Markus–Yamabe Conjecture for Asymptotic Stability of Planar Piecewise Linear Systems

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Markus–Yamabe conjecture for asymptotic stability of planar piecewise linear systems

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Abstract We present in this paper a detailed study on the Markus–Yamabe conjecture in planar piecewise linear systems. We consider discontinuous piecewise linear systems with two zones separated by a straight line, in which every subsystem is asymptotically stable. We prove the existence of limit cycles under explicit parameter conditions and give more different counterexamples to the Markus-Yamabe conjecture in addition to the counterexamples given by Llibre and Menezes. In particular, we consider continuous planar piecewise linear systems. For such a system with $n+1$ zones separated by n parallel straight lines in phase space, we prove that if each of subsystems is asymptotically stable, then this system has a globally asymptotically stable equilibrium point, therefore the Markus–Yamabe conjecture still holds. Some examples are given to illustrate the main results.

Keywords Piecewise linear systems · Limit cycle · sliding segment · Filippov systems · Markus–Yamabe conjecture

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1 Introduction and main results

The Markus–Yamabe conjecture [2,8,9,13,19,20,34,36,37] is that if all eigenvalues of the Jacobian matrix $Df(\mathbf{x})$ have negative real part for any $\mathbf{x} \in \mathbb{R}^n$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^1$ vector field with $f(\mathbf{0}) = \mathbf{0}$, then the origin is globally asymptotically stable. The conjecture is true when $n \leq 2$, which was completely solved by Fessler [13], Gutierrez [21] and Glutsyuk [20], and false [1, 11] when $n \geq 3$. For two-dimensional cases, it is of interest to study whether the Markus-Yamabe conjecture still holds if the differentiability condition of the vector field is weakened. Following Llibre and Menezes [28], we call a planar piecewise linear (PWL) Markus–Yamabe system if every subsystem is asymptotically stable.

For continuous planar PWL Markus–Yamabe systems with two zones separated by a straight line, the authors in [5] proved that the unique equilibrium point in the separation straight line is globally asymptotically stable and the authors in [40] proved that the Markus-Yamabe conjecture still holds no matter where the unique equilibrium point is. Recently, Llibre and Menezes [28] proved that discontinuous planar PWL Markus–Yamabe systems with two zones separated by a straight line have at most one limit cycle and gave some examples with one limit cycle, thus presented a counterexample to the Markus-Yamabe conjecture. As shown in [28], the theory of

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limit cycles [3,4,6,12,16–18,22–25,27,30–33] is an effective tool in studying the Markus–Yamabe conjecture.

Unlike [28] where a unique singularity of PWL systems lies on the switched line, we investigate the discontinuous planar PWL Markus–Yamabe systems containing no boundary singularities and obtain the explicit parameter conditions for the existence of limit cycles, thus giving more counterexamples to the Markus-Yamabe conjecture. Finally, we will prove that if a planar PWL Markus-Yamabe system with $n+1$ zones separated by n parallel straight lines is continuous, then this system has a unique equilibrium point as a global attractor, therefore the Markus–Yamabe conjecture still holds.

Now, we firstly consider the following discontinuous PWL system

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A}(\mathbf{x}-\mathbf{p}), & \text{if } x_1 > 0, \\ \mathbf{B}(\mathbf{x}-\mathbf{q}), & \text{if } x_1 < 0, \end{cases} \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2)^T, \mathbf{p} = (p_1, p_2)^T, \mathbf{q} = (q_1, q_2)^T \in \mathbb{R}^2$, and $\mathbf{A} = (a_{ij}), \mathbf{B} = (b_{ij}) \in \mathbb{R}^{2\times 2}$. The line $\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$ is a set of points of discontinuity.

If the transversal components of the two sub-vector fields at a point in Σ have the same sign, its trajectory crosses the boundary. On the contrary, if the transversal components of the two sub-vector fields are of opposite sign, the positive trajectory will therefore remain on Σ. Hence the sliding sets can be given as follows: the attractive sliding segment:

$$\Sigma_{as} = \{\mathbf{x} \in \Sigma \mid \mathbf{e}_1^T\mathbf{A}(\mathbf{x}-\mathbf{p}) < 0 < \mathbf{e}_1^T\mathbf{B}(\mathbf{x}-\mathbf{q})\},$$

the repulsive sliding segment:

$$\Sigma_{rs} = \{\mathbf{x} \in \Sigma \mid \mathbf{e}_1^T\mathbf{B}(\mathbf{x}-\mathbf{q}) < 0 < \mathbf{e}_1^T\mathbf{A}(\mathbf{x}-\mathbf{p})\}$$

where $\mathbf{e}_i, i = 1, 2$ denote the standard orthogonal basis of $\mathbb{R}^2$ and $\mathbf{e}_1^T$ denote the transpose of $\mathbf{e}_1$. The point $\bar{\mathbf{x}} = (0, \bar{x}_2)$ is said to be an invisible fold point for the right subsystem if

$$\mathbf{e}_1^T\mathbf{A}(\bar{\mathbf{x}}-\mathbf{p}) = 0, \ a_{12}\mathbf{e}_2^T\mathbf{A}(\bar{\mathbf{x}}-\mathbf{p}) < 0.$$

Similarly, the point $\bar{\mathbf{y}} = (0, \bar{y}_2)$ is said to be an invisible fold point for the left subsystem if

$$\mathbf{e}_1^T\mathbf{B}(\bar{\mathbf{y}}-\mathbf{q}) = 0, \ b_{12}\mathbf{e}_2^T\mathbf{B}(\bar{\mathbf{y}}-\mathbf{q}) > 0.$$

A matrix is said to be Hurwitz if all of its eigenvalues have negative real part. If the left or right subsystem has a boundary node, then system (1.1) has no periodic orbits. The case that one of subsystems has a boundary focus was discussed in [28]. Note that if $a_{12} = 0$ or $b_{12} = 0$, then the component of the corresponding vector field normal to Σ has the same sign. Thus system (1.1) can not have any periodic orbits. In this paper we consider the case that each subsystem has no boundary singularities and the main results are shown in the following:

Theorem 1 *For system* (1.1)*, suppose that all of the following conditions hold:*

(i) $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are Hurwitz and* $a_{12}b_{12} > 0$*;*
(ii) $p_1 < 0, q_1 > 0$, $\bar{\boldsymbol{x}} = (0, \bar{x}_2)$ *is an invisible fold point of the right subsystem and* $\bar{\boldsymbol{y}} = (0, \bar{y}_2)$ *is an invisible fold point of the left subsystem.*

Then the following holds,

(a) if $a_{12}(\bar{x}_2 - \bar{y}_2) < 0$*, then this system has only one periodic orbit which is a stable limit cycle and a repulsive sliding segment;*
(b) if $a_{12}(\bar{x}_2 - \bar{y}_2) = 0$*, then each trajectory tends to a pseudo-equilibrium point as* $t \to +\infty$*;*
(c) if $a_{12}(\bar{x}_2 - \bar{y}_2) > 0$*, then each trajectory enters an attractive sliding segment for some positive time and tends to a stable pseudo-equilibrium point as* $t \to +\infty$*.*

Theorem 1 will be proved in Section 2. This theorem shows that system (1.1) undergoes a pseudo-Hopf bifurcation [7] at $\bar{x}_2 = \bar{y}_2$.

Secondly, we consider continuous planar PWL systems with each subsystem asymptotically stable. Recall that if the planar PWL system is continuous on the dividing line and each of subsystems is asymptotically stable, then this system has a globally asymptotically stable equilibrium point [40]. This shows that the Markus-Yamabe conjecture is true. Now we consider more general case: is the Markus-Yamabe conjecture is true for a continuous planar PWL system with $n+1$ zones separated by n parallel straight lines?

Precisely, let $\Gamma_i, i = 1, 2, \cdots, n+1$ denote the $n+1$ zones separated by n parallel straight lines $\Sigma_i,\ i = 1, 2, \cdots, n$, and consider the following continuous PWL system

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A}_1(\mathbf{x}-\mathbf{p}_1), \text{ if } \mathbf{x} \in \Gamma_1, \\ \mathbf{A}_2(\mathbf{x}-\mathbf{p}_2), \text{ if } \mathbf{x} \in \Gamma_2, \\ \vdots, \\ \mathbf{A}_{n+1}(\mathbf{x}-\mathbf{p}_{n+1}), \text{ if } \mathbf{x} \in \Gamma_{n+1}, \end{cases} \tag{1.2}$$

where $\mathbf{x} = (x_1, x_2)^T, \mathbf{p}_i \in \mathbb{R}^2, \mathbf{A}_i \in \mathbb{R}^{2\times 2}, i = 1, 2, \cdots, n+1$. It is easy to see that system (1.2) is continuous if $\mathbf{A}_i(\mathbf{x}-\mathbf{p}_i) = \mathbf{A}_{i+1}(\mathbf{x}-\mathbf{p}_{i+1})$ for any point $\mathbf{x} \in \Sigma_i,\ i = 1, 2, \cdots, n$.

