Multiscale stochastic volatility for variance swaps with constant elasticity of variance

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Multiscale stochastic volatility for variance swaps with constant elasticity of variance

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Abstract Variance swap is one of volatility derivatives popularly used for the risk management of financial instruments traded in volatile market. An appropriate choice of a volatility model should be an important part of the risk management. In this paper, we consider a hybrid model of constant elasticity of variance and multiscale (fast and slow) stochastic volatility. Based on this framework, we obtain a closed form solution formula for the approximate fair strike values of continuous time variance swaps and compute the solution. The theoretical formula is validated by numerical solution given by Monte-Carlo simulation. The multiscale effect and the predictability of the strike price movements are investigated for a given information about the underlying asset and volatility.

Keywords Variance swap · Stochastic volatility · Constant elasticity of variance · Multiscale

1 Introduction

Volatility itself began to be considered as an asset in 1993 after the Chicago Options Exchange (CBOE) Volatility Index (VIX) was introduced. Volatility trading volume has been increased rapidly in real markets and a variety of volatility derivatives have been produced since then. As volatility markets have developed, volatility and variance swaps have dominated the markets among other things. The purpose of volatility or variance swaps are predicting

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the future levels of volatility, trading spreads between the implied and realized volatilities, or hedging the risks provoked by volatility movements. Refer to Demeterfi et al. (1999) and Carr and Madan (2001) for general understanding of volatility and variance swaps.

A variance swap is a forward contract for the future realized variance of an underlying asset. This product can allow an investor to speculate about the spread between its realized and implied variances or hedge its volatility risk. One leg of the swap pays a fixed amount called a strike, while the other leg pays an amount reflecting the realized variance of the underlying asset returns. The difference between these two amounts would be net income for the counterparties settled in cash at the expiration of the contract. The question of how to determine the fair value of the strike for the variance swap has been an active research topic in mathematical finance community since its inception in 1998. Little and Pant (2001) developed a finite difference method for the price of discretely sampled variance swaps in the local volatility model of Dupire (1994) and Derman and Kani (1994). Zhu and Lian (2011) extended the work of Little and Pant (2001) by incorporating the Heston stochastic volatility model. Wang et al. (2015) obtained a closed form approximate solution under the constant elasticity of variance (CEV) model. Kim and Kim (2019) derived analytic solutions for variance swaps with a double-mean-reverting volatility model. Cao et al. (2018) obtained a semi-closed form approximation based on a hybrid CEV and stochastic volatility model introduced by Choi et al. (2015).

In this paper, we use a more general hybrid framework combining the CEV and stochastic volatility diffusions than the model of Choi et al. (2015). In our model, the stochastic volatility is assumed to be driven by not only a fast fluctuating process but also a slowly varying process. Market data show the need for the multiscale framework as shown in, for example, Fouque et al. (2011) and Seo and Kim (2022). The fast scale process is driven by a general ergodic process that has an invariant distribution. It could be the Ornstein-Uhlenbeck process as used in Choi et al. (2015), the CIR process or any other ergodic process. The purpose of this paper is to obtain a multiscale stochastic volatility correction to the CEV fair strike price of variance swaps.

This paper is organized as follows. Section 2 describes an underlying model for pricing variance swaps and formulates a pricing problem in terms of a partial differential equation (PDE). Section 3 solves the PDE problem for a fast scale volatility correction to the CEV variance strike in an asymptotic manner. Also, some examples of the fast scale correction are provided. In Section 4, we extend the results in Section 3 to a multiscale stochastic volatility problem by adding a slow scale variation process. Section 5 concludes.

2 Model and problem formulation

2.1 Multiscale stochastic volatility

Let X_t be the price of a risky asset on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ at time t , where $\mathbb{Q}$ is the risk-neutral probability measure. Then the CEV model, which was first considered by Cox (1996), for X_t is given by the stochastic differential equation (SDE)

$$dX_t = rX_t dt + \sigma X_t^{1-\beta} dW_t^x, \quad (1)$$

where r is risk-free rate of interest and σ (volatility coefficient) and β (the elasticity of variance) are positive constants. If $\beta = 0$ in (1), then the CEV model becomes the Black-Scholes model (Black and Scholes 1973). The inequality $0 < \beta < 1$ is required to satisfy the absorbing condition at $X_t = 0$.

The CEV model suggests that an underlying asset and its volatility have a definite, either positive or negative, correlation depending on the choice of the elasticity of variance parameter. Empirical studies such as Harvey (2001) and Ghysels et al. (1996), however, show that the elasticity of variance is time varying. Kim et al. (2014) shows that the elasticity of variance is randomly fluctuating and the CEV model has a difficulty of capturing the implied volatility smile, one of main features of volatility observed in the market. So, it is required that the volatility has a random source other than the underlying asset itself. That is, a hybrid model giving a stochastic volatility correction the CEV model is necessary. There are two possible approaches. The first one is to replace the constant elasticity of variance with a function of a diffusion process as developed by Kim et al. (2014) and (2015). The second approach is to multiply the CEV volatility by a function of a diffusion process as done by Choi et al. (2015). We choose a hybrid model of the second approach but the model in this article is a more general framework than the model of Choi et al. (2015).

We change the constant σ in (1) into a function, f , of not only a fast scale diffusion process, Y_t , but also a slow scale diffusion process Z_t .

$$\sigma \longrightarrow f(Y_t, Z_t)$$

Here, Y_t and Z_t satisfy the SDEs

$$\begin{aligned} dY_t &= \frac{1}{\varepsilon} \alpha(Y_t) dt + \frac{1}{\sqrt{\varepsilon}} \gamma(Y_t) dW_t^y, \\ dZ_t &= \delta c(Z_t) dt + \sqrt{\delta} g(Z_t) dW_t^z, \\ d\langle W^x, W^y \rangle_t &= \rho_{xy} dt, \quad d\langle W^y, W^z \rangle_t = \rho_{yz} dt, \quad d\langle W^z, W^x \rangle_t = \rho_{zx} dt, \end{aligned} \quad (2)$$

respectively, where the functions α , γ , c and g are assumed to satisfy the existence and uniqueness of solutions to these SDEs and ε and δ are small positive parameters. The processes Y_t and Z_t are assumed to satisfy the necessary conditions stated in Fouque et al. (2016), in particular, about ergodicity and

existence of unique invariant distribution. So, the volatility of the underlying asset is given by $X_t^{-\beta} f(Y_t, Z_t)$. It is driven by three processes X_t , Y_t and Z_t . It is thought of as a CEV-like model with fast and slow variation factors of stochastic volatility.

2.2 Problem formulation

In terms of the strike, K , and the realized variance, $V_{t,T}^{re}$, of a continuous variance swap over the time interval $[t, T]$, the payoff, $h_{t,T}(K)$, of the swap is given by

$$h_{t,T}(K) = (V_{t,T}^{re} - K) \times N_{var},$$

where N_{var} is the given notional amount of the variance swap. Then the value, $V_t^{var}(K)$, of the variance swap at time t is

$$V_t^{var}(K) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} h_{t,T}(K) | \mathcal{F}_t].$$

Since the fair strike, denoted by K_t^{var} , at time t is defined by the condition that the contract is fair to both parties, K_t^{var} should satisfy $V_t^{var}(K_t^{var}) = 0$ and so

$$K_t^{var} = \mathbb{E}^{\mathbb{Q}}[V_{t,T}^{re} | \mathcal{F}_t].$$

As in Brodie and Jain (2008), we define the realized variance $V_{t,T}^{re}$ by

$$V_{t,T}^{re} = \frac{1}{T-t} \int_t^T V_t dt,$$

where V_t denotes the instantaneous variance which is given by

$$V_t = f^2(Y_t, Z_t) X_t^{-2\beta}$$

in our model. In practice, the real variance is discretely sampled and thus the continuous sampling assumption may cause some approximation errors. However, the daily monitoring of term sheets of this kind of contract generally reduces the discrepancy considerably. For example, the relative difference between continuous and daily sampled variance swaps is around $0.1 \sim 0.2\%$ under the exponential Ornstein-Uhlenbeck volatility model as shown in Kim and Kim (2019). So, we keep the idealized contract variance based on continuously sampled realized variance without loss of generality. Then, using the Ito lemma, the fair strike price K_t^{var} is now expressed as

$$\begin{aligned} K_t^{var} &= \frac{1}{T-t} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T f^2(Y_s, Z_s) X_s^{-2\beta} ds | \mathcal{F}_t \right] \\ &= \frac{2}{T-t} \mathbb{E}^{\mathbb{Q}} \left[\left(\int_t^T r ds + f(Y_s, Z_s) X_s^{-\beta} dW_s^x \right) - \ln \left(\frac{X_T}{X_t} \right) | \mathcal{F}_t \right] \quad (3) \\ &= 2r - \frac{2}{T-t} \mathbb{E}^{\mathbb{Q}} [\ln X_T | \mathcal{F}_t] + \frac{2}{T-t} \ln X_t. \end{aligned}$$

