Trigonometric Shock Waves in the Kaup-Boussinesq System

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Trigonometric shock waves in the Kaup-Boussinesq system

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Abstract We consider the modulationally stable version of the Kaup-Boussinesq system which models propagation of nonlinear waves in various physical situations. It is shown that the Whitham modulation equations for this model have a new type of solutions which describe trigonometric shock waves. In the Gurevich-Pitaevskii problem of evolution of an initial discontinuity these solutions correspond to a non-zero wave excitation on one of the sides of the discontinuity. As a result, the problem of evolution of a trigonometric shock propagating along a rarefaction wave is developed. Our analytical results are confirmed by numerical calculations.

Keywords Solitons · Wave Breaking · Dispersive Shock Waves · Whitham Modulation Equations · Kaup-Boussinesq System

1 Introduction

Dispersion in nonlinear systems can dramatically affect wave's profile evolution leading to a host of new physical wave structures such as solitons and dispersive shock waves. In particular, it is well known now that in such systems a typical evolution of an initial pulse with a fairly smooth and large initial profile is accompanied by a gradual steepening of the profile followed by the wave breaking and formation of a dispersive shock wave (DSW). Theoretically, DSWs, also called undular bores in fluid mechanics applications, can be represented as modulated nonlinear periodic waves and then the process of their formation

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and evolution is described in the Gurevich-Pitaevskii approximation [1] by the Whitham theory of modulations [2] (for reviews see [3,4]).

The original formulation of Gurevich and Pitaevskii approach was applied to description of expanding collisionless shocks (plasma analogs of DSWs) in framework of the Whitham-averaged equations for the integrable Korteweg-de Vries (KdV) equation [5,6]. Due to universality of the KdV equation, this approach can naturally be applied to many other physical situations and it was extended to many other nonlinear wave equations. For example, when the condition of unidirectional propagation of the KdV approximation is relaxed, shallow water waves are described by various forms of the Boussinesq equations [7]. The most convenient for our purposes form has been derived by Kaup [8]; this is the so-called Kaup-Boussinesq (KB) system which is also integrable by the inverse scattering transform method. Periodic solutions of the KB system were obtained in Ref. [9] and the corresponding Gurevich-Pitaevskii theory was extended to the KB case in Refs. [10–12].

In applications of Gurevich-Pitaevskii approach to concrete water wave problems, the KB equations with negative dispersion are used. However, the dispersion relation for this kind of equations corresponds to a dynamical instability of small wavelength perturbations over a fluid of constant depth h_0 . There exists another form of the KB system with positive dispersion

$$\begin{aligned} h_t + (hu)_x - \frac{1}{4}u_{xxx} &= 0, \\ u_t + uu_x + h_x &= 0, \end{aligned} \quad (1)$$

where h is the local height of the water layer and u is a local mean flow velocity. For this equation the dispersion relation of linear waves reads

$$\omega^2 = h_0 k^2 + \frac{1}{4}k^4, \quad (2)$$

and the system (1) does not suffer from this kind of deficiency. Moreover, it appears as an approximation to the nonlinear polarization dynamics of a two-component Bose-Einstein condensate [13] as well as to dynamics of magnetization in magnetics with easy-plane anisotropy, so it deserves thorough investigation.

In Ref. [12] the Riemann problem of evolution of initial discontinuities was studied for the system (1). Here we consider the initial states of a different type: we assume that on one side of the initial discontinuity the profiles are represented by periodic solutions rather than by uniform distributions, as it was supposed in the standard Riemann problem. This means that our theory describes spreading out of the front of the nonlinear wave excitation along the rarefaction wave. This opens a new route to analytical description of wave structures arising from more complex initial states.

2 Periodic solutions and Whitham equations

In this section, we review briefly the results from the theory of the periodic wave solutions of the system (1) in necessary for us form and present the

Whitham equations governing their modulation dynamics (more details can be found in Refs. [9, 10, 12]).