We have affirmative answer to the above question.

Theorem 2 *System* (1.2) *has a globally asymptotically stable equilibrium point if for each* $i \in \{1, 2, \cdots, n\}$, $\boldsymbol{A}_i$ *is Hurwitz and* $\boldsymbol{A}_i(\boldsymbol{x}-\boldsymbol{p}_i) = \boldsymbol{A}_{i+1}(\boldsymbol{x}-\boldsymbol{p}_{i+1})$ *for any point* $\boldsymbol{x} \in \Sigma_i$.

The proof of this theorem is given in Section 3. Some examples and simulations are provided in Section 4.

2 Proofs of Theorem 1

2.1 Preliminaries

To pave the way for the proof of Theorem 1, we need some preliminaries.

System (1.1) can be classified according to the type of the singular point [29]: a focus (F); a diagonalizable node (N) with distinct eigenvalues; an improper node (iN), i.e. a non-diagonalizable node. The right subsystem has a boundary singularity if $p_1 = 0$, a virtual singularity if $p_1 < 0$ and a real singularity if $p_1 > 0$. The subscript v for the singularity type represents a virtual singularity, e.g., F_v a virtual focus. Denote by $\varphi_{\mathbf{A}}(t, \mathbf{x}_0)$ and $\varphi_{\mathbf{B}}(t, \mathbf{x}_0)$ the solutions generated by the right and left subsystems given by

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}-\mathbf{p}), \mathbf{x} \in \mathbb{R}^2,$$

$$\dot{\mathbf{x}} = \mathbf{B}(\mathbf{x}-\mathbf{q}), \mathbf{x} \in \mathbb{R}^2,$$

with initial condition $\varphi_{\mathbf{A}}(0, \mathbf{x}_0) = \mathbf{x}_0$ and $\varphi_{\mathbf{B}}(0, \mathbf{x}_0) = \mathbf{x}_0$, respectively.

Without loss of generality, set $a_{12} > 0$. Let

$$\begin{aligned} I_1 =& \{(0, x_2) \mid \bar{y}_2 < x_2 < +\infty\},\ I_2 = \{(0, x_2) \mid -\infty < x_2 < \bar{y}_2\}, \\ I_3 =& \{(0, x_2) \mid -\infty < x_2 < \bar{x}_2\},\ I_4 = \{(0, x_2) \mid \bar{x}_2 < x_2 < \infty\}. \end{aligned}$$

It is easy to see that the positive trajectories of the points in I_2 under the flow of the left subsystem intersect Σ first in I_1 and the positive trajectories of the points in I_4 under the flow of the right subsystem intersect Σ first in I_3.

Let $P_{\mathbf{B}}$ denote the map that associates the points in I_2 with their points of first return to Σ under the flow of the left subsystem, which is given by

$$P_{\mathbf{B}} : (0, x_2) \mapsto\ (0, h_1(x_2)) = \varphi_{\mathbf{B}}(\tau(x_2), (0, x_2)),$$

where $\tau(x_2) > 0$ is the time of first return of the point $(0, x_2)$ to Σ. Similarly, the map $P_{\mathbf{A}}$ that associates the points in I_4 with their points of first return to Σ under the flow of the right subsystem can be given by

$$P_{\mathbf{A}} : (0, z_2) \mapsto\ (0, h_2(z_2)) = \varphi_{\mathbf{A}}(\tau(z_2), (0, z_2)),\ \tau(z_2) > 0. \tag{2.1}$$

Set

$$\mu_{\mathbf{B}}(x_2) = \frac{dh_1}{dx_2}(x_2),\ \mu_{\mathbf{A}}(z_2) = \frac{dh_2}{dz_2}(z_2).$$

If $\bar{y}_2 \geq \bar{x}_2$, then $I_1 \subset I_4, I_3 \subset I_2$ and a Poincaré map can be defined by

$$\begin{aligned} P_{\mathbf{A}} \circ P_{\mathbf{B}} : I_2 &\rightarrow I_2 \\ (0, x_2) &\mapsto (0, P(x_2)) \end{aligned} \tag{2.2}$$

where $P(x_2) = h_2(h_1(x_2))$.

Based on the chain rule, we have

$$\frac{dP(x_2)}{dx_2} = \frac{dh_2}{dz_2}(h_1(x_2))\frac{dh_1}{dx_2}(x_2) = \mu_{\mathbf{A}}(h_1(x_2))\mu_{\mathbf{B}}(x_2).$$

In order to prove the statement (a) of Theorem 1, we first prove the following proposition.

Proposition 1 *Suppose that $a_{12} > 0$. If the left subsystem has a stable singularity of type L and the right subsystem of system* (1.1) *has a stable singularity of type R where $L, R \in \{F_v, N_v, iN_v\}$, then*

$$0 < \frac{dP(x_2)}{dx_2} < 1, x_2 < \bar{x}_2.$$

The proof of this proposition is completed if we show the following several results.

Lemma 1 *$\mu_{\boldsymbol{B}}(x_2)$ and $\mu_{\boldsymbol{A}}(z_2)$ are invariant under any vertical lines-preserving linear change of variables, time-rescaling or translation along the y axis.*

The proof of this lemma is straightforward and so is omitted.

Lemma 2 *Suppose that $a_{12} > 0$. If the right subsystem of system* (1.1) *has a stable singularity of type R where $R \in \{F_v, N_v, iN_v\}$, then $-1 < \mu_{\boldsymbol{A}}(z_2) < 0, z_2 > \bar{x}_2$.*

Proof If the right subsystem of system (1.1) has a stable singularity, then $\mathbf{A}$ is Hurwitz. According to Proposition 5 of [29], the right subsystem of (1.1) can be transformed into the following simple normal form

$$\dot{\mathbf{x}} = \mathbf{M}\begin{pmatrix} x_1 - p_1 \\ x_2 \end{pmatrix}, \mathbf{M} \in \{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$$

where

$$\begin{aligned} \mathbf{M}_1 &= \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix}, \text{ with } a < -1; \\ \mathbf{M}_2 &= \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix}, \text{ with } a < 0; \\ \mathbf{M}_3 &= \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}, \text{ with } \lambda = -1, \end{aligned} \tag{2.3}$$

by a vertical lines-preserving linear transformation which means a linear transformation of variables in the plane preserving the vertical lines.

If $\mathbf{M} = \mathbf{M}_1$, then it is easy to get that

$$\varphi_{\mathbf{M}}(t, \mathbf{x}_0) = e^{at}\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}(\mathbf{x}_0 - \mathbf{p}) + \mathbf{p}.$$

From (2.1), we have

$$h_2(z_2) = \frac{p_1(e^{at_z} - \cosh t_z)}{\sinh t_z}$$

where t_z satisfies

$$z_2 = \frac{p_1(\cosh t_z - e^{-at_z})}{\sinh t_z}.$$

According to the proof of Proposition 3 in [40], we have

$$\mu_{\mathbf{M}_1}(z_2) = -\frac{1 + e^{at_z(z_2)}(a \sinh t_z(z_2) - \cosh t_z(z_2))}{1 - e^{-at_z(z_2)}(a \sinh t_z(z_2) + \cosh t_z(z_2))},$$

and

$$-1 < \mu_{\mathbf{M}_1}(z_2) < 0.$$

If $\mathbf{M} = \mathbf{M}_2$, then

$$h_2(z_2) = \frac{p_1(e^{at_z} - \cos t_z)}{\sin t_z}$$

where t_z satisfies

$$z_2 = \frac{p_1(\cos t_z - e^{-at_z})}{\sin t_z}.$$

From the proof of Proposition 4 in [40], we have

$$\mu_{\mathbf{M}_2}(z_2) = -\frac{1 + e^{at_z(z_2)}(a \sin t_z(z_2) - \cos t_z(z_2))}{1 - e^{-at_z(z_2)}(a \sin t_z(z_2) + \cos t_z(z_2))},$$

and

$$-1 < \mu_{\mathbf{M}_2}(z_2) < 0.$$

If $\mathbf{M} = \mathbf{M}_3$, then

$$h_2(z_2) = \frac{p_1(e^{\lambda t_z} - 1)}{\lambda t_z}$$

and t_z satisfies

$$z_2 = \frac{p_1(1 - e^{-\lambda t_z})}{\lambda t_z}.$$

Based on the proof of Proposition 5 in [40], we have

$$\mu_{\mathbf{M}_3}(z_2) = -\frac{1 + e^{\lambda t_z(z_2)}(\lambda t_z(z_2) - 1)}{1 - e^{-\lambda t_z(z_2)}(\lambda t_z(z_2) + 1)},$$

and

$$-1 < \mu_{\mathbf{M}_3}(z_2) < 0.$$

According to Lemma 1, $\mu_{\mathbf{A}}(z_2)$ is invariant under any vertical lines-preserving linear change of variables, time-rescaling or translation along the y axis. Therefore, this lemma is proved.

