Fermatean Hesitant Fuzzy Sets with Medical Decision Making Application

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Fermatean Hesitant Fuzzy Sets with Medical Decision Making Application

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Abstract

Fermatean fuzzy set idea obtained by combining fermatean fuzzy sets and hesitant fuzzy sets can be used in practice to simplify the solution of complicated multi-criteria decision-making (MCDM) problems. Initially, the notion of fermatean hesitant fuzzy set is given and the operations related to this concept are presented. Aggregation operators according to fermatean hesitant fuzzy sets are given and basic properties of these operators are studied. To choose the best alternative in practice, a novel MCDM method that is obtained with operators has been created. Finally, an example of infectious diseases was examined to indicate the effectiveness of the suggested techniques.

1. Introduction

The reasoning and DM processes of people in the face of daily events are studied by many disciplines, including psychology, philosophy, cognitive science, and artificial intelligence. These processes are generally tried to be described based on various mathematical and statistical models. In this process, the problem of decision-making arises. DM is defined as the operation of selecting one or more of the alternative forms of behavior faced by a person or an institution in order to achieve a specific goal. Research shows that while it is sufficient to make many daily decisions intuitively, this path alone is not enough for complex and vital decisions. MCDM is a collection of analytical approaches that appreciate the benefits and deficits of alternatives based on many criteria. MCDM methods are used to support the DM process and to select one or more alternatives from a set of alternatives with different characteristics according to conflicting criteria or to rank these alternatives. In other words, in MCDM methods, decision-makers(DMR) rank the alternatives with different characteristics by evaluating them according to many criteria. MCDM is a set of methods that are frequently used in all areas of life and at all levels.

In general, uncertainty is the situation in which a given event may have different consequences and there is no information about the probabilities of those consequences. Therefore, uncertainty is a very important notion for the DM process. It is not easy to know the probabilities of events happening in real-life. Therefore, the DM process occurs under uncertainty. Fuzzy logic theory [33] proposes a strong logical inference structure in the face of uncertain and imprecise knowledge. Fuzzy logic theory gives computers the ability to process people's linguistic data and work using people's experiences. While gaining this ability, it uses symbolic expressions instead of numerical expressions. These symbolic expressions are called fuzzy sets(FS). It is understood that the elements of fuzzy sets are actually decision variables containing probability states. Instead of probability values of possibilities, fuzzy sets arise by assigning membership degrees to each of them objectively.

In the FS A, the degree of belonging of an element to the set is ζ_A, while the degree of not belonging is $1-\zeta_A$. Therefore, the sum of the degrees of belonging and not belonging is equal to 1. However, this situation is insufficient to explain the uncertainty in some problems. For this reason, Atanassov [1] proposed the intuitionistic fuzzy set (IFS) theory, which is the generalization of FS. IFSs consist of membership degree(MD) and non-membership degree(ND) whose sum is less than or equal to 1. Yager [29] defined Pythagorean fuzzy set(PFS) as a more general and more comprehensive set than IFS. PFS is defined as the sum of the squares of MD and ND less than or equal to 1. There is an extensive diversity of studies on FS, IFS, and PFS such as [2]-[5], [9]-[12], [16], [17], [23], [28], [30], [31], [35].

Yager [32] introduced the q-step orthopair fuzzy set. The basic rule in this set theory is that the sum of MD with ND should not be greater than 1. Based on this idea, Senepati and Yager [19] introduced the Fermatean fuzzy set(FFS) and examined its basic features. In [20], Fermatean arithmetic mean, division, and subtraction which are new transactions for FFS, are defined and some of their properties are examined. In [21], new weighted aggregated operators related to FFSs are defined. [13] have defined fermatean fuzzy soft set(FFSS) and entropy measures. Shahzadi and Akram [22] offered a new decision support algorithm with respect to the FFSS and defined the new aggregated operators. Garg et al. [6] new FFS type aggregated operators defined by t-norm and t-conorm have been defined.

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The FS notion was generalized to the hesitant fuzzy set(HFS) notion by Torra [24]. This new set of the FS can handle the situations that the complexity in building the MD does not get up from a margin of error or a certain probability distribution of the probable values, however, originates from hesitation among a few several values [34]. Hence the HFS can more precisely reflect the people's hesitancy in stating their preferences over objects, compared to the FS and its other generalizations. Later, HFS and IFS were combined to obtain a new HFS which is called intuitionistic hesitant fuzzy set(IHFS) [18]. The fundamental notion is to form the situation in which instead of a individual MD and ND, human beings hesitate among a set of MD and ND and they require to symbolize such a hesitation. In [36], the notion of a dual HFS was improved and was given some properties. As an extension of the dual IVHFS, the HIVIFS approach was given [14]. IIn [15], the notion of IHFS to group DM problems using fuzzy cross-entropy was applied. The Pythagorean HFS(PFHS) was initially given by Khan et al [7]. PHFS compensates the case that the sum of its MDs is less than 1.

In this study, a new HFS which is called the fermatean hesitant fuzzy set will be given and investigated some properties. A new score function will be defined for comparison between fermatean hesitant fuzzy numbers(FHN). In addition, aggregation operators connected to FHS will be studied and the MCDM method related to FHFS will be introduced. An example of medical decision-making will be studied that illustrates how the method works. Finally, the proposed method in this study will be compared with previously known methods.

2. Preliminaries

General information about the fermatean fuzzy environment and hesitant fuzzy sets is the subject of this section.

Throughout the paper, U, Σ as the initial universe and parameters sets, respectively will be denoted.

For $\zeta_N : U \to [0,1]$ and $\eta_N : U \to [0,1]$, the FFS N is indicated by $N = \{(u, \zeta_N(u), \eta_N(u)) : u \in U\}$. For the FFS, the condition $0 \leq \zeta_N^3(u) + \eta_N^3(u) \leq 1$ [19] is holds.

The degree of indeterminacy of u to N is described as $\theta_N(u) = \sqrt[3]{1-(\zeta_N^3(u)+\eta_N^3(u))}$, for any FFS N and $u \in U$.

For FFSs $N = \{\zeta_N, \eta_N\}$, $N_1 = \{\zeta_{N_1}, \eta_{N_1}\}$ and $N_2 = \{\zeta_{N_2}, \eta_{N_2}\}$, some operations as follows [19]:

i. $N_1 \cap N_2 = (\min\{\zeta_{N_1}, \zeta_{N_2}\}, \max\{\eta_{N_1}, \eta_{N_2}\})$;
ii. $N_1 \cup N_2 = (\max(\zeta_{N_1}, \zeta_{N_2}), \min(\eta_{N_1}, \eta_{N_2}))$;
iii. $N^t = (\eta_N, \zeta_N)$;
iv. $N_1 \boxplus N_2 = \left(\sqrt[3]{\zeta_{N_1}^3 + \zeta_{N_2}^3 - \zeta_{N_1}^3 \zeta_{N_2}^3}, \eta_{N_1}\eta_{N_2}\right)$;
v. $N_1 \boxtimes N_2 = \left(\zeta_{N_1}^3 \zeta_{N_2}^3, \sqrt[3]{\eta_{N_1}^3 + \eta_{N_2}^3 - \eta_{N_1}^3 \eta_{N_2}^3}\right)$;
vi. $\alpha N = \left(\sqrt[3]{1-(1-\zeta_N^3)^\alpha}, \eta_N^\alpha\right)$;
vii. $N^\alpha = \left(\zeta_{N_1}^3, \sqrt[3]{1-(1-\eta_N^3)^\alpha}\right)$.

The properties of complement of FFS as follows [19]:

i. $(N_1 \cap N_2)^c = N_1^c \cup N_2^c$;
ii. $(N_1 \cup N_2)^c = N_1^c \cap N_2^c$;
iii. $(N_1 \boxplus N_2)^c = N_1^t \boxtimes N_2^c$;
iv. $(N_1 \boxtimes N_2)^c = N_1^c \boxplus N_2^c$;
v. $\alpha(N)^c = (N^\alpha)^c$;
vi. $(N^c)^\alpha = (\alpha N)^c$.

Definition 2.1. *[19] Choose a FFS $N = \{\zeta_N, \eta_N\}$. For FFS N,*

$$SF = \zeta_N^3 - \eta_N^3. \tag{2.1}$$

is said to be a score function.

The function SF is in $[-1,1]$.

Take the two FFSs $N_1 = \{\zeta_{N_1}, \eta_{N_1}\}$ and $N_2 = \{\zeta_{N_2}, \eta_{N_2}\}$. If the following condition (A) is hold, then it is called a natural quai-ordering concerning the FFS [19]:
(A) $\qquad N_1 \geq N_2 \quad \Leftrightarrow \quad m_{N_1} \leq n_{N_2}.$

For the two FFSs N_1 and N_2;
(a) $SF_{N_1} < SF_{N_2} \quad \Rightarrow \quad N_1 < N_2$,
(b) $SF_{N_1} > SF_{N_2} \quad \Rightarrow \quad N_1 > N_2$,
(c) $SF_{N_1} = SF_{N_2} \quad \Rightarrow \quad N_1 \sim N_2$.

Definition 2.2. *[19] For a FFS $N = \{\zeta_N, \eta_N\}$, the accuracy function of FFS N is defined as:*

$$AF = \zeta_N^3 + \eta_N^3. \tag{2.2}$$

Then, $AF \in [0,1]$. Clearly, $0 \le AF = \zeta_N^3 + \eta_N^3 \le 1$. Further, $\theta_N^3 + AF = 1$. The larger the AF, the higher the accuracy of the FFS N. The lower the h_N, the higher the accuracy of the FFS N.

For the two FFSs $N_1 = (\zeta_{N_1}, \eta_{N_1})$ and $N_2 = (\zeta_{N_2}, \eta_{N_2})$;
(a) $SF_{N_1} < SF_{N_2} \Rightarrow N_1 < N_2$,
(b) $SF_{N_1} > SF_{N_2} \Rightarrow N_1 > N_2$,
(c) $SF_{N_1} = SF_{N_2} \Rightarrow N$ then

(i) $AF_{N_1} < AF_{N_2} \Rightarrow N_1 < N_2$,

(ii) $AF_{N_1} > AF_{N_2} \Rightarrow N_1 > N_2$,

(iii) $AF_{N_1} = AF_{N_2} \Rightarrow N_1 = N_2$.