Therefore, the pricing of the fair variance strike (based on a known information about the value of X_t , Y_t and Z_t) becomes a calculation of the conditional expectation

$$P^{\varepsilon, \delta, \beta}(t, x, y, z) := \mathbb{E}^{\mathbb{Q}}[\ln X_T | X_t = x, Y_t = y, Z_t = z], \quad (4)$$

where the Markov property of the joint process (X_t, Y_t, Z_t) has been applied. This can be thought of as a pricing problem for a European derivative but without discount factor. Using the Markov property of the underlying processes, we solve this problem after applying the well known Feynman-Kac theorem to change the conditional expectation into a PDE problem given by

$$\begin{aligned} \mathcal{L}^{\varepsilon, \delta, \beta} P^{\varepsilon, \delta, \beta}(t, x, y, z) &= 0, \quad t < T, \\ \mathcal{L}^{\varepsilon, \delta, \beta} &= \frac{\partial}{\partial t} + \mathcal{L}_{X, Y, Z}^{\varepsilon, \delta, \beta}, \\ P^{\varepsilon, \delta, \beta}(T, x, y, z) &= \ln x, \end{aligned} \quad (5)$$

where $\mathcal{L}_{X, Y, Z}^{\varepsilon, \delta, \beta}$ is an infinitesimal generator of the joint diffusion process (X_t, Y_t, Z_t) given by

$$\begin{aligned} \mathcal{L}_{X, Y, Z}^{\varepsilon, \delta, \beta} &= \frac{1}{2} f^2(y) x^{2(1-\beta)} \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} \\ &+ \frac{1}{\sqrt{\varepsilon}} \left(\rho_{xy} \gamma(y) f(y) x^{1-\beta} \frac{\partial^2}{\partial x \partial y} + \sqrt{\delta} \rho_{yz} \gamma(y) g(z) \frac{\partial^2}{\partial y \partial z} \right) \\ &+ \frac{1}{\varepsilon} \left(\frac{1}{2} \gamma^2(y) \frac{\partial^2}{\partial y^2} + \alpha(y) \frac{\partial}{\partial y} \right) \\ &+ \sqrt{\delta} \rho_{zx} f(y) g(z) x^{1-\beta} \frac{\partial^2}{\partial z \partial x} + \delta \left(\frac{1}{2} g^2(z) \frac{\partial^2}{\partial z^2} + c(z) \frac{\partial}{\partial z} \right). \end{aligned}$$

3 Pricing with fast scale stochastic volatility

In this section, we obtain the fair price of strike of a variance swap when the volatility is driven by the fast scale process Y_t so that $c = g = 0$ and $\rho_{yz} = \rho_{zx} = 0$ hold in (2). So, $\mathcal{L}_{X, Y, Z}^{\varepsilon, \delta, \beta}$ is given by

$$\begin{aligned} \mathcal{L}_{X, Y}^{\varepsilon, \delta, \beta} &= \frac{1}{2} f^2(y) x^{2(1-\beta)} \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} + \frac{1}{\sqrt{\varepsilon}} \left(\rho_{xy} \gamma(y) f(y) x^{1-\beta} \frac{\partial^2}{\partial x \partial y} \right) \\ &+ \frac{1}{\varepsilon} \left(\frac{1}{2} \gamma^2(y) \frac{\partial^2}{\partial y^2} + \alpha(y) \frac{\partial}{\partial y} \right). \end{aligned}$$

3.1 Asymptotic expansion

Since it is very difficult to find an exact solution of equation (5) for general function f , it is inevitable to seek an approximate solution for the problem.

Fortunately, the assumption that the parameters ε (the inverted mean reversion rate of volatility) and β (the elasticity of variance) are small is well known to meet the market findings in literature. Refer to, for example, Fouque et al. (2011) and Kim et al. (2014).

Since the operator $\mathcal{L}^{\varepsilon, \beta}$ consists of bundles of like powers of $\frac{1}{\sqrt{\varepsilon}}$, it is better to split the operator into three parts as follows.

$$\begin{aligned}\mathcal{L}^{\varepsilon, \beta} &= \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1^\beta + \mathcal{L}_2^\beta, \\ \mathcal{L}_0 &:= \frac{1}{2} \gamma^2(y) \frac{\partial^2}{\partial y^2} + \alpha(y) \frac{\partial}{\partial y}, \\ \mathcal{L}_1^\beta &:= \rho_{xy} \gamma(y) f(y) x^{1-\beta} \frac{\partial^2}{\partial x \partial y}, \\ \mathcal{L}_2^\beta &:= \frac{\partial}{\partial t} + \frac{1}{2} f^2(y) x^{2(1-\beta)} \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x}.\end{aligned}$$

The operators $\mathcal{L}_1^\beta$ and $\mathcal{L}_2^\beta$ involve the small parameter β . The Taylor expansions of $x^{1-\beta}$ and $x^{2(1-\beta)}$ in β provide

$$\begin{aligned}\mathcal{L}_1^\beta &= \sum_{i=0}^{\infty} \beta^i \mathcal{L}_{1i}, \\ \mathcal{L}_{10} &:= \rho_{xy} \gamma(y) f(y) x \frac{\partial^2}{\partial x \partial y}, \\ \mathcal{L}_{11} &:= -\rho_{xy} \gamma(y) f(y) x \ln x \frac{\partial^2}{\partial x \partial y}, \\ \mathcal{L}_{12} &:= \frac{\rho_{xy}}{2} \gamma(y) f(y) x (\ln x)^2 \frac{\partial^2}{\partial x \partial y},\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_2^\beta &= \sum_{i=0}^{\infty} \beta^i \mathcal{L}_{2i}, \\ \mathcal{L}_{20} &:= \frac{\partial}{\partial t} + \frac{1}{2} f^2(y) x^2 \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x}, \\ \mathcal{L}_{21} &:= -f^2(y) x^2 \ln x \frac{\partial^2}{\partial x^2}, \\ \mathcal{L}_{22} &:= f^2(y) x^2 (\ln x)^2 \frac{\partial^2}{\partial x^2},\end{aligned}$$

respectively. We are interested in $P^{\varepsilon, \beta}$ of the form

$$P^{\varepsilon, \beta} = \sum_{i=0}^{\infty} \beta^i P_i^\varepsilon.$$

Plugging this expansion into equation (5), we obtain a hierarchy of equations as follows.

$$\beta^0\text{-term} : \left(\frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{10} + \mathcal{L}_{20} \right) P_0^\varepsilon = 0, \quad (6)$$

$$\beta\text{-term} : \left(\frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{10} + \mathcal{L}_{20} \right) P_1^\varepsilon + \left(\frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{11} + \mathcal{L}_{21} \right) P_0^\varepsilon = 0, \quad (7)$$

$$\begin{aligned} \beta^2\text{-term} : & \left(\frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{10} + \mathcal{L}_{20} \right) P_2^\varepsilon + \left(\frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{11} + \mathcal{L}_{21} \right) P_1^\varepsilon \\ & + \left(\frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{12} + \mathcal{L}_{22} \right) P_0^\varepsilon = 0. \end{aligned} \quad (8)$$

...