Based on Lemma 1 and Lemma 2, we have the following result.

Lemma 3 *Suppose that $b_{12} > 0$. If the left subsystem of system* (1.1) *has a stable singularity of type L where $L \in \{F_v, N_v, iN_v\}$, then $-1 < \mu_{\boldsymbol{B}}(x_2) < 0, x_2 < \bar{y}_2$.*

Using Lemma 2 and Lemma 3, Proposition 1 can be proved easily.

2.2 Proof of Theorem 1

We now give the proof of Theorem 1.

Proof Without loss of generality, assume that $a_{12} > 0$.

If $\bar{y}_2 > \bar{x}_2$, then it is easy to find that the set $\{(0, x_2) \mid \bar{x}_2 < x_2 < \bar{y}_2\}$ is a repulsive sliding segment. Moreover, we have from Proposition 1 that for each point $\mathbf{x}_0 \in I_3$, the sequence $\{(P_{\mathbf{A}} \circ P_{\mathbf{B}})^n(\mathbf{x}_0)\}_{n=0}^{+\infty}$ has a limit which corresponds to the fixed point of $P_{\mathbf{A}} \circ P_{\mathbf{B}}$ where $P_{\mathbf{A}} \circ P_{\mathbf{B}}$ is given by (2.2). Based on Banach Contraction Theorem [10], the continuous map $P_{\mathbf{A}} \circ P_{\mathbf{B}}$ has only one fixed point in I_3. This means that system (1.1) has only a periodic orbit. It follows from Proposition 1 that this periodic orbit is a stable limit cycle. Thus the statement (a) is proved.

If $\bar{y}_2 = \bar{x}_2$, then the invisible fold point of the left subsystem is also the invisible fold point of the right subsystem. According to the definition of Filippov vector field [15], we define the Filippov vector field at $\bar{\mathbf{x}}$ as $f^0(\bar{\mathbf{x}}) = \mathbf{0}$ which means $\bar{\mathbf{x}}$ is a pseudo-equilibrium point of system (1.1) and $P_{\mathbf{A}} \circ P_{\mathbf{B}}(\bar{\mathbf{x}}) = \bar{\mathbf{x}}$. Similar to the proof of statement (a), we have that $P_{\mathbf{A}} \circ P_{\mathbf{B}}$ has only one fixed point and this point is asymptotically stable. Therefore, each trajectory tends to the equilibrium point $\bar{\mathbf{x}}$ as $t \to +\infty$.

To prove the statement (c), let

$$\begin{aligned} I_{\mathbf{A},0} &= \{(0, x_2) \mid \bar{y}_2 < x_2 \le \bar{x}_2\}, \\ I_{\mathbf{B},0} &= \{(0, x_2) \mid \bar{y}_2 \le x_2 < \bar{x}_2\}, \\ I_{\mathbf{A},i+1} &= P_{\mathbf{A}}^{-1}(I_{\mathbf{B},i}), \\ I_{\mathbf{B},i+1} &= P_{\mathbf{B}}^{-1}(I_{\mathbf{A},i}), i \in \mathbb{N}. \end{aligned} \tag{2.4}$$

Similar to the proof of Theorem 1 in [40], we get that each trajectory intersects $I_{\mathbf{A},i}$ or $I_{\mathbf{B},i}$, $i \in \mathbb{N}$ for some positive time and will eventually fall into the segment $\{(0, x_2) \in \mathbb{R}^2 \mid \bar{y}_2 \le x_2 \le \bar{x}_2\}$ at a large time and remain for all later times. According to Filippov convex method [14],[15],[26], along the attractive sliding set,

$$\Sigma_{as} = \{(0, x_2) \in \mathbb{R}^2 \mid \bar{y}_2 < x_2 < \bar{x}_2\}$$

the sliding motion can be defined by

$$\dot{\mathbf{x}} = f^0(\mathbf{x}) = (1 - \alpha(\mathbf{x}))\mathbf{A}(\mathbf{x} - \mathbf{p}) + \alpha(\mathbf{x})\mathbf{B}(\mathbf{x} - \mathbf{q}) = (0, f_2^0(\mathbf{x}))^T, \ \mathbf{x} \in \Sigma_{as} \tag{2.5}$$

where

$$\alpha(\mathbf{x}) = \frac{\mathbf{e}_1^T \mathbf{A}(\mathbf{x} - \mathbf{p})}{\mathbf{e}_1^T (\mathbf{A}(\mathbf{x} - \mathbf{p}) - \mathbf{B}(\mathbf{x} - \mathbf{q}))}.$$

Let

$$f_2^0(\bar{\mathbf{x}}) = \mathbf{e}_2^T \mathbf{A}(\mathbf{x} - \mathbf{p}), f_2^0(\bar{\mathbf{y}}) = \mathbf{e}_2^T \mathbf{B}(\mathbf{x} - \mathbf{q}).$$

It is easy to find that

$$f_2^0(\bar{\mathbf{x}}) \cdot f_2^0(\bar{\mathbf{y}}) < 0. \tag{2.6}$$

From (2.5) we get that the numerator of $f_2^0(\mathbf{x})$ is polynomial of degree 2 in x_2. According to (2.6), we have that $f_2^0(\mathbf{x})$ has two zeros and only one zero $\mathbf{x}_e = (0, x_{e,2})^T$ lies in Σ_{as} which is a pseudo-equilibrium point of system (1.1). Since

$$\begin{aligned} f_2^0(\mathbf{x}) < 0, \text{ for } \mathbf{x} = (0, x_2), x_2 \in (x_{e,2}, \bar{x}_2], \\ f_2^0(\mathbf{x}) > 0, \text{ for } \mathbf{x} = (0, x_2), x_2 \in [\bar{y}_2, x_{e,2}), \end{aligned}$$

we obtain that $\mathbf{x}_e$ is asymptotically stable and each trajectory in the sliding segment tends to $\mathbf{x}_e$ as $t \to +\infty$.

Therefore, Theorem 1 is proved.

3 Proof of Theorem 2

3.1 preliminaries

It is easy to find that the linear transformation of variables can be done to change the angle of n parallel lines such that these n lines are parallel to the y axis. Let $\Sigma_i = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = a_i\}$, $i = 1, 2, \cdot, n$ be the n switched lines where $a_n < a_{n-1} < \cdots < a_2 < a_1$. Therefore, system (1.2) can be transformed into the following system

$$\dot{\mathbf{x}} = \mathbf{A}_i(\mathbf{x} - \mathbf{p}_i), \text{ if } \mathbf{x} \in \Gamma_i, \ i = 1, 2, \cdots, n+1 \tag{3.1}$$

where $\mathbf{p}_i = (p_{i,1}, p_{i,2})^T \in \mathbb{R}^2, \mathbf{A}_i \in \mathbb{R}^{2\times 2}$ and

$$\begin{aligned} \Gamma_1 =& \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > a_1\}, \\ \Gamma_i =& \{(x_1, x_2) \in \mathbb{R}^2 \mid a_i < x_1 < a_{i-1}\}, \ i = 2, 3, \cdots, n \\ \Gamma_{n+1} =& \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < a_n\}. \end{aligned}$$

Denote by $\Phi(t, \mathbf{x}_0)$ the solutions of system (3.1) with initial condition $\Phi(0, \mathbf{x}_0) = \mathbf{x}_0$. The singular point $\mathbf{p}_i$ is called a real singular point when $\mathbf{p} \in \Gamma_i$, a virtual singular point when $\mathbf{p}_i \notin \Gamma_i$.

To complete the proof of Theorem 2, we need to prove the following propositions.

Proposition 2 *System* (3.1) *has a unique equilibrium point if for each* $i \in \{1, 2, \cdots, n\}$, $\boldsymbol{A}_i$ *is Hurwitz and* $\boldsymbol{A}_i(\boldsymbol{x} - \boldsymbol{p}_i) = \boldsymbol{A}_{i+1}(\boldsymbol{x} - \boldsymbol{p}_{i+1})$ *for any point* $\boldsymbol{x} \in \Sigma_i$.