For two FFS N, M, from a binary relation $\le_{(SF,AF)}$, it may be shown as $N \le_{(SF,AF)} M$ iff the condition (B) holds:

(B) $$(SF_N < SF_M) \vee (SF_N = SF_M \wedge AF_N \le AF_M).$$

Definition 2.3. *[25] The set*

$$T = \{(u, t_T(u)) : u \in U\} \tag{2.3}$$

is called HFS, where $t_T(u)$ indicates the set of some values in unit interval, that is probable MD of $u \in U$ to T.

From now on, HFN will be used as $t = t_T(u)$ throughout the paper.

Definition 2.4. *The following operations are hold for three HFNs t, t_1, t_2:*

1. $t^c = \cup_{\delta \in t}\{1 - \delta\}$;
2. $t_1 \cap t_2 = \cup_{\delta_1 \in t_1, \delta_2 \in t_2} \min\{\delta_1, \delta_2\}$;
3. $t_1 \cup t_2 = \cup_{\delta_1 \in t_1, \delta_2 \in t_2} \max\{\delta_1, \delta_2\}$.

Definition 2.5. *The set*

$$P_T = \{(u, \zeta_{P_T}(u), \eta_{P_T}(u)) : u \in U\} \tag{2.4}$$

is called PHFS in U, where $(\zeta_{P_T}(u), \eta_{P_T}(u))$ are functions from U to $[0,1]$, showing a probable MD and ND of $u \in U$ in P_T respectively. Further, for each element $u \in U$
(i.) $\forall t_{P_T}(u) \in \zeta_{P_T}(u)$, $\exists t'_{P_T}(x) \in \eta_{P_T}(x)$, such that $0 \le t^2_{P_T}(u) + t'^2_{P_T}(u) \le 1$
(ii.) $\forall t'_{P_T}(u) \in \eta_{P_T}(u)$, $\exists t_{P_T}(u) \in \zeta_{P_T}(u)$, such that $0 \le t^2_{P_T}(u) + t'^2_{P_T}(u) \le 1$.

3. Fermatean Hesitant Fuzzy Sets

Definition 3.1. *The set*

$$F_T = \{(u, \zeta_{F_T}(u), \eta_{F_T}(u)) : u \in U\} \tag{3.1}$$

is called a fermatean hesitant fuzzy set($\mathfrak{FH}$), where
(i.) For each element $u \in U$, $\zeta_{F_T}(u)$, $\eta_{F_T}(u)$ are functions from U to $[0,1]$, demonstrating a likely MD and ND of element $u \in U$ in F_T respectively,
(ii.) $\forall \quad t_{F_T}(u) \in \zeta_{F_T}(u)$, $\exists \quad t'_{F_T}(u) \in \eta_{F_T}(u)$, such that $0 \le t^3_{F_T}(u) + t'^3_{F_T}(u) \le 1$,
(iii.) $\forall \quad t'_{F_T}(u) \in \eta_{F_T}(u)$, $\exists \quad t_{F_T}(u) \in \zeta_{F_T}(u)$, such that $0 \le t^3_{F_T}(u) + t'^3_{F_T}(u) \le 1$.

From this stage on, the set of all elements belonging to $\mathfrak{FH}$'s will be denoted by $\mathfrak{FH}(U)$. If U has only $(u, \zeta_{F_T}(u), \eta_{F_T}(u))$, it is called a fermatean hesitant fuzzy number($\mathfrak{FHN}$ (represented by $\tilde{t} = \{\zeta_{\tilde{t}}, \eta_{\tilde{t}}\}$).

Example 3.2. *For U, consider a set F_H in U given by*

$$F_T = \{(u_1, [0.80, 0.92, 0.85, 1], [0.1, 0.64, 0.56]), (u_2, [0.08, 0.77, 0.84], [0.23, 0.81, 0.90]), (u_3, [0.0, 0.25, 0.58, 0.63], [0.61, 0.941])\}$$

Then, $0.8^3 + 0.0^3 = 0.512$, $0.92^3 + 0.0^3 = 0.78$, $0.85^3 + 0.0^3 = 0.614$, $1^3 + 0.0^3 = 1$.
Secondly, $0.08^3 + 0.23^3 = 0.0127$, $0.77^3 + 0.23^3 = 0.47$, $0.84^3 + 0.23^3 = 0.605$
and $0.0^3 + 0.61^3 = 0.227$, $0.25^3 + 0.61^3 = 0.243$, $0.58^3 + 0.61^3 = 0.422$, $0.63^3 + 0.61^3 = 0.477$.
Calculations are also made according to other values. It is clear from here that $\{\zeta_{\tilde{t}}(u_1), \eta_{\tilde{t}}(u_1)\}$, $\{\zeta_{\tilde{t}}(u_2), \eta_{\tilde{t}}(u_2)\}$ and $\{\zeta_{\tilde{t}}(u_3), \eta_{\tilde{t}}(u_3)\}$ are $\mathfrak{FH}$s.

We note that if $\zeta_{F_T}(u)$ and $\eta_{F_T}(u)$ have only one element, then the $\mathfrak{FH}$ become a FFS. Further, if the ND is $\{0\}$, then $\mathfrak{FH}$ become a HFS.

$$\theta_{F_T}(u) = \bigcup_{t_{F_T}(u) \in \zeta_{F_T}(u), t'_{F_H}(u) \in \eta_{F_T}(u)} \sqrt[3]{1 - t^3_{F_T} - t'^3_{F_T}}$$

is said to be a indeterminacy degree of u to F_T, where $1 - t^3_{F_T} - t'^3_{F_H} \ge 0$ with for any $\mathfrak{FH}$ $F_T = \{(u, \zeta_{F_T}(u), \eta_{F_H}(u)) : u \in U\}$.

Definition 3.3. *The basic operations for three $\mathfrak{FHN}$s $\widetilde{t} = \{\zeta_{\widetilde{t}}, \eta_{\widetilde{t}}\}$, $\widetilde{t}_1 = \{\zeta_{\widetilde{t}_1}, \eta_{\widetilde{t}_1}\}$, $\widetilde{t}_2 = \{\zeta_{\widetilde{t}_2}, \eta_{\widetilde{t}_2}\}$ and $\alpha > 0$ are as follows: For three $\mathfrak{FHN}$s $\widetilde{t} = \{\zeta_{\widetilde{t}}, \eta_{\widetilde{t}}\}$, $\widetilde{t}_1 = \{\zeta_{\widetilde{t}_1}, \eta_{\widetilde{t}_1}\}$, $\widetilde{t}_2 = \{\zeta_{\widetilde{t}_2}, \eta_{\widetilde{t}_2}\}$ and $\alpha > 0$, then*

i. $\widetilde{t}_1 \cup \widetilde{t}_2 = (\max\{\zeta_{\widetilde{t}_1} . \zeta_{\widetilde{t}_2}\}, \min\{\eta_{\widetilde{t}_1}, \eta_{\widetilde{t}_2}\})$,

ii. $\widetilde{t}_1 \cap \widetilde{t}_2 = (\min\{\zeta_{\widetilde{t}_1}, \zeta_{\widetilde{t}_2}\}, \max\{\eta_{\widetilde{t}_1}, \eta_{\widetilde{t}_2}\})$.

iii. $\widetilde{t}^c = (\eta_{\widetilde{t}}, \zeta_{\widetilde{t}})$.

iv. $\widetilde{t}_1 \boxplus \widetilde{t}_2 = \left(\bigcup_{t_{\widetilde{t}_1} \in \zeta_{\widetilde{t}_1}, t_{\widetilde{t}_2} \in \zeta_{\widetilde{t}_2}} \left\{ \sqrt[3]{t^3_{\widetilde{t}_1} + t^3_{\widetilde{t}_2} - t^3_{\widetilde{t}_1} t^3_{\widetilde{t}_2}} \right\}, \bigcup_{t'_{\widetilde{t}_1} \in \eta_{\widetilde{t}_1}, t'_{\widetilde{t}_2} \in \eta_{\widetilde{t}_2}} \left\{ t'_{\widetilde{t}_1}, t'_{\widetilde{t}_2} \right\} \right)$.

v. $\widetilde{t}_1 \boxtimes \widetilde{t}_2 = \left(\bigcup_{t_{\widetilde{t}_1} \in \zeta_{\widetilde{t}_1}, t_{\widetilde{t}_2} \in \zeta_{\widetilde{t}_2}} \left\{ t_{\widetilde{t}_1}, t_{\widetilde{t}_2} \right\}, \bigcup_{t'_{\widetilde{t}_1} \in \eta_{\widetilde{t}_1}, t'_{\widetilde{t}_2} \in \eta_{\widetilde{t}_2}} \left\{ \sqrt[3]{\widetilde{t}_1'^3 + \widetilde{t}_2'^3 - \widetilde{t}_1'^3 \widetilde{t}_2'^3} \right\} \right)$.

vi. $\alpha \widetilde{t} = \left(\bigcup_{t_{\widetilde{t}} \in \zeta_{\widetilde{t}}} \left\{ \sqrt[3]{1 - (1 - (t_{\widetilde{t}})^3)^\alpha} \right\}, \bigcup_{t'_{\widetilde{t}} \in \eta_{\widetilde{t}}} \left\{ \left(t'_{\widetilde{t}} \right)^\alpha \right\} \right), \alpha > 0.$

vii. $\widetilde{t}^\alpha = \left(\bigcup_{t_{\widetilde{t}} \in \zeta_{\widetilde{t}}} \left\{ t^\alpha_{\widetilde{t}} \right\}, \bigcup_{t'_{\widetilde{t}} \in \eta_{\widetilde{t}}} \left\{ \sqrt[3]{1 - \left(1 - (t'^3_{\widetilde{t}})^\alpha\right)} \right\} \right), \alpha > 0.$

It should be noted that the results from these processes are also an $\mathfrak{FHN}$.

