On Injectivity of an Integral Operator Connected to Riemann Hypothesis

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On Injectivity of an Integral Operator Connected to Riemann Hypothesis *

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Abstract

Using the equivalent formulation of RH given by Beurling ([4], 1955), Alcantara-Bode showed ([2], 1993) that Riemann Hypothesis holds if and only if the integral operator on the Hilbert space $L^2(0,1)$ having the kernel defined by fractional part function of the expression between brackets $\{y/x\}$, is injective.

Since then, the injectivity of the integral operator used in equivalent formulation of RH has not been addressed nor has been dissociated from RH and, a pure mathematics solution for RH is not ready yet.

Here is a numerical analysis approach of the injectivity of the linear bounded operators on separable Hilbert spaces addressing the problems like the one presented in [2]. Apart of proving the injectivity of the Beurling - Alcantara-Bode integral operator, we obtained the following result: every linear bounded operator (or its associated Hermitian), strict positive definite on a dense family of including approximation subspaces in $L^2(0,1)$ built on simple functions, is injective if the rate of convergence to zero of its unbounded sequence of inverse condition numbers on approximation subspaces is $o(n^{-s})$ for some $s \geq 0$. When s = 0, the sequence is inferior bounded by a not null constant, that is the case in the Beurling - Alcantara-Bode integral operator.

In the Theorem 4.1 we addressed with numerical analysis tools the injectivity of the integral operator in [2] claiming that - even if a solution in pure mathematics is desired, together with the Theorem 1, pg. 153 in [2], the RH holds.

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§1. Introduction.

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In the paper of Alcantara-Bode [2], the author proved (Theorem 1, pg. 152) that RH holds if and only if the following Hilbert-Schmidt integral operator T_ρ defined on $L^2(0,1)$ by:

$$(T_\rho u)(y) = \int_0^1 \{\frac{y}{x}\} u(x) dx \tag{1}$$

has its null space $N_{T\rho} = \{0\}$ where the kernel function is the fractional part function of the expression between brackets.

The linear bounded integral operator has been introduced in [4] (Beurling, 1955) for providing first equivalent formulation of RH in terms of functional analysis. This equivalent formulation of RH has been used in [2] (Alcantara-Bode, 1993) for providing second equivalent formulation of Riemann Hypothesis in terms of injectivity of the operator in (1) on $L^2(0,1)$.

We address in this paper the injectivity of linear bounded operators on separable Hilbert spaces. The Theorem 2.1 from §2 on generic separable Hilbert spaces as well its versions adapted for the integral operators on $L^2(0,1)$ (§3, Lemmas 3.1 and 3.2), are the criteria using the operator approximations on a dense family of approximation subspaces.

Let H be a separable Hilbert space. The norm considered on H is the norm induced by its inner product. Because a linear bounded operator T and its associated Hermitian and positive definite operator (T^*T) have the same null space N_T, we will consider its associated Hermitian operator when we do not have enough information about positivity of T on H. Then, we take T or (T^*T) in order to have its positivity $\langle Tu, u\rangle \geq 0$ on H as we need. In the following T is positive definite on H and our aim is to provide methods for investigating its strict positivity on H, in other words, in finding if its null space $N_T = \{0\}$.

We will see later that a weaker property than the positivity on H is enough and is mandatory for investigating the injectivity of the linear operators on H, that is the strict positivity of the operators on the dense family of approximation subspaces (a-positivity property).

Let $B := \{u \in \text{H}; \parallel u \parallel = 1\}$ be the unit sphere in H.
Because a not null element u is in N_T if and only if $u/ \parallel u \parallel \in N_T$, we will consider the normalized elements of N_T for convenience.

A family of finite dimension subspaces $\Im := \{S_n, n \in N\} \subset$ H is a proper family of approximation subspaces if the following properties hold on the subspaces of the family:

(a) $S_n \subset S_{n+1}, n \in N$

(b) $\beta_n(u) := \|$u - u$_n$ $\| \to 0$, for n $\to \infty$
for every u $\in B$ and with u$_n$ its orthogonal projection on S$_n$, $\quad (\forall)$ n $\in N$.