Proof According to $\mathbf{A}_{i+1}(\mathbf{x}-\mathbf{p}_{i+1}) = \mathbf{A}_i(\mathbf{x}-\mathbf{p}_i)$ for $\mathbf{x} \in \Sigma_i$, we obtain

$$\mathbf{A}_{i+1}\mathbf{e}_2 = \mathbf{A}_i\mathbf{e}_2. \tag{3.2}$$

If $\mathbf{e}_1^T\mathbf{A}_i\mathbf{e}_2 \neq 0$, then we find from (3.2) that

$$p_{i+1,1} - a_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A}_{i+1})}(p_{i,1} - a_i) \tag{3.3}$$

where $\det(\mathbf{A}_i)\det(\mathbf{A}_{i+1}) > 0$ since $\mathbf{A}_{i+1}$ and $\mathbf{A}_i$ are Hurwitz. If $\mathbf{e}_1^T\mathbf{A}_i\mathbf{e}_2 = 0$, then from (3.2) we have

$$p_{i+1,1} - a_i = \frac{\mathbf{e}_1^T\mathbf{A}_i\mathbf{e}_1}{\mathbf{e}_1^T\mathbf{A}_{i+1}\mathbf{e}_1}(p_{i,1} - a_i). \tag{3.4}$$

Combining (3.3) and (3.4), we obtain that

$$(p_{i+1,1} - a_i)(p_{i,1} - a_i) > 0, \text{ or } p_{i+1,1} = a_i = p_{i,1}. \tag{3.5}$$

If $\mathbf{p}_i, i = 1, 2, \cdots, n$ are all virtual singular points, then according to (3.5) we get that

$$\begin{aligned}
&p_{1,1} - a_1 < 0, \ p_{2,1} - a_1 < 0,\\
&p_{2,1} - a_2 < 0, \ p_{3,1} - a_2 < 0,\\
&\vdots\\
&p_{n-1,1} - a_{n-1} < 0, \ p_{n,1} - a_{n-1} < 0,\\
&p_{n,1} - a_n < 0, \ p_{n+1,1} - a_n < 0,
\end{aligned}$$

which implies that system (3.1) has a real singular point $\mathbf{p}_{n+1}$. Thus the case that all subsystems have virtual singular points can not happen, that is system (3.1) has at least one real singular point.

If the subsystem of system (3.1) has a real equilibrium point $\mathbf{p}_{k_1}$ in Γ_{k_1} where $1 \le k_1 \le n+1$, then

i) for $k_1 = 1$, we have

$$p_{j,1} > a_{j-1} > a_j, \ 1 < j \le n, \ p_{n+1,1} > a_n,$$

which implies that $\mathbf{p}_i, i = 2, 3, \cdots, n+1$ are virtual singular points,

ii) for $k_1 = n+1$, we have

$$p_{j,1} < a_j, \ 1 \le j \le n,$$

which means that $\mathbf{p}_i, i = 1, 2, 3, \cdots, n$ are virtual singular points.

iii) for $1 < k_1 \le n$, we have

$$\begin{aligned}
&p_{j,1} > a_{j-1} > a_j, \ 1 \le j < k_1,\\
&p_{j,1} < a_j, \ k_1 < j \le n,\\
&p_{n+1,1} < a_n.
\end{aligned}$$

It follows that $\mathbf{p}_i, 1 \le i \le n+1, i \neq k_1$ are virtual singular points.

Similarly, we get that if the subsystem of system (3.1) has a boundary singular point $\mathbf{p}_{k_2}$ where $\mathbf{p}_{k_2} = \mathbf{p}_{k_2+1}$ in Σ_{k_2} where $1 \le k_2 \le n$, then $\mathbf{p}_i, 1 \le i \le n+1, i \notin \{k_2, k_2+1\}$ are virtual singular points.

Therefore, system (3.1) has a unique real singular point.

Proposition 3 *System* (3.1) *has no periodic orbits if for each* $i \in \{1, 2, \cdots, n\}$, $\boldsymbol{A}_i$ *is Hurwitz and* $\boldsymbol{A}_i(\boldsymbol{x}-\boldsymbol{p}_i) = \boldsymbol{A}_{i+1}(\boldsymbol{x}-\boldsymbol{p}_{i+1})$ *for any point* $\boldsymbol{x} \in \Sigma_i$.

Proof Suppose that system (3.1) has a periodic orbit γ. Let R^γ be the bounded region limited by γ and let

$$\begin{aligned}
F_1(\mathbf{x}) =& \mathbf{e}_1^T\mathbf{A}_i(\mathbf{x}-\mathbf{p}_i), \ \text{if } \mathbf{x} \in \Gamma_i, \ i = 1, 2, \cdots, n+1,\\
F_2(\mathbf{x}) =& \mathbf{e}_2^T\mathbf{A}_i(\mathbf{x}-\mathbf{p}_i), \ \text{if } \mathbf{x} \in \Gamma_i, \ i = 1, 2, \cdots, n+1.
\end{aligned}$$

Then we have

$$\oint_{\gamma} F_1(\mathbf{x})dx_2 - F_2(\mathbf{x})dx_1 = 0. \tag{3.6}$$

where

$$F_1(\mathbf{x})dx_2 - F_2(\mathbf{x})dx_1 = F_1(\mathbf{x})\dot{x}_2 dt - F_2(\mathbf{x})\dot{x}_1 dt \equiv 0, \text{ for any } \mathbf{x} \in \gamma$$

since γ is a periodic orbit of system (3.1).

Since R^γ can be separated into m regions $R_1^\gamma, R_2^\gamma, \cdots, R_m^\gamma$ by switched lines $\Sigma_i,\ i = 1, 2, \cdots, n$ and $F_1(\mathbf{x}), F_2(\mathbf{x})$ are continuous, we have

$$\oint_{\gamma} F_1(\mathbf{x})dx_2 - F_2(\mathbf{x})dx_1 = \sum_{i=1}^{m} \oint_{\partial R_i^\gamma} F_1(\mathbf{x})dx_2 - F_2(\mathbf{x})dx_1.$$

According to the Green's theorem [31,35], we have

$$\oint_{\partial R_i^\gamma} F_1(\mathbf{x})dx_2 - F_2(\mathbf{x})dx_1 = \iint_{R_i^\gamma} \frac{\partial F_1(\mathbf{x})}{\partial x_1} + \frac{\partial F_2(\mathbf{x})}{\partial x_2} d\mathbf{x} = \iint_{R_i^\gamma} \mathrm{tr}(\mathbf{A}_{n_i})d\mathbf{x},$$

where $\mathrm{tr}(\cdot)$ denotes the trace of matrix and $\{n_i, i = 1, 2, \cdots, m\}$ is a subsequence of $\{1, 2, 3, \cdots, n+1\}$. Since $\mathbf{A}_i,\ i = 1, 2, 3, \cdots, n+1$ are all Hurwitz, we get $\mathrm{tr}(\mathbf{A}_i) < 0,\ i = 1, 2, 3, \cdots, n+1$. Thus we obtain

$$\oint_{\gamma} F_1(\mathbf{x})dx_2 - F_2(\mathbf{x})dx_1 = \sum_{i=1}^{m} \oint_{\partial R_i^\gamma} F_1(\mathbf{x})dx_2 - F_2(\mathbf{x})dx_1 < 0. \tag{3.7}$$

Combining (3.6) and (3.7) gives a contradiction. Therefore, the proof of Proposition 3 is completed.

Proposition 4 *Consider a general linear system*

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{p}) \tag{3.8}$$

where $\boldsymbol{x} = (x_1, x_2)^T, \boldsymbol{p} = (p_1, p_2)^T \in \mathbb{R}^2, \boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{2\times 2}$, *let* $\varphi_{\boldsymbol{A}}(t, \boldsymbol{x}_0)$ *denote the solution of system* (3.8) *with initial condition* $\varphi_{\boldsymbol{A}}(0, \boldsymbol{x}_0) = \boldsymbol{x}_0$. *For any* $m_2 > m_1 > p_1$, *let* $\Sigma_1^0 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = m_1\}, \Sigma_2^0 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = m_2\}$ *be two lines parallel to the y-axes and* Γ_0 *be the region between* Σ_1^0 *and* Σ_2^0. *If* $\boldsymbol{A}$ *is Hurwitz, then for each point* $\boldsymbol{x}_0 \in \Sigma_2^0$, *there exists a* $t_1 > 0$ *such that* $\varphi_{\boldsymbol{A}}(t_1, \boldsymbol{x}_0) \in \Sigma_1^0$. *Moreover, if* $\boldsymbol{e}_1^T \boldsymbol{A}(\boldsymbol{x}_0 - \boldsymbol{p}) \le 0,\ \boldsymbol{x}_0 \in \Sigma_2^0$, *then there exists a* $t_2 > 0$ *such that* $\varphi_{\boldsymbol{A}}(t, \boldsymbol{x}_0) \in \Gamma_0$ *for* $0 < t < t_2$ *with* $\varphi_{\boldsymbol{A}}(t_2, \boldsymbol{x}_0) \in \Sigma_1^0$.

