Forced Sliding Mode Control for Chaotic Systems Synchronization

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Research Article

Keywords: chaos synchronization, robust control, synergetic control theory, Arneodo chaotic system, sliding mode control, forced sliding mode

Posted Date: November 29th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-1108040/v1

Nonlinear Dynamics manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract Synchronization of chaotic systems is considered to be a common engineering problem. However, the proposed laws of synchronization control do not always provide robustness towards the parametric perturbations. The purpose of this article is to show the use of synergy-cybernetic approach for the construction of robust law for Arneodo chaotic systems synchronization. As the main method of design of robust control, the method of design of control with forced sliding mode of the synergetic control theory is considered. To illustrate the effectiveness of the proposed law it's compared in this article with the classical sliding mode control. The distinctive features of suggested robust control law are the more good compensation of parametric perturbations (better performance indexes — the root-mean-square error, average absolute value of error) without designing perturbation observers, the ability to exclude the chattering effect, less energy-consuming and a simpler analysis of the stability of a closed-loop system. Offered approach will allow a new consideration for the design of robust control laws for chaotic systems, taking into account the ideas of directed self-organization and robust control. It can be used for synchronization other chaotic systems.

Keywords chaos synchronization · robust control · synergetic control theory · Arneodo chaotic system · sliding mode control · forced sliding mode

Mathematics Subject Classification (2020) 93B35 · 93B52 · 93C30 · 93D21 · 34H10

1 Introduction

Chaotic system study is the field of modern nonlinear science that is extremely relevant and is of great interest to both theorists and practitioners. Literature review shows

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that these systems are widely used in the development of technical systems (oscillation generators, lasers, cryptosystems, secure data transfer systems, neural networks, and robot technical systems), biology, chemistry, ecology, medicine, economics, etc. [26], [5], [1], [3], [32], [33], [8], [10], [34], [12], [9], [13], [30], [4], [37].

The most frequently studied and implemented chaotic systems are the chaotic systems of Lorenz, Chua, Chen, Lu, Rössler, etc. Also, new types of chaotic systems are periodically proposed. This article considers the Arneodo chaotic systems [3], [32], [33]. Note that, a model of the Arneodo system with a different type of the right part of the third equation of the system is presented in the literature, for example, in [24], [11]. This is the so-called fractional-order Arneodo's system.

The common engineering problem of chaotic systems control is the synchronization problem – it is necessary to ensure the synchronous behavior of two mutually connected chaotic systems with similar different chaotic attractors with different initial conditions [26], [5], [1], [3], [32], [33], [8], [10], [34], [12], [9], [13], [30], [4], [37]. That is, taking into account the behavior of the master system, it is necessary to control the slave system in a way to provide for their synchronization.

Traditionally the problem of chaotic system synchronization is solved by the methods of active control [1], [8], [33], methods of adaptive control [10], [37], [11], [40], method of design of the sliding mode control (SMC) [39], [7], [38], [37], [23], [17], [36], [29], [2], backstepping [32], [34], [11] and etc. [12], [9], [13], [30], [4]. Furthermore, the developers are interested not only in solving the synchronization problem, but also the solving the problem of ensuring the robustness of the closed-loop system to perturbations. In the presented works, robustness is ensured by constructing an observer or SMC. But when constructing an observer, the complexity and dimension of the control system increases, and systems with SMC have a negative chattering effect. Thus, *the main problem of this article* is the problem of increasing robustness and eliminating a chattering effect when synchronizing two Arneodo chaotic systems. Also, unfortunately, there are no attempts in the literature to combine the methods of modern control theory with the ideas of synergetics to solve the tasks of chaotic systems synchronization.

The main purpose of this paper is to demonstrate the new approach of modern control theory to ensuring a system's robustness and eliminating a chattering effect in case of synchronization of two Arneodo chaotic systems. This approach is the synergy-cybernetic approach proposed by prof. A.A. Kolesnikov [14], [16], [31] and also known as the synergetic control theory (SCT). It guarantees the stability of the object's motion towards the target attractors (invariant manifolds) due to the appropriate design of nonlinear control laws that ensure both the fulfillment of control objectives (invariants) and compensation of external and parametric perturbations.

The synergy-cybernetic approach have the main strengths:

- can possess the properties of order reduction and decoupling in the design procedure [27];
- directed self-organization [14];
- the design of scalar and vector control laws with the nonlinearity and high dimensionality of the control object's model [14], [28], [25];

- includes methods for the synthesis of adaptive [18], [6], robust [22] and SMC control laws [19], [21], [15];
- synergetic laws are well suited for digital control implementation [31], [27].