The KB system (1) is completely integrable and it can be represented as a compatibility condition of two linear equations (see [8]) with a free spectral parameter λ :

$$\psi_{xx} = \mathcal{A}\psi, \quad \psi_t = -\frac{1}{2}\mathcal{B}_x\psi + \mathcal{B}\psi_x, \quad (3)$$

where

$$\mathcal{A} = h - \left(\lambda - \frac{1}{2}u\right)^2 \quad \text{and} \quad \mathcal{B} = -\left(\lambda + \frac{1}{2}u\right). \quad (4)$$

If $h(x, t)$ and $u(x, t)$ correspond to a one-phase periodic in x and t solution of the system (1), then it is parameterized most conveniently by the values $\lambda_i, i = 1, 2, 3, 4$, which determine the structure of the spectrum of the second-order differential equation in Eqs. (3). We introduce the polynomial

$$P(\lambda) = \prod_{i=1}^4 (\lambda - \lambda_i) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4, \quad (5)$$

where s_i are the standard symmetric functions of its four zeros λ_i ,

$$\begin{aligned} s_1 &= \sum_i \lambda_i, \quad s_2 = \sum_{i<j} \lambda_i \lambda_j, \quad s_3 = \sum_{i<j<k} \lambda_i \lambda_j \lambda_k, \\ s_4 &= \lambda_1 \lambda_2 \lambda_3 \lambda_4. \end{aligned} \quad (6)$$

Then the physical variables are expressed in terms of the function $\mu(x, t)$,

$$\begin{aligned} u(x, t) &= s_1 - 2\mu(x, t), \\ h(x, t) &= \frac{1}{4}s_1^2 - s_2 - 2\mu^2(x, t) + s_1\mu(x, t), \end{aligned} \quad (7)$$

provided $\mu = \mu(\theta)$, $\theta = x - Vt$, satisfies the equation

$$\mu_\theta = 2\sqrt{-P(\mu)} \quad (8)$$

where

$$V = \frac{1}{2}s_1 = \frac{1}{2} \sum_{i=1}^4 \lambda_i. \quad (9)$$

The real zeros λ_i are ordered according to the inequalities

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \quad (10)$$

and solutions of Eq. (8) can be expressed in standard way in terms of elliptic functions. The real solution of Eq. (8) corresponds to oscillations of μ in one of two possible intervals, $\lambda_1 \leq \mu \leq \lambda_2$ or $\lambda_3 \leq \mu \leq \lambda_4$. The wavelength of these solutions is equal to

$$L = \int_{\lambda_1}^{\lambda_2} \frac{d\mu}{\sqrt{-P(\mu)}} = \frac{2K(m)}{\sqrt{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)}}, \quad (11)$$

where

$$m = \frac{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)} \quad (12)$$

and $K(m)$ is the complete elliptic integral of the first kind [14]. We shall not list here all possible solutions and confine ourselves to some limiting cases necessary for further discussion.

Let μ oscillate in the interval

$$\lambda_1 \leq \mu \leq \lambda_2 \quad (13)$$

and the other two zeroes are equal to each other, $\lambda_3 = \lambda_4$. Then the elliptic functions reduce to the trigonometric ones and we obtain

$$\begin{aligned} \mu(\theta) &= \lambda_2 - \frac{(\lambda_2 - \lambda_1) \cos^2 W}{1 + \frac{\lambda_2 - \lambda_1}{\lambda_4 - \lambda_2} \sin^2 W}, \\ W &= \sqrt{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)} \theta. \end{aligned} \quad (14)$$

If we take the limit $\lambda_2 - \lambda_1 \ll \lambda_4 - \lambda_1$ in this solution, then we get the small-amplitude limit of harmonic oscillations

$$\mu(\theta) \cong \lambda_2 - \frac{1}{2}(\lambda_2 - \lambda_1) \cos[(\lambda_3 - \lambda_1)\theta]. \quad (15)$$

On the other hand, if we take the limit $\lambda_2 \rightarrow \lambda_3 = \lambda_4$, then the argument of the trigonometric functions becomes small and we can approximate them by the first terms of their series expansions. This corresponds to an algebraic soliton of the form

$$\mu(\theta) = \lambda_2 - \frac{\lambda_2 - \lambda_1}{1 + (\lambda_2 - \lambda_1)^2 \theta^2}. \quad (16)$$