Proof If $a_{12} = 0$, then the proof is trivial and so is omitted. If $a_{12} \neq 0$, according to Proposition 5 of [29], system (3.8) can be transformed into a simple normal form. Hence it suffices to consider the case that $\mathbf{A} \in \{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ where $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ are given by (2.3). Suppose that $\mathbf{x}_0 = (x_{0,1}, x_{0,2}) \in \Sigma_2^0$ where

$$x_{0,1} = m_2 > m_1 > p_1.$$

For convenience, let

$$\delta_1 = x_{0,1} - p_1, \delta_2 = x_{0,2} - p_2.$$

If $\mathbf{A} = \mathbf{M}_1$, it is easy to obtain that the x_1-component of solution $\varphi_{\mathbf{A}}(t, \mathbf{x}_0)$ which is given by

$$g_1(t) = p_1 + \frac{\delta_1 + \delta_2}{2} e^{(a+1)t} + \frac{\delta_1 - \delta_2}{2} e^{(a-1)t}.$$

Considering the derivative of $g(t)$ with respect to t, we get

$$g_1'(t) = \frac{1}{2} e^{(a-1)t} [(\delta_1 + \delta_2)(a+1)e^{2t} + (\delta_1 - \delta_2)(a-1)].$$

If $\delta_2 > -a\delta_1$, then $g_1'(t)$ has a unique zero with $g_1(0) > 0$. It follows that $g_1(t)$ increases first and then decreases with respect to t. In consideration of $g_1(t) \to p_1$ as $t \to +\infty$, there exists a $t_1 > 0$ such that $g_1(t_1) = m_1$.

If $\delta_2 < -\delta_1$, then $g_1'(t)$ has a unique zero

$$\bar{t} = \frac{1}{2} \ln \frac{(\delta_1 - \delta_2)(1-a)}{(\delta_1 + \delta_2)(1+a)}.$$

Thus we get the minimum of $g_1(t)$ as follows

$$g_1(\bar{t}) = p_1 + \frac{\delta_1 - \delta_2}{2(1+a)} e^{(a-1)\bar{t}} < p_1.$$

If $-\delta_1 \le \delta_2 \le -a\delta_1$, we have $g_1'(t) < 0$ which means that $g_1(t)$ decreases with respect to t. Therefore, if $\delta_2 \le -a\delta_1$ equivalent to the condition $\mathbf{e}_1^T \mathbf{A}(\mathbf{x}_0 - \mathbf{p}) \le 0$, then there exists a $t_2 > 0$ such that

$$g_1(t_2) = m_1,\ m_1 < g_1(t) < m_2,\ \text{for every } 0 < t < t_2.$$

If $\mathbf{A} = \mathbf{M}_2$, it is easy to find that the x_1-component of the solution $\varphi_{\mathbf{A}}(t, \mathbf{x}_0)$ is

$$g_2(t) = p_1 + e^{at}(\delta_1 \cos t + \delta_2 \sin t).$$

Owing to the property of trigonometric function $\delta_1 \cos t + \delta_2 \sin t$, we obtain that there exists a $t_3 > 0$ such that $g_2(t_3) = m_1$.

Taking the derivative of $g_2(t)$ gives

$$g_2'(t) = e^{at}[(a\delta_1 + \delta_2)\cos t + (-\delta + a\delta_2)\sin t] = e^{at}\sqrt{(1+a^2)(\delta_1^2 + \delta_2^2)}\,\sin(t+\theta)$$

where

$$\sin\theta = \frac{a\delta_1 + \delta_2}{\sqrt{(1+a^2)(\delta_1^2+\delta_2^2)}},\ \cos\theta = \frac{-\delta_1 + a\delta_2}{\sqrt{(1+a^2)(\delta_1^2+\delta_2^2)}}.$$

According to the monotonicity of $g_2'(t)$, we get that if $a\delta_1 + \delta_2 \le 0$ equivalent to $\mathbf{e}_1^T \mathbf{A}(\mathbf{x}_0 - \mathbf{p}) \le 0$, then $-\pi \le \theta < 0$ and $g_2(t)$ decreases until $t = -\theta$ which is a minimum point of $g_2(t)$. Since

$$g_2(-\theta) = p_1 - e^{-a\theta}\sqrt{\frac{\delta_1^2 + \delta_2^2}{1+a^2}} < p_1,$$

there exists a $t_4 > 0$ such that

$$g_2(t_4) = m_1,\ m_1 < g_2(t) < m_2,\ \text{for every } 0 < t < t_4.$$

If $\mathbf{A} = \mathbf{M}_3$, it can be easily seen that the x_1-component of the solution $\varphi_{\mathbf{A}}(t, \mathbf{x}_0)$ is

$$g_3(t) = p_1 + (\delta_1 + \lambda\delta_2 t)e^{\lambda t}$$

and the derivative of $g_3(t)$ is

$$g_3'(t) = \lambda(\delta_1 + \delta_2 + \lambda\delta_2 t)e^{\lambda t}. \tag{3.9}$$

If $\delta_2 < -\delta_1$, then we find from (3.9) that $g_3(t)$ increases first and then decreases. Note that $g_3(t) \to p_1$ as $t \to +\infty$ which implies that there exists a $t_5 > 0$ such that $g_3(t_5) = m_1$. If $-\delta_1 \le \delta_2 < 0$, then $g_3(t)$ decreases with respect to t. If $\delta_2 > 0$, then $g_3(t)$ decreases first and has a minimum at $\bar{t} = -\frac{\delta_1 + \delta_2}{\lambda\delta_2}$ where

$$g_3(\bar{t}) = p_1 - \delta_2 e^{\lambda\bar{t}} < p_1.$$

Thus we conclude that if $-\delta_1 \le \delta_2$ equivalent to $\mathbf{e}_1^T \mathbf{A}(\mathbf{x}_0 - \mathbf{p}) \le 0$, then there exists a $t_6 > 0$ such that

$$g_3(t_6) = m_1,\ m_1 < g_3(t) < m_2,\ \text{for every } 0 < t < t_6.$$

Therefore, the proof is completed.

3.2 Proof of Theorem 2

Proof If $\mathbf{e}_1^T\mathbf{A}_1\mathbf{e}_2 = 0$, then the conclusions are obviously true. Thus we only need to prove the case of $\mathbf{e}_1^T\mathbf{A}_1\mathbf{e}_2 \neq 0$. Without loss of generality, set $\mathbf{e}_1^T\mathbf{A}_1\mathbf{e}_2 > 0,\ \mathbf{e}_2^T\mathbf{A}_1\mathbf{e}_2 < 0$.

According to Proposition 2, system (3.1) has a unique equilibrium point. We shall prove the case $k_1 = 2$. That is system (3.1) has a unique equilibrium point $\mathbf{p}_2$ in $\varGamma_2$. The other cases can be proved similarly, so we omit it.

For convenience, we define the following regions:

$$\begin{aligned}
\mathrm{R}_2^s =& \{\mathbf{x}_0 \in \varGamma_2 \mid \text{the trajectory starting at } \mathbf{x}_0 \text{ remains in } \varGamma_2 \text{ for all positive time}\},\\
\mathrm{R}_i^r =& \{\mathbf{x}_0 \in \varGamma_i \mid \text{the positive trajectory starting at } \mathbf{x}_0 \text{ leaves } \varGamma_i \text{ through the line } \varSigma_{i-1}\},\\
\mathrm{R}_i^l =& \{\mathbf{x}_0 \in \varGamma_i \mid \text{the positive trajectory starting at } \mathbf{x}_0 \text{ leaves } \varGamma_i \text{ through the line } \varSigma_i\},
\end{aligned}$$

where $i = 2, 3, 4, \cdots, n$.

