Degeneration of Solitons for a the (3+1)-dimensional Generalized Nonlinear Evolution Equation for the Shallow Water Waves

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Degeneration of solitons for a the (3+1)-dimensional generalized nonlinear evolution equation for the shallow water waves

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Abstract A the (3+1)-dimensional generalized nonlinear evolution equation for the shallow water waves is investigated with different methods. Based on symbolic computation and Hirota bilinear form, N -soliton solutions are constructed. In the process of degeneration of N -soliton solutions, T -breathers are derived by taking complexication method. Then rogue waves will emerge during the degeneration of breathers by taking the parameter limit method. Through full degeneration of N -soliton, M -lump solutions are derived based on long wave limit approach. In addition, we also find out that the partial degeneration of N -soliton process can generate the hybrid solutions composed of soliton, breather and lump.

Keywords Shallow-water waves· T -breather· M -lump· Hybrid solution

1 Introduction

Shallow water waves refer to the water waves with large water wavelength relative to the depth [1], also called the long water waves. There are some natural problems are related to the shallow water waves, such as the tides, storms, atmospheric flows and tsunamis [2,3]. Nonlinear evolution equations (NLEEs) describe lots of physical and science phenomena, are fundamental models in nonlinear science, especially in nonlinear dynamics. The wave phenomenon of NLEEs are playing an essential role in studying various areas of physics and mathematics, such as nonlinear optics, plasmas, oceanography and Bose-Einstein condensates [4-9]. Solitons, breathers, lumps and rogue waves are typical wave models to study nonlinear scientific issues [10,11,12]. Lump wave is a special kind of rational function wave with the characteristics of energy concentration and localization property in the space [13-18]. In the train of the study, a series of methods has been quickly developed to obtain lump wave including the first mathematical description of lump wave [19], the long wave limit method is an effective method proposed to study the M -lump solutions of the integrable systems in [20-24], this positive method has attracted a lot of researchers' attention. the direct method [25-30], the parameter limit method [31-34], complexication method [35,36], bilinear neural network method [37], etc. Through these methods, so many valuable and interesting results have sprung

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up. Inspired by the previous studies, we mainly focus on the (3+1)-dimensional generalized nonlinear evolution equation for the shallow water waves to study the evolution and degeneration behaviour of N -solitons in this work. By applying the Horita bilinear method [34] and other systematic method, the structures of solutions for this equation are analyzed.

In this paper, we mainly investigate the (3+1)-dimensional generalized nonlinear evolution equation for the shallow water waves [35,36], the equation reads

$$l_1 u_{xz} + (l_2 u_t + l_3 u_{xxx} + l_4 u u_x)_y + l_5 (u_x \partial_x^{-1} u_y)_x = 0 \quad (1)$$

where $u(x, y, z, t)$ is a real differential function of the independent variables x, y, z and t , ∂_x^{-1} represents the integral with respect to x , $l_i (i = 1, 2, 3, 4, 5)$ are real nonzero constants. Eq.(1) is also a KdV-type equation which can be regard as a model for the incompressible [37].

Based on the previous researches, we mainly investigate the deformation behaviors of N -soliton, most of solutions obtained for Eq.(1) are new and may be very meaningful to explain some special nonlinear science phenomenon. In Section 2, N -soliton solution for Eq.(1) is given and T -breathers are derived. M -lumps are obtained by long wave limit method in Section 3. Hybrid solutions consisted of solitons and breathers or lumps are obtained in Section 4. A brief conclusion is given in Section 5 finally.

