An Initial-Controlled Double-Scroll Hyperchaotic Map

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Research Article

Keywords: hyperchaotic map, offset boosting, double-scroll orbit

Posted Date: December 21st, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-1177591/v1>

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Abstract

Initial condition-dominated offset boosting provides a special channel to arrange coexisting orbits. Due to the nonlinearity and inherent periodicity, sinusoidal function is often introduced into a dynamical system for multistability design. In this paper, an initial-controlled double-scroll hyperchaotic map is constructed based on two sine functions. Four patterns of the double-scroll hyperchaotic orbits are found as 0-degree, 90-degree, 45-degree and 135-degree. Consequently, different modes for attractor growing are demonstrated. Finally, hardware experiments based on STM32 are carried out to verify the theoretical analysis and numerical simulation.

1. Introduction

Multistable chaotic system is the hot topic in nonlinear community [1–3], where the chaotic system can be modified under various transformations including attractor self-reproducing, attractor doubling and symmetrization [4–6]. Many works associated with this topic in continuous systems has been addressed. It shows that the initial condition-dominated offset boosting is the key entrance to polarity adjust and attractor control. All the modified systems have phase orbits under different positions with the same shapes [7–11]. Attractor growing is another issue in attractor control under the unified Lyapunov exponents. Many regimes of attractor hatching have been reported including those cases triggered by conditional symmetry and self-reproducing [15–18]. However, discrete chaotic map shows its unique properties compared to continuous systems when offset boosting is introduced for orbit controlling. Offset boosting in a discrete chaotic map does not like the operation in a continuous system. Kong et al. discussed some partial functions of offset boosting in a 2-D discrete chaotic map [12]; Bao et al. proposed a sine map where the initial condition was applied for selecting the sequence with different average values [13]. Strikingly, it has been reported that even a simple chaotic map with sinusoidal nonlinearity can obtain its partial amplitude control [14].

What makes the case even more bizarre is that an initial condition may control the direction for self-reproducing and the amplitude of the coexisting sequences. In this paper, we aim to find a discrete map with self-reproducing, and its properties are summarized as follows:

1. The hyperchaotic map has two sine functions and is of inversion symmetry, where an initial condition can extract out a double-scroll hyperchaotic orbit in four different directions.
2. The double-scroll orbit is a whole one rather than a pair of coexisting ones, which reside only in a connected basin of attraction, and thus a specific pattern for lattice construction is found.
3. The amplitude of the sequence from the double-scroll orbit is thus controlled by the initial condition. Four patterns with the amplitude control are found in the sequences x and y including 0-degree, 90-degree, 45-degree and 135-degree.
- (4) Circuit implementation based on STM32 are built to verify the compound lattice dynamics, including attractor self-reproducing and attractor growing.

The left of the paper is organized as, in Section 2, the 2-D hyperchaotic map is proposed with basic analysis including the stability of fixed points and bifurcation. In Section 3, offset boosting-dominated lattice dynamics is exhaustively studied, including double-scroll orbit forming and attractor growing. In Section 4, STM32-based circuit realization is set up for further verification. The conclusion is presented in the last section.

2. A 2-d Hyperchaotic Map And Basic Analysis

2.1 Mathematical model

Inspired by periodic function-triggered offset boosting [13], the following 2D discrete hyperchaotic map is constructed:

$$\begin{cases} x_{n+1} = -x_n + a\sin(x_n + y_n) \\ y_{n+1} = -y_n + b\sin(x_n), \end{cases}$$

1

Where n is a natural number, and the n -th iteration states are represented by x_n and y_n respectively. System parameters are represented by a, b giving various dynamics. System (1) is of inversion symmetry since it is invariant under the transformation $x \rightarrow -x, y \rightarrow -y$. When $a = 3, b = -1.6$ and $(x_0, y_0) = (1, 1)$, map (1) has hyperchaotic solutions with two positive Lyapunov exponents (0.2506, 0.07074). Basic phase orbits are plotted as shown in Fig. 1.