Theorem 3.4. *For three $\mathfrak{FHN}$s $\widetilde{t} = \{\zeta_{\widetilde{t}}, \eta_{\widetilde{t}}\}$, $\widetilde{t}_1 = \{\zeta_{\widetilde{t}_1}, \eta_{\widetilde{t}_1}\}$, $\widetilde{t}_2 = \{\zeta_{\widetilde{t}_2}, \eta_{\widetilde{t}_2}\}$, the following hold:*

1) $\widetilde{t}_1 \boxplus \widetilde{t}_2 = \widetilde{t}_2 \boxplus \widetilde{t}_1$

2) $\widetilde{t}_1 \boxtimes \widetilde{t}_2 = \widetilde{t}_2 \boxtimes \widetilde{t}_1$

3) $\alpha(\widetilde{t}_1 \boxplus \widetilde{t}_2) = \alpha \widetilde{t}_1 \boxplus \alpha \widetilde{t}_2, \quad \alpha > 0$

4) $(\alpha_1 + \alpha_2)\widetilde{t} = \alpha_1 \widetilde{t} \boxplus \alpha_2 \widetilde{t}, \ \alpha_1, \quad \alpha_2 > 0$

5) $(\widetilde{t}_1 \boxtimes \widetilde{t}_2)^\alpha = \widetilde{t}_1^\alpha \boxtimes \widetilde{t}_2^\alpha, \quad \alpha > 0$

6) $\widetilde{t}^{\alpha_1 + \alpha_2} = \widetilde{t}^{\alpha_1} \boxtimes \widetilde{t}^{\alpha_2}, \quad \alpha_1, \alpha_2 > 0$

Proof. We will only prove item 1. Since other items can be proved in a similar way, they are not included here.

$$
\begin{aligned}
\textbf{1)} \quad \widetilde{t}_1 \boxplus \widetilde{t}_2 &= \left(\bigcup_{t_{\widetilde{t}_1} \in \zeta_{\widetilde{t}_1}, t_{\widetilde{t}_2} \in \zeta_{\widetilde{t}_2}} \left\{ \sqrt[3]{t^3_{\widetilde{t}_1} + t^3_{\widetilde{t}_2} - t^3_{\widetilde{t}_1} t^3_{\widetilde{t}_2}} \right\}, \bigcup_{t'_{\widetilde{t}_1} \in \eta_{\widetilde{t}_1}, t'_{\widetilde{t}_2} \in \eta_{\widetilde{t}_2}} \left\{ t'_{\widetilde{t}_1}, t'_{\widetilde{t}_2} \right\} \right) \\
&= \left(\bigcup_{t_{\widetilde{t}_2} \in \zeta_{\widetilde{t}_2}, t_{\widetilde{t}_1} \in \zeta_{\widetilde{t}_1}} \left\{ \sqrt[3]{t^3_{\widetilde{t}_2} + t^3_{\widetilde{t}_1} - t^3_{\widetilde{t}_2} t^3_{\widetilde{t}_1}} \right\}, \bigcup_{t'_{\widetilde{t}_2} \in \eta_{\widetilde{t}_1}, t'_{\widetilde{t}_1} \in \eta_{\widetilde{t}_1}} \left\{ t'_{\widetilde{t}_2}, t'_{\widetilde{t}_1} \right\} \right) \\
&= \widetilde{t}_1 \boxplus \widetilde{t}_2
\end{aligned}
$$

□

Example 3.5. *Let $\widetilde{t}_1 = (\{0.71, 0.86\}, \{0.55, 0.64\})$ and $\widetilde{t}_2 = (\{0.59, 0.78\}, \{0.35, 0.45\})$be two $\mathfrak{FHN}$s. For $\alpha > 0$,*

$$
\begin{aligned}
\widetilde{t}_1 \boxplus \widetilde{t}_2 &= (\{0.788, 0.873, 0.893, 0.932\}, \{0.1925, 0.2475, 0.224, 0.288\}) \\
\widetilde{t}_1 \boxtimes \widetilde{t}_2 &= (\{0.5867, 0.624, 0.5926, 0.648\}, \{0.42, 0.554, 0.5074, 0.671\}) \\
\alpha \widetilde{t}_1 &= (\{0.903, 0.954\}, \{0.3025, 0.41\}) \\
\alpha \widetilde{t}_2 &= (\{0.712, 0.897\}, \{0.1225, 0.2025\})
\end{aligned}
$$

Definition 3.6. *Let $\widetilde{t} = \{\zeta_{\widetilde{t}}, \eta_{\widetilde{t}}\}$ be a $\mathfrak{FHN}$. Then,*

$$
S(\widetilde{t}) = \frac{1}{E_{t_{\widetilde{t}}} \in \zeta_{\widetilde{t}}} \sum_{t_{\widetilde{t}} \in \zeta_{\widetilde{t}}} t^3_{\widetilde{t}} - \frac{1}{E_{t'_{\widetilde{t}}} \in \eta_{\widetilde{t}}} \sum_{t'_{\widetilde{t}} \in \eta_{\widetilde{t}}} t'^3_{\widetilde{t}} \tag{3.2}
$$

is said to be a score function of $\widetilde{t}$ ($S(\widetilde{t}) \in [-1, 1]$).

In this definition, $E_{t_{\widetilde{t}}}$ and $E_{t'_{\widetilde{t}}}$ denote the number of elements in $\zeta_{\widetilde{t}}$ and $\eta_{\widetilde{t}}$, respectively.

Let $\widetilde{t}_1 = \{\zeta_{\widetilde{t}_1}, \eta_{\widetilde{t}_1}\}$, $\widetilde{t}_2 = \{\zeta_{\widetilde{t}_2}, \eta_{\widetilde{t}_2}\}$ be two $\mathfrak{FHN}$'s. If the score functions are chosen as $S(\widetilde{t}_1)$ of $\widetilde{t}_1$ and $S(\widetilde{t}_2)$ of $\widetilde{t}_2$, we have

1) If $S(\widetilde{t}_1) < S(\widetilde{t}_2) \quad \Rightarrow \quad \widetilde{t}_1 < \widetilde{t}_2$.

2) If $S(\widetilde{t}_1) > S(\widetilde{t}_2) \quad \Rightarrow \quad \widetilde{t}_1 > \widetilde{t}_2$.

3) If $S(\widetilde{t}_1) = S(\widetilde{t}_2) \quad \Rightarrow \quad \widetilde{t}_1 \sim \widetilde{t}_2$.

Example 3.7. *Let $\widetilde{t}_1 = (\{0.59, 0.78, 0.93\}, \{0.15, 0.54, 0.67\})$, $\widetilde{t}_2 = (\{0.20, 0.80, 0.95\}, \{0.10, 0.84, 0.95\})$, $\widetilde{t}_3 = (\{0.00, 0.44, 0.81, 0.92\}, \{0.09, 0.79, 0.88\})$ be three $\mathfrak{FHN}$'s. From Definition 3.2, $S(\widetilde{t}_1) = 0.36$, $S(\widetilde{t}_2) = 0.025$ and $S(\widetilde{t}_3) = -0.045$. Hence, $S(\widetilde{t}_1) > S(\widetilde{t}_2) > S(\widetilde{t}_3)$.*

Example 3.8. *Let $\widetilde{t}_1 = (\{0.7, 0.6\}, \{0.5, 0.699, 0.6\})$, $\widetilde{t}_2 = (\{0.5303, 0.78\}, \{0.2, 0.8\})$ be two $\mathfrak{FHN}$'s. $S(\widetilde{t}_1) = 0.52$ and $S(\widetilde{t}_2) = 0.52$. That is $S(\widetilde{t}_1) = S(\widetilde{t}_2)$.*

When $S(\widetilde{t}_1) = S(\widetilde{t}_2)$, it cannot possible to differentiate between $\widetilde{t}_1$ and $\widetilde{t}_2$. Naturally, such situations are very common in real life. The following definition can be given as a solution to this situation.

Definition 3.9. *Let $\widetilde{t} = (\zeta_{\widetilde{t}}, \eta_{\widetilde{t}})$ be a $\mathfrak{FHN}$. Then,*

$$A(\widetilde{t}) \quad = \quad \frac{1}{E_{t_{\widetilde{t}}} \in \zeta_{\widetilde{t}}} \sum_{t_{\widetilde{t}} \in \zeta_{\widetilde{t}}} t_{\widetilde{t}}^{3} + \frac{1}{E_{t'_{\widetilde{t}}} \in \eta_{\widetilde{t}}} \sum_{t'_{\widetilde{t}} \in \eta_{\widetilde{t}}} t_{\widetilde{t}}^{\prime 3} \tag{3.3}$$

is called the accuracy degree of $\widetilde{t}$.