2 From N -soliton solutions to T -order breather solutions

In the following, we will obtain N -soliton solutions for Eq.(1) by Hirota bilinear method, and according to taking the conjugate complex of the parameters of the N -soliton ($N = 2T$) solutions, T -breather solutions also can be obtained. Firstly, we take the transformation

$$u(x, y, z, t) = \frac{6l_3}{l_4} (\ln f)_{xx}, \quad (2)$$

and

$$l_4 = l_5,$$

where f is an unknown function about x, y, z, t . Eq.(1) can be converted to the bilinear operator D as follows

$$(l_1 D_x D_z + l_2 D_y D_t + l_3 D_x^3 D_y) f \cdot f = 0 \quad (3)$$

The Hirota bilinear operator $D_x^m D_t^n$ are defined by [31] ($n, m \geq 0$).

$$D_x^l D_y^m D_z^n D_t^r f(x, t) \cdot g(x, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^r f(x, y, z, t) g(x', y', z', t')|_{x'=x, y'=y, z'=z, t'=t}.$$

On the basis of [30,31], we choose the following test functions

$$f = f_N = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^N \mu_i \eta_i + \sum_{i<j}^N A_{ij} \mu_i \mu_j\right) \quad (4)$$

where $\eta_i = k_i(x + p_i y + q_i y + r_i t) + \eta_i^{(0)}$, $k_i \neq 0$, $p_i \neq 0$, $q_i \neq 0$, $\eta_i^{(0)}$ ($i = 1, 2, 3, \dots$) are arbitrary real constants. The first $\sum$ is the summation of all possible combinations of $\mu_1 = 0, 1$, $\mu_2 = 0, 1, \dots$, $\mu_N = 0, 1$. Another $\sum$ means over all possible combinations of the N parts with the particular form $i < j$. According to Eq.(3) and (4), we get

$$\begin{cases} r_i = \frac{-l_3 k_i^2 p_i - l_1 q_i}{l_2 p_i}, \\ e^{A_{ij}} = \frac{-3l_3 k_i k_j (p_i + p_j) + l_3 k_i^2 (2p_i + p_j) + l_3 k_j^2 (p_i + 2p_j) - l_2 (p_i - p_j)(r_i - r_j)}{3l_3 k_i k_j (p_i + p_j) + l_3 k_i^2 (2p_i + p_j) + l_3 k_j^2 (p_i + 2p_j) - l_2 (p_i - p_j)(r_i - r_j)}. \end{cases} \quad (5)$$

Here N is a positive integer, the positive constant $e^{A_{ij}}$ is not necessary component of the soliton, but it is very convenient to the calculation of breathers.

Generally, the function expressed in Eq.(4) can not generate a breather solution of Eq.(1) if only one of above constants takes a complex number. However, when taking the paired complex value of above real parameters, smooth breathers of Eq.(1) can be yielded on a zero background. This idea named the paired complex valuation of solitons has a long history which goes back to the works for few decades ago [6,8,9]. We also note that there are some constraints on $e^{A_{ij}}$ to guarantee reality and smoothness of the breathers generated by the expression in Eq.(4).

Based on the idea of taking complex value mentioned above, we take

$$\begin{cases} k_{2j-1} = k_{j1} + ik_{j2}, k_{2j} = k_{j1} - ik_{j2}, \\ p_{2j-1} = p_{j1} + ip_{j2}, p_{2j} = p_{j1} - ip_{j2}, \\ q_{2j-1} = q_{j1} + iq_{j2}, q_{2j} = q_{j1} - iq_{j2}, \\ \eta_j^{(0)} = 0, j = 1, 2, \dots, \frac{N}{2}. \end{cases} \quad (6)$$

$k_{j1}, k_{j2}, p_{j1}, p_{j2}, q_{j1}, q_{j2}$ are real constants and N is even. Now Eq.(4) will be rewritten as

$$f = f_{2T} = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{2T} \mu_i \eta_i + \sum_{i < j}^{2T} A_{ij} \mu_i \mu_j\right) \quad (7)$$