2.2 Fixed Points

The fixed points of map (1) are denoted as $F = (x_e, y_e)$ [25–26]. The Jacobian matrix of the map (1) at the fixed points can be written as

$$J = \begin{bmatrix} -1 + a\cos(x_e + y_e) & a\cos(x_e + y_e) \\ b\cos(x_e) & -1 \end{bmatrix}$$

2

$$\det(\lambda E - J) = \lambda^2 - \text{tr}(J)\lambda + \det(J) = 0 \quad (3)$$

The characteristic equation can be derived,

$$(\lambda + 1)^2 - a(\lambda + 1)\cos(x + y) - ab\cos x \cos(x + y) = 0$$

Therefore the eigenvalues can be solved as,

$$R_1 = \frac{\sqrt{a^2 - 4x^2}}{2} + \frac{\sqrt{a^2 - 4x^2} + 4\sqrt{a^2 - 4x^2}\sqrt{b^2 - 4y^2}}{2} \quad (5)$$

$$R_2 = \frac{\sqrt{a^2 - 4x^2}}{2} - \frac{\sqrt{a^2 - 4x^2} + 4\sqrt{a^2 - 4x^2}\sqrt{b^2 - 4y^2}}{2} \quad (6)$$

when $a = 3$, and b varies in $[1, 1.7]$, $\lambda_1 > 1$, $\lambda_2 \leq 1$, fixed points are unstable.

2.3 Bifurcation analysis

More solutions can be found by bifurcation analysis. When $a = 3$, and b varies in $[-1.7, 1.7]$, the map (1) undergoes symmetric dynamical evolvments including period, chaos, and hyperchaos [27–30], as shown in Fig. 2. Interestingly, symmetric Lyapunov exponents are accompanied by asymmetric bifurcation, which is caused by the five orbit types from different initial conditions. When b varies in $[-1.7, -1.494]$, map (1) exhibits hyperchaos; when b is in $[-1.44, -1.332]$, map (1) drops in chaos; when b increases further, map (1) gives periodic solutions. In the positive region, when b increases in $[0, 1.7]$, map (1) outputs the above solutions in a symmetric evolvment. Two chaotic orbits are found under different Lyapunov exponents. Four types of dynamical behaviors in map (1) are depicted in Table 1, corresponding typical phase trajectories are shown in Fig. 3.

Table 1
Dynamical behavior of map (1) $a = 3$ and $(x_0, y_0) = (1, 1)$.

b	Attractor type	Lyapunov exponents
– 1.1	Quasi-periodicity	0, – 0.4644
– 1.32	Quasi-periodicity	0, – 0.2735
– 1.37	Chaos	0.065, – 0.025
– 1.427	Periodic points	– 0.002, – 0.02692
– 1.544	Chaos	0.1837, 0
– 1.60	Hyperchaos	0.2506, 0.07074

3. Offset Boosting-dominated Lattice Dynamics

3.1 Coexisting bifurcations

Two periodic functions are prone to produce lattice of coexisting orbits. Here it was found that a single initial condition extracts out a symmetric attractor characterized with the structure of inversion symmetry, which are the elements for attractor self-reproducing. For example, when $a = 3$, $b = -1.6$, there are totally five of inversion symmetry according to x -dimension, A: attractor self-symmetry, B: 0 degree, C: 90 degrees, D: 45 degrees, and E: 135 degrees, typical phase orbits under a couple of different initial values are plotted as shown in Fig. 4(a). Here five patterns of coexisting phase trajectories are collected, which share unified Lyapunov exponents are shown in Fig. 4(b) [31–33]. Any attractor in its basin can be controlled to evolve with a unified bifurcation process, as shown in Fig. 5.

3.2 Four patterns of initial-controlled double-scroll orbit

Unlike those coexisting attractors in continuous system, here coexisting orbits of any pattern can be visited by initial condition. As shown in Fig. 6 and Fig. 7, with the increase of x_0 , and y_0 , hyperchaotic orbits are visited in the x -dimension and y -dimension. Correspondingly, the sequences with different offsets can be obtained according to the initial condition. Four patterns of the double-scroll hyperchaotic orbits are summarized as:

Pattern I: 0-degree with initial condition $IC = (1 + 2k\pi, 1)$;

Pattern II: 90-degree with initial condition $IC = (1, 1 + 2l\pi)$;

Pattern III: 45-degree with initial condition $IC = (1 + 2k\pi, 1 + 2l\pi)$;

Pattern IV: 135-degree with initial condition $IC = (1 - 2k\pi, 1 + 2l\pi)$;

where k, l are integers.