$S(\widetilde{t})$ and $A(\widetilde{t})$ in this definition can be understood as the mean value and standard deviation in statistics, respectively. Based on this idea, the following definition can be given:

Definition 3.10. *Take two $\mathfrak{FHN}$'s $\widetilde{t}_1 = \{\zeta_{\widetilde{t}_1}, \eta_{\widetilde{t}_1}\}$, $\widetilde{t}_2 = \{\zeta_{\widetilde{t}_2}, \eta_{\widetilde{t}_2}\}$. Let $S(\widetilde{t}_1)$, $A(\widetilde{t}_1)$ and $S(\widetilde{t}_2)$ $A(\widetilde{t}_2)$ be denoted the score functions and the deviation degrees of $\widetilde{t}_1$ and $\widetilde{t}_2$, respectively. Wee have,*

1) *If $S(\widetilde{t}_1) < S(\widetilde{t}_2) \Rightarrow \widetilde{t}_1 < \widetilde{t}_2$.*
2) *If $S(\widetilde{t}_1) > S(\widetilde{t}_2) \Rightarrow \widetilde{t}_1 > \widetilde{t}_2$.*
3) *If $S(\widetilde{t}_1) = S(\widetilde{t}_2) \Rightarrow \widetilde{t}_1 \sim \widetilde{t}_2$.*
    a. *If $A(\widetilde{t}_1) < A(\widetilde{t}_2) \Rightarrow \widetilde{t}_1 < \widetilde{t}_2$.*
    b. *If $A(\widetilde{t}_1) > A(\widetilde{t}_2) \Rightarrow \widetilde{t}_1 > \widetilde{t}_2$.*
    c. *If $A(\widetilde{t}_1) = A(\widetilde{t}_2) \Rightarrow \widetilde{t}_1 \sim \widetilde{t}_2$.*

Example 3.11. *Take the values in Example 3.8. Then, $A(\widetilde{t}_1) = 0.621$ and $A(\widetilde{t}_2) = 0.5723$. From Definition 3.10, while $S(\widetilde{t}_1) = S(\widetilde{t}_2)$, $A(\widetilde{t}_1) > A(\widetilde{t}_2) \Rightarrow \widetilde{t}_1 > \widetilde{t}_2$.*

It can be seen from Example 3.11 that for the two $\mathfrak{FHN}$s $\widetilde{t}_1$ and $\widetilde{t}_2$, the number of elements of the corresponding fermatean MD is not equal. However, if the measurement of the distance between them is computed, the corresponding number of elements will be equal.

Definition 3.12. *Let $\widetilde{t}_1 = \{\zeta_{\widetilde{t}_1}, \eta_{\widetilde{t}_1}\}$, $\widetilde{t}_2 = \{\zeta_{\widetilde{t}_2}, \eta_{\widetilde{t}_2}\}$, $\widetilde{t}_3 = \{\zeta_{\widetilde{t}_3}, \eta_{\widetilde{t}_3}\}$ be three $\mathfrak{FHN}$'s in U. If the cases*

D1. $0 \leq D(\widetilde{t}_1, \widetilde{t}_2) \leq 1$,
D2. $D(\widetilde{t}_1, \widetilde{t}_2) = D(\widetilde{t}_2, \widetilde{t}_1)$,
D3. $D(\widetilde{t}_1, \widetilde{t}_2) = 0$ *if and only if* $\widetilde{t}_1 = \widetilde{t}_2$.

are hold, then the function $D : \mathfrak{FH} \times \mathfrak{FH} \to [0,1]$ ($D(\widetilde{t}_1, \widetilde{t}_2)$) is called a distance measure.

Definition 3.13. *Let $\widetilde{t}_1$, $\widetilde{t}_2$ be two $\mathfrak{FHN}$'s. Then,*

$$D(\widetilde{t}_1, \widetilde{t}_2) = \frac{1}{2}\left[\frac{1}{E_{u_i}} \sum_{i=1}^{E_{u_i}} \left|(t_{\widetilde{t}_1}^{\beta(k)}(u_i))^3 - (t_{\widetilde{t}_2}^{\beta(k)}(u_i))^3\right| + \frac{1}{E_{u_i}} \sum_{i=1}^{E_{u_i}} \left|(t_{\widetilde{t}_1}^{\prime\beta(k)}(u_i))^3 - (t_{\widetilde{t}_2}^{\prime\beta(k)}(u_i))^3\right|\right] \tag{3.4}$$

is defined as a distance measure of $\widetilde{t}_1$, $\widetilde{t}_2$, where $t_{\widetilde{t}_1}^{\beta(k)}, t_{\widetilde{t}_2}^{\beta(k)}, t_{\widetilde{t'}_1}^{\beta(k)}, t_{\widetilde{t'}_2}^{\beta(k)}$ be the i^{th} largest value in $\widetilde{t}_1$, $\widetilde{t}_2$ respectively.

The number of elements in $\mathfrak{FHN}$s can be different. In this case, they can be made equivalent by adding elements to the $\mathfrak{FHN}$, which has fewer elements. The lowest factor according to the pessimistic principle is added while the contrary situation will be embraced in the optimistic principle.

Now let's give an example of distance measure:

Example 3.14. *Take the $\mathfrak{FHN}$s $\widetilde{t}_1 = (\{0.68, 0.75, 92\}, \{0.56, 0.72, 0.84\})$, $\widetilde{t}_2 = (\{0.47, 0.65, 0.93\}, \{0.38, 0.78, 0.88\})$. Then,*

$$\begin{aligned} D(\widetilde{t}_1, \widetilde{t}_2) &= \frac{1}{2}\left[\frac{1}{3}\left(\left|0.68^2 - 0.47^2\right| + \left|0.75^2 - 0.65^2\right| + \left|0.92^2 - 0.93^2\right|\right) + \frac{1}{3}\left(\left|0.56^2 - 0.38^2\right| + \left|0.72^2 - 0.78^2\right| + \left|0.84^2 - 0.88^2\right|\right)\right] \\ &= \frac{1}{2}[0.043 + 0.104] = 0.0735 \end{aligned}$$

4. Aggregation Operator

Definition 4.1. *For a number of $\mathfrak{FHN}$s $\widetilde{t}_i = \{\zeta_{\widetilde{t}_i}, \eta_{\widetilde{t}_i}\}$ $(1 \leq i \leq n)$, a fermatean hesitant fuzzy weighted average($\mathfrak{FHWA}$) operator is a function $\mathfrak{FHWA} : \mathfrak{FHN}^n \to \mathfrak{FHN}$, where*

$$\mathfrak{FHWA}(\widetilde{t}_1, \widetilde{t}_2, \cdots, \widetilde{t}_n) \quad = \quad \omega_1 \widetilde{t}_1 \boxplus \omega_2 \widetilde{t}_2 \boxplus \cdots \boxplus \omega_n \widetilde{t}_n \tag{4.1}$$

where ω_i is a weight vector of $\widetilde{t}_i$ ($\sum_{i=1}^{n} \omega_i = 1$).

Theorem 4.2. *For $i = 1, 2, \cdots, n$, take $\mathfrak{FHN}$s $\widetilde{t}_i = \{\zeta_{\widetilde{t}_i}, \eta_{\widetilde{t}_i}\}$. The $\mathfrak{FHWA}$ operator is defined as:*

$$\mathfrak{FHWA}(\widetilde{t}_1, \widetilde{t}_2, \cdots, \widetilde{t}_n) = \left(\bigcup_{t_{\widetilde{t}_1} \in \zeta_{\widetilde{t}_1}, t_{\widetilde{t}_2} \in \zeta_{\widetilde{t}_2}, \cdots, t_{\widetilde{t}_n} \in \zeta_{\widetilde{t}_n}} \sqrt[3]{1 - \prod_{i=1}^{n} \left(1 - t_{\widetilde{t}_i}^3\right)^{\omega_i}}, \bigcup_{t'_{\widetilde{t}_1} \in \eta_{\widetilde{t}_1}, t'_{\widetilde{t}_2} \in \eta_{\widetilde{t}_2}, \cdots, t'_{\widetilde{t}_n} \in \eta_{\widetilde{t}_n}} \prod_{i=1}^{n} \left(t'_{\widetilde{t}_i}\right)^{\omega_i} \right) \tag{4.2}$$

Proof. Mathematical Induction Method will be used in the proof of this theorem. For $n=1$, according to Theorem 3.4, since

$$\mathfrak{FHWA}(\tilde{t}_1)=\sqrt[3]{1-\left(1-t^3_{\tilde{t}_1}\right)^{\omega_i}}=\sqrt[3]{\left(\tilde{t}^3_1\right)^{\omega_i}}=\tilde{t}_1.$$

This result shows that Equation 4.2 is satisfied for $n=1$. In this step, let's assume that for $n=k$, Equation 4.2 is compensated. Thus, it should be shown that for $n=k+1$, Equation 4.2 also compensated. Therefore,

$$\begin{aligned}
\omega_1\tilde{t}_1 \boxplus \omega_2\tilde{t}_2 \boxplus \cdots \boxplus \omega_k\tilde{t}_k \boxplus \omega_{k+1}\tilde{t}_{k+1} &= \left(\bigcup_{t_{\tilde{t}_1}\in\zeta_{\tilde{t}_1}, t_{\tilde{t}_2}\in\zeta_{\tilde{t}_2},\cdots,t_{\tilde{t}_k}\in\zeta_{\tilde{t}_k}} \sqrt[3]{1-\prod_{i=1}^{k}\left(1-t^3_{\tilde{t}_i}\right)^{\omega_i}}, \bigcup_{t'_{\tilde{t}_1}\in\eta_{\tilde{t}_1}, t'_{\tilde{t}_2}\in\eta_{\tilde{t}_2},\cdots,t'_{\tilde{t}_k}\in\eta_{\tilde{t}_k}} \prod_{i=1}^{k}\left(t'_{\tilde{t}_i}\right)^{\omega_i}\right)\\
&\boxplus \left(\bigcup_{t_{\tilde{t}_{k+1}}\in\zeta_{\tilde{t}_{k+1}}} \sqrt[3]{1-\left(1-t^3_{\tilde{t}_{k+1}}\right)^{\omega_{k+1}}}, \bigcup_{t'_{\tilde{t}_{k+1}}\in\eta_{\tilde{t}_{k+1}}} \left(t'_{\tilde{t}_{k+1}}\right)^{\omega_{k+1}}\right)\\
&= \left(\bigcup_{t_{\tilde{t}_1}\in\zeta_{\tilde{t}_1}, t_{\tilde{t}_2}\in\zeta_{\tilde{t}_2},\cdots,t_{\tilde{t}_{k+1}}\in\zeta_{\tilde{t}_{k+1}}} \sqrt[3]{1-\prod_{i=1}^{k+1}\left(1-t^3_{\tilde{t}_i}\right)^{\omega_i}}, \bigcup_{t'_{\tilde{t}_1}\in\eta_{\tilde{t}_1}, t'_{\tilde{t}_2}\in\eta_{\tilde{t}_2},\cdots,t'_{\tilde{t}_{k+1}}\in\eta_{\tilde{t}_{k+1}}} \prod_{i=1}^{k+1}\left(t'_{\tilde{t}_i}\right)^{\omega_i}\right).
\end{aligned}$$