Inserting Eq.(6) into Eq.(7), then

$$\begin{aligned} \eta_{2T-1} &= \eta_{2T}^* = \eta_{j1} + i\eta_{j2}, \\ \eta_{j1} &= k_{j1}x + (k_{j1}p_{j1} - k_{j2}p_{j2})y + (k_{j1}q_{j1} - k_{j2}q_{j2})z + \left(\frac{l_3}{l_2}k_{j1}(3k_{j2}^2 - k_{j1}^2) + \frac{l_1}{l_2} \frac{k_{j2}(p_{j1}q_{j2} - p_{j2}q_{j1}) - k_{j1}(p_{j1}q_{j1} + p_{j2}q_{j2})}{p_{j1}^2 + p_{j2}^2}\right)t, \\ \eta_{j2} &= k_{j2}x + (k_{j2}p_{j1} + k_{j1}p_{j2})y + (k_{j2}q_{j1} + k_{j1}q_{j2})z + \left(\frac{l_3}{l_2}k_{j2}(k_{j2}^2 - 3k_{j1}^2) + \frac{l_1}{l_2} \frac{k_{j1}(p_{j2}q_{j1} - p_{j1}q_{j2}) - k_{j2}(p_{j1}q_{j1} + p_{j2}q_{j2})}{p_{j1}^2 + p_{j2}^2}\right)t, \\ j &= 1, 2, \dots, T. \end{aligned} \quad (8)$$

Case 1. T=1 A 1-order breather $u_b^{[1]} = \frac{6l_3}{l_4}(\ln f_b^{[1]})_{xx}$ is constructed by

$$f_b^{[1]} = 1 + 2e^{\eta_{11}} \cos(\eta_{12}) + e^{A_{12}} e^{2\eta_{11}} \quad (9)$$

and

$$e^{A_{12}} = \frac{3l_3 k_{12}^2 p_{11}(p_{11}^2 + p_{12}^2) + l_3 k_{11} k_{12} p_{12}(p_{11}^2 + p_{12}^2) + l_1 p_{12}(p_{11} q_{12} - p_{12} q_{11})}{-3l_3 k_{11}^2 p_{11}(p_{11}^2 + p_{12}^2) + l_3 k_{11} k_{12} p_{12}(p_{11}^2 + p_{12}^2) + l_1 p_{12}(p_{11} q_{12} - p_{12} q_{11})} > 0.$$

η_{11} and η_{12} are real component and imaginary component of η_1 respectively. For a given time t , the trajectory of 1-order breather is a line l_1 on (x, y) -plane, $l_1: \eta_{11} = k_{11}x + (k_{11}p_{11} - k_{12}p_{12})y + (k_{11}q_{11} - k_{12}q_{12})z + (\frac{l_3}{l_2}k_{11}(3k_{12}^2 - k_{11}^2) + \frac{l_1}{l_2} \frac{k_{12}(p_{11}q_{12} - p_{12}q_{11}) - k_{11}(p_{11}q_{11} + p_{12}q_{12})}{p_{11}^2 + p_{12}^2})t = 0$, and periodic along the direction of the line l_2 on (x, y) -plane, $l_2: \eta_{12} = k_{12}x + (k_{12}p_{11} + k_{11}p_{12})y + (k_{12}q_{11} + k_{11}q_{12})z + (\frac{l_3}{l_2}k_{12}(k_{12}^2 - 3k_{11}^2) + \frac{l_1}{l_2} \frac{k_{11}(p_{12}q_{11} - p_{11}q_{12}) - k_{12}(p_{11}q_{11} + p_{12}q_{12})}{p_{11}^2 + p_{12}^2})t = 0$. That is to say, the location depends on k_{11} and $k_{11}p_{11} - k_{12}p_{12}$, the period depends on k_{12} and $k_{12}p_{11} + k_{11}p_{12}$ on (x, y) -plane. Three forms of 1-order breather for Eq.(1) are constructed with different values of k_{11} , k_{12} , p_{11} and p_{12} in Eq.(9). When taking $k_{11} = 0$ and $k_{11}p_{11} - k_{12}p_{12} \neq 0$, Fig.1(a) displays the 1-order breathers are parallel to the x -axis on the (x, y) -plane as $k_1 = \frac{1}{6}i, k_2 = -\frac{1}{6}i, p_1 = \frac{1}{4} + i, p_2 = \frac{1}{4} - i, q_1 = \frac{1}{10} + i, q_2 = \frac{1}{10} - i$. When taking $k_{11} \neq 0$ and $k_{11}p_{11} - k_{12}p_{12} = 0$, $k_1 = \frac{1}{4} + i, k_2 = \frac{1}{4} - i, p_1 = \frac{1}{6} + \frac{1}{24}i, p_2 = \frac{1}{6} - \frac{1}{24}i, q_1 = \frac{1}{10} + i, q_2 = \frac{1}{10} - i$, the 1-order breathers are parallel to the y -axis shown in Fig.1(b). When taking $k_{11} \neq 0$ and $k_{11}p_{11} - k_{12}p_{12} \neq 0$, the 1-order breathers are parallel to the the line $k_{11}x + (k_{11}p_{11} - k_{12}p_{12})y = 0$ shown in Fig.1(c) with $k_1 = \frac{1}{4} + i, k_2 = \frac{1}{4} - i, p_1 = \frac{1}{6} + \frac{1}{6}i, p_2 = \frac{1}{6} - \frac{1}{6}i, q_1 = \frac{1}{10} + i, q_2 = \frac{1}{10} - i$.

