal{R}(M)$ in $V(Q_T)$, where $M(x, t) = \sum_{j=1}^n \sum_{r=1}^K m_{j,r}^{n,K}(x, t)$.

THEOREM 2.1. *A system (0.1)–(0.4) is null-controllable at the time $T > \pi$, if as well as (1.8) for the function $M(x, t) = \sum_{j=1}^n \sum_{r=1}^K m_{j,r}^{n,K}(x, t)$, $\forall n, K \in \mathbb{N}$ the condition hold*

$$\begin{aligned} m_{n,K}^{n,K}(x, t) &+ \sum_{l=1}^{\frac{K-2}{2}} m_{n,K-2l}^{n,K} \left[x, t - 2l \frac{T}{K} \right] - \sum_{l=1}^{\frac{K}{2}} m_{n,K-2l+1}^{n,K} \left[x, t - 2(l+1) \frac{T}{K} \right] + \sum_{r=1}^{\frac{n-2}{2}} m_{n-2i,K}^{n,K} \left[x - 2r \frac{\pi}{n}, t \right] + \\ &+ \sum_{r=1}^{\frac{n-2}{2}} \sum_{l=1}^{\frac{K-2}{2}} m_{n-2r,K-2l}^{n,K} \left[x - 2r \frac{\pi}{n}, t - 2l \frac{T}{K} \right] - \sum_{r=1}^{\frac{n-2}{2}} \sum_{l=1}^{\frac{K}{2}} m_{n-2r,K-2l+1}^{n,K} \left[x - 2r \frac{\pi}{n}, t - 2(l+1) \frac{T}{K} \right] - \\ &- \sum_{r=1}^{\frac{n}{2}} m_{n-2r+1,K}^{n,K} \left[x - 2(r+1) \frac{\pi}{n}, t \right] - \sum_{r=1}^{\frac{n}{2}} \sum_{l=1}^{\frac{K-2}{2}} m_{n-2r+1,K-2l}^{n,K} \left[x - 2(r+1) \frac{\pi}{n}, t - 2l \frac{T}{K} \right] + \\ &+ \sum_{r=1}^{\frac{n}{2}} \sum_{l=1}^{\frac{K}{2}} m_{n-2r+1,K-2l+1}^{n,K} \left[x - 2(r+1) \frac{\pi}{n}, t - 2(l+1) \frac{T}{K} \right] = 0. \end{aligned} \quad (2.1)$$

Proof. Let the system (0.1)–(0.4) is null-controllable. Then, according to the definition 1.2, 0 belongs to the set $\mathcal{R}(M)$. That is, for all controls (1.9), in addition to (1.18), for a finite function $m_{n,K}^{n,K}(x, t) = M(x, t) \left[H \left(x - \frac{\pi(n-1)}{n} \right) - H(x - \pi) \right] \left[H \left(t - \frac{T(K-1)}{K} \right) - H(t - T) \right]$ the condition

$$\sum_{k \in Z} \mathcal{T}_{2Tk} (\mathcal{T}_{\pi(2k+1)} - \mathcal{T}_{\pi(2k-1)}) OE \left(g_{n,K}^{n,K} \right) (x) = m_{n,K}^{n,K}(x, t) \quad (2.2)$$

must be satisfied.

Taking into account 2π -periodicity by x and $2T$ -periodicity by t , and that the function $M(x, t)$ is odd, we rewrite the condition (2.2) as:

$$g_{n-1,K-1}^{n,K}(x - \frac{\pi}{n} - \frac{T}{K}) + g_{n,K-1}^{n,K}(-x + 2\pi - \frac{\pi}{n} - \frac{T}{K}) + g_{n-1,K}^{n,K}(-x + \frac{\pi}{n} + 2T - \frac{T}{K}) + g_{n,K}^{n,K}(x - 2\pi + \frac{\pi}{n} + 2T - \frac{T}{K}) = m_{n,K}^{n,K}(x, t). \quad (2.3)$$

Substituting (1.9) in (2.3) and performing algebraic transformations taking into account (1.15), we obtain (2.1). The theorem is proved. $\square$

In Theorem 2.1 it is proved that the condition (2.1) ensures null-controllability of the system (0.1)–(0.3) in $V(Q_T)$ provided that the system is approximately null-controlled.

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