In a similar way, in the second case μ oscillates in the interval

$$\lambda_3 \leq \mu \leq \lambda_4, \quad (17)$$

and in the limit $\lambda_1 = \lambda_2$ we arrive at the nonlinear trigonometric solution

$$\begin{aligned} \mu(\theta) &= \lambda_3 + \frac{(\lambda_4 - \lambda_3) \cos^2 W}{1 + \frac{\lambda_4 - \lambda_3}{\lambda_3 - \lambda_1} \sin^2 W}, \\ W &= \sqrt{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} \theta. \end{aligned} \quad (18)$$

Again for $\lambda_4 - \lambda_3 \ll \lambda_4 - \lambda_1$ we obtain the small-amplitude harmonic waves limit

$$\mu(\theta) \cong \lambda_3 - \frac{1}{2}(\lambda_4 - \lambda_3) \cos[(\lambda_3 - \lambda_1)\theta]. \quad (19)$$

At last, for $\lambda_3 \rightarrow \lambda_1$ the solution (18) reduces to the algebraic soliton:

$$\mu(\theta) = \lambda_1 + \frac{\lambda_4 - \lambda_1}{1 + (\lambda_4 - \lambda_1)^2 \theta^2}. \quad (20)$$

In strictly periodic solutions the parameters λ_i are constant and in slightly modulated waves they become slow varying functions of x and t which change little in one wavelength and one period. Evolution of λ_i is governed by the Whitham equations

$$\frac{\partial \lambda_i}{\partial t} + v_i \frac{\partial \lambda_i}{\partial x} = 0, \quad i = 1, 2, 3, 4. \quad (21)$$

The Whitham velocities v_i can be computed by means of the formulas

$$v_i(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(1 - \frac{L}{\partial_{\lambda_i} L} \partial_{\lambda_i}\right) V, \quad i = 1, 2, 3, 4. \quad (22)$$

Explicit expressions for Whitham's velocities v_i can be easily obtained after substitution of the phase velocity V and the wavelength L given by Eqs. (9) and (11) (see, e.g., [12]). We need only the limiting formulas here. In the limit $\lambda_2 \rightarrow \lambda_1$ (i.e., $m \rightarrow 0$) we obtain

$$\begin{aligned} v_1 = v_2 &= 2\lambda_1 + \frac{(\lambda_4 - \lambda_3)^2}{2(\lambda_3 + \lambda_4 - 2\lambda_1)}, \\ v_3 &= \frac{1}{2}(3\lambda_3 + \lambda_4), \quad v_4 = \frac{1}{2}(\lambda_3 + 3\lambda_4), \end{aligned} \quad (23)$$

and in another limit $m \rightarrow 0$, i.e. $\lambda_3 \rightarrow \lambda_4$, we have

$$\begin{aligned} v_1 &= \frac{1}{2}(3\lambda_1 + \lambda_2), \quad v_2 = \frac{1}{2}(\lambda_1 + 3\lambda_2), \\ v_3 = v_4 &= 2\lambda_4 + \frac{(\lambda_2 - \lambda_1)^2}{2(\lambda_1 + \lambda_2 - 2\lambda_4)}. \end{aligned} \quad (24)$$

Having received the basic equations, we can now proceed to the description of nonlinear trigonometric solutions for the KB system (1).