Based on Proposition 4 and asymptotic stability of subsystems, we get the following results, as illustrated in Figure 1:

1) each trajectory starting in $\varGamma_1$ will enter one of the regions $\mathrm{R}_2^s, \mathrm{R}_2^r, \mathrm{R}_2^l$ for some positive time;
2) each positive trajectory starting in R_i^l will enter either R_{i+1}^r or R_{i+1}^l for every $i \in \{2, 3, \cdots, n\}$;
3) each positive trajectory starting in $\varGamma_{n+1}$ can only enter R_n^r and each positive trajectory starting in the region R_i^r must enter R_{i-1}^r where $i = 4, 5, \cdots, n$. Thus each trajectory starting in $\varGamma_{n+1}$ or $\bigcup_{i=3}^n \mathrm{R}_i^r$ will enter R_2^s or R_2^r for some positive time;
4) each trajectory starting in R_2^s converges to $\mathbf{p}_2$ as $t \to +\infty$.

$$\begin{tikzcd}
& & & & & \mathrm{R}_2^s & \\
\Gamma_{n+1} \arrow[r] & \mathrm{R}_n^r \arrow[r] & \cdots\cdots \arrow[r] & \mathrm{R}_4^r \arrow[r] & \mathrm{R}_3^r \arrow[r] \arrow[ur] & \mathrm{R}_2^r \arrow[r, shift left] & \Gamma_1 \arrow[l, shift left] \arrow[ul] \arrow[dl] \\
& \mathrm{R}_n^l \arrow[ul] & \cdots\cdots \arrow[l] \arrow[ul] & \mathrm{R}_4^l \arrow[l] \arrow[ul] & \mathrm{R}_3^l \arrow[l] \arrow[ul] & \mathrm{R}_2^l \arrow[l] \arrow[ul] &
\end{tikzcd}$$

Fig. 1: Each trajectory of system (3.1) staring in one region may enter the region that the arrow points to for some positive time.

Note that if there exists an arbitrarily large positively invariant set, then according to Poincaré–Bendixson Theorem [39] each trajectory starting in this set approaches either the equilibrium point $\mathbf{p}_2$ or a periodic orbit as $t \to +\infty$. According to Proposition 3, system (3.1) has no periodic orbits. Therefore, each trajectory starting in $\mathbb{R}^2\backslash\{\mathbf{p}_2\}$ approaches the equilibrium point $\mathbf{p}_2$ as $t \to +\infty$.

To complete the proof, it suffices to prove that for any compact set $\mathcal{K}$ containing the equilibrium point $\mathbf{p}_2$, there exists a positively invariant set $\mathcal{M}_{\mathcal{K}} \supset \mathcal{K}$. Now we give the construction process of the positively invariant sets.

We need to define several maps.

Recall that for any given positive definite matrices $\mathbf{W}_i,\ i = 1, 2, 3, \cdots, n+1$, there exist positive definite matrices $\mathbf{Q}_i$, such that

$$\mathbf{A}_i^T\mathbf{Q}_i + \mathbf{Q}_1\mathbf{A}_i = -\mathbf{W}_i.$$

It is easy to check that

$$V_i(\mathbf{x}) = (\mathbf{x} - \mathbf{p}_i)^T\mathbf{Q}_i(\mathbf{x} - \mathbf{p}_i)$$

is a quadratic Lyapunov function of i-th subsystem. Let

$$V_i^{-1}(c) = \{\mathbf{x} \in \mathbb{R}^{2\times 2} \mid V_i(\mathbf{x}) = c\}, c > 0$$

be the level set. It is easy to see that $V_i^{-1}(c)$ is convex. See [38] for more details.

For any point $\mathbf{x}_1 = (x_{1,1}, x_{1,2})^T \in \Sigma_n$ satisfying $\mathbf{e}_1^T \mathbf{A}_n(\mathbf{x}_1 - \mathbf{p}_n) > 0$, according to Proposition 4 the positive trajectory of $\mathbf{x}_1$ intersects Σ_i at some positive time for $i = 2, 3, \cdots, n-1$. Precisely, there exist

$$0 < \tau_1(\mathbf{x}_1) < \tau_2(\mathbf{x}_1) < \cdots < \tau_{n-2}(\mathbf{x}_1)$$

such that

$$\Phi(\tau_k(\mathbf{x}_1), \mathbf{x}_1) \in \Sigma_{n-k},\ k = 1, 2, \cdots, n-2.$$

Define $\mathcal{G}_1$ and γ_1 as

$$\mathcal{G}_1(\mathbf{x}_1) = \Phi(\tau_{n-2}(\mathbf{x}_1), \mathbf{x}_1),\ \gamma_1(\mathbf{x}_1) = \{\Phi(t, \mathbf{x}_1) \mid 0 \le t \le \tau_{n-2}\}.$$

Note that if $x_{1,2}$ is large enough, then the set $V_2^{-1}(V_2(\mathcal{G}_1(\mathbf{x}_1))) \cap \Gamma_2$ is composed of two curves according to the convexity of the level set. Let $\gamma_2(\mathbf{x}_1)$ be one of the two curves containing $\mathcal{G}_1(\mathbf{x}_1)$ as a endpoint and $\mathcal{G}_2(\mathbf{x}_1) \in \Sigma_1$ be the another endpoint of $\gamma_2(\mathbf{x}_1)$. Let $\gamma_3(\mathbf{x}_1)$ be the curve $V_1^{-1}(V_1(\mathcal{G}_2(\mathbf{x}_1))) \cap \Gamma_1$ and $\mathcal{G}_3(\mathbf{x}_1) \in \Sigma_1$ be the another endpoint of $\gamma_3(\mathbf{x}_1)$.

For any point $\mathbf{x}_2 = (x_{2,1}, x_{2,2}) \in \Sigma_2$ satisfying $\mathbf{e}_1^T \mathbf{A}_2(\mathbf{x}_2 - \mathbf{p}_2) < 0$ and $x_{2,2}$ is small enough, the set $V_2^{-1}(V_2(\mathbf{x}_2)) \cap \Gamma_2$ is composed of two curves, one of which denoted by $\gamma_5(\mathbf{x}_2)$ connects $\mathbf{x}_2$ and $\mathcal{G}_4(\mathbf{x}_2) \in \Sigma_1$. Let

$$\mathcal{G}_5(\mathbf{x}_2) = (a_n, x_{2,2}) \in \Gamma_n,\ \gamma_6(\mathbf{x}_2) = \{(x_1, x_2) \mid a_n \le x_1 \le a_2, x_2 = x_{2,2}\}.$$

Let $\gamma_7(\mathbf{x}_2)$ be the curve $V_{n+1}^{-1}(V_{n+1}(\mathcal{G}_5(\mathbf{x}_2))) \cap \Gamma_{n+1}$ connecting $\mathcal{G}_5(\mathbf{x}_2)$ and the point in Σ_n denoted by $\mathcal{G}_6(\mathbf{x}_2)$.

It can be easily seen that for any given point $\mathbf{x}_1 \in \Sigma_n$ where $\mathbf{e}_1^T \mathbf{A}_n(\mathbf{x}_1 - \mathbf{p}_n) > 0$, there exist $\delta_0 > 0$ and $\mathbf{x}_2 = (x_{2,1}, x_{2,2}) \in \Sigma_2$ satisfying $x_{2,2} < -\delta_0$ such that

$$\begin{aligned}&\mathbf{e}_1^T \mathbf{A}_2(\mathbf{x}_2 - \mathbf{p}_2) < 0, \mathbf{e}_2^T(\mathbf{x}_1 - \mathcal{G}_6(\mathbf{x}_2)) > 0,\ \mathbf{e}_2^T(\mathcal{G}_3(\mathbf{x}_1) - \mathcal{G}_4(\mathbf{x}_2)) > 0,\\ &\mathbf{e}_2^T \mathbf{A}_k(\mathbf{x} - \mathbf{p}_k) > 0,\ \text{for every } \mathbf{x} \in \gamma_6(\mathbf{x}_2).\end{aligned}$$

Let $\gamma_4(\mathbf{x}_1, \mathbf{x}_2)$ be the line between $\mathcal{G}_3(\mathbf{x}_1)$ and $\mathcal{G}_4(\mathbf{x}_2)$, and $\gamma_8(\mathbf{x}_1, \mathbf{x}_2)$ be the line between $\mathbf{x}_1$ and $\mathcal{G}_6(\mathbf{x}_2)$. It is easy to verify that the region $\mathcal{R}(\mathbf{x}_1, \mathbf{x}_2)$ bounded by

$$\{\gamma_1(\mathbf{x}_1), \gamma_2(\mathbf{x}_1), \gamma_3(\mathbf{x}_1), \gamma_4(\mathbf{x}_1, \mathbf{x}_2), \gamma_5(\mathbf{x}_2), \gamma_6(\mathbf{x}_2), \gamma_7(\mathbf{x}_2), \gamma_8(\mathbf{x}_1, \mathbf{x}_2)\}$$

is a positively invariant set, as shown in Figure 2.