Fig. 1: Different spatial structures of 1-order breather $u_b^{[1]}$ as $l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0$.

Case 2. T=2 A 2-order breather $u_b^{[2]} = \frac{6l_3}{l_4} (\ln f_b^{[2]})_{xx}$ is constructed by

$$f_b^{[2]} = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_4} + e^{A_{12}} e^{\eta_1 + \eta_2} + e^{A_{13}} e^{\eta_1 + \eta_3} + e^{A_{14}} e^{\eta_1 + \eta_4} + e^{A_{23}} e^{\eta_2 + \eta_3} + e^{A_{24}} e^{\eta_2 + \eta_4} + e^{A_{34}} e^{\eta_3 + \eta_4} \\ + e^{A_{12}} e^{A_{13}} e^{A_{23}} e^{\eta_1 + \eta_2 + \eta_3} + e^{A_{12}} e^{A_{14}} e^{A_{24}} e^{\eta_1 + \eta_2 + \eta_4} + e^{A_{13}} e^{A_{14}} e^{A_{34}} e^{\eta_1 + \eta_3 + \eta_4} + e^{A_{23}} e^{A_{24}} e^{A_{34}} e^{\eta_2 + \eta_3 + \eta_4} \\ + e^{A_{12}} e^{A_{13}} e^{A_{14}} e^{A_{23}} e^{A_{24}} e^{A_{34}} e^{\eta_1 + \eta_2 + \eta_3 + \eta_4}. \quad (10)$$

with $\eta_1 = \eta_2^* = \eta_{11} + \eta_{12}i$, $\eta_3 = \eta_4^* = \eta_{21} + \eta_{22}i$, $e^{A_{ij}} = \frac{-3l_3 k_i k_j (p_i + p_j) + l_3 k_i^2 (2p_i + p_j) + l_3 k_j^2 (p_i + 2p_j) - l_2 (p_i - p_j)(r_i - r_j)}{3l_3 k_i k_j (p_i + p_j) + l_3 k_i^2 (2p_i + p_j) + l_3 k_j^2 (p_i + 2p_j) - l_2 (p_i - p_j)(r_i - r_j)}$, $r_i = \frac{-l_3 k_i^2 p_i - l_1 q_i}{l_2 p_i}$, and the definitions in Eq.(6) and Eq.(8). When $\eta_1 \neq \eta_3$, $l_1 = 2$, $l_2 = 1$, $l_3 = 1$, $l_4 = 3$, $q_1 = q_2 = q_3 = q_4 = -3$, $z = 0$, $t = 0$, the breathers are parallel pattern (Fig.2(a)), and the intersectant pattern is shown in Fig.2(b). When $\eta_1 = \eta_3$, the breather degenerates to 1-order breather (Fig.2(c)).