The frequency of the sequence from a single attractor is lower than that of the sequence from a symmetric pair of attractors. Note that, here the attractor with two pairs is not the same as symmetric pair of coexisting attractors in continuous system. Here the double-scroll hyperchaotic orbits is a whole one rather than a pair of coexisting ones. Continuously being visited phase orbits can also be indicated by a scissor-mouth-like bifurcation diagram, as plotted in Fig. 8. Here the distance between two petals in an attractor is controlled by the initial condition.

3.3 Attractor growing

Periodic sinusoidal function gives the chance of attractor self-reproducing and also makes the system easy to exhibit attractor growing. Four coexisting modes for attractor growing are found as 0-degree, 90-degree, 45-degree, 135-degree. When $a = 2$, $b = -1.999$, attractor growing in the y -dimension, as shown in Fig. 9. Mode I: 0-degree with initial condition $IC = (1 + 2k\pi, 1)$, where the turquoise attractor grows simultaneously in both positive and negative y -dimension, purple double-scroll attractors, one growing along the positive y -axis and the other along the negative y -axis, and gold double-scroll attractors, one growing along the positive y -axis while the other grows along the negative y -axis.

When $a = 3$, $b = -1.7623$, different regimes of attractor growing show up, as plotted in Fig. 10. Mode I: 0-degree with initial condition $IC = (1 + 2k\pi, 1)$, as plotted in Fig. 10(a)-(b), where the turquoise attractor grows in the second and fourth quadrants at the same time, purple double-scroll attractors, one of which grows in the second quadrant and the other in the fourth, and gold double-scroll attractors, the first grows in the second quadrant, whereas the second grows in the fourth quadrant. Mode II: 90-degree with initial condition $IC = (1 + 2/\pi, 1)$, as plotted in Fig. 10(c)-(d). An arrow of the corresponding color indicates the attractor's growth direction.

4. Stm32-based Hardware Implementation

The STM32F103 microcontroller is employed as the core in this section to implement chaotic map capture and transmission, and for digital-to-analog conversion, the TLV5618 module is employed. On the oscilloscope of the SDS1102X, the necessary waveforms and phase trajectories are displayed. To begin, two arrays for variable calculations are defined. Second, after initializing the first element of the array, iterative calculations are done, with the elements of the array being updated according to other discrete equations. Scaling up is used to determine the array's values. Through the MOSI transmission line of SPI, the chaotic signal is sent to the TLV5618 module. The TLV5618 module, as a dual-channel digital-to-analog converter, can transform the gathered chaotic digital signal into an analog voltage and pass it to the oscilloscope's x/y axis signal input through two separate channels A and B.

5. Conclusions

In this paper, a novel hyperchaotic map is derived by introducing sinusoidal functions. Due to the unique structure, the hyperchaotic map outputs a uniformly distributed lattice in x and y dimensions. The emergence of offset-boosting gives different types of double-scroll hyperchaotic orbits. The double-scroll orbits are arranged in different directions under different initial conditions, namely 0-degree, 90-degree, 45-degree and 135-degree, which makes a specific style for lattice construction. By this way, the initial conditions control the amplitude of the sequence from the double-scroll orbit. Finally, based on the STM32 platform, the specific lattice dynamics are proven including initial-controlled amplitude and attractor growing. How to apply this new proposed hyperchaotic map in security systems is the promising future work [34–38].

Declarations

Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Acknowledgements

This work was supported financially by the National Natural Science Foundation of China (Grant No.: 61871230), and the Provincial Key Training Programs of Innovation and Entrepreneurship for Undergraduates (Grant No.: 201910300053Z).

Conflict of interest

We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Figures

Figure 1

Phase orbit of map (1) with $a = 3$, $b = -1.6$ and $(x_0, y_0) = (1, 1)$: (a) phase orbit, (b) discrete sequences.

Figure 2

Dynamical evolution of map (1) with $a = 3$ when b varies in $[-1.7, 1.7]$ under $(x_0, y_0) = (1, 1)$: (a) Lyapunov exponents, (b) bifurcation diagram.

Figure 3

Typical phase trajectories of map (1) with $a = 3$ and $(x_0, y_0) = (1, 1)$: (a) $b = -1.1$, (b) $b = -1.32$, (c) $b = -1.37$, (d) $b = -1.427$, (e) $b = -1.544$, (f) $b = -1.6$.

Figure 4

Lattice of hyperchaos in map (1) with $a = 3$, $b = -1.6$. Here green: initial condition (IC) = $(1, 1)$, red: IC = $(1 + 2\pi, 1)$, magenta: IC = $(1, 1 + 2\pi)$, cyan: IC = $(1 + 2\pi, 1 + 2\pi)$, yellow: IC = $(1 - 2\pi, 1 + 2\pi)$: (a) hyperchaotic orbits, (b) unified Lyapunov exponents.