This result is desired. □

Definition 4.3. *For $i=1,2,\cdots,n$, choose $\mathfrak{FHN}$s $\tilde{t}_i=\{\zeta_{\tilde{t}_i},\eta_{\tilde{t}_i}\}$ and for $(\omega_i\geq 0)$ take weight vector of $\tilde{t}_i$ ω. Then, a Fermatean hesitant fuzzy weighted geometric ($\mathfrak{FHWG}$) operator is a mapping $\mathfrak{FHWG}:\mathfrak{FHN}^n\rightarrow\mathfrak{FHN}$, where*

$$\mathfrak{FHWG}(\tilde{t}_1,\tilde{t}_2,\cdots,\tilde{t}_n) \quad = \quad \tilde{t}_1^{\omega_1}\boxtimes\tilde{t}_2^{\omega_2}\boxtimes\cdots\boxtimes\tilde{t}_n^{\omega_n} \tag{4.3}$$

Theorem 4.4. *For $\mathfrak{FHN}$s $\tilde{t}_i=\{\zeta_{\tilde{t}_i},\eta_{\tilde{t}_i}\}$ $(1\leq i\leq n)$, the $\mathfrak{FHWG}$ operator*

$$\mathfrak{FHWG}(\tilde{t}_1,\tilde{t}_2,\cdots,\tilde{t}_n)=\left(\bigcup_{t_{\tilde{t}_1}\in\zeta_{\tilde{t}_1}, t_{\tilde{t}_2}\in\zeta_{\tilde{t}_2},\cdots,t_{\tilde{t}_n}\in\zeta_{\tilde{t}_n}} \prod_{i=1}^{n}\left(t_{\tilde{t}_i}\right)^{\omega_i}, \bigcup_{t'_{\tilde{t}_1}\in\eta_{\tilde{t}_1}, t'_{\tilde{t}_2}\in\eta_{\tilde{t}_2},\cdots,t'_{\tilde{t}_n}\in\eta_{\tilde{t}_n}} \sqrt[3]{1-\prod_{i=1}^{n}\left(1-t'^3_{\tilde{t}_i}\right)^{\omega_i}},\right). \tag{4.4}$$

Theorem 4.5. *For $\mathfrak{FHN}$s $\tilde{t}_i=\{\zeta_{\tilde{t}_i},\eta_{\tilde{t}_i}\}$ $i=1,2,\cdots,n$,*

$$\mathfrak{FHWG}(\tilde{t}_1,\tilde{t}_2,\cdots,\tilde{t}_n) \quad \leq \quad \mathfrak{FHWA}(\tilde{t}_1,\tilde{t}_2,\cdots,\tilde{t}_n) \tag{4.5}$$

Since $\prod_{i=1}^{n}(\tilde{t}_i)^{\omega_i}\leq\sum_{i=1}^{n}\omega_i\tilde{t}_i$, for $\tilde{t}_i>0$, $\omega_i>0$, and $\sum_{i=1}^{n}\omega_i=1$ (the equality holds iff $\tilde{t}_1=\tilde{t}_2=\cdots=\tilde{t}_n$) [26], this theorem can be easily proved.

Theorem 4.6. *The $\mathfrak{FHWA}$ and $\mathfrak{FHWG}$ operators provide the boundedness property. That is, for a collection of $\mathfrak{FHN}$s $\tilde{t}_i=\{\zeta_{\tilde{t}_i},\eta_{\tilde{t}_i}\}$ and $1\leq i\leq n$, if $\zeta^-=\min\{\zeta_i\}$, $\zeta^+=\max\{\zeta_i\}$, $\eta^-=\min\{\eta_i\}$, $\eta^+=\max\{\eta_i\}$, then*

$$\begin{aligned}
(\zeta^-,\eta^+)\leq\mathfrak{FHWA}(\tilde{t}_1,\cdots,\tilde{t}_n)\leq(\zeta^+,\eta^-)\\
(\zeta^-,\eta^+)\leq\mathfrak{FHWG}(\tilde{t}_1,\cdots,\tilde{t}_n)\leq(\zeta^+,\eta^-)
\end{aligned}$$

Proof. We will only do proof for $\mathfrak{FHWA}$.

For a collection of $\mathfrak{FHN}$s $\tilde{t}_i=\{\zeta_{\tilde{t}_i},\eta_{\tilde{t}_i}\}$ $(1\leq i\leq n)$, we can take $\zeta^-\leq\zeta_i\leq\zeta^+$, $\eta^-\leq\eta_i\leq\eta^+$. Suppose that $\tilde{t}_{\min}=\{\zeta^-,\eta^+\}$, $\tilde{t}_{\max}=\{\zeta^+,\eta^-\}$.

$$\sum_{i=1}^{n}\omega_i\zeta^-\leq\sum_{i=1}^{n}\omega_i\zeta_i\leq\sum_{i=1}^{n}\omega_i\zeta^+\leq\sum_{i=1}^{n}\omega_i\eta^-\leq\sum_{i=1}^{n}\omega_i\eta_i\leq\sum_{i=1}^{n}\omega_i\eta^+. \tag{4.6}$$

$$\begin{aligned}
S(\tilde{t}_{\min}) &= \left(\frac{1}{E_{t_{\tilde{t}}}\in\zeta_{\tilde{t}}}\sum_{t_{\tilde{t}}\in\zeta_{\tilde{t}}}\omega_i\zeta^-\right)^3-\left(\frac{1}{E_{t'_{\tilde{t}}}\in\eta_{\tilde{t}}}\sum_{t'_{\tilde{t}}\in\eta_{\tilde{t}}}\omega_i\eta^+\right)^3\\
S(\tilde{t}_{\max}) &= \left(\frac{1}{E_{t_{\tilde{t}}}\in\zeta_{\tilde{t}}}\sum_{t_{\tilde{t}}\in\zeta_{\tilde{t}}}\omega_i\zeta^+\right)^3-\left(\frac{1}{E_{t'_{\tilde{t}}}\in\eta_{\tilde{t}}}\sum_{t'_{\tilde{t}}\in\eta_{\tilde{t}}}\omega_i\eta^-\right)^3\\
S(\mathfrak{FHWA}(\tilde{t}_1,\cdots,\tilde{t}_n)) &= \left(\frac{1}{E_{t_{\tilde{t}}}\in\zeta_{\tilde{t}}}\sum_{t_{\tilde{t}}\in\zeta_{\tilde{t}}}\omega_i\zeta_i\right)^3-\left(\frac{1}{E_{t'_{\tilde{t}}}\in\eta_{\tilde{t}}}\sum_{t'_{\tilde{t}}\in\eta_{\tilde{t}}}\omega_i\eta_i\right)^3
\end{aligned}$$

From here, $S(\tilde{t}_{\min})\leq S(\mathfrak{FHWA})\leq S(\tilde{t}_{\max})$ is obtained. □

Theorem 4.7. *The $\mathfrak{FHWA}$ and $\mathfrak{FHWG}$ operators provide the idempotency property. If all $\widetilde{t}_i$ ($1 \le i \le n$) are equal and $\widetilde{t}_i = \widetilde{t} = \{\zeta, \eta\}$ with $\sum_{i=1}^{n} \omega_i = 1$, then*

$$\begin{aligned} \mathfrak{FHWA}(\widetilde{t}_1, \cdots, \widetilde{t}_n) &= \widetilde{t}, \\ \mathfrak{FHWG}(\widetilde{t}_1, \cdots, \widetilde{t}_n) &= \widetilde{t}. \end{aligned}$$

Proof. We will only do proof for $\mathfrak{FHWA}$.

Since $\widetilde{t}_i = \widetilde{t} = \{\zeta, \eta\}$, then

$$\mathfrak{FHWA}(\widetilde{t}_1, \cdots, \widetilde{t}_n) = \mathfrak{FHWA}(\widetilde{t}, \cdots, \widetilde{t}) = \left(\sum_{i=1}^{n} \omega_i \zeta, \sum_{i=1}^{n} \omega_i \eta \right) = (\zeta, \eta) = \widetilde{t},$$

for all $1 \le i \le n$. □

Theorem 4.8. *The $\mathfrak{FHWA}$ and $\mathfrak{FHWG}$ operators provide the monotonicity property. That is, For two collections of $\mathfrak{FHN}$ $\widetilde{t}_i = \{\zeta_{\widetilde{t}_i}, \eta_{\widetilde{t}_i}\}$ and $\widetilde{u}_i = \{\zeta_{\widetilde{u}_i}, \eta_{\widetilde{u}_i}\}$, if $\zeta_{\widetilde{u}_i} \ge \zeta_{\widetilde{t}_i}$ and $\eta_{\widetilde{u}_i} \le \eta_{\widetilde{t}_i}$ for all i, then*

$$\begin{aligned} \mathfrak{FHWA}(\widetilde{t}_1, \cdots, \widetilde{t}_n) &\le \mathfrak{FHWA}(\widetilde{u}_1, \cdots, \widetilde{u}_n) \\ \mathfrak{FHWG}(\widetilde{t}_1, \cdots, \widetilde{t}_n) &\le \mathfrak{FHWG}(\widetilde{u}_1, \cdots, \widetilde{u}_n) \end{aligned}$$

Proof. Since $\zeta_{\widetilde{u}_i} \ge \zeta_{\widetilde{t}_i}$, $\eta_{\widetilde{u}_i} \le \eta_{\widetilde{t}_i}$ for all i, we have

$$\sum_{i=1}^{n} \omega_i \zeta_{\widetilde{t}_i} \le \sum_{i=1}^{n} \omega_i \zeta_{\widetilde{u}_i}, \quad \sum_{i=1}^{n} \omega_i \eta_{\widetilde{u}_i} \le \sum_{i=1}^{n} \omega_i \eta_{\widetilde{t}_i}.$$