The novelty of this paper is to combine synergy-cybernetic approach (i. e. SCT) and SMC techniques (forced sliding mode (FSM)) in order to synchronize two Arneodo chaotic system and to ensure robustness and eliminating a chattering effect. As the main method of design of robust control, the method of design of control with FSM of the synergetic control theory is considered. The effectiveness of the proposed control law is illustrated by an example of comparison with the classical SMC. Obtained results have features: the more good compensation of parametric perturbations (better performance indexes — the root-mean-square error, average absolute value of error) without the design of perturbation observers, in the possibility of eliminating the chattering effect in SMC, in less energy-consuming and in a simpler analysis of the stability of a closed-loop system.

This work is a development of the work [21] in which the problem of synchronization of chaotic Sprott's systems is considered. The rest of this paper is organized as follows. Section 2 provides object and control problem. Section 3 describes the procedure for designing the control law with FSM. In Section 4 synthesis example for two Arneodo chaotic system is demonstrated. Simulation results and discussion are presented in Section 5. Finally, conclusion and future study are drawn in Section 6.

2 Problem statement

Arneodo chaotic system model [33], [32]:

$$\begin{aligned} \dot{x}_1(t) &= x_2; \\ \dot{x}_2(t) &= x_3; \\ \dot{x}_3(t) &= ax_1 - bx_2 - x_3 - x_1^2, \end{aligned} \tag{1}$$

where x_i are the state variables; and a, b are the constant parameters.

Fig. 1 shows the strange attractor of the Arneodo system (1) with nominal parameters $a = 7.5$, $b = 3.8$.

In the problem of the chaotic systems synchronization, the mathematical model of a control object with two Arneodo systems includes a model of a master system in the form of (1) and a model of a slave system [32]:

$$\begin{aligned} \dot{y}_1(t) &= y_2; \\ \dot{y}_2(t) &= y_3; \\ \dot{y}_3(t) &= ay_1 - by_2 - y_3 - y_1^2 + u, \end{aligned} \tag{2}$$

where y_i are the state variables of the slave system and u is the control.

By introducing the new variables $e_i(t) = y_i(t) - x_i(t)$, $i = \overline{1,3}$ are the synchronization errors, we represent the combined dynamics of master and slave systems (1),

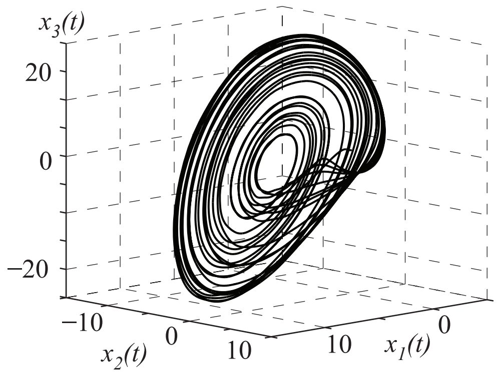

Fig. 1 Strange attractor of the Arneodo system.

(2) by the following system [32]:

$$\begin{aligned} \dot{e}_1(t) &= e_2; \\ \dot{e}_2(t) &= e_3; \\ \dot{e}_3(t) &= ae_1 - be_2 - e_3 - (y_1 + x_1)e_1 + u. \end{aligned} \tag{3}$$

Then for the system (3), the problem is set to construct the control law u that provides for asymptotical synchronization of chaotic systems (1) and (2), i. e. the near-zero errors of synchronization $e_i(t) \to 0$, $i = \overline{1,3}$ and robustness to parametric perturbations acting on the master system (1). That is, in the slave system (2) the parameters a, b do not change and are equal to nominal values.

3 Method of control design with FSM

It is possible to speed up the process of sliding by organizing a new sliding along the surface of lower dimensionality, i.e. by organizing FSM [15]. Introduction to the FSM system not only accelerates the transient process, but also endows the system with the properties of invariance to changes of object parameters in a wide range and to external unmatched disturbances, and also makes it easier to solve the problem of analyzing the stability of a closed-loop system [21].

If, after organizing the first sliding in the system, it was possible to achieve the required dynamic properties, then there is no need to introduce the next sliding into the system. If it was not possible, then a second sliding should be organized, etc. up to and including the $(n-1)$ sliding, where n is the control object's dimensionality. Thus, a series of successive transitions from one sliding surface to another is organized.