A proper family subspaces is dense in H: $\overline{\cup_{n=1}^{\infty} S_n}$ = H, property easy to show. The main reason in defining a dense family of subspaces in H being a proper family, is to evidentiate the two properties used extensively in this paper: the inclusion of the approximation subspaces and, the increasing monotone convergence in norm to 1 of the sequence of the projections on approximation subspaces defined for every element from unit sphere B $\subset$ H.

We say that the bounded linear operator T is almost strict positive definite on H or a-positive, if there exists a proper family $\Im$ on which, for every subspace S$_n \in \Im$, there exists $\alpha_n > 0$ such that:

$$\langle Tv, v\rangle \geqq \alpha_n \|v\|^2, \quad \forall v \in S_n$$

We name the set $\{\alpha_n, n \geq 1\}$, the set of a-positive parameters associated to T on $\Im$. In facts, a-positive parameters are the smallest eigenvalues of the operator restrictions to the corresponding subspaces. We will use the term a-positive also associated to the family $\Im$ for convenience.

Let $L_{\Im}$ be the set of linear bounded operators on H that are a-positive with respect to the family $\Im$.

Taking $\Im^0 = \cup_{n=1}^{\infty} S_n$ we define H$_{\Im^0}$:= H $\setminus \Im^0$ and $B_0 := B \cap$ H$_{\Im^0}$.

Let T $\in L_{\Im}$. If there exists u $\in B \cap N_T$ then u $\notin S_n$ for every n ≥ 1, i.e. u $\notin \Im^0$. For this reason we will consider in our analysis only u $\in$ H that are eligible, in the sense that these elements from H could be the normalised zeros of T, u $\in B_0$.

We will refer in this paper to B_0 as the set of eligible zeros of the a-positive operators given a proper family $\Im$.

Observations: we point out two extreme situations that we will exclude from our analysis:

1.) - If T is positive definite on H but is not a-positive on a proper family, then T should have a zero inside the family: there exists S$_n \in \Im$ and v $\in$ S$_n$ not null, such that $\langle Tv, v\rangle = 0$. So T is not injective and the investigations on its injectivity ends here. If T is not positive definite on H then the observation is valid for its associate Hermitian.
That means, a-positivity property of an linear bounded operator is mandatory for its injectivity.

2.) - If T is a-positive and the approximation subspaces of $\Im$ are invariant subspaces of T, then T is injective and so, no investigations for its injectivity needed.
As a consequence, from now on the linear bounded operators in considera-

tion would be a-positive operators satisfying $T(S_n) \nsubseteq S_n$, $(\forall) n \geq 1$

§2. Injectivity Criteria.

Let $\Im$ be a proper family of approximation subspaces on H and T $\in L_{\Im}$.

Suppose that there exists w $\in S_n^{\perp}$ for which Tw $\notin S_n^{\perp}$. For a such element w and for every v $\in S_n$ both not null, the inner product $\langle Tw, v\rangle$ is well defined in H. Now, because an strict positive operator on a finite dimension subspace has the same eigenvalues as its adjoint, in our case $T^{\star}_{|S_n}$ and $T_{|S_n}$ have both the same largest eigenvalue λ_n^{max} on S_n, we obtain the following estimation for the considered inner product:

$\mid \langle Tw, v\rangle \mid = \mid \langle w, T^{\star} v\rangle \mid \leq \lambda_n^{max} \parallel w \parallel \parallel v \parallel$

If there exists u $\in N_T \cap B_0$, we take w := (u_n - u) $\in S_n^{\perp}$ and $v = u_n$, the projection of u on S_n. Obviously, Tw = T$u_n \notin S_n^{\perp}$ (because of the a-positivity of T $\langle Tu_n, u_n\rangle \neq 0$ and, because there exists an index $n_0(u)$ from which u is not orthogonal to S_n from the density of the proper family). Then:

$\alpha_n \parallel u_n \parallel^2 \leq \langle Tu_n, u_n\rangle = \mid \langle T(u_n - u), u_n\rangle \mid$ and, together with the previous relations we obtain:

$$\mu_n \parallel u_n \parallel \leq \beta_n(u) \tag{2}$$

where $\mu_n := \lambda_n^{min} / \lambda_n^{max}$ is the ratio between the smallest and largest eigenvalues of the restriction of T to the subspace S_n, inequality obtained using the a-positivity property and replacing α_n with λ_n^{min}.