3 Trigonometric shock wave

Here we consider formation of the trigonometric shock wave pattern. In case of conventional initial conditions with uniform distributions on both sides of the initial step-like discontinuity such a structure is not generated [12]. However, there exist quite natural initial conditions for which this kind of solutions does appear. Let the initial conditions have the following form. On the one side of the point $x = 0$ we have constant distributions of h and u , and on the other side there is a non-modulated nonlinear periodic wave described by four constant parameters λ_i . This means that we are interested in evolution of an initially sharp border of the wavy region. Generally speaking, evolution of an initial discontinuity between wavy states, that is one-phase solutions of the equation under consideration, belongs to the class of two-phase problems which demand development of the modulation theory of quasi-periodic solutions. Such a problem in its full generality is very difficult and although some its

Fig. 1 Sketches of the behavior of the Riemann invariants in trigonometric dispersive shock wave solutions of the Whitham equations with $\lambda_3 = \lambda_4$. Vertical dashed lines indicated by s_- and s_+ define the edges of a trigonometric shock wave. Corresponding wave structure is shown in Fig. 2.

Fig. 2 Evolution of a trigonometric shock wave for $h_L = 0.3$, $u_L = 0$ and $\lambda_3^R = \lambda_4^R = 0.85$. Figures show the initial state (left column) and wave profiles for depth h and flow velocity u for $t = 100$ (middle column) and $t = 200$ (right column). Red (thick) curves show the result of numerical calculations, and a blue (thin) one shows the analytical solution. Dashed lines illustrate envelopes of wave structure, vertical dashed lines indicate the edges of the trigonometric shock wave (x_- and x_+). We have here dark solitons of elevation h and bright solitons of flow velocity u at the soliton edge of the shock. The corresponding diagram of Riemann invariants is shown in Fig. 1.

general principles were established long ago (see, e.g., [15]), it has not find real applications yet. Instead, some degenerate particular problem were studied like propagation of solitons along wavy background (see, e.g., [16–18]). Here we shall also consider a degenerate problem of developing trigonometric shock along varying rarefaction wave. Before turning to oscillatory solutions, let us consider first the dispersionless limit.

For smooth enough wave distributions we can neglect the last dispersion term in the first equation of the system (1) and arrive at the dispersionless equations

$$h_t + (hu)_x = 0, \quad u_t + uu_x + h_x = 0, \quad (25)$$

which coincide with the well-known shallow water equations. First of all, this system admits a trivial solution for which $h = \text{const}$ and $u = \text{const}$. We shall call such a solution a “plateau”. We introduce the so-called Riemann invariants

$$\lambda_{\pm} = \frac{u}{2} \pm \sqrt{h}. \quad (26)$$

Using these dispersionless Riemann invariants, equations (25) can be written in the following diagonal form

$$\frac{\partial \lambda_{\pm}}{\partial t} + v_{\pm}(\lambda_{-}, \lambda_{+}) \frac{\partial \lambda_{\pm}}{\partial x} = 0, \quad (27)$$

where

$$v_{\pm}(\lambda_{-}, \lambda_{+}) = \frac{1}{2}(3\lambda_{\pm} + \lambda_{\mp}). \quad (28)$$

These dispersionless variables are required for the correct choice of the initial state.

We assume that at the initial moment the profile is divided into two parts by the point $x = 0$. We consider two types of profile. First, at $x < 0$ there is a plateau characterized by the constant dispersionless invariants $\lambda_{-} = \text{const}$, $\lambda_{+} = \text{const}$, and for $x > 0$ there is a non-modulated wave described by the Eq. (14) (see Fig. 1 and left column in Fig. 2). We shall denote the parameters of this periodic wave as λ_1^R , λ_2^R , and $\lambda_3^R = \lambda_4^R$. The second type of initial state is similar: there is a non-modulated wave described by Eq (18) with $\lambda_1^L = \lambda_2^L = \text{const}$, $\lambda_3^L = \text{const}$, $\lambda_4^L = \text{const}$ at $x < 0$ and a plateau with $\lambda_{-} = \text{const}$, $\lambda_{+} = \text{const}$ at $x > 0$ (see Fig. 3 and left column in Fig. 4).