Figure 5

Coexisting bifurcations in map (1) with $b = -1.6$ and a varies in $[1, 3]$: (a) $(x_0, y_0) = (x_0, 1)$, (b) $(x_0, y_0) = (1, y_0)$.

Figure 6

Coexisting orbits in map (1) with $a = 3$, $b = -1.6$: (a) Pattern I: turquoise: initial condition (IC) = $(1, 1)$, purple: IC = $(1 + 2\pi, 1)$, gold: IC = $(1 + 4\pi, 1)$, (b) Pattern II: springgreen: IC = $(1, 1)$, hotpink: IC = $(1, 1 + 2\pi)$, skyblue: IC = $(1, 1 + 4\pi)$.

Figure 7

Coexisting orbits in map (1) with $a = 3$, $b = -1.6$: (a) Pattern III: green: initial condition (IC) = $(1, 1)$, magenta: IC = $(1 + 2\pi, 1 + 2\pi)$, cyan: IC = $(1 + 4\pi, 1 + 4\pi)$, (b) Pattern IV: forestgreen: IC = $(1, 1)$, brown: IC = $(1 - 2\pi, 1 + 2\pi)$, blue: IC = $(1 - 4\pi, 1 + 4\pi)$.

Figure 8

Bifurcation diagrams of initial conditions in map (1) with $a = 3$, $b = -1.6$: (a) $(x_0, y_0) = (x_0, 1)$, (b) $(x_0, y_0) = (1, y_0)$.

Figure 9

Coexisting phase orbits of attractor growing in map (1) with $a = 2$, $b = -1.999$, Mode I: where turquoise: IC = $(1, 1)$, purple: IC = $(1 + 2\pi, 1)$, gold: IC = $(1 + 4\pi, 1)$.

Figure 10

Coexisting phase orbits of attractor growing in map (1) with $a = 3$, $b = -1.764$, (a) Mode I: turquoise: initial condition (IC) = $(1, 1)$, purple: IC = $(1 + 2\pi, 1)$, gold: IC = $(1 + 4\pi, 1)$, (b) Mode II: springgreen: IC = $(1, 1)$, hotpink: IC = $(1, 1 + 2\pi)$, skyblue: IC = $(1, 1 + 4\pi)$.

Figure 11

The hardware implementation based on STM32 for map (1) with $a = 3$, $b = -1.6$ and $IC = (1 + 2\pi, 1 + 2\pi)$.

Figure 12

The waveform diagram of hardware-implemented in map (1) with $a = 3$ and $(x_0, y_0) = (1, 1)$: (a) $b = -1.37$, (b) $b = -1.6$.

Figure 13

Typical hyperchaotic phase trajectories of map (1) from oscilloscope with $a = 3$, $b = -1.6$: (a) $IC = (1 + 2\pi, 1)$, (b) $IC = (1, 1 + 2\pi)$, (c) where yellow: $IC = (1 + 2\pi, 1 + 2\pi)$, magenta: $IC = (1, 1)$, (d) $IC = (1 - 2\pi, 1 + 2\pi)$.

Figure 14

Coexisting orbits of map (1) from oscilloscope with $a = 3$, $b = -1.6$: (a) turquoise: initial condition (IC) = $(1, 1)$, purple: IC = $(1 + 2\pi, 1)$, gold: IC = $(1 + 4\pi, 1)$, (b) springgreen: IC = $(1, 1)$, hotpink: IC = $(1, 1 + 2\pi)$, skyblue: IC = $(1, 1 + 4\pi)$.

Figure 15

Lattice of hyperchaos of map (1) from oscilloscope with $a = 3$, $b = -1.6$. Here green: initial condition (IC) = $(1, 1)$, red: IC = $(1 + 2\pi, 1)$, magenta: IC = $(1, 1 + 2\pi)$, cyan: IC = $(1 + 2\pi, 1 + 2\pi)$, yellow: IC = $(1 - 2\pi, 1 + 2\pi)$.

Figure 16

Attractor growing of map (1) from oscilloscope with $(x_0, y_0) = (1, 1)$: (a) $a = 2$, $b = -1.999$, (b) $a = 3$, $b = -1.764$.