Pointing out: if there exists u $\in N_T \cap B_0$, then (2) holds on every $S_n \in \Im$ n $\geq n_0(u)$ where n$_0(u)$ is the finite index from where the orthogonal projections of u are not null - in virtue of the properties of the proper family given u $\in B_0$.

So, the property (2) for an eligible u $\in N_T$, is valid starting from a finite index n$_0(u)$; however, it is validated also for indexes n $< n_0(u)$ because for such indexes $u_n = 0$ and $\beta_n(u) = 1$.

For T $\in \Im$, given u $\in B_0$ we define the expression $\theta_n(u) := \theta_n^T(u)$ on every subspace of the proper family S$_n$, n $\geq n_0(u)$ by:

$$\theta_n(u) := \beta_n^2(u) - \mu_n^2 \parallel u_n \parallel^2 = 1 - (\mu_n^2 + 1) \parallel u_n \parallel^2 \tag{3}$$

with u_n the orthogonal projection of u on S$_n$. The last member in the equality is obtained using the relationship for every u $\in B$ and its orthogonal projections on S$_n$ and S$_n^{\perp}$: $\parallel u_n \parallel^2 + \parallel (u - u_n) \parallel^2 = \parallel u \parallel^2 = 1$.

We will split the set of eligible elements B_0 in two disjoint subsets:

1) $B_0^1 := B_0^1(T) = \quad \{u \in B_0; \quad \theta_n(u) \geq 0, n \geq n_0(u)\}$;

for which, $\quad B_0^1 \supset B_0 \cap N_T$

2) $B_0^2 := B_0^2(T) = \quad \{u \in B_0; \quad \theta_n(u) < 0, n \geq n_0(u)\};$
for which, $\quad B_0^2 \bigcap N_T = \{\emptyset\}$

Note. Because for every normalised u $\in N_T$ the inequality (2) holds globally on the proper family $\Im$, this behaviour will be propagated from now on for each statement involving (2); it is the reason we would omit carrying out every time the index specifications referring to all subspaces of the family, when it is evident from context. We would do it until the end of the paragraph.

Theorem 2.1. (Injectivity Criteria.) Let T$\in L_{\Im}$.

A.) If the sequence $\{\mu_n\}, n \geq 1$ is inferior bounded by a strict positive constant independent of n, $\quad \mu_n \geq C > 0, \quad$ for every n ≥ 1 then T is injective.

B.) If Q is a linear, bounded and injective operator on H such that Q (or its associate Hermitian) verifies $\mu_n(Q) \leq \mu_n(T), n \geq 1$ then T is injective.

Proof of A.)
Suppose that the sequence $\{\mu_n\}, n \geq 1$ is inferior bounded by a constant C independent of n, strict positive.
Let u $\in B_0$.

1) If there exists an infinite subsequence of subspaces from the family for which $\theta_{n_m}(u) \geq 0$, n$_m \geq n_0(u)$ we obtain:

$$\| u_{n_m} \|^2 \leq 1 \diagup (1 + \mu_{n_m}^2) < 1 \diagup (1 + C^2) < 1$$

and $\lim_{m \to \infty} \| u_{n_m} \|^2 < 1 \diagup (1 + C^2) < 1$.

Or, $\{u_{n_m}\}$, n$_m \geq n_0(u)$ should converge in norm to 1 on the approximation subspaces for every u $\in$ B. Thus, the inequality $\theta_{n_m}(u) \geq 0$ could not take place on an infinity of subspaces of the family. We should have instead at most only a finite number of subspaces verifying it.

If there exists a finite number of subspaces on which $\theta_{n_m}(u) \geq 0$, $n_m \geq n_0(u)$ then we will shrink the family including these subspaces inside of the subspaces already verifying the opposite relationship, obtaining u $\notin B_0^1$ when the sequence $\{\mu_n\}, n \geq 1$ is inferior bounded by a not null constant.

