An Elementary Proof to $e^{\gamma} \log \log n > \Psi(n)/n$ for any $n > 30$

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An Elementary Proof to $e^\gamma \log \log n > \frac{\psi(n)}{n}$ for any $n > 30$

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Abstract

Let $\mathcal{P}$ be the set of all primes, $\psi(n) = n \prod_{p \in \mathcal{P}, p|n} \left(1 + \frac{1}{p}\right)$ be the Dedekind psi function, we unconditionally show that $e^\gamma \log \log n > \frac{\psi(n)}{n}$ for any $n > 30$, where γ is Euler's constant.

1 Propositions

Proposition 1. Denote by p_k , the k^{th} prime

$$p_k \geq k \left(\log k + \log \log k - 1 + \frac{\log \log k - 2.1}{\log k} \right) \text{ for } k \geq 3$$

Proof. View [2] ■

Proposition 2. Exists an k_0 such that, for each $k \geq k_0$ we have

$$\log \log 3^{k+2} > \left(1 + \frac{1}{p_k}\right) \log \log 3^{k+1}$$

Proof. We have for each $k \geq 57$

$$k \left(\log k + \log \log k - 1 + \frac{\log \log k - 2.1}{\log k} \right) > \frac{\log(k+1) + \log \log 3}{\log(k+2) - \log(k+1)}$$

then, by Propositions 1

$$\begin{aligned} p_k &> \frac{\log(k+1) + \log \log 3}{\log(k+2) - \log(k+1)} \\ p_k \log(k+2) - p_k \log(k+1) &> \log(k+1) + \log \log 3 \end{aligned}$$

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We add to both sides of the inequality $p_k \log(k+1) + p_k \log \log 3$ and regrouping,

$$\log \log 3^{k+2} > \left(1 + \frac{1}{p_k}\right) \log \log 3^{k+1} \text{ for any } k \geq 57$$

where $k_0 = 57$, verifying directly by computer it is found that for any $k_0 > 15$. $\blacksquare$

Proposition 3. Given any $n \geq 210$ and an $\xi_0 > 0.632$ we have

$$e^\gamma \log \log n > \xi_0^{-1} \log \log 3^{k+2}$$

where $k = \sum_{p|n} 1$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ and $N_k = p_1 p_2 \dots p_k$, (the product of the k primes divisors of n), then $n \geq N_k$. The minimum product of 4 different primes is 210. From this, is easy to proof that: $e^\gamma \log \log n \geq e^\gamma \log \log N_k > \xi_0^{-1} \log \log 3^{k+2}$ with $\xi_0 > 0.632 \approx 0.631535$ and $n \geq 210$. $\blacksquare$

2 Theorems

Theorem 1. For each $k \geq 1$ and $0 < \xi_1 < 0.664$ we have

$$\xi_1^{-1} \log \log 3^{k+2} > \prod_{i=1}^k \left(1 + \frac{1}{p_i}\right) \quad (1)$$

Proof. For all $1 \leq k \leq k_0$, with $k_0 = 56$ it is verified directly by computer.

For $k = 56$ we have

$$\xi_1^{-1} \log \log 3^{56+2} > \prod_{i=1}^{56} \left(1 + \frac{1}{p_i}\right) \quad (2)$$

For $k = 57$ we use Proposition 2 and inequality (2).

$$\xi_1^{-1} \log \log 3^{57+2} > \left(1 + \frac{1}{p_{57}}\right) \xi_1^{-1} \log \log 3^{(57-1)+2} > \left(1 + \frac{1}{p_{57}}\right) \prod_{i=1}^{56} \left(1 + \frac{1}{p_i}\right)$$

$$\xi_1^{-1} \log \log 3^{57+2} > \prod_{i=1}^{57} \left(1 + \frac{1}{p_i}\right)$$

Suppose this is true for any $57 < k < m$, Then, for $k = m - 1$ we have

$$\xi_1^{-1} \log \log 3^{(m-1)+2} > \prod_{i=1}^{m-1} \left(1 + \frac{1}{p_i}\right) \quad (3)$$

then, for $k = m$ we use Proposition 2 and inequality (3).

$$\xi_1^{-1} \log \log 3^{m+2} > \left(1 + \frac{1}{p_m}\right) \xi_1^{-1} \log \log 3^{(m-1)+2} > \left(1 + \frac{1}{p_m}\right) \prod_{i=1}^{m-1} \left(1 + \frac{1}{p_i}\right)$$

$$\xi_1^{-1} \log \log 3^{m+2} > \prod_{i=1}^m \left(1 + \frac{1}{p_i}\right)$$

therefore, for any $k \geq 1$ and $0 < \xi_1 < 0.664$ the inequality (1) is correct. $\blacksquare$

Theorem 2.

$$e^\gamma \log \log n > \frac{\psi(n)}{n} \text{ for any } n > 30 \quad (4)$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ and $N_k = p_1 p_2 \dots p_k$ (the product of the k primes divisors of n), then $n \geq N_k$, by properties of the Dedekind function, we have

$$\frac{\psi(n)}{n} = \frac{\psi(N_k)}{N_k} = \prod_{i=1}^k \left(1 + \frac{1}{p_i}\right) \quad (5)$$

For all $30 < n < 210$, it is verified directly by computer the inequality (4). For any $n \geq N_k \geq 210$, using the Proposition 3, Theorem 1, equality (5) and a ξ such that $0.632 < \xi_0 \leq \xi \leq \xi_1 < 0.664$, then

$$e^\gamma \log \log n > e^\gamma \log \log N_k > \xi^{-1} \log \log 3^{k+2} > \prod_{i=1}^k \left(1 + \frac{1}{p_i}\right) = \frac{\psi(n)}{n}$$

therefore, for any $n > 30$ the inequality (4) is correct. $\blacksquare$

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