We turn now to the study of situations when the dispersion effects are taken into account. In order to satisfy the matching conditions, the only possible solution arising from a given initial state may be a trigonometric shock wave. Trigonometric shock wave can be represented approximately as a modulated nonlinear periodic wave in which parameters λ_i change slowly along the wave structure. In such a modulated periodic solution two equal Riemann invariants are changing and the other two remain constant along the entire shock. Thus, in our case the constant Riemann invariants and the Riemann invariants at the boundaries of the structure should have equal values. This situation resembles the so called ‘contact discontinuity’ which plays an important role in the theory of viscous shocks (see, e.g., [19]). This type of DSW was first reported in Ref. [20] where the evolution of a step problem was studied for the focusing modified KdV equation. In Ref. [21] these (trigonometric) DSWs were called contact DSWs. The trigonometric shock waves for the KB system are described by the modulated finite-amplitude nonlinear periodic solutions (14) or (18). The evolution of the trigonometric shock wave is determined by the Whitham

Fig. 3 Sketches of the behavior of the Riemann invariants in trigonometric dispersive shock wave solutions of the Whitham equations with $\lambda_1 = \lambda_2$. Vertical dashed lines indicated by s_- and s_+ define the edges of a trigonometric shock wave. Corresponding wave structure is shown in Fig. 4.

Fig. 4 Evolution of a trigonometric shock wave $h_R = 0.3$, $u_R = 0$ and $\lambda_1^L = \lambda_2^L = -0.85$. Figures show the initial state (left column) and wave profiles for depth h and flow velocity u for $t = 100$ (middle column) and $t = 200$ (right column). Red (thick) curves show the result of numerical calculations, and a blue (thin) one shows the analytical solution. Dashed lines illustrate envelopes of wave structure, vertical dashed lines indicate the edges of the trigonometric shock wave (x_- and x_+). On the contrary to the case of Fig. 2, now we get dark solitons of both elevation h and flow velocity u at the soliton edge of the shock. The corresponding diagram of Riemann invariants is shown in Fig. 3.

equations (21). In our case of the step-like initial conditions we have to find self-similar solutions for which all Riemann invariants depend only on $\xi = x/t$, and the Whitham equations reduce to

$$\frac{d\lambda_i}{d\xi} \cdot [v_i(\lambda_1, \lambda_2, \lambda_3, \lambda_4) - \xi] = 0, \quad i = 1, 2, 3, 4. \quad (29)$$

If, for instance, we consider the first type of initial condition then in this situation, shown in Fig. 1, trigonometric shock wave has two equal parameters $\lambda_3 = \lambda_4$ and the invariants λ_1 and λ_2 are constant along the whole wave pattern including shock region. We assume that there is a plateau with dispersionless Riemann invariants λ_-^L and λ_+^L to the left of the trigonometric shock wave. Thus we have $v_3(\lambda_-^L, \lambda_+^L, \lambda_4(\xi), \lambda_4(\xi)) = v_4(\lambda_-^L, \lambda_+^L, \lambda_4(\xi), \lambda_4(\xi)) = \xi$. Consequently, we obtain

$$\begin{aligned} \lambda_1 &= \lambda_-^L, & \lambda_2 &= \lambda_+^L, \\ v_4 &= 2\lambda_4 - \frac{(\lambda_+^L - \lambda_-^L)^2}{\lambda_+^L + \lambda_-^L - 2\lambda_4} = \xi, \end{aligned} \quad (30)$$

where the last formula determines the dependence of λ_4 on ξ , which can be represented in the explicit form

$$\lambda_4(\xi) = \frac{1}{4} \left[\xi + \lambda_+^L + \lambda_-^L + \sqrt{(\xi - \lambda_+^L - \lambda_-^L)^2 + 2(\lambda_+^L - \lambda_-^L)^2} \right]. \quad (31)$$

Here ξ varies within the interval $s_- \leq \xi \leq s_+$ with

$$s_- = \frac{3\lambda_+^L + \lambda_-^L}{2}, \quad s_+ = 2\lambda_4^R + \frac{(\lambda_+^L - \lambda_-^L)^2}{2(\lambda_+^L + \lambda_-^L - 2)}, \quad (32)$$

where $\lambda_4^R = \lambda_3^R$ is the maximum value of $\lambda_4(\xi)$ defined by the right boundary condition of trigonometric shock wave. The wavelength in this case is given by the formula

$$L = \frac{2\pi}{\sqrt{(\lambda_4(\xi) - \lambda_-^L)(\lambda_4(\xi) - \lambda_+^L)}}. \quad (33)$$