The description of the main SCT method was presented in [14], [16], [31], [27], and here we will describe in detail the modification of this method named as FSM. The model of the system has the form [21], [20], [15]:

$$\begin{aligned} \dot{x}_j(t) &= f_j(x_1,...,x_n) + a_{j+1}x_{j+1}, \quad j = \overline{1,n-1}; \\ \dot{x}_n(t) &= f_n(x_1,...,x_n) + u, \end{aligned} \tag{4}$$

where $\mathbf{x} = [x_1, ..., x_n]^T$ is the vector of state variables; dim $\mathbf{x} = n \times 1$; $u = u(\mathbf{x})$ is the scalar control; $f_i(x_1, ..., x_n)$, $i = \overline{1,n}$ are the continuous differentiable functions (in general, they are nonlinear).

For the system (4), the problem of design a sliding control with an FSM is stated: it's required to define such a control in the function of the state variables of the object (4), which ensures the transfer of the representing point (RP) of the object from an arbitrary initial state (in a certain admissible region) to a given state determined by the desired invariant – the goal of control.

At kth stage of the control design with FSM, we will always consider a manifold of the form that depends on the vector of state variables of this stage

$$\psi_k(\mathbf{x^{(k)}}) = \sum_{j=1}^{n_k-1} \beta_{kj} \left|x_j\right| + |s_k| = 0. \tag{5}$$

For example, for first stage there is $\mathbf{x^{(k)}} = \mathbf{x}$.

The structure (5) includes the sliding surface

$$s_k = \sum_{j=1}^{n_k-1} \alpha_{kj} x_j + x_{n_k} + u_k\left(x_1, ..., x_{n_k-1}\right), \tag{6}$$

where α_{kj}, β_{kj} are the parameters, due to the choice of which the motion of the system (4) in the sliding mode is given the required dynamic properties; n_k is the dimension of the system at the kth stage; $u_k\left(x_1, ..., x_{n_k-1}\right)$ is the continuous function, unknown at this stage, which plays the role of "internal" control for the decomposed system of the next $(k+1)$th stage. Note that in terms of SCT, an invariant manifold is an attractor.

Under the control $u = u(\mathbf{x})$ and due to $s_1 = 0$ $(k = 1)$ from (6) the behavior of system (4) will be described by a system of lower dimensionality — $(k+1)$th stage's a decomposed system:

$$\begin{aligned} \dot{x}_j(t) &= f_j(x_1, ..., x_j) + a_{j+1} x_{j+1}, \quad j = \overline{1, n-2}; \\ \dot{x}_{n-1}(t) &= f_{n-1}(x_1, ..., x_{n-1}) - a_n \sum_{j=1}^{n-1} \alpha_{1j} x_j - a_n u_1\left(x_1, ..., x_{n_k-1}\right). \end{aligned} \tag{7}$$

At each kth stage of the design, the basic functional equation of the SCT is considered [14], [16], [19], [21], [15]:

$$T_k \dot{\psi}_k(t) + \psi_k = 0, \tag{8}$$

where $\dot{\psi}_k(t) = \sum_{i=1}^{n_k} \dfrac{\partial \psi_k}{\partial x_i} \dot{x}_i(t)$.

Let us substitute into (8) the expression of the manifold (5) and its derivative obtained by virtue of the original equations of the object (4). From the obtained equation we express the desired control

$$u = -\sum_{j=1}^{n-1}\left(\alpha_{1j} + \frac{\partial u_1}{\partial x_j}\right)\left(f_j\left(x_1,...,x_n\right) + a_{j+1}x_{j+1}\right) - f_n\left(x_1,...,x_n\right) - \\ -\left(\sum_{j=1}^{n-1}\beta_j\left(f_j\left(x_1,...,x_n\right) + a_{j+1}x_{j+1}\right)\operatorname{sign}\, x_j + \frac{1}{T_1}\psi_1\right)\operatorname{sign}\, s_1. \tag{9}$$

The detailed finding of the expression (9) is shown in Appendix A. Also the expression (9) can be represented in the form of

$$u = u_{eq} + u_{SMC},$$

where $u_{eq} = -\sum\limits_{j=1}^{n-1}\left(\alpha_{1j} + \frac{\partial u_1}{\partial x_j}\right)\left(f_j\left(x_1,...,x_n\right) + a_{j+1}x_{j+1}\right) - f_n\left(x_1,...,x_n\right)$ is the equivalent control;
$u_{SMC} = -\left(\sum\limits_{j=1}^{n-1}\beta_j\left(f_j\left(x_1,...,x_n\right) + a_{j+1}x_{j+1}\right)\operatorname{sign}\, x_j + \frac{1}{T_1}\psi_1\right)\operatorname{sign}\, s_1 =$
$= -M\left(\mathbf{x}\right)\operatorname{sign}\, s_1$ is the discontinuous control, in which $M\left(\mathbf{x}\right)$ is the nonlinear function analogous to the large SMC gain [35].