3.2 Asymptotic analysis

We employ the asymptotic analysis of Fouque et al. (2011) to obtain an approximate solution for $P^{\varepsilon, \beta}(t, x, y)$. In this analysis, notation $\langle \cdot \rangle$ defined by

$$\langle \xi \rangle = \int \xi(y) \Phi(y) dy \quad (9)$$

is going to be hired many times, where Φ denotes the invariant distribution of the ergodic process Y_t , and the following lemma, called the centering condition, for the operator $\mathcal{L}_0$ is useful.

Lemma 1 *The Poisson equation*

$$\mathcal{L}_0 p(t, x, y) + \xi(t, x, y) = 0,$$

has a solution only if the function ξ is centered with respect to the invariant distribution Φ of the process Y_t . i.e.,

$$\langle \xi(t, x, \cdot) \rangle = 0.$$

Proof Refer to Section 3.2 in Fouque et al. (2011).

The solution of $\mathcal{L}_0 p(t, x, y) = 0$ is assumed to satisfy the growth condition of the following lemma for the operator $\mathcal{L}_0$.

Lemma 2 *Assume that equation $\mathcal{L}_0 p(t, x, y) = 0$ admits only solutions that do not grow as much as*

$$\frac{\partial p}{\partial y} \sim e^{\frac{y^2}{2}}, \quad y \rightarrow \infty.$$

Then the solution p does not depend on variable y .

Proof Refer to Theorem 4.1 in Choi et al. (2013).

We need the following lemma for the operator $\mathcal{L}_{20}$.

Lemma 3 *The solution of the final value problem*

$$\begin{aligned}\langle \mathcal{L}_{20} \rangle p(t, x) &= 0, \quad x > 0, \quad 0 \leq t < T, \\ p(T, x) &= g(x),\end{aligned}$$

where

$$\langle \mathcal{L}_{20} \rangle = \frac{\partial}{\partial t} + \frac{1}{2} \langle f^2 \rangle x^2 \frac{\partial^2}{\partial x^2} + rx \frac{\partial}{\partial x},$$

can be expressed as $p(t, x) = \mathbb{E}[g(e^\eta)]$ for all $x > 0$ and $t \in [0, T]$, where η is a random variable normally distributed with mean $\ln x + (r - \frac{1}{2} \langle f^2 \rangle)(T - t)$ and variance $\langle f^2 \rangle (T - t)$.

Proof Refer to Proposition 4.2 in Cao et al. (2021).

Also, we need the following lemma for the operator $\mathcal{L}_{20}$.

Lemma 4 *The solution of the final value problem*

$$\begin{aligned}\langle \mathcal{L}_{20} \rangle p(t, x) &= g(t, x), \quad x > 0, \quad 0 \leq t < T, \\ p(T, x) &= 0.\end{aligned}$$

is given by

$$h(t, x) = - \int_0^\tau \int_{-\infty}^\infty \frac{e^{-\frac{(y-z)^2}{2\langle f^2 \rangle(\tau-s)}}}{\sqrt{2\pi \langle f^2 \rangle (\tau-s)}} g\left(T - \tau, e^{y - (r - \frac{1}{2} \langle f^2 \rangle)s}\right) dy ds,$$

where $\tau = T - t$ and $z = \ln x + (r - \frac{1}{2} \langle f^2 \rangle)(T - t)$.

Proof Refer to Proposition 4.3 in Cao et al. (2021).

First, we deal with equation (6) with respect to the parameter ε . If we expand P_0^ε as

$$P_0^\varepsilon = \sum_{j=0}^{\infty} \varepsilon^{\frac{j}{2}} P_{0,j}, \quad (10)$$

then, from equation (6), we have a system of PDEs in the following form.

$$\mathcal{L}_0 P_{0,k} + \mathcal{L}_{10} P_{0,k-1} + \mathcal{L}_{20} P_{0,k-2} = 0, \quad k = 0, 1, 2, \dots, \quad (11)$$

where $P_{0,-2} = P_{0,-1} = 0$ are defined. From equation (11) with $k = 0$ and $P_{0,-2} = P_{0,-1} = 0$, we have $\mathcal{L}_0 P_{0,0} = 0$. Then, from Lemma 2, $P_{0,0}$ is y -independent so that $P_{0,0} = P_{0,0}(t, x)$. For $k = 1$, we have $\mathcal{L}_0 P_{0,1} = 0$ since $P_{0,0}$ is independent of variable y . From Lemma 2 again, $P_{0,1}$ is independent of y so that $P_{0,1} = P_{0,1}(t, x)$. Then, for $k = 2$, equation (11) becomes

$$\mathcal{L}_0 P_{0,2} + \mathcal{L}_{20} P_{0,0} = 0. \quad (12)$$

From Lemma 1 (the centering condition), this equation has a solution only if $\langle \mathcal{L}_{20} P_{0,0} \rangle = 0$. We can rewrite this as $\langle \mathcal{L}_{20} \rangle P_{0,0} = 0$ since $P_{0,0}$ is independent of y . So, we consider the following final value problem for $P_{0,0}$.

$$\begin{aligned} \langle \mathcal{L}_{20} \rangle P_{0,0}(t, x) &= 0, \quad x > 0, \quad 0 \leq t < T, \\ P_{0,0}(T, x) &= \ln x. \end{aligned}$$

Applying Lemma 3 to this problem, we obtain

$$P_{0,0}(t, x) = \ln x + \left(r - \frac{1}{2} \langle f^2 \rangle \right) (T - t). \quad (13)$$

Inserting this $P_{0,0}$ into equation (12) gives

$$\mathcal{L}_0 P_{0,2}(t, x, y) = \frac{1}{2} (f^2(y) - \langle f^2(y) \rangle). \quad (14)$$

According to Fouque et al. (2016), the averaged terminal condition $\langle P_{0,2}(T, x, \cdot) \rangle = 0$ is the appropriate final condition for the second order term $P_{0,2}$ (not $P_{0,2}(T, x, y) = 0$). Let ψ be the solution of the Poisson equation

$$\mathcal{L}_0 \psi = f^2 - \langle f^2 \rangle.$$

Then $P_{0,2} = -\frac{1}{2} \psi(y) + F_{0,2}(t, x)$ for some function $F_{0,2}$ independent of y . There is no explicit formula for ψ here but we will find the solution ψ using a numerical method (the Gauss-Legendre quadrature) later. For $k = 3$, equation (11) becomes $\mathcal{L}_0 P_{0,3} + \mathcal{L}_{10} P_{0,2} + \mathcal{L}_{20} P_{0,1} = 0$. Then the centering condition implies $\langle \mathcal{L}_{20} \rangle P_{0,1} = -\langle \mathcal{L}_{10} P_{0,2} \rangle$ and so we obtain

$$\begin{aligned} \langle \mathcal{L}_{20} \rangle P_{0,1}(t, x) &= 0, \quad x > 0, \quad 0 \leq t < T, \\ P_{0,1}(T, x) &= 0, \end{aligned}$$

which leads to $P_{0,1}(t, x) \equiv 0$.