Thus

$$\begin{aligned} S(\mathfrak{FHWA}(\widetilde{t}_1, \cdots, \widetilde{t}_n)) &= \left(\frac{1}{E_{t_{\widetilde{t}}} \in \zeta_{\widetilde{t}}} \sum_{t_{\widetilde{t}} \in \zeta_{\widetilde{t}}} \omega_i \zeta_{\widetilde{t}_i} \right)^3 - \left(\frac{1}{E_{t'_{\widetilde{t}}} \in \eta_{\widetilde{t}}} \sum_{t'_{\widetilde{t}} \in \eta_{\widetilde{t}_i}} \omega_i \eta_{\widetilde{t}_i} \right)^3, \\ S(\mathfrak{FHWA}(\widetilde{u}_1, \cdots, \widetilde{u}_n)) &= \left(\frac{1}{E_{u_{\widetilde{u}}} \in \zeta_{\widetilde{u}}} \sum_{u_{\widetilde{u}} \in \zeta_{\widetilde{u}}} \omega_i \zeta_{\widetilde{u}_i} \right)^3 - \left(\frac{1}{E_{u'_{\widetilde{u}}} \in \eta_{\widetilde{u}}} \sum_{u'_{\widetilde{u}} \in \eta_{\widetilde{u}_i}} \omega_i \eta_{\widetilde{u}_i} \right)^3. \end{aligned}$$

From here, $\mathfrak{FHWA}(\widetilde{t}_1, \cdots, \widetilde{t}_n) \le \mathfrak{FHWA}(\widetilde{u}_1, \cdots, \widetilde{u}_n)$ is obtained. □

5. Method based on FHFSs

In $\mathfrak{MCDM}$, decision matrix (DCMX) techniques are employed to characterize attributes, weight them, and properly sum the weighted attributes to offer a relative ranking among design alternatives. This matrix shows the assessment info of all alternatives according to an attribute. A DCMX is formed of rows and columns that consent to the assessment of alternatives relative to different decision criteria. That is, for m alternatives and n attributes, a DCMX is a $m \times n$ matrix with each element t_{ij} being the j-th attribute value of the i-th alternative.

Consider there is an anonymous $\mathfrak{MCDM}$ with different m alternatives $\mathbf{U} = \{U_1, U_2, \cdots, U_m\}$. Choose the universe of discourse including the attributes as $\mathbf{U}$ and the set of all attributes as $\mathbf{O} = \{O_1, O_2, \cdots, O_n\}$.

To assess the efficiency of the ith alternative U_i under the jth attribute O_j, the DMR is supposed to ensure not only the info that the alternative U_i compensates for the attribute O_j, but also the info that the alternative U_i does not compensate the attribute O_j. This two-part info may be stated by ζ_{ij} and η_{ij} that demonstrates the MD that the alternative U_i compensates the attribute O_j, ND that the alternative U_i does not compensate the attribute O_j, therefore, the efficiency of the alternative U_i under the attribute O_j may be stated by a $\mathfrak{FHN}$ $t_{ij} = \{\zeta_{ij}, \eta_{ij}\}$ with the condition that $\forall t_{ij} \in \zeta_{ij}$, $\exists t'_{ij} \in \eta_{ij} \Rightarrow 0 \le t_{ij}^3 + t'^3_{ij} \le 1$, and $\forall t_{ij} \in \eta_{ij}$, $\exists t'_{ij} \in \zeta_{ij} \Rightarrow 0 \le t_{ij}^3 + t'^3_{ij} \le 1$. The FHF DCMX $\mathfrak{T}$:

$$\mathfrak{T} = \begin{bmatrix} \mathrm{t}_{11} & \mathrm{t}_{12} & \cdots & \mathrm{t}_{1n} \\ \mathrm{t}_{21} & \mathrm{t}_{22} & \cdots & \mathrm{t}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{t}_{m1} & \mathrm{t}_{m2} & \cdots & \mathrm{t}_{mn} \end{bmatrix}$$

Since the attributes will have different degrees of importance, $0 \le \omega_j \le 1$ and $\sum_{j=1}^{m} \omega_j$ for each attribute, $\omega = \{\omega_1, \omega_2, \cdots, \omega_n\}^T$ is a weight vector for all attributes specified by the DMRs. Generally, the DMRs require to identify degrees of importance of attributes. Hence because of the confusion and ambiguity of DM problems and naturally nominative structure of human consideration, info concerning attribute weights is generally incomplete. Consider that the DMRs ensure the attribute weight info can be offered in undermentioned formats [8]:

1. If $\{\omega_i \ge \omega_j\}$, then the ranking is weak.
2. If $\{\omega_i - \omega_j \ge \beta_i (> 0)\}$, the ranking is strict.
3. If $\{\omega_i \ge \beta_i \omega_j\}$, $0 \le \beta_i \le 1$, then the ranking is by multiples.
4. If $\{\lambda_i \le \omega_i \le \lambda_i + \beta_i\}$, $0 \le \lambda_i \le \lambda_i + \beta_i \le 1$, then it is in the form of an interval.

5. If $\{\omega_i - \omega_j \geq \omega_k - \omega_l\}$, for $j \neq k \neq l$, then the ranking is differences.

for $i \neq j$.
What has happened so far is explained to define the problem. Now, the optimal weights of the attributes will be determined by maximizing the deviation method.

The assessment of the attribute weights is crucial for $\mathfrak{MCDM}$. A maximizing deviation technique to define the attribute weights for solving $\mathfrak{MCDM}$ problems with quantitative info is suggested by Wang [27]. Here, an optimization model based on the maximizing deviation method to define the optimal weight of attribute under $\mathfrak{FHF}$ framework has been built. The deviation of U_i relative to all other alternatives can be given as:
For the attribute $O_j \in \mathbf{O}$, the deviation of the alternative U_i to all the other alternatives can be expressed as: For $1 \leq i \leq n$ and $1 \leq j \leq n$, and

$$d(t_{ij}, t_{kj}) = \frac{1}{2}\left[\frac{1}{v}\sum_{\lambda=1}^{v}\left|(t_{ij}^{\beta(\lambda)})^2 - (t_{kj}^{\beta(\lambda)})^2\right| + \frac{1}{v}\sum_{\lambda=1}^{v}\left|(t_{ij}^{\prime\beta(\lambda)})^2 - (t_{kj}^{\prime\beta(\lambda)})^2\right|\right] \tag{5.1}$$

$$D_{ij}(\omega) = \sum_{k=1}^{m} \omega_j d(t_{ij}, t_{kj})$$

denotes the $\mathfrak{FHF}$ distance between $\mathfrak{PHN}$s t_{ij}, t_{kj} defined as in.

Definition 5.1. *For $1 \leq j \leq n$,*

$$\begin{aligned} D_j(\omega) &= \sum_{k=1}^{m} D_{ij}(\omega) \\ &= \sum_{i=1}^{m}\sum_{k=1}^{m}\omega_j\left(\frac{1}{2}\frac{1}{v}\sum_{\lambda=1}^{v}\left[\left|(t_{ij}^{\beta(\lambda)3} - t_{kj}^{\beta(\lambda)}3)\right|\right] + + \frac{1}{v}\sum_{\lambda=1}^{v}\left[\left|(t_{ij}^{\prime\beta(\lambda)3}) - t_{kj}^{\prime\beta(\lambda)3}\right|\right]\right) \end{aligned} \tag{5.2}$$

then the deviation value of all alternatives from the other alternatives, for the Oj attribute, is $\mathbf{D}_j(\omega)$. By selecting the weight vector ω that maximizes all deviation values, a nonlinear programming model for all the attributes can be obtained:

$$(M-1) \quad s.t. \begin{cases} \max D(\omega) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\omega_j d(t_{ij}, t_{kj}) \\ \omega_j \geq 0, \quad 1\ leq j \leq n, \quad \sum_{j=1}^{n}\omega_j = 1 \end{cases} \tag{5.3}$$

Model (M - 1) is solved as:

$$L(\omega, \delta) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\omega_j d(t_{ij}, t_{kj}) + \frac{\delta}{2}\left(\sum_{j=1}^{n}\omega_j - 1\right) = 0$$

The (M-1) model here is a constrained optimization problem and this model is called the Lagrange function, such that n is a real number representing the Lagrange multiplier variable.

$$\frac{\partial L}{\partial \omega_j} = \sum_{i=1}^{m}\sum_{k=1}^{m} d(t_{ij}, t_{kj}) + \delta\omega_j = 0 \tag{5.4}$$

$$\frac{\partial L}{\partial \delta} = \frac{1}{2}\left(\sum_{j=1}^{n}\omega_j - 1\right) = 0 \tag{5.5}$$

are partial derivatives for $\mathbf{L}$. Then, for $1 \leq j \leq n$,

$$\omega_j = \frac{-\sum_{i=1}^{m}\sum_{j=1}^{n} d(t_{ij}, t_{kj})}{\delta} \tag{5.6}$$

and we have

$$\delta = -\sqrt[3]{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} d(t_{ij}, t_{kj})\right)^3}. \tag{5.7}$$

$\delta > 0$, $\sum_{i=1}^{m}\sum_{j=1}^{n} d(t_{ij}, t_{kj})$ signify the sum of deviations of all the alternatives according to the jth attribute, and $\sqrt[3]{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} d(t_{ij}, t_{kj})\right)^3}$ signifies the sum of deviations of all the alternatives according to all the attributes.