Control (9) transits the RP systems (4) from an arbitrary initial state to the manifold $\psi_1 = 0$ — to the attractor whose structure includes a sliding surface (6) [14], [16]. Since the motion relative to $\psi_1 = 0$ is asymptotically stable at $T_1 > 0$ according (8). This means that the RP inevitably falls on the submanifold (6), that is, on the sliding surface $s_1 = 0$.

At the second stage ($k = 2$) we design the control law, which provides the FSM along the selected sliding surface of the first stage (6). Also at this stage, we define the expression of the "internal" control $u_1\left(x_1,...,x_{n-1}\right)$. If we repeat the procedure for the design of the first stage, then a sliding of the second order, or FSM, appears in the original system. In other words, a new sliding is organized on the first-order sliding surface, decreasing the dimensionality of the original system by one. Thus, at the second stage of the design for the decomposed system (7), a manifold of the form of (5) is introduced. Similarly, based on functional equations of the form (8), we find the "internal" control $u_1\left(x_1,...,x_{n-1}\right)$.

The specified procedure of the second stage, if necessary, can be repeated $(n-1)$ times until the order of the original system becomes equal to one, for example, of the form

$$\dot{x}_1\left(t\right) = f_1\left(x_1\right) - a_2 u_q\left(x_1\right) \tag{10}$$

or until we provide the desired target invariant (or target invariants in case of vector control design).

The expression $u_q\left(x_1\right)$ for the finishing decomposed system (10) is selected or found using the SCT methods or some other methods of modern control theory. Then the obtained equation $u_q\left(x_1\right)$ is substituted to the previous control $u_{q-1}\left(x_1,x_2\right)$, which, in turn, is also substituted to $u_{q-2}\left(x_1,x_2,x_3\right)$, and so on, up to the control $u_1\left(x_1,...,x_{n-1}\right)$, which is directly included in the control law (9).

Please note, that in order to exclude the differentiation of the function $\text{sign}(\cdot)$ included in each "internal" control, it is recommended to replace it with one of the equivalent continuous functions, for example, $\tanh(A\cdot)$ or $(2/\pi)\arctan(A\cdot)$, where $A > 0$ is the large coefficient, and $(\cdot)$ is the argument of the function $\text{sign}(\cdot)$.

The conditions for the asymptotic stability of the closed-loop system in SCT [14], [16] consist of the stability conditions for functional equations of the form (8), that is, $T_i > 0$, and stability conditions for the finishing decomposed system, the dimensionality of which is significantly lower than the dimensionality of the original system (4), for example, (10). Thus, we see that the stability analysis is quite simple, since there is no need to analyze the entire initial system.

At each k-stage of design, also it's necessary to check the condition for the occurrence of the sliding mode [35]:

$$s_k \cdot \dot{s}_k(t) < 0. \tag{11}$$

To analyze the condition (11) we need take into account:

- there is expression for the sliding surface s_k. It expression is (6);
- it's expedient to express the derivative of the sliding surface from the corresponding equation of the form (8) – we need to insert (5) into (8). In common case we have

$$\dot{s}_k = -\Big(\sum_{j=1}^{n_k-1} \beta_{kj}\,\text{sign}(x_j)\dot{x}_j(t) + \frac{1}{T_k}\psi_k\Big)\,\text{sign}(s_k).$$

4 Design of the robust control law for Arneodo chaotic systems

At the first stage of the design of control u for the system (3), we define an invariant manifold of the form (5):

$$\psi_1(\mathbf{x}^{(1)}) = \beta_{11}|e_1| + \beta_{12}|e_2| + |s_1| = 0, \tag{12}$$

where $s_1 = \alpha_{11}e_1 + \alpha_{12}e_2 + e_3 + u_1(e_1,e_2)$, $\mathbf{x}^{(1)} = [e_1, e_2, e_3]^T$.