Now, we analyze equation (7) with respect to the parameter ε . If we expand P_1^ε as

$$P_1^\varepsilon = \sum_{j=0}^{\infty} \varepsilon^{\frac{j}{2}} P_{1,j}, \quad (15)$$

with final conditions $P_{1,0}(T, x, y) = 0$ and $P_{1,1}(T, x, y) = 0$, then we can find that $P_{1,0}$ and $P_{1,1}$ are independent of y by the same argument as above. Also, we can obtain

$$\mathcal{L}_0 P_{1,2} + \mathcal{L}_{20} P_{1,0} + \mathcal{L}_{21} P_{0,0} = 0. \quad (16)$$

Inserting the solution $P_{0,0}$ in (13) into (16) and using the centering condition, we can induce

$$\begin{aligned} \langle \mathcal{L}_{20} \rangle P_{1,0}(t, x) &= -\langle f^2 \rangle \ln x, \quad x > 0, \quad 0 \leq t < T, \\ P_{1,0}(T, x) &= 0. \end{aligned} \quad (17)$$

Then, from Lemma 4 with $g(t, x) = -\langle f^2 \rangle \ln x$, we can obtain the following solution of equation (17) for $P_{1,0}$.

$$P_{1,0}(t, x) = \langle f^2 \rangle \ln x (T - t) + \frac{1}{2} \langle f^2 \rangle \left(r - \frac{1}{2} \langle f^2 \rangle \right) (T - t)^2.$$

By plugging $P_{0,0}$ and $P_{1,0}$ into equation (16), we have

$$P_{1,2}(t, x, y) = -\psi(y) \left(2 \ln x - \langle f^2 \rangle (T - t) \right) + k(t, x), \quad (18)$$

for some function k , which is obtained by the following equation

$$\mathcal{L}_0 P_{1,2} = \frac{1}{2} \left(\langle f^2 \rangle - f^2(y) \right) \left(2 \ln x - \langle f^2 \rangle (T - t) \right).$$

We also have the PDE

$$\mathcal{L}_0 P_{1,3} + \mathcal{L}_{10} P_{1,2} + \mathcal{L}_{11} P_{0,2} + \mathcal{L}_{20} P_{1,1} + \mathcal{L}_{21} P_{0,1} = 0.$$

Then the centering condition

$$\langle \mathcal{L}_{10} P_{1,2} + \mathcal{L}_{11} P_{0,2} + \mathcal{L}_{20} P_{1,1} + \mathcal{L}_{21} P_{0,1} \rangle = 0$$

implies that $\langle \mathcal{L}_{20} P_{1,1} \rangle = 2\rho \langle \gamma f \psi' \rangle$. So, we have a final-value problem

$$\begin{aligned} \langle \mathcal{L}_{20} \rangle P_{1,1} &= 2\rho \langle \gamma f \psi' \rangle, \quad x > 0, \quad 0 \leq t < T, \\ P_{1,1}(T, x) &= 0. \end{aligned} \quad (19)$$

Applying Lemma 4 with $g(x, t) = 2\rho \langle \gamma f \psi' \rangle$, we have a solution of equation (19) as

$$P_{1,1}(t, x) = -2\rho \langle \gamma f \psi' \rangle (T - t).$$

Finally, we substitute P_0^ε , P_1^ε and the expansion

$$P_2^\varepsilon = P_{2,0} + \sqrt{\varepsilon} P_{2,1} + \varepsilon P_{2,2} + \varepsilon \sqrt{\varepsilon} P_{2,3} + \cdots$$

into equation (8), and implement a similar analysis to what we have done earlier. Under the condition in Lemma 2 for $P_{2,0}$ and $P_{2,1}$ are functions independent of y . Also, we have the equation

$$\mathcal{L}_0 P_{2,2} + \mathcal{L}_{20} P_{2,0} + \mathcal{L}_{21} P_{1,0} + \mathcal{L}_{22} P_{0,0} = 0. \quad (20)$$

By the centering condition, we have $\langle \mathcal{L}_{20} P_{2,0} \rangle = -\langle f^2 \rangle^2 \ln x (T - t) + \langle f^2 \rangle (\ln x)^2$ which in turn yields the final-value problem

$$\begin{aligned} \langle \mathcal{L}_{20} \rangle P_{2,0} &= -\langle f^2 \rangle^2 \ln x (T - t) + \langle f^2 \rangle (\ln x)^2, \quad x > 0, \quad 0 \leq t < T, \\ P_{2,0}(T, x) &= 0. \end{aligned} \quad (21)$$

By applying Lemma 4 to equation (21), we obtain

$$\begin{aligned} P_{2,0}(t, x) = & -\langle f^2 \rangle (\ln x)^2 (T-t) + \langle f^2 \rangle \left\{ \left(-r + \frac{3}{2} \langle f^2 \rangle \right) \ln x - \frac{1}{2} \langle f^2 \rangle \right\} (T-t)^2 \\ & + \frac{1}{3} \langle f^2 \rangle \left(r - \frac{1}{2} \langle f^2 \rangle \right) \left(2 \langle f^2 \rangle - r \right) (T-t)^3. \end{aligned} \quad (22)$$

We put the above results together to obtain the following approximation formula for the fair strike price from (3).

Proposition 1 *If the underlying asset price X_t follows the risk-neutral model (1) with the fast scale stochastic volatility dynamics (2) and the required assumptions stated in Section 2 and the growth condition described in Section 3 are satisfied, then the fair strike price at time t is approximated by K_t^{fast} which is*

$$\begin{aligned} K_t^{fast} = & \langle f^2 \rangle - \beta \langle f^2 \rangle \left[2 \ln x + \left(r - \frac{1}{2} \langle f^2 \rangle \right) (T-t) \right] \\ & + 4\beta \sqrt{\varepsilon} \rho \langle \gamma f \psi' \rangle - \frac{2}{T-t} \varepsilon \psi(y) \\ & + 2\beta^2 \left[\langle f^2 \rangle (\ln x)^2 - \langle f^2 \rangle \left\{ \left(-r + \frac{3}{2} \langle f^2 \rangle \right) \ln x - \frac{1}{2} \langle f^2 \rangle \right\} (T-t) \right. \\ & \quad \left. - \frac{2}{3} \langle f^2 \rangle \left(r - \frac{1}{2} \langle f^2 \rangle \right) \left(\langle f^2 \rangle - \frac{1}{2} r \right) (T-t)^2 \right], \end{aligned} \quad (23)$$

where ψ satisfies

$$\left(\gamma^2(y) \frac{\partial^2}{\partial y^2} + 2\alpha(y) \frac{\partial}{\partial y} \right) \psi = (f^2(y) - \langle f^2(y) \rangle).$$

Remark As a special case of the result in Proposition 1, the CEV strike price, K_t^{cev} , at time t is given by

$$\begin{aligned} K_t^{cev} \approx & \langle f^2 \rangle - \beta \langle f^2 \rangle \left[2 \ln x + \left(r - \frac{1}{2} \langle f^2 \rangle \right) (T-t) \right] \\ & + 2\beta^2 \langle f^2 \rangle \left[(\ln x)^2 - \left\{ \left(-r + \frac{3}{2} \langle f^2 \rangle \right) \ln x - \frac{1}{2} \langle f^2 \rangle \right\} (T-t) \right. \\ & \quad \left. - \frac{2}{3} \left(r - \frac{1}{2} \langle f^2 \rangle \right) \left(\langle f^2 \rangle - \frac{1}{2} r \right) (T-t)^2 \right]. \end{aligned} \quad (24)$$

The approximation formula in Proposition 1 has been derived by using a formal analysis. Since the payoff function of our pricing problem is a smooth

function $\ln x$ and the approximation is a first-order one in ε , a proof of the accuracy is relatively easy and the error is $\mathcal{O}(\beta^2 + \varepsilon)$. The proof is omitted here. Refer to Fouque et al. (2011) for details. Later, we will confirm the accuracy numerically.

3.3 Specific models

In this section, we specify the underlying model in several ways and confirm the theoretical formula in Proposition 1 using Monte-Carlo (MC) simulation and check the fair strike price behavior of each model with respect to the mean reversion rate of the fast varying process Y_t .

For MC simulations, 100,000 sample paths are used for the variance swap strike price. We calculate the square root of the fair strike price as a percentage (SRVS(%)) for different sets of values ε and β . We compute the values based on the S&P 500 market data in 2020 and we choose six different values of the initial underlying value X_0 between 2500 and 3500. In Figure 1-3, the line represents the theoretical strike price and the circle represents the MC simulation outcome.

3.3.1 The OU volatility

We consider that X_t and Y_t are the processes given by

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t)X_t^{1-\beta} dW_t^x, \\ dY_t &= \frac{1}{\varepsilon}(\theta - Y_t)dt + \frac{1}{\sqrt{\varepsilon}}(\sqrt{2\nu})dW_t^y, \end{aligned} \quad (25)$$

where $f(y) = 0.45(\frac{\tan^{-1}(y) + \frac{\pi}{2}}{\pi}) + 0.05$, which allows the model to reflect the implied volatility of the market, $\theta = 0$, $\nu = 1$, $r = 0.05$ and W_t^x and W_t^y are Brownian motions with correlation coefficient $\rho_{xy} = -0.4$. The parameters are cited from Cao et al. (2021). Also, β is chosen to be 0.01. ε takes the four different values 0.005, 0.01, 0.03 and 0.05.