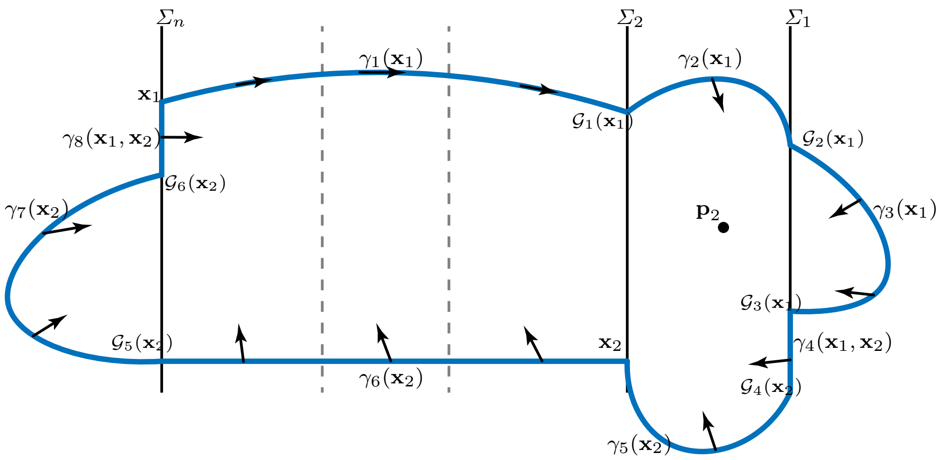
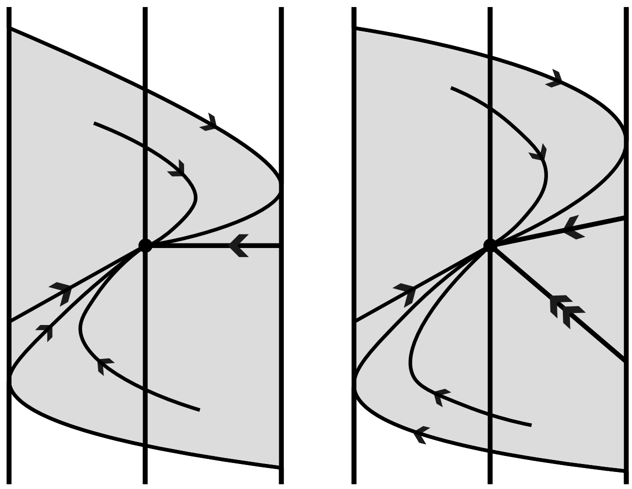


Fig. 2: The positively invariant set $\mathcal{R}(\mathbf{x}_1, \mathbf{x}_2)$ determined by points $\mathbf{x}_1$ and $\mathbf{x}_2$.

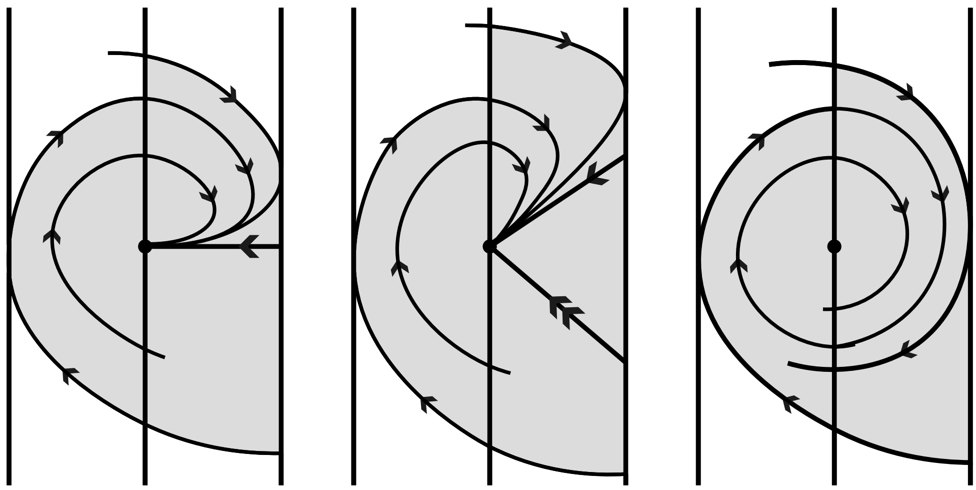
According to the above construction process of positively invariant sets, we obtain that for any compact set $\mathcal{K}$ containing the equilibrium point $\mathbf{p}_2$, there exist $\mathbf{x}_1 = (x_{1,1}, x_{1,2}) \in \Sigma_n$ and $\mathbf{x}_2 = (x_{2,1}, x_{2,2}) \in \Sigma_2$, such that the region $\mathcal{R}(\mathbf{x}_1, \mathbf{x}_2) \supset \mathcal{K}$ is positively invariant.

We conclude that system (3.1) has a globally asymptotically stable equilibrium point. Therefore, the proof of Theorem 2 is completed.

We remark that if system (3.1) has a unique equilibrium point $\mathbf{p}_{k_1}$ in Γ_{k_1} where $1 \leq k_1 \leq n+1$, then each trajectory starting in zone Γ_i, $i \neq k_1$ leaving Γ_i at some point must enter Γ_{k_1} at a large time and remain for all positive time. It is easy to notice that solutions remaining in Γ_{k_1} for all positive times will approach the equilibrium point $\mathbf{p}_{k_1}$ asymptotically as $t \to \infty$. If the subsystem of system (3.1) has a boundary equilibrium point $\mathbf{p}_{k_2}$ in Σ_{k_2} where $1 \leq k_2 \leq n$, then each positive trajectory shall enter a positively invariant set in $\Gamma_{k_2} \cup \Sigma_{k_2} \cup \Gamma_{k_2+1}$ at a large time and tends to the equilibrium point $\mathbf{p}_{k_2}$ asymptotically as $t \to \infty$, as shown in the Figure 3.



(a) $iN_b - iN_b$ (b) $iN_b - N_b$ (c) $N_b - N_b$



(d) $F_b - iN_b$ (e) $F_b - N_b$ (f) $F_b - F_b$

Fig. 3: Sketches of the phase portrait of system (3.1) with $\mathbf{p}_{k_2}$ in Σ_{k_2} satisfying the conditions of Theorem 2. The bold straight lines with arrows are invariant under corresponding sub-vector field. The grey regions are positively invariant. The subscript b for the singularity type represents a boundary singularity.

4 Examples and Simulations

To illustrate the results, we give some examples.

4.1 Example 1

Consider the following discontinuous PWL system

$$\dot{\mathbf{x}} = \begin{cases} (x_2 - x_1 - 6, x_1 - 3x_2 + 10), \text{ if } x_1 > 0, \\ (2x_2 - 3x_1 + 3 - 2q_2, x_1 - 2x_2 - 1 + 2q_2), \text{ if } x_1 < 0. \end{cases} \tag{4.1}$$

It is easy to see that $(-4, 2)$ is the virtual stable node of the right subsystem, $(1, q_2)$ is the virtual node of the left subsystem, $(0, 6)$ is the invisible fold point of the right subsystem and $(0, q_2 - \frac{3}{2})$ is the invisible fold point of the left subsystem.

If $q_2 = 8$, then it is easy to obtain that system (4.1) has only one limit cycle. The trajectory passing through $(-0.1, 6)$ and the trajectory passing through $(-0.1, 5)$ both approach the limit cycle as $t \to +\infty$, as shown in Figure 4a. By numerical calculation, we obtain that the limit cycle intersects Σ at $(0, 6.8901)$ and $(0, 5.3141)$ approximately.

If $q_2 = 6$, then the attractive sliding set is $\Sigma_{as} = \{(0, x_2) \in \mathbb{R}^2 \mid \frac{9}{2} < x_2 < 6\}$ and each trajectory will enter the sliding segment for some positive time, as shown in Figure 4b. Moreover, the trajectory on the sliding segment will approach to a new asymptotically stable equilibrium point $(0, 3 + \sqrt{3})$ as $t \to +\infty$.

If $q_2 = \frac{15}{2}$, then the invisible fold points of the left and right subsystem are the same point and each trajectory converges to this pseudo-equilibrium point as $t \to +\infty$, as shown in Figure 4c.