Let us substitute (12) into the functional SCT equation (8). From which we obtain, by virtue of the equations of the object (3), the control law:

$$\begin{aligned} u = &-(a-e_1)e_1 - \left(\alpha_{11} - b + \frac{\partial u_1}{\partial e_1}\right)e_2 - \left(\alpha_{12} - 1 + \frac{\partial u_1}{\partial e_2}\right)e_3 - \\ &- \left(\beta_{11}e_2\,\text{sign}\;e_1 + \beta_{12}e_3\,\text{sign}\;e_2 + \frac{1}{T_1}\psi_1\right)\text{sign}\;s_1. \end{aligned} \tag{13}$$

Under the action of control (13), the RP of system (3) falls into a neighborhood of the manifold (12). The motion along which, by virtue of $s_1 = 0$, is described by a decomposed system:

$$\begin{aligned} &\dot{e}_1(t) = e_2; \\ &\dot{e}_2(t) = -\alpha_{11}e_1 - \alpha_{12}e_2 - u_1(e_1,e_2). \end{aligned} \tag{14}$$

To find the "internal" control $u_1(e_1, e_2)$ for the system (14) at the second stage of the design, we similarly define a manifold of the form

$$\psi_2(\mathbf{x}^{(2)}) = \beta_{21}|e_1| + |s_2| = 0, \tag{15}$$

where $s_2 = \alpha_{21}e_1 + e_2$, $\mathbf{x}^{(2)} = [e_1, e_2]^T$.

Substituting this expression into the functional equation (8), we obtain, by virtue of the equations of the object (14), the law of "internal" control:

$$u_1(e_1, e_2) = -\alpha_{11}e_1 - (\alpha_{12} - \alpha_{21})e_2 + \left(\beta_{21}e_2 \operatorname{sign}\, e_1 + \frac{1}{T_2}\psi_2\right)\operatorname{sign}\, s_2. \tag{16}$$

Thus, substituting (16) in (13), we get the final expression for the control. Then the motion of the system (14) under the action of the control (16) will be described by the equation:

$$\dot{e}_1(t) = -\alpha_{21}e_1,$$

which is stable at $\alpha_{21} > 0$, and the choice of the value of this parameter can provide the desired dynamics of the transient process. To these conditions we add the stability conditions for the equations (8): $T_i > 0$, $i = 1, 2$. Thus, the stability conditions for the closed-loop system (3), (13) are $\alpha_{21} > 0$, $T_i > 0$, $i = 1, 2$.

Let's check the condition (11) for the occurrence of a sliding mode for each stage of the design. At the second stage of the design, the second sliding surface $s_2 = \alpha_{21}e_1 + e_2$ enters in (15). Then its derivative, expressed from the functional equation (8) for the manifold (15), has the form

$$\dot{s}_2(t) = -\left(\beta_{21}e_2 \operatorname{sign}\, e_1 + \frac{1}{T_2}\psi_2\right)\operatorname{sign}\, s_2.$$

Hence it can be seen that the condition (11) for the second sliding surface has form

$$-(\alpha_{21}e_1 + e_2)\left(\beta_{21}e_2 \operatorname{sign}\, e_1 + \frac{1}{T_2}\psi_2\right)\operatorname{sign}\, s_2 < 0$$

and is satisfied at $\alpha_{21} > 0$, $\beta_{21} > 0$, $T_2 > 0$.

Similarly, taking into account (12), we obtain the expression for the first sliding surface of the first stage of design:

$$s_1 = \alpha_{11}e_1 + \alpha_{12}e_2 + e_3 + u_1(e_1, e_2) = e_3 + \left(\beta_{21}e_2 \operatorname{sign}\, e_1 + \frac{1}{T_2}\psi_2\right)\operatorname{sign}\, s_2 + \alpha_{12}e_2,$$

and from (8), taking into account (12), we express its derivative

$$\dot{s}_1(t) = -\left(\beta_{11}e_2 \operatorname{sign}\, e_1 + \beta_{12}e_3 \operatorname{sign}\, e_2 + \frac{1}{T_1}\psi_1\right)\operatorname{sign}\, s_1.$$

From this, it can be seen that the condition for the occurrence of the sliding mode (11) for the first sliding surface is directly provided by the choice of parameters $\alpha_{12} > 0$, $\beta_{11} > 0$, $\beta_{12} > 0$, $\beta_{23} > 0$, $T_1 > 0$, $T_2 > 0$.

Thus, the analysis of the fulfillment of the condition of occurrence of the sliding mode (11) in the proposed methods is quite simple.