Using Equation 5.6 and Equation 5.7, we have

$$\omega_j = \frac{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\omega_j d(t_{ij},t_{kj})}{\sqrt[3]{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} d(t_{ij},t_{kj})\right)^3}}.$$

By normalizing ω_j, we obtain

$$\omega_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m}\omega_j\left(\frac{1}{v}\sum_{\lambda=1}^{v}\left[\left|(t_{ij}^{\beta(\lambda)3}-t_{kj}^{\beta(\lambda)}3)\right|\right]+\frac{1}{v}\sum_{\lambda=1}^{v}\left[\left|(t_{ij}^{\prime\beta(\lambda)3})-t_{kj}^{\prime\beta(\lambda)3}\right|\right]\right)}{\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{k=1}^{m}\omega_j\left(\frac{1}{v}\sum_{\lambda=1}^{v}\left[\left|(t_{ij}^{\beta(\lambda)3}-t_{kj}^{\beta(\lambda)}3)\right|\right]+\frac{1}{v}\sum_{\lambda=1}^{v}\left[\left|(t_{ij}^{\prime\beta(\lambda)3})-t_{kj}^{\prime\beta(\lambda)3}\right|\right]\right)\right]} \tag{5.8}$$

There are also cases where the weight vector can be known partially but not completely. For these cases, using the set of known weight information, Δ, a new constrained optimization model can be constructed as follows:

$$(M-2)\begin{cases}\max D(\omega)=\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{k=1}^{m}\omega_j\left(\frac{1}{v}\sum_{\lambda=1}^{v}\left[\left|(t_{ij}^{\beta(\lambda)3}-t_{kj}^{\beta(\lambda)}3)\right|\right]+\frac{1}{v}\sum_{\lambda=1}^{v}\left[\left|(t_{ij}^{\prime\beta(\lambda)3})-t_{kj}^{\prime\beta(\lambda)3}\right|\right]\right)\right]\\ s.t\quad \omega\in\Delta,\quad \omega_j\geq 0,\quad j=1,2,\cdots,n,\quad \sum_{j=1}^{n}\omega_j=1\end{cases} \tag{5.9}$$

where Δ is a set of constraint conditions that the weight value ω_j should compensate with respect to the necessities in real cases. This model must be solved to obtain the optimal solution $\omega=\{\omega_1,\omega_2,\cdots,\omega_n\}$.

5.1. Algorithm

A convenient approach can be given in this subsection, which will be helpful in solving $\mathfrak{MCDM}$ problems where the info about the attribute weights is incomplete known or not completely unknown, and the attribute values take the form of $\mathfrak{FHF}$ info.

Step 1: For decision ($t_{ij}=\{\zeta_{ij},\eta_{ij}\}$, $1\leq i\leq n; 1\leq j\leq m$), build the $\mathfrak{FHF}$ decision matrices $C=(h_{ij})_{m\times n}$.

If two types of the attribute exist, then the $\mathfrak{FH}$ DCMX can be converted into the normalized $\mathfrak{FHF}$ DCMX $D_N=(\mu)_{m\times n}$, where

$$(\mu)_{m\times n}=\begin{cases}t_{ij}, & \textit{if the attribute is of benefit type}\\ t_{ij}^{c}, & \textit{if the attribute is of cost type.}\end{cases} \tag{5.10}$$

In 5.10, $t_{ij}^{c}=\{\eta_{ij},\zeta_{ij}\}$ $1\leq i\leq n; 1\leq j\leq m$. When all attributes are identical, DCMX is not normalized.

Step 2: When the info concerning the attribute weights is completely unknown, from Equation 5.8, the attribute weights are obtained. When the info concerning the attribute weights is partly known, with the solving of model (M - 2) attribute weights are obtained.

Step 3: Employ the improved aggregation operators to acquire the $\mathfrak{FHN}$ t_i $1\leq i\leq n$ for the alternatives U_i. This is improved operators to maintain the collective whole preference values t_i $i=1,2,\cdots,n$ of alternative U_i, where $\omega=\{\omega_1,\omega_2,\cdots,\omega_n\}^T$ the weighting vector of the attributes.

Step 4: By utilization Equation 3.2, the scores $S(t_i)$ and the deviation degree $\beta(t_i)$ of all the whole values t_i are computed.

Step 5: Rank the alternatives U_i and then choose the best one.

6. Application to Infectious Diseases

We will do a study on diseases and the symptoms of these diseases. Let the diseases be given by the set $U=\{viral fever, malaria, typhoid, stomach problem, chest problem\}=\{u_1,u_2,u_3,u_4,u_5\}$ and the symptoms by the set $A=\{temprature, headache, stomachpain, cough\}=\{O_1,O_2,O_3,O_4\}$. The five possible alternatives U_i are to be assessed utilization the $\mathfrak{FHF}$ info of three DMRs as shown in $T=(t_{ij})$ $(1\leq i\leq 5)$.

Consider that the info on the attribute weights is partially known, and the known weight info is presented as:

$$\Delta=\{0.18\leq\omega_1\leq 0.25,\quad 0.22\leq\omega_2\leq 0.30,\quad 0.27\leq\omega_3\leq 0.35,\quad 0.33\leq\omega_4\leq 0.40,\}$$

for $\sum_{j=1}^{4}=1$. Apparently, the counts of values in distinct $\mathfrak{FHN}$s are distinct. To more accurately compute the distance between two $\mathfrak{FH}$s, the shorter one needs to be expanded until both have an identical length when compared. Looking at the desired regulations, it can be thought that the DMRs are pessimistic. In this case, $\mathfrak{FHF}$ data is given again in Table 2 by adding their minimum values.

Step 1. Normalize the DCMX. Since the measurements are identical, no normalization is performed.

Step 2. Run the model (M - 2) and build the individual objective model:

	O_1	O_2	O_3	O_4
U_1	((0.8, 0.7), (0.6, 0.5)))	((0.3, 0.5), (0.7, 0.9))	((0.6), (0.6, 0.7, 0.8))	((0.8, 0.9), (0.4, 0.5))
U_2	((0.6, 0.5, 0.7), (0.8))	((0.5, 0.6), (0.6, 0.7))	((0.8, 0.9), (0.4, 0.5))	((0.7, 0.8), (0.4, 0.6))
U_3	((0.6, 0.7), (0.6, 0.8, 0.9))	((0.6, 0.7, 0.9), (0.3, 0.4))	((0.5, 0.7), (0.5, 0.6))	((0.4, 0.5), (0.7, 0.9))
U_4	((0.9), (0.3, 0.4, 0.6))	((0.6, 0.7), (0.6))	((0.5, 0.6, 0.4), (0.7, 0.8))	((0.6, 0.5), (0.8))
U_5	((0.3, 0.4), (0.8, 0.9))	((0.7, 0.8), (0.5, 0.7))	((0.4, 0.6), (0.6, 0.7))	((0.7, 0.8, 0.9), (0.5))

Table 1: $\mathfrak{FH}$ DCMX

	O_1	O_2	O_3	O_4
U_1	({0.8, 0.8, 0.7}, {0.6, 0.6, 0.5})	({0.3, 0.3, 0.5}, {0.7, 0.7, 0.9})	({0.6, 0.6, 0.6}, {0.6, 0.7, 0.8})	({0.8, 0.8, 0.9}, {0.4, 0.4, 0.5})
U_2	({0.6, 0.5, 0.7}, {0.8, 0.8, 0.8})	({0.5, 0.5, 0.6}, {0.6, 0.6, 0.7})	({0.8, 0.8, 0.9}, {0.4, 0.4, 0.5})	({0.7, 0.7, 0.8}, {0.4, 0.4, 0.6})
U_3	({0.6, 0.6, 0.7}, {0.6, 0.8, 0.9})	({0.6, 0.7, 0.9}, {0.3, 0.3, 0.4}))	({0.5, 0.5, 0.7}, {0.5, 0.5, 0.6)}	({0.4, 0.4, 0.5}, {0.7, 0.7, 0.9})
U_4	({0.9, 0.9, 0.9}, {0.3, 0.4, 0.6})	({(0.6, 0.6, 0.7}, {0.6 0.6, 0.6})	({0.5, 0.6, 0.4}, {0.7, 0.7, 0.8})	({0.6, 0.6, 0.5}, {0.8, 0.8, 0.8})
U_5	({0.3, 0.3, 0.4}, {0.8, 0.8, 0.9})	({0.7, 0.7, 0.8}, {0.5, 0.5, 0.7})	({0.4, 0.4, 0.6}, {0.6, 0.6, 0.7})	({0.7, 0.8, 0.9}, {0.5, 0.5, 0.5})

Table 2: F$\mathfrak{FH}$ DCMX

$$(M-2)\begin{cases}\max D(\omega) = 5.1091\omega_1 + 5.1254\omega_2 + 4.368\omega_3 + 5.326\omega_4 \\ s.t \quad \omega \in \Delta, \quad \omega_j \geq 0, \quad 1 \leq j \leq 4\end{cases} \tag{6.1}$$

To acquire the optimal weight vector $\omega = \{0.15, 0.27, 0.33, 0.22\}^T$, this model will be solved.

Step 3. The decision info endowed in matrix $T = (t_{ij})_{m\times n}$ and the $\mathfrak{FHWA}$ operator to acquire the whole preferences values $\widetilde{t}_i$ of the alternatives U_i $(i = 1,2,3,4,5)$ are benefited. Therefore,

$\widetilde{t}_1 = (\{0.6676, 0.7261, 0.708, 0.643, 0.6821, 0.66, 0.72, 0.74\}, \{0.581, 0.6113, 0.639, 0.6247, 0.6542, 0.684, 0.6102, 0.671,$
$0.642, 0.653, 0.6871, 0.718, 0.5653, 0.6, 0.6216, 0.5937, 0.6247, 0.6528, 0.605, 0.6365, 0.6652, 0.6354, 0.6686, 0.7\})$
$\widetilde{t}_2 = (\{0.7, 0.72, 0.7657, 0.68, 0.71, 0.732, 0.775, 0.684, 0.713, 0.76, 0.78, 0.7, 0.725, 0.77, 0.7887, 0.7036, 0.7304, 0.774,$
$0.8, 0.7162, 0.72, 0.7413, 0.7824, 0.8\}, \{0.509, 0.5565, 0.55, 0.6, 0.531, 0.58, 0.5712, 0.6244\})$
$\widetilde{t}_3 = (\{0.53, 0.544, 0.61, 0.62, 0.5731, 0.585, 0.64, 0.6482, 0.712, 0.718, 0.748, 0.753, 0.555, 0.57, 0.6263, 0.636, 0.6, 0.605,$
$0.655, 0.663, 0.723, 0.729, 0.757, 0.762\}, \{0.5, 0.532, 0.523, 0.565, 0.52, 0.5622, 0.552, 0.6, 0.514, 0.555, 0.546, 0.59, 0.543, 0.59, 0.577, 0.623, 0.5$
$\widetilde{t}_4 = (\{0.671, 0.689, 0.659, 0.658, 0.677, 0.6452, 0.7, 0.71, 0.6835, 0.683, 0.7, 0.671\}, \{0.62, 0.643, 0.643, 0.6716, 0.683, 0.714\})$
$\widetilde{t}_5 = (\{0.6, 0.64, 0.703, 0.645, 0.68, 0.735, 0.61, 0.65, 0.712, 0.66, 0.7, 0.743, 0.6, 0.64, 0.705, 0.65, 0.682, 0.737,$
$0.612, 0.652, 0.715, 0.66, 0.7, 0.745\}, \{0.582, 0.612, 0.64, 0.67, 0.6, 0.623, 0.68, 0.682\})$

Step 4. Compute the scores $S(\widetilde{t}_i)$ of whole $\mathfrak{FHN}$s $\widetilde{t}_i$: $S(\widetilde{t}_1) = 0.0716$, $S(\widetilde{t}_2) = 0.2266$, $S(\widetilde{t}_3) = 0.0251$, $S(\widetilde{t}_4) = 0.021$, $S(\widetilde{t}_5) = 0.05053$.