Then, we are in the situation $\theta_n(u) < 0$ for every n $\geq n_0(u)$ on the $\Im$, i.e. u $\in B_0^2$.

2) If $\theta_n(u) < 0$ holds for n $\geq n_0(u)$, then $\beta_n(u) < \mu_n \| u_n \|, n \geq n_0(u)$. But, u satisfying such relationship could not be in N_T because a reverse inequality given in (2) holds for eligible u in N_T. The only restriction we put on u $\in$ H, has been its eligibility, u $\in B_0$. Thus, if $\{\mu_n\}$, n ≥ 1 is

inferior bounded, does not exists u $\in B_0$ that is in N_T. Or, B_0 contains all normalized zeros of T, so $N_T = \{0\}$.

Proof of B.)

Because Q is injective, then Q or ($Q^\star$Q) is strict positive on H and so, is a-positive on $\Im$. Let $\mu_n(Q), n \geq 1$ be the injectivity parameters of Q or of its associate Hermitian.

For every u $\in B_0$, u $\in B_0^2(Q)$ having $\theta_n^Q(u) < 0$ otherwise, u should be in N_Q that is a null set.
Let u $\in B_0$.
From $\mu_n(Q) \leq \mu_n(T)$ we obtain $\theta_n^T(u) \leq \theta_n^Q(u) < 0$, valid for n ≥ 1 showing that u $\in B_0^2(T)$. Because u is chosen an arbitrary eligible element, $B_0 \subseteq B_0^2(T)$ that means $B_0^1(T) = \emptyset$, showing that T is injective. ■

§3. On Approximations of Integral Operators. On the separable Hilbert space L^2(0,1), we will take in the following paragraphs the proper family $\Im_\chi := \{\mathrm{S}_h, nh = 1, n \geq 2\}$, where $\mathrm{S}_h := span\{\chi_{n,k}, k = 1, n, nh = 1\}$ built on simple functions that are linear combinations of indicator functions of disjoint intervals

$$\chi_{n,k} := \chi_{n,k}(t) = \begin{cases} 1 & t \in \triangle_k := ((k-1)h, kh] \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

is dense in L^2(0,1) and meets the requests imposed by the definition of a proper family of approximation subspaces introduced in the previous paragraph.
Citing (Buescu & Paixa~o, 2007), the family of functions $\{\chi_{n,k}\}$, k = 1,n, nh =1, is defining a trace class integral operator with the kernel

$$r_h(x, y) = h^{-1} \sum_{k=1}^{n} \chi_{n,k}(x)\chi_{n,k}(y)$$

that is an orthogonal projection in L^2(0,1) on S$_h$ having the orthogonal eigenfunctions $\{\chi_{n,k}\}, k = 1, n$. In fact, $\mathrm{S}_h \in \Im_\chi$ are generated by the families of the orthogonal eigenfunctions of the projection operators $\{P_h := P_{|S_h}\}$, nh =1, $n \geq 2$.
Moreover, because the orthogonality of eigenfunctions is dictated by the disjoint interval support, $\chi_{n,k}(x)\chi_{n,j}(y) = \delta_{k,j}$ the entries in the matrix representation (5) bellow, of the operator restrictions on approximation subspaces will be zero outside the diagonal and, it explain the form of the discrete approximations of the kernel functions on approximation subspaces defined bellow.

Thus, the linear integral operator (like in (1))

$$T_\rho u := (T_\rho u)(y) = \int_0^1 \rho(y,x)u(x)dx$$

has the discrete approximations of the kernel function ρ like in [5],[6]:

$$\rho_h(y,x) = h^{-1}\sum_{k=1}^{n} \chi_{n,k}(x)\rho(y,x)\chi_{n,k}(y)$$

Obviously, $(\forall)$ u $\in$ H, its projection on S_h take the form of a simple function: $\mathrm{P}_{|_{S_h}} u = \sum_{k=1}^{n}\langle u, \chi_{n,k}\rangle\chi_{n,k}$.