As one can see in Figure 1, the theoretical results match well those from the MC simulation. Also, for any fixed ε , SRVS(%) is dwindling as the initial underlying asset price X_0 increases. For any fixed X_0 , SRVS(%) decreases as the value of ε decreases.

3.3.2 The CIR volatility

We consider that X_t and Y_t satisfy

$$\begin{aligned} dX_t &= rX_t dt + \sqrt{Y_t}X_t^{1-\beta} dW_t^x, \\ dY_t &= \frac{1}{\varepsilon}(\theta - Y_t)dt + \frac{1}{\sqrt{\varepsilon}}(\sqrt{2Y_t}\sigma)dW_t^y, \end{aligned} \quad (26)$$

Fig. 1: Square root of the fair strike price(%) against X_0 for the OU stochastic volatility when $\beta = 0.01$

respectively, where $\theta = 0.07597$, $\sigma = 0.48665$, $r = 0.05$ and W_t^x and W_t^y are Brownian motions with correlation coefficient $\rho_{xy} = -0.67099$. These parameters come from Mrazek et al. (2017). The parameters are selected in such a way that the initial volatility value is the same as that of the OU volatility model.

Fig. 2: Square root of the fair strike price(%) against X_0 for the CIR stochastic volatility when $\beta = 0.01$

The theoretical results match well those from the MC simulation also in Figure 2. For any fixed ε , $\text{SRVS}(\%)$ decreases as X_0 increases. For any fixed X_0 , $\text{SRVS}(\%)$ increases as the value of ε decreases in this case. On the other hand, the leverage effect appears to be the same regardless of the ε value because β is fixed.

3.3.3 The ExpOU volatility

We consider the case that X_t and Y_t are given by

$$\begin{aligned} dX_t &= rX_t dt + me^{Y_t} X_t^{1-\beta} dW_t^x, \\ dY_t &= \frac{1}{\varepsilon}(\theta - Y_t) dt + \frac{1}{\sqrt{\varepsilon}}(\sqrt{2\nu}) dW_t^y, \end{aligned} \quad (27)$$

respectively, where $\theta = 0$, $m = 0.275$, $\nu = 0.3239$, $r = 0.05$ and W_x^1 and W_t^y are Brownian motions with correlation coefficient $\rho_{xy} = -0.45$. The parameters are cited from Kim et al. (2019) and this choice of parameters provides the same initial value of the volatility among the three volatility models.

Fig. 3: Square root of the fair strike price(%) against X_0 for the ExpOU stochastic volatility when $\beta = 0.01$

The theoretical results match well those from the MC simulation also in Figure 3. For any fixed ε , $\text{SRVS}(\%)$ decreases as X_0 increases. For any fixed X_0 , $\text{SRVS}(\%)$ increases as the value of ε decreases.

3.4 Volatility model comparison

In this section, we compare the above three specific models which have different invariant distributions of Y_t , which may cause different shifts in the strike price of the variance swap. For comparison, we use S&P 500 market data in 2020 and 2021 and choose five different values of X_0 between 3200 and 4000 and two different values, 0.005 and 0.05, of ε . Since β affects only the process X_t , we fix $\beta = 0.01$. In Table 1, (+) indicates that the initial value Y_0 is greater than the long-term mean (θ) of Y_t while (−) implies that the Y_0 is less than that.

Table 1: Comparison of the three volatility models with respect to the square root of the fair strike price (%) with $\beta = 0.01$

Models	ε	X_0				
		3200	3400	3600	3800	4000
Model 1	0.005	27.4514	27.4356	27.4208	27.4068	27.3936
	0.05(+)	31.7836	31.7701	31.7573	31.7452	31.7337
	0.05(−)	23.5572	23.5389	23.5216	23.5053	23.4899
Model 2	0.005	29.3166	29.3034	29.2910	29.2793	29.2682
	0.05	25.5074	25.4922	25.4780	25.4645	25.4517
Model 3	0.005	28.5181	28.5015	28.4858	28.4710	28.4570
	0.05	28.2911	28.2742	28.2585	28.2436	28.2294

Model 1 in Table 1 refers to the CEV model with the OU stochastic volatility (25). Recall that Y_t is normally distributed and the volatility of X_t is given by $(0.45(\frac{\tan^{-1}(Y_t) + \frac{\pi}{2}}{\pi}) + 0.05)X_t^{-\beta}$ in this model. The case of $\varepsilon = 0.005$ corresponds to fast mean reversion of Y_t while $\varepsilon = 0.05$ implies slow mean reversion. The fair strike prices in Table 1 support numerically this mean reversion speed difference well.

Model 2 in Table 1 refers to the CEV model with the CIR stochastic volatility in which Y_t follows the CIR process and the volatility of X_t is given by $\sqrt{Y_t}X_t^{-\beta}$. As ε becomes smaller from 0.05 to 0.005, the fair strike price increases. It has an opposite direction in Model 1 if Y_0 is greater than the long-term mean (θ) of Y_t .

In the CEV model with the ExpOU stochastic volatility, named Model 3, Y_t follows the OU process and the volatility of X_t is $me^{Y_t}X_t^{-\beta}$. There is no significant change in the fair strike price between $\varepsilon = 0.05$ and $\varepsilon = 0.005$ under this model, which is not like Model 1 and Model 2.

3.5 Comparison of the CEV, SV and SVCEV models

In this section, we consider three underlying asset models which are the (pure) stochastic volatility (SV) model, i.e., the local volatility model (25) with $\beta = 0$,

and the CEV model, i.e., the model (25) with $\varepsilon = 0$, and the hybrid stochastic volatility and CEV (SVCEV) model (25). In the CEV model, β is chosen to be a non-zero fixed constant 0.01. In the SVCEV model, β and ε are the same non-zero fixed constant 0.01.

Table 2: Comparison of the SV, CEV and SVCEV models with respect to the square root of the fair strike prices (%)

Models	X_0					
	2500	2700	2900	3100	3300	3500
SV	30.2716	30.2716	30.2716	30.2716	30.2716	30.2716
CEV	26.9438	26.9234	26.9044	26.8867	26.8701	26.8546
SVCEV	28.1761	28.1565	28.1384	28.1215	28.1056	28.0907

As shown in Table 2, the fair strike prices with the SV model are higher than the corresponding prices with the CEV model and the SVCEV prices are between those two prices.

4 Multiscale stochastic volatility correction

In this section, we obtain a multiscale volatility correction to the CEV variance strike price by incorporating a slow scale volatility process to the fast scale volatility and making full use of the model (2).

4.1 Asymptotic expansion

First, we reorganize $\mathcal{L}_{X,Y,Z}^{\varepsilon,\delta,\beta}$ as follows with respect to the small parameter δ .

$$\mathcal{L}_{X,Y,Z}^{\varepsilon,\delta,\beta} = \left(\frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1^\beta + \mathcal{L}_2^\beta \right) + \sqrt{\delta} \left(\frac{1}{\sqrt{\varepsilon}} \mathcal{M}_3 + \mathcal{M}_1^\beta \right) + \delta \mathcal{M}_2,$$

where

$$\begin{aligned}
\mathcal{L}_0 &:= \frac{1}{2}\gamma^2(y)\frac{\partial^2}{\partial y^2} + \alpha(y)\frac{\partial}{\partial y}, \\
\mathcal{L}_1^\beta &:= \rho_{xy}\gamma(y)f(y)x^{1-\beta}\frac{\partial^2}{\partial x\partial y}, \\
\mathcal{L}_2^\beta &:= \frac{\partial}{\partial t} + \frac{1}{2}f^2(y)x^{2(1-\beta)}\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x}, \\
\mathcal{M}_1^\beta &:= \rho_{zx}f(y)g(z)x^{1-\beta}\frac{\partial^2}{\partial z\partial x}, \\
\mathcal{M}_2 &:= \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2} + c(z)\frac{\partial}{\partial z} \\
\mathcal{M}_3 &:= \rho_{yz}\gamma(y)g(z)\frac{\partial^2}{\partial y\partial z}.
\end{aligned}$$