Step 5. Rank all the alternatives U_i according to the scores $S(\widetilde{t}_i)$ of overall $\mathfrak{FHN}$s $\widetilde{t}_i$: We have $S(\widetilde{t}_2) > S(\widetilde{t}_1) > S(\widetilde{t}_5) > S(\widetilde{t}_3) > S(\widetilde{t}_4)$ which shows that $U_2 > U_1 > U_5 > U_3 > U_4$. From this, it is understood that the best choice is U_2.

Furthermore, when $\mathfrak{FHWG}$ operator is used for the same problem, in the new approach here, the same operations will be performed starting from Step 3.

7. Comparison

First, a comparison will be made between the FFN-specific $\mathfrak{MCDM}$ method described by Senapati and Yager [19] and the method proposed in this study.

FFNs may be taken into account as a specific situation of $\mathfrak{FHN}$s when there is only one element in MD and ND. $\mathfrak{FHN}$s can be converted to FFNs by computing the average value of the MD and MD. Later, the new info may be represented in Table 3.

	O_1	O_2	O_3	O_4
U_1	({0.5, 0.5})	({0.7, 0.4})	({0.8, 0.7})	({0.9, 0.4})
U_2	({0.7, 0.5})	({0.8, 0.6})	({0.9, 0.2})	({0.6, 0.5})
U_3	({0.6, 0.8})	({0.2, 0.9})	({0.6, 0.5})	({0.8, 0.4})
U_4	({0.9, 0.3})	({0.5, 0.7})	({0.7, 0.8})	({0.7, 0.7})
U_5	({0.8, 0.7})	({0.9, 0.5})	({0.6, 0.6})	({0.7, 0.6})

Table 3: FF DCMX

With the comprehensive evaluation values using the FFWA, the score values are: $S(U_1) = 0.343$, $S(U_2) = 0.5216$, $S(U_3) = 0.102$, $S(U_4) = 0.095$, $S(U_5) = 0.247$. Then $S(U_2) > S(U_1) > S(U_5) > S(U_3) > S(U_4)$ which shows that $U_2 > U_1 > U_5 > U_3 > U_4$. That is the most desirable alternative is U_2.

Second, a comparison will be made between the HFN-specific $\mathfrak{MCDM}$ method and the method proposed in this study.

HFNs may be taken into account as a specific situation of $\mathfrak{FFN}$s, if DMRs solely think MDs in assessment. Then, $\mathfrak{FFN}$s may be converted to HFNs by remaining solely the MDs, and the HF info may be shown in Table 4.

	O_1	O_2	O_3	O_4
U_1	{0.7, 0.8}	{0.3, 0.4}	{0.7}	{0.9, 0.8}
U_2	{0.6, 0.5, 0.7}	{0.4, 0.5}	{0.9, 0.8}	{0.7, 0.8}
U_3	{0.8}	{0.6, 0.7}	{0.4, 0.6, 0.5}	{0.5, 0.6}
U_4	({0.3, 0.4})	({0.7, 0.8})	({0.5, 0.6})	({0.6, 0.7, 0.8})
U_5	{0.5, 0.6}	{0.7, 0.8, 0.9}	{0.6, 0.7}	{0.3, 0.4}

Table 4: HF DCMX

If the hesitant fuzzy weighted average operator (HFWA) [25] is employed, score values are: $S(U_1) = 0.6345$, $S(U_2) = 0.6891$, $S(U_3) = 0.5297$, $S(U_4) = 0.5813$, $S(U_5) = 0.5674$. Then $S(U_2) > S(U_1) > S(U_5) > S(U_4) > S(U_3)$ which shows that $U_2 > U_1 > U_5 > U_4 > U_3$. That is the most desirable alternative is U_2.

Now, a comparison will be made between the intuitionistic hesitant fuzzy numbers and the proposed method.

IHFNs may be taken into account as a specific situation of PHFNs if DMRs are limited to the definition of PHFS. The PHFNs may be converted to IHFNs by limiting the $MD^2 + ND^2 \leq 1$, to the $MD + ND \leq 1$, and the IHF info may be shown in Table 5.

	O_1	O_2	O_3	O_4
U_1	({0.3, 0.4}, {0.5, 0.6})	({0.5, 0.6}, {0.4, 0.3})	({0.6}, {0.5, 0.4, 0.3})	({0.5, 0.6}, {0.4, 0.3})
U_2	({0.3, 0.5, 0.4}, {0.5})	({0.3, 0.4}, {0.6, 0.5})	({0.6, 0.7}, {0.4, 0.6})	({0.7, 0.6}, {0.6, 0.6})
U_3	({0.5, 0.6}, {0.4, 0.6, 0.8})	({0.5, 0.6, 0.7}, {0.3, 0.4}))	({0.5, 0.7}, {0.5, 0.6)}	({0.4, 0.4}, {0.7, 0.9})
U_4	({0.9}, {0.3, 0.4, 0.6})	({0.6, 0.7}, {0.7})	({0.4, 0.6, 0.8}, {0.7, 0.8})	({0.3, 0.4}, {0.6})
U_5	({0.3, 0.4}, {0.8, 0.9})	({ 0.7, 0.8}, {0.5, 0.7})	({0.4, 0.6}, {0.6, 0.7})	({0.3, 0.4, 0.5}, {0.5})

Table 5: IHF DCMX

If the IHF weighted average operator (IHFWA) [15] is employed, score values are: $S(U_1) = 0.16035$, $S(U_2) = 0.1967$, $S(U_3) = 0.092$, $S(U_4) = -0.0868$, $S(U_5) = -0.798$. Then $S(U_2) > S(U_1) > S(U_3) > S(U_4) > S(U_5)$ which shows that $U_2 > U_1 > U_3 > U_4 > U_5$. That is the most desirable alternative is U_2.

A comparison will be made between the PHFNs and the presented method. The values in Table 6 [7] will be used for this comparison.

	O_1	O_2	O_3	O_4
U_1	({0.7, 0.8}, {0.5, 0.6})	({0.3, 0.4}, {0.8, 0.9})	({0.7}, {0.5, 0.6, 0.7})	({0.8, 0.9}, {0.3, 0.4})
U_2	({0.5, 0.6, 0.7}, {0.7})	({0.4, 0.5}, {0.7, 0.8})	({0.8, 0.9}, {0.3, 0.4})	({0.7, 0.8}, {0.5, 0.6})
U_3	({0.5, 0.6}, {0.6, 0.7, 0.8})	({0.7, 0.8, 0.9}, {0.3, 0.4}))	({0.6, 0.7}, {0.5, 0.6)}	({0.3, 0.4}, {0.7, 0.9})
U_4	({0.8}, {0.3, 0.4, 0.5})	({0.6, 0.7}, {0.7})	({0.4, 0.5, 0.6}, {0.7, 0.8})	({0.5, 0.6}, {0.7})
U_5	({0.3, 0.4}, {0.8, 0.9})	({ 0.7, 0.8}, {0.5, 0.6})	({0.4, 0.5}, {0.6, 0.7})	({0.6, 0.7, 0.8}, {0.5})

Table 6: PHF DCMX [7]

If the PHF weighted average operator (PHFWA) [7] is employed, the score values are: $S(U_1) = 0.2757$, $S(U_2) = 0.3047$, $S(U_3) = 0.0402$, $S(U_4) = -0.0538$, $S(U_5) = 0.0439$. Then $S(U_2) > S(U_1) > S(U_5) > S(U_3) > S(U_4)$ which shows that $U_2 > U_1 > U_3 > U_4 > U_5$. That is the most desirable alternative is U_2.

Comparisons made in this section are given in the table below:

8. Conclusion

In this study, a $\mathfrak{FH}$ was obtained by combining the FFS and the HFS. For $\mathfrak{FHN}$s, the operations and comparison techniques were given. To solve the $\mathfrak{MCDM}$ problems under the $\mathfrak{FH}$ environment, the $\mathfrak{FHWA}$ and $\mathfrak{FHWG}$ operators were put forward to aggregate the $\mathfrak{FHN}$s given by the DMR. A $\mathfrak{MCDM}$ technique combined with the presented operators is built to solve the $\mathfrak{MCDM}$ problems in different situations. Subsequently, a numerical example based on infectious diseases is provided to indicate the applications and advantages of the proposed methods.

Method	ranking
for proposed method	$U_2 > U_1 U_5 > U_3 > U_4$
for FFNs	$U_2 > U_1 U_5 > U_3 > U_4$
for HFNs	$U_2 > U_1 U_5 > U_4 > U_3$
for IHFNs	$U_2 > U_1 U_3 > U_4 > U_5$
for PHFNs	$U_2 > U_1 U_5 > U_3 > U_4$

Table 7: Comparison HF DCMX

Contributions

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Conflict of interests

The authors declare that they have no competing interests.

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This article does not contain any studies with human participants or animals performed by any of the authors.

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