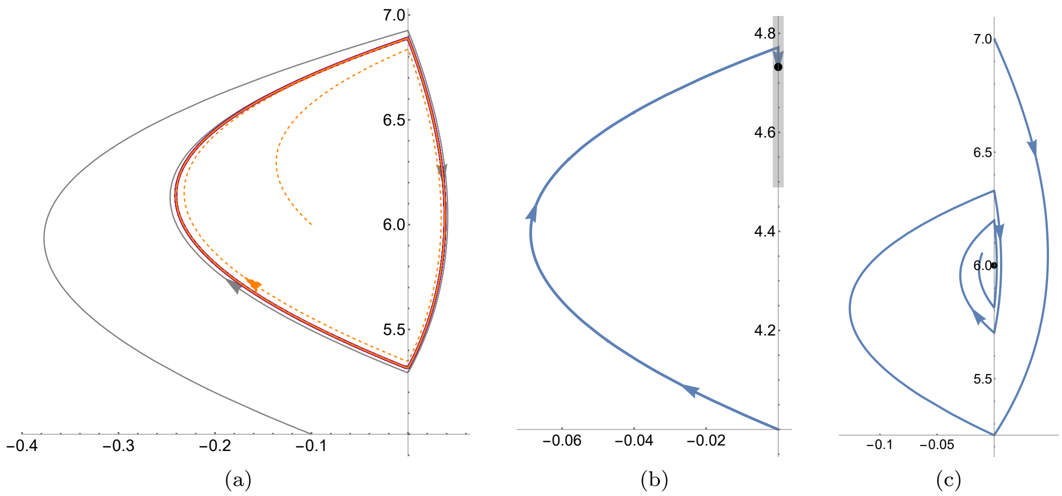


Fig. 4: (a)The trajectories of solutions of system (4.1) with $q_2 = 8$ and initial conditions $\mathbf{x}_0 = (0, 6.89), (-0.1, 6), (-0.1, 5)$ respectively. (b) The trajectory of system (4.1) with $q_2 = 6, \mathbf{x}_0 = (0, 4)$. (c) The trajectory of system (4.1) with $q_2 = 15/2, \mathbf{x}_0 = (0, 7)$.

4.2 Example 2

Consider the following discontinuous PWL system

$$\dot{\mathbf{x}} = \begin{cases} (3x_2 - 4x_1 - 10, x_2 - 2x_1 - 4), \text{ if } x_1 > 0, \\ (x_2 - 2x_1 + 4 - q_2, -x_1 - 4x_2 + 2 + 4q_2), \text{ if } x_1 < 0. \end{cases} \tag{4.2}$$

Note that $(-1, 2)$ is the virtual stable node of the right subsystem, $(2, q_2)$ is the virtual improper node of the left subsystem, $(0, \frac{10}{3})$ is the invisible fold point of the right subsystem and $(0, q_2 - 4)$ is the invisible fold point of the left subsystem.

If $q_2 = 9$, then it is easy to get that system (4.2) has only one limit cycle. The trajectory passing through $(-0.1, 5)$ and the trajectory passing through $(0.1, 5)$ will both approach the limit cycle as $t \to +\infty$, as shown in Figure 5a. By means of numerical calculation, we find that the limit cycle intersects Σ at $(0, 6.3694)$ and $(0, 3.0330)$ approximately.

If $q_2 = 7$, then the attractive sliding set is $\Sigma_{as} = \{(0, x_2) \in \mathbb{R}^2 \mid 3 < x_2 < 10/3\}$ and each trajectory enters the sliding segment for some positive time and eventually approaches to a new asymptotically stable point $(0, \frac{137-\sqrt{2545}}{26})$ as $t \to +\infty$, as shown in Figure 5b.

If $q_2 = \frac{22}{3}$, then the invisible fold points of the left and right subsystem are identical and each trajectory converges to this pseudo-equilibrium point as $t \to +\infty$, as shown in Figure 5c.

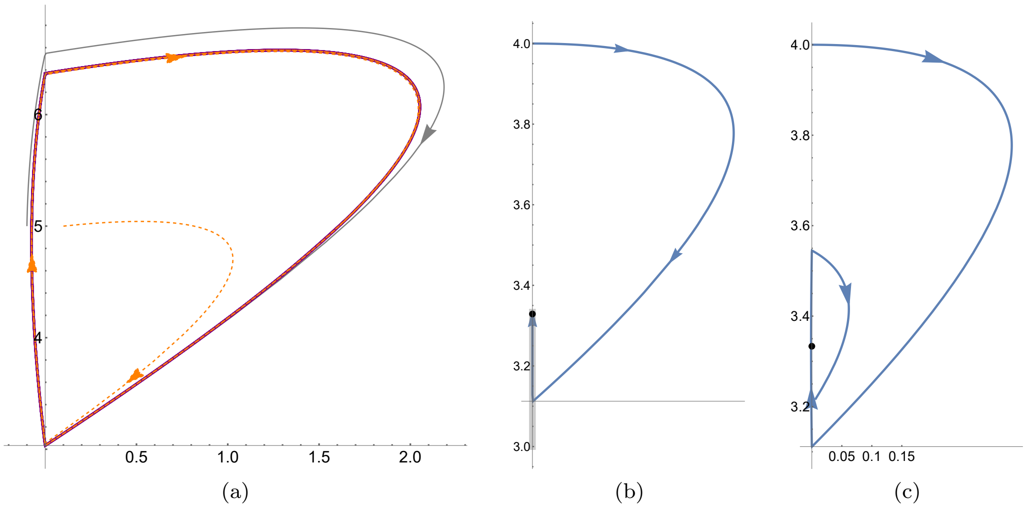


Fig. 5: (a) The trajectories of solutions of system (4.2) with $q_2 = 9$ and initial conditions $\mathbf{x}_0 = (0, 6.37), (-0.1, 5), (0.1, 5)$ respectively. (b) The trajectory of system (4.2) with $q_2 = 7, \mathbf{x}_0 = (0, 4)$. (c) The trajectory of system (4.2) with $q_2 = \frac{22}{3}, \mathbf{x}_0 = (0, 4)$.

4.3 Example 3

Consider the following continuous PWL system

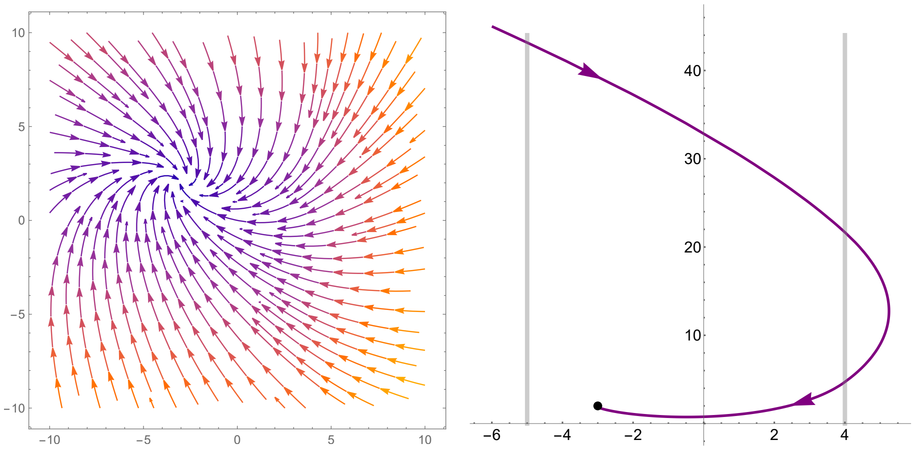
$$\dot{\mathbf{x}} = \begin{cases} (x_2 - 3x_1 + 3, -x_1 - 2x_2 + 1), \text{ if } x_1 > 4, \\ (x_2 - x_1 - 5, -x_1 - 2x_2 + 1), \text{ if } -5 < x_1 < 4, \\ (x_2 - x_1 - 5, \dfrac{1}{4}(-x_1 - 8x_2 + 19)), \text{ if } x_1 < -5. \end{cases} \tag{4.3}$$

It is easy to see that $(1, 0)$ is the virtual stable node of the right subsystem, $(-3, 2)$ is the real stable focus of the middle system and $(-\frac{7}{3}, \frac{8}{3})$ is the virtual stable improper node of the left subsystem. In addition, $(4, 9)$ is the invisible fold point of the right subsystem and $(-5, 0)$ is the invisible fold point of the left subsystem.

From the phase portraits as shown in Figure 6a, we get that all trajectories converge to an asymptotically stable equilibrium point as $t \to +\infty$. For example, we obtain the trajectory of system (4.3) with the initial value $x(0) = (-6, 45)$. This trajectory starting in the zones $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < -5\}$ will cross the switching lines $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = -5\}$ and $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 4\}$, enter the zone $\{(x_1, x_2) \in \mathbb{R}^2 \mid -5 < x_1 < 4\}$ and approach the equilibrium point $(-3, 2)$ as $t \to +\infty$, as shown in Figure 6b.

5 Conclusions

In this paper, we have investigated piecewise linear system with two zones separated by a straight line with each subsystem asymptotically stable and obtained the explicit parameter conditions for the existence of limit cycle and stable sliding segment. Besides, we have proved that the Markus–Yamabe conjecture is true for the continuous planar piecewise linear system with $n+1$ zones separated by n parallel straight, in which each of subsystems is asymptotically stable.



(a) (b)

Fig. 6: (a) The phase portraits of system (4.3). (b) A trajectory of the solution of system (4.3) with $\mathbf{x}_0 = (-0.1, -5)$.

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