The operators $\mathcal{L}_1^\beta$, $\mathcal{L}_2^\beta$, and $\mathcal{M}_1^\beta$ involve the small parameter β . The expansions of $\mathcal{L}_1^\beta$, $\mathcal{L}_2^\beta$ with respect to β are the same as in Section 3 and the Taylor expansion of $\mathcal{M}_1^\beta$ with respect to β is given by

$$\begin{aligned}
\mathcal{M}_1^\beta &= \sum_{i=0}^{\infty} \beta^i \mathcal{M}_{1i}, \\
\mathcal{M}_{10} &:= \rho_{zx}f(y)g(z)x\frac{\partial^2}{\partial z\partial x}, \\
\mathcal{M}_{11} &:= -\rho_{zx}f(y)g(z)x\ln x\frac{\partial^2}{\partial z\partial x}, \\
\mathcal{M}_{12} &:= \frac{1}{2}\rho_{zx}f(y)g(z)x(\ln x)^2\frac{\partial^2}{\partial z\partial x},
\end{aligned}$$

We are interested in $P^{\varepsilon,\delta,\beta}$ of the form

$$P^{\varepsilon,\delta,\beta} = \sum_{i=0}^{\infty} \sqrt{\delta^i} P_i^{\varepsilon,\beta}.$$

Inserting this expansion into equation (5) and collecting terms of like-powers of $\sqrt{\delta}$, we find that

$$\delta^0\text{-term} : \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1^\beta + \mathcal{L}_2^\beta \right) P_0^{\varepsilon,\beta} = 0, \quad (28)$$

$$\sqrt{\delta}\text{-term} : \left(\frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3 + \mathcal{M}_1^\beta \right) P_0^{\varepsilon,\beta} + \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1^\beta + \mathcal{L}_2^\beta \right) P_1^{\varepsilon,\beta} = 0, \quad (29)$$

$$\delta\text{-term} : \mathcal{M}_2 P_0^{\varepsilon,\beta} + \left(\frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3 + \mathcal{M}_1^\beta \right) P_1^{\varepsilon,\beta} + \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1^\beta + \mathcal{L}_2^\beta \right) P_2^{\varepsilon,\beta} = 0. \quad (30)$$

...

4.2 Asymptotic analysis

We use the same asymptotic analysis as done in Section 3 to obtain an approximate solution for $P^{\varepsilon, \delta, \beta}(t, x, y, z)$. For equation (28), we expand $P_0^{\varepsilon, \beta}$ as

$$P_0^{\varepsilon, \beta} = \sum_{i=0}^{\infty} \beta^i P_{0,i}^{\varepsilon},$$

where $P_{0,i}^{\varepsilon}$ is expanded as

$$P_{0,i}^{\varepsilon} = \sum_{j=0}^{\infty} \sqrt{\varepsilon^j} P_{0,i,j}.$$

Then $P_{0,i,j}$ are exactly the same values as found in the fast-scale volatility case and they are

$$\begin{aligned} P_{0,0,0}(t, x, \bar{z}) &= P_{0,0}(t, x) = \ln x + \left(r - \frac{1}{2} \langle f^2 \rangle\right) (T - t), \\ P_{0,0,1}(t, x, \bar{z}) &= P_{0,1}(t, x) = 0, \\ P_{0,0,2}(t, x, \bar{z}) &= P_{0,2}(t, x) = \psi, \\ P_{0,1,0}(t, x, \bar{z}) &= P_{1,0}(t, x) = \langle f^2 \rangle \ln x (T - t) + \frac{1}{2} \langle f^2 \rangle \left(r - \frac{1}{2} \langle f^2 \rangle\right) (T - t)^2, \\ P_{0,1,1}(t, x, \bar{z}) &= P_{1,1}(t, x) = -2\rho_{xy} \langle \gamma f \psi' \rangle (T - t), \\ P_{0,2,0}(t, x, \bar{z}) &= P_{2,0}(t, x) = -\langle f^2 \rangle (\ln x)^2 (T - t) \\ &\quad + \langle f^2 \rangle \left\{ \left(-r + \frac{3}{2} \langle f^2 \rangle\right) \ln x - \frac{1}{2} \langle f^2 \rangle \right\} (T - t)^2 \\ &\quad + \frac{1}{3} \langle f^2 \rangle \left(r - \frac{1}{2} \langle f^2 \rangle\right) (2 \langle f^2 \rangle - r) (T - t)^3, \end{aligned}$$

where $\bar{z} = \langle f \rangle$.

Now, we analyze equation (29) with respect to the parameters β and ε . If we expand $P_0^{\varepsilon, \beta}$ and $P_1^{\varepsilon, \beta}$ as

$$P_0^{\varepsilon, \beta} = \sum_{i=0}^{\infty} \beta^i P_{0,i}^{\varepsilon}, \quad P_1^{\varepsilon, \beta} = \sum_{i=0}^{\infty} \beta^i P_{1,i}^{\varepsilon},$$

we have a hierarchy as follows.

$$\beta^0\text{-term} : \left(\frac{1}{\sqrt{\varepsilon}} \mathcal{M}_3 + \mathcal{M}_{10} \right) P_{0,0}^{\varepsilon} + \left(\frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{10} + \mathcal{L}_{20} \right) P_{1,0}^{\varepsilon} = 0, \quad (31)$$

$$\begin{aligned} \beta^1\text{-term} : & \left(\frac{1}{\sqrt{\varepsilon}} \mathcal{M}_3 + \mathcal{M}_{10} \right) P_{0,1}^{\varepsilon} + \mathcal{M}_{11} P_{0,0}^{\varepsilon} \\ & + \left(\frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{10} + \mathcal{L}_{20} \right) P_{1,1}^{\varepsilon} + \left(\frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{11} + \mathcal{L}_{21} \right) P_{1,0}^{\varepsilon} = 0. \end{aligned} \quad (32)$$

...

We insert the extensions of $P_{0,0}^\varepsilon$ and $P_{1,0}^\varepsilon$ into equation (31) and gather terms of like powers of $\sqrt{\varepsilon}$. Then a system of PDEs is obtained as

$$\mathcal{M}_3 P_{0,0,k-1} + \mathcal{M}_{10} P_{0,0,k-2} + \mathcal{L}_0 P_{1,0,k} + \mathcal{L}_{10} P_{1,0,k-1} + \mathcal{L}_{20} P_{1,0,k-2} = 0, \quad (33)$$

where $k = 0, 1, 2, \dots$ and $P_{0,0,-1} = P_{0,0,-2} = P_{1,0,-1} = P_{1,0,-2} = 0$ are defined. From equation (33) with $k = 0$ and $k = 1$, we have $\mathcal{L}_0 P_{1,0,0} = 0$ and $\mathcal{L}_0 P_{1,0,1} = 0$, respectively. By Lemma 2, $P_{1,0,0}$ and $P_{1,0,1}$ are independent of y and so $P_{1,0,0} = P_{1,0,0}(t, x, z)$ and $P_{1,0,1} = P_{1,0,1}(t, x, z)$. Then, for $k = 2$, equation (33) turns into

$$\mathcal{M}_{10} P_{0,0,0} + \mathcal{L}_0 P_{1,0,2} + \mathcal{L}_{20} P_{1,0,0} = 0. \quad (34)$$

Because of the centering condition and simple computations, the equation becomes

$$\langle \mathcal{L}_{20} \rangle P_{1,0,0} = 2\rho_{zx} g \langle f \rangle^2 (T - t).$$

Applying Lemma 4 to this equation with a fixed $\bar{z} = \langle f \rangle$ and $P_{1,0,0}(T, x, z) = 0$, we obtain

$$P_{1,0,0}(t, x, \bar{z}) = -\rho_{zx} g \langle f \rangle^2 (T - t)^2. \quad (35)$$