Fig. 2: The 2-order breather $u_b^{[2]}$ as (a) parallel pattern with $k_1 = \frac{1}{6} + \frac{1}{6}i$, $k_2 = \frac{1}{6} - \frac{1}{6}i$, $k_3 = \frac{1}{4} + \frac{1}{4}i$, $k_4 = \frac{1}{4} - \frac{1}{4}i$, $p_1 = \frac{1}{2} + i$, $p_2 = \frac{1}{2} - i$, $p_3 = \frac{1}{2} + i$, $p_4 = \frac{1}{2} - i$, (b) intersectant pattern with $k_1 = 1 - 2i$, $k_2 = 1 + 2i$, $k_3 = 2 - i$, $k_4 = 2 + i$, $p_1 = -2 + \frac{5}{2}i$, $p_2 = -2 - \frac{5}{2}i$, $p_3 = -\frac{1}{2} + 2i$, $p_4 = -\frac{1}{2} - 2i$, (c) degenerated pattern with $k_1 = \frac{1}{6} + \frac{1}{6}i$, $k_2 = \frac{1}{6} - \frac{1}{6}i$, $k_3 = \frac{1}{6} + \frac{1}{6}i$, $k_4 = \frac{1}{6} - \frac{1}{6}i$, $p_1 = \frac{1}{2} + i$, $p_2 = \frac{1}{2} - i$, $p_3 = \frac{1}{2} + i$, $p_4 = \frac{1}{2} - i$.

Case 3. T=3 The 3-order breathers with different shapes are shown in Fig.3 with $l_1 = 2$, $l_2 = 1$, $l_3 = 1$, $l_4 = 3$, $q_1 = q_2 = q_3 = q_4 = -3$, $z = 0$, $t = 0$. In case of $\eta_1 \neq \eta_3 \neq \eta_5$. An intersectant pattern of breather is displayed in Fig.3(a). In case of $\eta_1 = \eta_3 = \eta_5$, $p_1 \neq p_3 \neq p_5$, a mixed pattern of breather is displayed in Fig.3(b); while a parallel pattern of breather with appropriate value is displayed in Fig.3(c). In case of $\eta_1 = \eta_3 = \eta_5$, $p_1 = p_3 = p_5$, the 3-order breather degenerate to a 1-order breather completely

(Fig.3(d)). From Fig.3, it means that the parameters in Eq.(6) affect and control the period, amplitude and phase of the breather, the interaction behaviours involved high-order breathers are elastic.

Fig. 3: The 3-order breather $u_b^{[3]}$ as (a) intersectant pattern with $k_1 = 1 - 2i, k_2 = 1 + 2i, k_3 = 1 - i, k_4 = 1 + i, k_5 = 3i, k_6 = -3i, p_1 = -2 + \frac{5}{2}i, p_2 = -2 - \frac{5}{2}i, p_3 = -\frac{1}{2} + 2i, p_4 = \frac{1}{2} - i, p_5 = -5 + i, p_6 = -5 - i$, (b) mixed pattern with $k_1 = 1 - \frac{3}{2}i, k_2 = 1 + \frac{3}{2}i, k_3 = 1 - \frac{3}{2}i, k_4 = 1 + \frac{3}{2}i, k_5 = 1 - \frac{3}{2}i, k_6 = 1 + \frac{3}{2}i, p_1 = -\frac{1}{2} + i, p_2 = -\frac{1}{2} - i, p_3 = -\frac{1}{2} + \frac{3}{2}i, p_4 = -\frac{1}{2} - \frac{3}{2}i, p_5 = -2 + 4i, p_6 = -2 - 4i$, (c) parallel pattern with $k_1 = 1 - \frac{3}{2}i, k_2 = 1 + \frac{3}{2}i, k_3 = 1 - \frac{3}{2}i, k_4 = 1 + \frac{3}{2}i, k_5 = 1 - \frac{3}{2}i, k_6 = 1 + \frac{3}{2}i, p_1 = -\frac{1}{2} + i, p_2 = -\frac{1}{2} - i, p_3 = -\frac{1}{2} + \frac{3}{2}i, p_4 = -\frac{1}{2} - \frac{3}{2}i, p_5 = -\frac{1}{2} + \frac{7}{4}i, p_6 = -\frac{1}{2} - \frac{7}{4}i$, (d) degenerated pattern with $k_1 = 1 - \frac{3}{2}i, k_2 = 1 + \frac{3}{2}i, k_3 = 1 - \frac{3}{2}i, k_4 = 1 + \frac{3}{2}i, k_5 = 1 - \frac{3}{2}i, k_6 = 1 + \frac{3}{2}i, p_1 = -\frac{1}{2} + i, p_2 = -\frac{1}{2} - i, p_3 = -\frac{1}{2} + i, p_4 = -\frac{1}{2} - i, p_5 = -\frac{1}{2} + i, p_6 = -\frac{1}{2} - i$.