Accordingly, taking any $v_h \in \mathrm{S}_h$ not null, of the form $v_h(\mathrm{t}) = \sum_{k=1}^{n} \mathrm{c}_k\chi_{n,k}(\mathrm{t})$ the discrete approximation of the integral operator, verifies the following equality:

$$\begin{aligned}\langle T_\rho^h v_h, v_h\rangle &= \int_0^1\int_0^1 \rho_h(y,x)v_h(x)\overline{v}_h(y)dxdy \\ &= h^{-1}\mathbf{c}_h^T M_h \overline{\mathbf{c}}_h \end{aligned} \tag{5}$$

Here the (n $\times$ n) diagonal matrix $M_h = [d_{kk}^h]_{k=1,n}$ is the matrix representation of the restriction T_ρ^h of the integral operator T_ρ to the subspace S_h and the vector $\mathbf{c}_h$ is the vector of coordinates $\{\mathrm{c}_k$, k=1,n$\}$ of v_h in C^n verifying $\|v_h\|^2 = \mathrm{h}\|\mathbf{c}_h\|^2_{C^n}$.

Taking a look at (5) and at the matrix representation M_h of the T_ρ^h on S_h, if the matrix is strict positive definite, we have: $\lambda_{min}(M_h) = \min_{k=1,n} d_{kk}^h$, from where:

$$\langle T_\rho^h v_h, u_h\rangle \geq h^{-2}\lambda_{min}(M_h)\|v_h\|^2$$

So, the a-positivity parameters associated to T_ρ should have the form like $\alpha_n = h^{-2}\lambda_{min}(M_h)$, nh = 1, n $\geq$ 1.

Remark 3.1) The linear bounded integral operator T_ρ is a-positive on $\Im_\chi$ if and only if on approximation subspaces its kernel function is verifying

$$d_{kk}^h := \int_{\triangle_k}\int_{\triangle_k} \rho(y,x)dxdy > 0 \tag{6}$$

for k = 1,n, nh = 1, i.e. the diagonal entries in the diagonal matrices M_h are strict positive.

Here is the main theorem (Theorem 2.1.A) reformulated for integral operators on $\mathrm{L}^2(0,1)$ taking the proper family $\Im_\chi$.

Lemma 3.1. (*Injectivity Criteria for Integral Operators.*) Let $T_\rho \in L_{\Im_\chi}$. If the sequence $\mu_h(T_\rho), nh = 1, n \geq 1$

$$\mu_h(T_\rho) = \frac{min_{(k=1,n)} d^h_{kk}}{max_{(k=1,n)} d^h_{kk}} \tag{7}$$

is inferior bounded, $\mu_h(T_\rho) > C$ by a strict positive constant C independent of n, nh = 1, then T_ρ is injective.
Proof.

In the view of the previous paragraphs, if T $\in L_{\Im_\chi}$ is a linear bounded integral operator its a-positivity parameters are given by $\alpha_n = h^{-2} \min_{k=1,n} d^h_{kk}$, nh =1, n$\geq$ 2. Thus, the injectivity criteria (Theorem 2.1.A) could be used in determining the injectivity of T_ρ observing that in this case $\{\mu_n, n \geq 1\}$ are given as in (7). ■

We will introduce a method for analysing the the injectivity of apositive operators having the injectivity parameters sequence inferior unbounded.

Let define the I-multiplier operator, the operator obtained as a product of identity by a power of x, $\{I_s\}, s \in N$ on $L^2(0,1)$ as follows: for u $\in L^2(0,1)$

$$(I_s u)(x) = x^s u(x) \quad i.e. \quad I_s : u(x) \to x^s u(x) \tag{8}$$

It is easy to show that for every s, I_s is linear, bounded, positive definite and injective on $L^2(0,1)$ and so, a-positive on $\Im_\chi$. The injectivity and positivity properties on H are coming from $\langle I_s u, u\rangle = \int_0^1 w(x)\overline{w(x)}dx > 0$ for u $\neq 0$ where w(x) = $x^{s/2}u(x)$. Its injectivity parameters applying the formula (7), are for n ≥ 1

$$\mu_{n(I_s)} = 1/(\sum_{i=0,s} n^i (n-1)^{s-i}) = o(n^{-s})$$

Lemma 3.2 (*I-multipliers Rule.*) Let T $\in L_{\Im_\chi}$. If exists an I-multiplier operator I_s for some $s \geq 1$ such that

$$\mu_n(I_s) \leq \mu_n(T)$$

for n $\geq$ 1, then T is injective.
Proof

Suppose that the relationship exists. Then, taking Q = I_s in the Theorem 3.1.B, we obtain T injective. ■