Inserting the price $P_{0,0,0}$ and the value $\mathcal{L}_{20} P_{1,0,0}$ into equation (34), we have

$$\mathcal{L}_0 P_{1,0,2} = 2\rho_{zx} g \langle f \rangle (f - \langle f \rangle) (T - t).$$

Then from Lemma 4 with the final condition $P_{1,0,2}(T, x, y, z) = 0$ the solution is

$$P_{1,0,2}(t, x, y, z) = 4\rho_{zx} g \langle f \rangle \bar{\psi}(T - t), \quad (36)$$

where $\bar{\psi}$ satisfies

$$\mathcal{L}_0 \bar{\psi} = \frac{1}{2} (f - \langle f \rangle).$$

One can continue the similar process to the above argument for $k = 3$. We have

$$\mathcal{M}_3 P_{0,0,2} + \mathcal{L}_0 P_{1,0,3} + \mathcal{L}_{10} P_{1,0,2} + \mathcal{L}_{20} P_{1,0,1} = 0.$$

By Lemma 1, this equation becomes

$$\langle \mathcal{L}_{20} \rangle P_{1,0,1} = -2\rho_{yz} g \langle f \rangle \langle \gamma \bar{\psi} \rangle (T - t).$$

Applying Lemma 4, the solution $P_{1,0,1}$ of the above equation with terminal condition $P_{1,0,1}(T, x, z) = 0$ is

$$P_{1,0,1}(t, x, \bar{z}) = \rho_{yz} g \langle f \rangle \langle \gamma \bar{\psi} \rangle (T - t)^2. \quad (37)$$

Now, we analyze equation (32) to find $P_{1,1,0}$ and $P_{1,1,1}$. A system of PDEs is given by

$$\begin{aligned} \mathcal{M}_{10} P_{0,1,k-2} + \mathcal{M}_3 P_{0,1,k-1} + \mathcal{L}_0 P_{1,1,k} + \mathcal{L}_{10} P_{1,1,k-1} + \mathcal{L}_{11} P_{1,0,k-1} \\ + \mathcal{M}_{11} P_{0,0,k-2} + \mathcal{L}_{20} P_{1,1,k-2} + \mathcal{L}_{21} P_{1,0,k-2} = 0, \quad k = 0, 1, 2, \dots, \end{aligned} \quad (38)$$

where $P_{n,m,-2} = P_{n,m,-1} = 0$ is defined for $n, m = 0, 1$. For $k = 0$ and $k = 1$, $P_{1,1,0}$ and $P_{1,1,1}$ are y -independent functions. For $k = 2$, we have

$$\mathcal{M}_{10}P_{0,1,0} + \mathcal{M}_{11}P_{0,0,0} + \mathcal{L}_0P_{1,1,2} + \mathcal{L}_{20}P_{1,1,0} + \mathcal{L}_{21}P_{1,0,0} = 0. \quad (39)$$

By Lemma 1, we can induce

$$\langle \mathcal{L}_{20} \rangle P_{1,1,0} = -2\rho_{zx}g \langle f^2 \rangle \langle f \rangle^2 (T-t)^2 - 2\rho_{zx}g \langle f \rangle^2 \ln x (T-t).$$

Applying Lemma 4 with the final condition $P_{1,0,0}(T, x, z) = 0$, we obtain

$$\begin{aligned} P_{1,1,0}(t, x, \bar{z}) &= \frac{2}{3}\rho_{zx}g \langle f \rangle^2 \{ \langle f^2 \rangle - \ln x \} (T-t)^3 \\ &\quad - \frac{1}{4}\rho_{zx}g \langle f \rangle^2 \left(r - \frac{1}{2} \langle f^2 \rangle \right) (T-t)^4. \end{aligned} \quad (40)$$

Then from equation (39) we obtain the term $P_{1,1,2}$ as

$$\begin{aligned} P_{1,1,2}(t, x, \bar{z}) &= -\frac{2}{3}\rho_{zx}g \langle f \rangle^2 \psi(T-t)^3 \\ &\quad - 4\rho_{zx}g \langle f \rangle \bar{\psi}(T-t) \left\{ \langle f^2 \rangle (T-t) + \ln x \right\}. \end{aligned} \quad (41)$$

To find $P_{1,1,1}$, we consider $k = 3$ and obtain

$$\mathcal{M}_3P_{1,0,2} + \mathcal{M}_{10}P_{0,1,1} + \mathcal{L}_0P_{1,1,3} + \mathcal{L}_{10}P_{1,1,2} + \mathcal{L}_{20}P_{1,1,1} + \mathcal{L}_{11}P_{1,0,2} + \mathcal{L}_{21}P_{1,0,1} = 0.$$

From Lemma 1, this equation becomes

$$\begin{aligned} \langle \mathcal{L}_{20} \rangle P_{1,1,1} &= -8\rho_{yz}\rho_{xz}g^2 \langle f \rangle^2 \langle \gamma \bar{\psi}' \rangle (T-t)^2 \\ &\quad + 4\rho_{xy}\rho_{zx}g \langle f \rangle \langle \gamma f \bar{\psi}' \rangle (T-t). \end{aligned}$$

Applying Lemma 4 with $P_{1,1,1}(T, x, z) = 0$, we obtain

$$\begin{aligned} P_{1,1,1}(t, x, \bar{z}) &= \frac{8}{3}\rho_{yz}\rho_{xz}g^2 \langle f \rangle^2 \langle \gamma \bar{\psi}' \rangle (T-t)^3 \\ &\quad - 2\rho_{xy}\rho_{zx}g \langle f \rangle \langle \gamma f \bar{\psi}' \rangle (T-t)^2. \end{aligned} \quad (42)$$

Lastly, we analyze equation (30) to obtain $P_{2,0,0}$. We expand $P_0^{\varepsilon, \beta}$, $P_1^{\varepsilon, \beta}$ and $P_2^{\varepsilon, \beta}$ as

$$P_0^{\varepsilon, \beta} = \sum_{i=0}^{\infty} \beta^i P_{0,i}^{\varepsilon}, \quad P_1^{\varepsilon, \beta} = \sum_{i=0}^{\infty} \beta^i P_{1,i}^{\varepsilon}, \quad P_2^{\varepsilon, \beta} = \sum_{i=0}^{\infty} \beta^i P_{2,i}^{\varepsilon}.$$

Then we have

$$\begin{aligned} \beta^0\text{-term} : \mathcal{M}_2P_{0,0}^{\varepsilon} + \left(\frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3 + \mathcal{M}_{10} \right) P_{1,0}^{\varepsilon} \\ + \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_{10} + \mathcal{L}_{20} \right) P_{2,0}^{\varepsilon} = 0. \end{aligned} \quad (43)$$

Expanding $P_{0,0}^\varepsilon$, $P_{1,0}^\varepsilon$ and $P_{2,0}^\varepsilon$ with respect to ε as done before, we obtain a system of PDEs given by

$$\begin{aligned} \mathcal{M}_2 P_{0,0,k-2} + \mathcal{M}_3 P_{1,0,k-1} + \mathcal{M}_{10} P_{1,0,k-2} \\ + \mathcal{L}_0 P_{2,0,k} + \mathcal{L}_{10} P_{2,0,k-1} + \mathcal{L}_{20} P_{2,0,k-2} = 0, \quad k = 0, 1, 2, \dots, \end{aligned}$$

where $P_{n,0,-1} = P_{n,0,-2} = 0$ is defined for $n = 0, 1, 2$. For $k = 1$, Lemma 2 implies that $P_{2,0,0}$ and $P_{2,0,1}$ are functions independent of y so that $P_{2,0,0} = P_{2,0,0}(t, x, z)$ and $P_{2,0,1} = P_{2,0,1}(t, x, z)$. For $k = 2$, we have

$$\mathcal{M}_2 P_{0,0,0} + \mathcal{M}_{10} P_{1,0,0} + \mathcal{L}_0 P_{2,0,2} + \mathcal{L}_{20} P_{2,0,0} = 0.$$

Applying Lemma 1, this equation becomes

$$\langle \mathcal{L}_{20} \rangle P_{2,0,0} = g^2 \langle f \rangle^2 (T-t)^2 + c \langle f \rangle (T-t).$$

Using Lemma 4 with final condition $P_{2,0,0}(T, x, z) = 0$, the solution is given by

$$P_{2,0,0}(t, x, \bar{z}) = -\frac{1}{3} g^2 \langle f \rangle^2 (T-t)^3 - \frac{c}{2} \langle f \rangle (T-t)^2. \quad (44)$$

We put the above results together to obtain the following formula for the fair strike price with the multiscale stochastic volatility dynamics.