5 Simulation and discussion

To illustrate the effectiveness of the proposed control law (13) let's compare it with the control law that realizes classical SMC. It can be designed according to the SMC method, for example described in [35], [36]. Applying this method, we get

$$\begin{aligned} u = (y_1 + x_1)\, e_1 - \frac{(\alpha c_1 + a c_3)\, e_1}{c_3} - \frac{(c_1 + \alpha c_2 - b c_3)\, e_2}{c_3} - \\ - \frac{(c_2 + c_3\,(\alpha - 1))\, e_3}{c_3} - \frac{\beta}{c_3}\,\mathrm{sign}\,(S)\,, \end{aligned} \tag{17}$$

with sliding surface

$$S = c_1 e_1 + c_2 e_2 + c_3 e_3, \tag{18}$$

where α, β, c_j are control law constant parameters.

It is possible to provide the desired eigenvalues $p_{01} = p_{02} = p_{03} = p_0 < 0$ of the state matrix of the system (3) obtained taking into account control law (17) by choosing the coefficients

$$\alpha = -p_0;\ c_2 = -2c_1/p_0;\ c_3 = c_1/p_0^2.$$

In the simulation, we assume that the parameters of the control laws (13), (17) and the slave system (2) are unchanged and equal to the nominal: $a = a_0 = 7.5$, $b = b_0 = 3.8$, but the parameters of the master system (1) change as follows:

$$\begin{aligned} a &= \begin{cases} a_0, & t \leqslant 20; \\ 1.5\, a_0, & t > 20; \end{cases} \\ b &= \begin{cases} b_0, & t \leqslant 20; \\ 1.5\, b_0, & t > 20. \end{cases} \end{aligned} \tag{19}$$

Closed-loop system simulation was carried out in Matlab R2021a with the initial conditions $x_1(0) = 3$; $x_2(0) = 2$; $x_3(0) = 2$; $y_1(0) = 1$; $y_2(0) = -1$; $y_3(0) = 7$.

In Fig. 2-5 the simulating results of the closed-loop system (3) with perturbations (19) are shown: by the red line with control law (13) with parameters $T_1 = T_2 = 0.1$; $\beta_{11} = \beta_{12} = \beta_{21} = 1$; $\alpha_{11} = 3.8$; $\alpha_{12} = 1$; $\alpha_{21} = 40$; $A = 20$; by the black line with SMC law (17), (18) with parameters $p_0 = -10$, $c_1 = 10$, $\beta = 0.5$. The parameters of the laws are selected so that the closed-loop system has sufficiently similar dynamic characteristics of transients and control amplitudes. The time interval from 0 to 15 s omits on Fig. 2-5.

The Fig. 2, 3 show that up to $t = 20$ s, the synchronization errors are zero. And after $t = 20$ s, there is no synchronization — the errors are different from zero, and in the case of law (13), the errors are more small for similar amplitudes of controls.

In order to evaluate the effects of presented different controllers, two performance indexes are introdused:

- the root-mean-square error (RMSE): $RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n} e_i^2}$;
- (AVG): $AVG = \frac{1}{n}\sum_{i=1}^{n} |e_i|$.

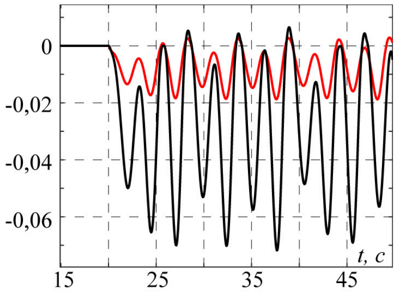

Fig. 2 Synchronization error $e_1(t)$.

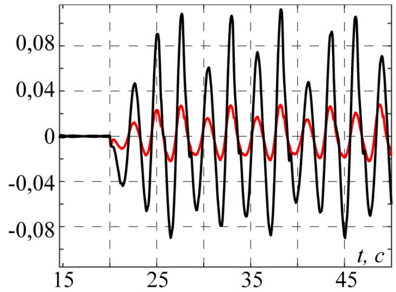

Fig. 3 Synchronization error $e_2(t)$.

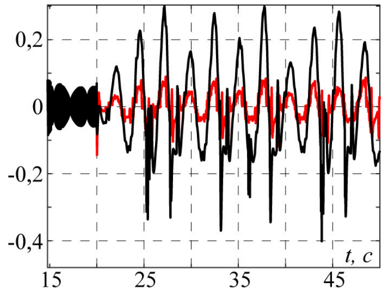

Fig. 4 Synchronization error $e_3(t)$.