Substitution of λ_i into Eq. (14) with subsequent substitution into Eq. (7) yields the modulated periodic solutions resulting in the trigonometric shock wave structure. Comparison of the obtained analytical solution with numerical calculations is shown in Fig. 2 for different values of time t . One can see that the trigonometric shock wave is located between the edges with coordinates $x_- = s_-t$ and $x_+ = s_+t$. This wave matches at its left edge with the left plateau and at its right edge with the non-modulated wave.

In a similar way, we can consider the second type of initial conditions. For this case the trigonometric shock wave has $\lambda_1 = \lambda_2$. An example of the diagram of Riemann invariants is shown in Fig. 3. Obviously, we get a symmetric situation, where the trigonometric shock wave matches the plateau characterised by the values λ_-^R and λ_+^R of the Riemann invariants at the right edge and the non-modulated wave with the values $\lambda_1^L = \lambda_2^L = \text{const}$, $\lambda_3^L = \lambda_-^R$ and $\lambda_4^L = \lambda_+^R$ of the Riemann invariants of the Whitham system at the left edge. The solution of the Whitham equations takes the form

$$\begin{aligned} v_1 &= v_2 = 2\lambda_1 + \frac{(\lambda_+^R - \lambda_-^R)^2}{2(\lambda_+^R + \lambda_-^R - 2\lambda_1)} = \xi, \\ \lambda_3 &= \lambda_-^R = \lambda_-^R, & \lambda_4 &= \lambda_+^R = \lambda_+^R, \end{aligned} \quad (34)$$

or

$$\lambda_1(\xi) = \frac{1}{4} \left[\xi + \lambda_+^R + \lambda_-^R - \sqrt{(\xi - \lambda_+^R - \lambda_-^R)^2 + 2(\lambda_+^R - \lambda_-^R)^2} \right] \quad (35)$$

where ξ varies in the interval $s_- \leq \xi \leq s_+$ with

$$s_- = -2\lambda_1^L + \frac{(\lambda_+^R - \lambda_-^R)^2}{2(\lambda_+^R + \lambda_-^R + 2)}, \quad s_+ = \frac{\lambda_+^R + 3\lambda_-^R}{2}. \quad (36)$$

Parameter $\lambda_1^L = \lambda_2^L$ is determined again by the initial conditions at the boundary with the non-modulated wave. The wavelength is given here by the formula

$$L = \frac{2\pi}{\sqrt{(\lambda_1(\xi) - \lambda_-^R)(\lambda_1(\xi) - \lambda_+^R)}}. \quad (37)$$

Our analytical results and numerical simulations for the corresponding wave structures are compared in Fig. 4. Now trigonometric shock wave is located between the edge points $x_- = s_-t$ and $x_+ = s_+t$ indicated by dotted lines. One can see that numerical calculations (red thick) agree with the analytical curve (blue thin) very well.

Although the wave patterns look similar, one should notice that in Fig. 2 the distribution of $u(x, t)$ tends to bright solitons at the soliton edge of the shock whereas in Fig. 4 we get dark solitons of $u(x, t)$ at this edge. In both cases the variable $h(x, t)$ takes negative values and that limits application of the developed theory to water waves. Nevertheless it is applicable to other physical situations such as nonlinear shocks in two-component Bose-Einstein condensates or in magnetics.

4 Conclusion

In this paper we have considered the cases in which the trigonometric shock waves arise when their evolution is governed by the integrable Kaup-Boussinesq system. The initial state contains a nonlinear wave which significantly extends the set of wave patterns which can be generated in various physical situations. Our results can find applications as approximation to the dynamics of polarization waves in two-component Bose-Einstein condensates and in magnetic systems with easy-plane anisotropy which are not constrained by the condition that h must be positive.

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Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

Author Contributions

All authors contributed equally to this work

Data Availability

The data are available from the corresponding author on reasonable request.

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