Proposition 2 *If the underlying asset price X_t follows the risk-neutral model (1) with the multiscale stochastic volatility dynamics (2) and the required assumptions stated in this section and the growth condition described in Section 3 are satisfied, then the fair strike price at time t is approximated by K_t^{multi} which is*

$$\begin{aligned} K_t^{multi} = & \langle f^2 \rangle - \beta \langle f^2 \rangle \left[2 \ln x + \left(r - \frac{1}{2} \langle f^2 \rangle \right) (T-t) \right] - \frac{2}{T-t} \varepsilon \psi(y) \\ & + 4\beta \sqrt{\varepsilon} \rho_{xy} \langle \gamma f \psi' \rangle + 2\sqrt{\delta} \rho_{zx} g \langle f \rangle^2 (T-t) \\ & + 2\beta^2 \langle f^2 \rangle \left[(\ln x)^2 - \left\{ \left(-r + \frac{3}{2} \langle f^2 \rangle \right) \ln x - \frac{1}{2} \langle f^2 \rangle \right\} (T-t) \right. \\ & \quad \left. - \frac{2}{3} \left(r - \frac{1}{2} \langle f^2 \rangle \right) \left(\langle f^2 \rangle - \frac{1}{2} r \right) (T-t)^2 \right] \\ & - 2\sqrt{\delta \varepsilon} \rho_{yz} g \langle f \rangle \langle \gamma \psi' \rangle (T-t) \\ & + \sqrt{\delta} \beta \rho_{zx} g \langle f \rangle^2 \left[\frac{4}{3} \left\{ \ln x - \langle f^2 \rangle \right\} + \frac{1}{2} \left(r - \frac{1}{2} \langle f^2 \rangle \right) (T-t) \right] (T-t)^2 \\ & - \sqrt{\delta \varepsilon} \beta \rho_{zx} g \langle f \rangle \left[\frac{16}{3} \rho_{yz} g \langle f \rangle \langle \gamma \bar{\psi}' \rangle (T-t) - 4\rho_{xy} \langle \gamma f \bar{\psi}' \rangle \right] (T-t) \\ & + \delta \langle f \rangle \left[\frac{2}{3} g^2 \langle f \rangle (T-t) + c \right] (T-t), \end{aligned} \quad (45)$$

where ψ and $\bar{\psi}$ satisfy

$$\begin{aligned} \left(\gamma^2(y) \frac{\partial^2}{\partial y^2} + 2\alpha(y) \frac{\partial}{\partial y} \right) \psi &= (f^2(y) - \langle f^2(y) \rangle), \\ \left(\gamma^2(y) \frac{\partial^2}{\partial y^2} + 2\alpha(y) \frac{\partial}{\partial y} \right) \bar{\psi} &= (f(y) - \langle f(y) \rangle), \end{aligned}$$

4.3 A specific model

We consider a specific multiscale volatility model given by

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t, Z_t) X_t^{1-\beta} dW_t^x, \\ dY_t &= \frac{1}{\varepsilon} (\theta - Y_t) dt + \frac{1}{\sqrt{\varepsilon}} (\nu_1 \sqrt{2}) dW_t^y, \\ dZ_t &= \delta (m - Z_t) dt + \sqrt{\delta} (\nu_2 \sqrt{2}) dW_t^z, \end{aligned} \tag{46}$$

where $f(y, z) = e^{y+z}$. This is thought of as a multiscale extension of Model 3 in Section 3.3. For computation, the parameters are cited from Fouque et al. (2004) and they are $r = 0.1$, $\theta = -0.8$, $\nu_1 = 0.5$, $\delta = 0.05$, $m = -0.8$ and $\nu_2 = 0.8$. The Brownian motions W_t^x , W_t^y and W_t^z are correlated by $\rho_{xy} = -0.2$, $\rho_{yz} = 0$ and $\rho_{zx} = -0.2$.

In Table (3), the numbers are the fair strike prices given by the formula and the numbers in parenthesis are the MC simulation results based on S&P500 market data in 2020 for four different values of ε . The theoretical and MC results are well matched regardless of the choice of ε . We also notice from the comparison between Table (1) and Table (3) that the addition of the slow scale volatility leads to more significant change of the fair strike value. Since the value of SRVS(%) converges slowly as slow scale is added and this makes a huge difference. This can be confirmed by the difference of about 1.23 in Table (3) when ε changes from 0.005 to 0.05, whereas model 3 in Table (1) shows a difference of 0.23. In Table 4, we fix $\varepsilon = 0.01$ and take the four values 0.03, 0.01, 0.005 and 0.001 of β . The table shows well the leverage effect which is one of stylized effects of volatility observed in market and it is more evident when β changes from 0.001 to 0.03. When β changes from 0.01 to 0.001, multiscale volatility also differ by 2.4 in SRVS (%) values. However, in the case of ExpOU volatility, the difference is 2.08, which is small. This is because only the X value affects by β , so there is less difference in the difference value than those values when ε changes.

5 Conclusion

In this paper, we employ an asymptotic analysis to derive a closed form solution formula for the approximate fair strike values of continuous time variance

Table 3: Comparison of the theoretical formula and MC simulation for the square root of the fair strike price (%) when $\beta = 0.01$

ε	X_0					
	2500	2700	2900	3100	3300	3500
0.005	33.2207 (33.2376)	33.1961 (33.2080)	33.1732 (33.2011)	33.1518 (33.1619)	33.1318 (33.1444)	33.1130 (33.1202)
0.01	33.2265 (33.2435)	33.2019 (33.2182)	33.1790 (33.1831)	33.1576 (33.1629)	33.1376 (33.1514)	33.1188 (33.1262)
0.03	32.2395 (32.2380)	32.2140 (32.2364)	32.1904 (32.1951)	32.1684 (32.1614)	32.1478 (32.1632)	32.1284 (32.1416)
0.05	31.9828 (31.9579)	31.9572 (31.9609)	31.9334 (31.9576)	31.9112 (31.9904)	31.8904 (31.9372)	31.8709 (31.8847)

Table 4: Comparison the square root of the fair strike price of the variance swap(%) with varies β and X_0 when $\varepsilon = 0.01$

β	X_0					
	2500	2700	2900	3100	3300	3500
0.03	27.9251 (27.9268)	27.8686 (27.9018)	27.8165 (27.8267)	27.7682 (27.7748)	27.7231 (27.7297)	27.6809 (27.6755)
0.01	32.7069 (32.6998)	32.6819 (32.6879)	32.6586 (32.6585)	32.6369 (32.6285)	32.6166 (32.6215)	32.5975 (32.5982)
0.005	33.9986 (33.9852)	33.9854 (33.9893)	33.9732 (33.9866)	33.9619 (33.9686)	33.9512 (33.9777)	33.9412 (33.9489)
0.001	35.1089 (35.1000)	35.1062 (35.0866)	35.1037 (35.0788)	35.1013 (35.0928)	35.0991 (35.1197)	35.0970 (35.1029)

swaps under a hybrid framework of constant elasticity of variance and multi-scale stochastic volatility in a risk-neutral market environment. Monte-Carlo simulation is also applied to obtain the values of the fair variance strike. Our analysis shows that the results produced by the theoretical formula match pretty well with those by the Monte-Carlo simulation. Moreover, given the invariant distribution of the processes driving fast and slow scale fluctuations of the volatility, the fair strike price of the variance swap can be obtained in an affordable manner. We have used the three specific examples of the OU, CIR and ExpOU stochastic volatility to compute the fair strike prices concretely. Our results can provide the prediction of the strike price movements from the information about the elasticity of variance (the leverage effect parameter) of the underlying asset return and the mean reversion rate of the volatility.

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