As a consequence of Lemmas 3.1 and 3.2, we could give the following suficient condition for injectivity of a linear bounded operator on $L^2(0,1)$

extending the range of s from s $\in N$ to s ≥ 0:

Remark 3.2) Let T $\in L_{\Im_\chi}$. If its injectivity parameters verifies $\mu_n(T) = o(n^{-s})$ for some $s \geq 0$, then T is injective.
The case s = 0 is specific for the operators having the sequence $\mu_n(T)$, n ≥ 1 inferior bounded (Lemma 3.1) and the cases for s > 0 are entering under the I-multipliers Rule (Lemma 3.2). ■
We remember that the a-positivity on a proper family of approximation subspaces is necessary for injectivity of the linear bounded operators on H. We do not have a response yet for the question if it is also a sufficient condition at least for operators $\in L_{\Im_\chi}$.

§4. On the injectivity of the integral operators with discontinue kernel functions. We will investigate the injectivity of two integral operators having the kernel functions discontinue on H := L^2(0,1). We will use the proper family of approximation subspaces introduced in previous paragraph, $\Im_\chi$ on which, for a-positivity of integral operators we will apply the Remark 3.1. The I-multiplier Rule is the right method to apply for the injectivity of the integral operator in Example 1 and, the Lemma 3.1 for the injectivity of the integral operator in the Example 2.

Example 1. Let the linear bounded integral operator T_σ defined by

$$(T_\sigma u)(y) := \int_0^1 \{\frac{y}{x}\}\{\frac{x}{y}\}u(x)dx$$

on L^2(0,1) having the kernel function $\sigma(y,x) = \{\frac{y}{x}\}\{\frac{x}{y}\}$. The brackets denote the fractional part function.

The integral operator is linear, bounded, with the kernel function discontinue, positive definite, Hermitian on H and, has no invariant subspaces in the proper family $\Im_\chi$.

The diagonal entries in the matrix representations of the operator discretiations on the subspaces of approximations are all strict positive (see Remark 3.1 for computing the matrix entries):

$$d_{11}^h = h^2\left(1 - \frac{\pi^2}{12}\right);\ d_{kk}^h = h^2\left(1/2 - k + k^2 ln\frac{k+1}{k}\right), k = 1, n, nh = 1$$

for k = 2,n, nh = 1. Modulo h^2, $d_{11} \approx 0.7836$ and $d_{22} \approx 0.12186$ showing that $d_{11}^h > d_{22}^h$. The decreasing monotony of diagonal entries comes from the following observation. The function f(t) = $\left(1/2 - t + t^2 ln\frac{t+1}{t}\right)$ defined on the interval [2, ∞) is monotone decreasing with the limit 0 for t converging to ∞. Its values for t = k, k $\geq$ 2 are the values of the diagonal entries in the matrices associated to the operator discretisations on approximation subspaces, modulo h^2. From the monotony of these entries we have on S_n: $d_{11}^h > d_{22}^h > ... > d_{nn}^h$, nh = 1.

The injectivity parameters sequence are given by (7):

$$\mu_n(T_\sigma) = (1/2 - n + n^2 ln((n+1)/n))/(1 - \pi^2/12)$$

n $\geq$ 2, nh = 1.

Hence T_σ is a-positive but, $\mu_n(T_\sigma)$ is not inferior bounded. Then we will apply the I-multipliers Rule (Lemma 3.2).

Let consider the denominator of $\mu_n(T_\sigma)$ as prototype for our considerations, being f(t) = $(1/2 - t + t^2 ln((t+1)/t)$, t $\in [1, \infty)$ that is converging to 0 when t $\to \infty$.