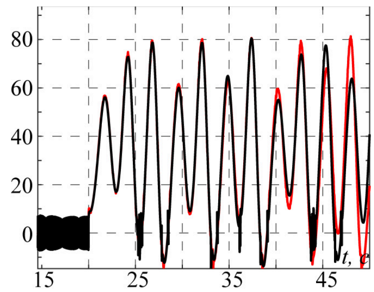

Fig. 5 Controls (13), (17).

To illustrate the simulation results and confirm the advantages of the proposed control law (13), the performance indexes are calculated, which are presented in Table 1.

Table 1 Performance indexes for parametric perturbations (19)

Error	Classical SMC (17)		Forced SMC (13)	
	RMSE	AVG	RMSE	AVG
$e_1, for\, t < 20$	0.2354	0.0592	1.17e-4	3.69e-5
$e_1, for\, t \geqslant 20$	0.107	0.1	0.0097	0.0078
$e_2, for\, t < 20$	0.2092	0.0648	6.38e-4	1.52e-4
$e_2, for\, t \geqslant 20$	0.0235	0.0199	0.014	0.012
$e_3, for\, t < 20$	0.1920	0.0640	0.0079	0.0034
$e_3, for\, t \geqslant 20$	0.041	0.026	0.039	0.033

Similar simulation results and values of performance indexes are obtained if parametric perturbations (19) are applied to the slave system (2).

Thus, the presented simulation results demonstrate that the control law with FSM (13), in comparison with the law (17) with a similar speed and control amplitude, provides:

– more accurate synchronization with parametric perturbations (19). It is clearly seen from the comparison of Fig. 2-4: the amplitudes of synchronization errors are 3-5 times smaller. Values of RMSE and AVG are significantly less (see Table 1);

– no chattering effect: considering the changes in the control graphs in Fig. 5 at an increased scale (Fig. 6), it can be seen that the control law (17), before the disturbance occurs, represents a high-frequency switching of the control signal with an amplitude of ± 6.5. And the control law (13) represents insignificant irregular changes in the interval ± 0.5. This suggests that the control law (13) is less energy-consuming.

Fig. 7 shows the phase trajectory of the system (3) with the control law (13) and the sliding surface (12). And Fig. 8 shows the phase trajectory of the decomposed system (14) with the control law (16) and the sliding surface (15).

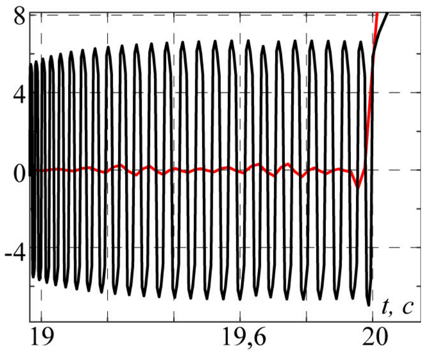

Fig. 6 Controls (13), (17) (an increased scale).

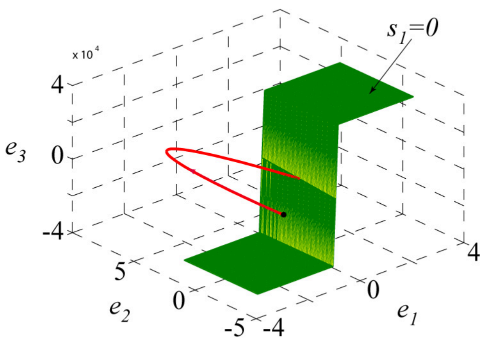

Fig. 7 Phase trajectory.

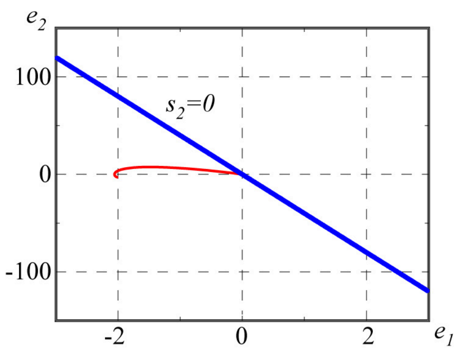

Fig. 8 Phase trajectory.

6 Conclusion

The paper demonstrates the use of a synergy-cybernetic approach for the design of a control law with FSM synchronization of Arneodo chaotic systems under conditions of parametric disturbances. The article's goal has been achieved, and the effectiveness of the proposed control law (13) is clearly demonstrated.

To summarize the results, it is necessary to underline the following advantages of the proposed approach:

- more good compensation of parametric perturbations without design of perturbation observers;
- values of RMSE and AVG are significantly less than with classical SMC;
- ability to eliminate the chattering effect without losing robustness;
- in general, the sliding surface is set implicitly, since it may contain an unknown function $u_k\left(\mathbf{x}^{(\mathbf{k})}\right)$. The sliding surfaces of the classical method are always set explicitly, as a rule, in the form of a linear combination of state variables (or errors);
- control law (13) is less energy-consuming, since the control amplitude is very small in the steady-state;
- simpler analysis of the closed-loop system's stability, since the analysis is reduced to the analysis of the stability of the final decomposed system, usually of significantly smaller dimension, and the analysis of the stability of functional equations (8). In the classical method, the stability of a closed-loop system is analyzed for system with initial dimension.

The approach presented in this work will allow a new consideration for the design of robust control laws for chaotic systems, taking into account the ideas of directed self-organization and nonlinear robust control.

As part of the development of this work, the results obtained will be compared with other methods of modern control theory, and the application of this approach to other chaotic systems will be continued.

Appendix A

Let's introduce derivative for module function:

$$\dot{|x|} = \frac{d|x|}{dt} = \dot{x}(t)\,\mathrm{sign}(x). \tag{A.1}$$

According (A.1) we find derivative of ψ_1 (5):

$$\dot{\psi}_1(t) = \sum_{j=1}^{n-1} \beta_{1j}\,\mathrm{sign}(x_j)\dot{x}_j(t) + \dot{s}_1\,\mathrm{sign}(s_1), \tag{A.2}$$

and derivative of s_1 (6):

$$\dot{s}_1 = \sum_{j=1}^{n-1} \alpha_{1j}\dot{x}_j(t) + \dot{x}_n(t) + \sum_{j=1}^{n-1} \frac{\partial u_1}{\partial x_j}\dot{x}_j(t). \tag{A.3}$$

And now we can substitute (A.2) and (A.3) into (8) for $k = 1$:

$$T_1 \left(\sum_{j=1}^{n-1} \beta_{1j} \operatorname{sign}(x_j) \dot{x}_j(t) + \left[\sum_{j=1}^{n-1} \alpha_{1j} \dot{x}_j(t) + \dot{x}_n(t) + \right. \right.$$
$$\left. \left. + \sum_{j=1}^{n-1} \frac{\partial u_1}{\partial x_j} \dot{x}_j(t) \right] \operatorname{sign}(s_1) \right) + \psi_1 = 0.$$

To simplify, we devide left part by T_1 and multiply by $\operatorname{sign}(s_1)$:

$$\begin{aligned} &\operatorname{sign}(s_1) \sum_{j=1}^{n-1} \beta_{1j} \operatorname{sign}(x_j) \dot{x}_j(t) + \sum_{j=1}^{n-1} \alpha_{1j} \dot{x}_j(t) + \dot{x}_n(t) + \\ &+ \sum_{j=1}^{n-1} \frac{\partial u_1}{\partial x_j} \dot{x}_j(t) + \frac{1}{T_1} \psi_1 \operatorname{sign}(s_1) = 0. \end{aligned} \tag{A.4}$$

We substitute $\dot{x}_j(t)$ and $\dot{x}_n(t)$ from (4) in (A.4)

$$\begin{aligned} &\operatorname{sign}(s_1) \sum_{j=1}^{n-1} \beta_{1j} \operatorname{sign}(x_j) \Big(f_j(x_1, \ldots, x_n) + a_{j+1} x_{j+1} \Big) + \\ &+ \sum_{j=1}^{n-1} \alpha_{1j} \Big(f_j(x_1, \ldots, x_n) + a_{j+1} x_{j+1} \Big) + f_n(x_1, \ldots, x_n) + u + \\ &+ \sum_{j=1}^{n-1} \frac{\partial u_1}{\partial x_j} \dot{x}_j(t) + \frac{1}{T_1} \psi_1 \operatorname{sign}(s_1) = 0. \end{aligned} \tag{A.5}$$

As a result, from (A.5) we can get the expression (9).

Statements and Declarations

The author declares that he have no conflict of interest. The author has no relevant financial or non-financial interests to disclose. No funding was received to assist with the preparation of this manuscript.

Data Availability Statements

The author declares that all data supporting the findings of this study are available within the article. The model, control laws and their parameters are fully presented in the article. It is not difficult to perform their modeling in Matlab.

Also data were generated in article are available on request from the author.

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