Observing that (f(t) - 1/t) and (f(t) - $1/t^2$), both $\to$ 0 for t $\to \infty$ and,

$1/t^2 < f(t) < 1/t$ for t $\gtrapprox$ 4 we could consider the following modulo the factor $(1 - \pi^2/12)$, as I-multipliers I_1 and I_2 (s =1 and s = 2 being the powers of x in the definition of the I-multiplier $I_{x^s}u(x) = x^s u(x)$ on $L^2(0,1)$. Then by Lemma 3.2, taking the I-multiplier I_s, s = 2, we proved that T_σ is injective.

Example 2: The proof of the injectivity of the integral operator used in RH Equivalence ([2]).
Let the linear bounded integral operator T_ρ defined by

$$(T_\rho u)(y) := \int_0^1 \{\frac{y}{x}\} u(x)dx \tag{9}$$

on $L^2(0,1)$ having the kernel function $\rho(y,x) := \{\frac{y}{x}\}$. This kernel function is the fractional part function of the quantity between brackets having jumps on the lines like y = k·x, k $\geq$ 1 resulting a discontinue function.

Because $(T_\rho \chi_{n,k})(y) \notin$ span($\{\chi_{n,k}\}, 1 \leq k \leq n, nh = 1$), the approximation subspaces are not invariant subspaces of T_ρ. Computing the entries of the diagonal matrices M_h, (see (6)) of the kernel approximations on the subspaces $\{S_h\}, nh = 1, n \geq 2$, we obtain:

$$d_{11}^h = \frac{h^2}{4}(3 - 2\gamma);\ d_{kk}^h = \frac{h^2}{2}(-1 + \frac{2k-1}{k-1} ln(\frac{k}{k-1})^{k-1}) \tag{10}$$

for k = 2,n, nh = 1 where $\gamma \cong 0.5772156...$ is the Euler-Mascheroni constant rounded here to 7 digits. The diagonal entries of matrix representations of T_ρ on each subspace S_h are strict positive valued and bounded by d_{11}^h and d_{22}^h. Moreover, the sequence $(-1 + \frac{2k-1}{k-1} ln(\frac{k}{k-1})^{k-1})$ is monotone and converges to 0.5 for k $\to \infty$. Approximated values modulo the factor h^2, of the diagonal entries are:

$$d_{1,1} \approx 0.461392, d_{2,2} \approx 0.636625, d_{3,3} \approx 0.506887, ...$$

Then, by Remark 3.1 T_ρ is a-positive on the approximation family having the a-positivity parameters given by $\alpha_n := h^{-2} d_{11}^h = h^{-2}\frac{1}{4}(3-2\gamma) \approx 0.461392\ h^{-2}, nh = 1$.
The injectivity parameters $\{\mu_n\}$ defined in (7) are $d_{11}^h / d_{22}^h, n \geq 2, nh = 1$ and will be all independent of h because all entries $d_{kk}^h, 1 \leq k \leq n$ are on the same order of h. The evaluations of $\{\mu_n\}, n \geq 1$ are:

$$\mu_n(T_\rho) = \frac{3-2\gamma}{-2+6ln(4/3)} \gtrapprox 0.724746, n \geq 2, nh = 1 \tag{11}$$

Hence the sequence $\mu_n(T_\rho)$, n ≥ 1, is bounded by a constant strict positive on every approximation subspace of the family.
By the Lemma 3.1 follows that the integral operator T_ρ is injective, equivalently, $N_{T_\rho} = \{0\}$.
We just proved the following theorem:

Theorem 4.1 The linear bounded integral operator T_ρ defined by

$$(T_\rho u)(y) := \int_0^1 \{\frac{y}{x}\} u(x) dx \tag{12}$$

on $L^2(0,1)$ having the kernel function $\rho(y,x) := \{\frac{y}{x}\}$ is injective.■

Claim: Based on the Theorem 4.1 and the Theorem 1, pg. 152 in [2], we could claim that the Riemann Hypothesis holds.

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