3 Evolution and degeneration from soliton to Lump solution

It is known that rogue wave or lump-type wave can be generated from breather by degeneration with parameter limit method [12,24]. By referring many studies, we also note that high-order lump can be generated in the process of degeneration of breather by long wave limit method. This section is consisted of two subsections, rogue wave or lump-type solution is obtained in subsection 3.1, M -lump is constructed in subsection 3.2.

3.1 Degeneration of breather and emergence of rogue wave or lump-type solution

Under the bilinear form in Eq.(3), we choose a new test function which is first applied to Eq.(1)

$$f(x, y, z, t) = \delta_1 \cosh \phi_1 + \delta_2 \cos \phi_2 + \delta_3 \cosh \phi_3 \quad (11)$$

where $\phi_i = k_i(a_i x + b_i y + c_i z + d_i t + \gamma_i)$, $k_i, a_i, b_i, c_i, d_i, \gamma_i (i = 1, 2, 3)$ are free real parameters to be

determined. Substituting Eq.(11) into Eq.(3) with computer algebra software Mathematic 12, we get two sets of parameter relations:

Case 1:

$$\begin{cases} a_1 = \frac{a_3 k_3}{k_1}, b_1 = 0, c_1 = 0, d_1 = \frac{d_3 k_3}{k_1}, \\ a_2 = 0, b_2 = -\frac{l_1 a_3 c_2}{l_2 d_3 + l_3 a_3^2 k_3^2}, c_2 = c_2, d_2 = 0, \\ a_3 = a_3, b_3 = 0, c_3 = 0, d_3 = d_3. \end{cases} \quad (12)$$

where $\delta_i k_i l_i \neq 0$ ($i = 1, 2, 3$). A multi-breather solitary solution is constructed by inserting Eq.(11) and Eq.(12) into Eq.(3) as follows

$$u(x, y, z, t) = \frac{6l_3}{l_4} \left(\frac{a_3^2 k_3^2 \delta_1 \cosh \phi_1 + a_3^2 k_3^2 \delta_3 \cosh \phi_3}{\delta_1 \cosh \phi_1 + \delta_2 \cos \phi_2 + \delta_3 \cosh \phi_3} - \frac{(a_3 k_3 \delta_1 \sinh \phi_1 + a_3 k_3 \delta_3 \sinh \phi_3)^2}{(\delta_1 \cosh \phi_1 + \delta_2 \cos \phi_2 + \delta_3 \cosh \phi_3)^2} \right) \quad (13)$$

where

$$\begin{cases} \phi_1 = k_1 \left(\frac{a_3 k_3}{k_1} x + \frac{d_3 k_3}{k_1} t + \gamma_1 \right), \\ \phi_2 = k_2 \left(-\frac{l_1 a_3 c_2}{l_2 d_3 + l_3 a_3^2 k_3^2} y + c_2 t + \gamma_2 \right), \\ \phi_3 = k_3 (a_3 x + d_3 t + \gamma_3). \end{cases} \quad (14)$$

The solution u expressed in Eq.(13) is a kind of double breather wave solution, which takes on the characteristics of double solitary waves with the variable ϕ_1 and ϕ_3 , and the characteristics of periodic solitary waves with the variable ϕ_2 . Besides, under the action of the function $\cos \phi_2$, u also shows the characteristics of local oscillation. Therefore, the breather wave solutions with these characteristics are named double breathers [38,39]. The oscillation behaviour of double breathers are presented in Fig.4(a). In order to construct the rogue wave or lump-type solutions from Eq.(11), the parameters δ_i ($i = 1, 2, 3$) must satisfy:

$$\lim_{k_i \rightarrow 0} (\delta_1 + \delta_2 + \delta_3) = 0 \quad (15)$$

so we take $k_2 = r k_1$, $\delta_1 = \cos(\tilde{i} m k_1)$, $\delta_2 = -2 \cosh(\tilde{i} n k_1)$, $\delta_3 = \cos(\tilde{i} s k_1)$, where $\tilde{i}$ is imaginary number, m, n, r, s are real number. Letting $k_1 \rightarrow 0$, lump solution are given by

$$u(x, y, z, t) = \frac{6l_3}{l_4} \left(\frac{4a_1^2}{m^2 + 2n^2 + s^2 + \theta_1^2 + 2r^2 \theta_2^2 + \frac{a_1^2}{a_3^2} \theta_3^2} - \frac{4(a_1 \theta_1 + \frac{a_1^2}{a_3} \theta_3)^2}{(m^2 + 2n^2 + s^2 + \theta_1^2 + 2r^2 \theta_2^2 + \frac{a_1^2}{a_3^2} \theta_3^2)^2} \right) \quad (16)$$

where

$$\lim_{k_i \rightarrow 0} \frac{\phi_i}{k_i} = \theta_i (i = 1, 2, 3).$$

$\theta_1 = a_1 x + \frac{a_1 d_3}{a_3} t + \gamma_1$, $\theta_2 = -\frac{l_1 l_3 a_3 c_2}{l_2 d_3} y + c_2 z + \gamma_2$, $\theta_3 = a_3 x + d_3 t + \gamma_3$. When $k_1 \rightarrow 0$, the characteristics of period and breather of solution Eq.(13) are degenerated as a rational function solution in Eq.(16). Besides, the solution u in Eq.(16) goes to 0 when $|x|$, $|y|$, $|z|$, $|t|$ tend to ∞ in any direction, and shows the characteristic of polynomial attenuation on all spatial variables. Meanwhile, we know that the solution is non-singular as the parameters satisfy $m^2 + 2n^2 + s^2 > 0$ from the expression of Eq.(16). From Fig.4(b),

the lump solution u has a upward peak and a downward valley, the lump solution with this structures is called rogue wave or lump-type solution [15,17].

Fig. 4: (a) Spatial structure of double breather in Eq.(13) as $l_1 = 2, l_2 = 1, l_3 = 2, l_4 = 12, \delta_1 = 2, \delta_2 = 4, \delta_3 = 5, k_1 = \frac{1}{4}, k_2 = 10, k_3 = \frac{1}{2}, a_3 = \frac{1}{4}, c_2 = 1, d_3 = 10, \gamma_i = 0 (i = 1, 2, 3), z = 10, t = 0$; (b) Spatial structure of lump wave in Eq.(16) as $l_1 = 1, l_2 = 1, l_3 = 1, l_4 = 12, a_3 = d_3 = r = 1, c_2 = -1, z = 1, t = 0$.

Case 2:

$$\begin{aligned} \delta_3 &= 0, a_2 = 0, b_1 = \frac{a_1 c_1 l_1 (\delta_2^2 - \delta_1^2)}{(l_2 d_1 + 4 l_3 a_1^3 k_1^2) \delta_1^2 - (l_2 d_1 + l_3 a_1^3 k_1^2) \delta_2^2}, \\ d_2 &= \frac{3 k_1^4 l_1 l_3 a_1^4 c_1 \delta_1^2}{l_2 k_2^2 b_2 (l_2 d_1 + 4 l_3 a_1^3 k_1^2) \delta_1^2 - (l_2 d_1 + l_3 a_1^3 k_1^2) \delta_2^2}, \\ c_2 &= -\frac{b_2 (l_2 d_1 + l_3 k_1^2 a_1^3)}{l_1 a_1} - \frac{3 \delta_1^2 k_1^4 l_1 l_3 a_1^4 c_1^2 (\delta_2^2 - \delta_1^2)}{k_2^2 b_2 ((l_2 d_1 + 4 l_3 a_1^3 k_1^2) \delta_1^2 - (l_2 d_1 + l_3 a_1^3 k_1^2) \delta_2^2)^2}, \end{aligned} \quad (17)$$

so we get

$$f(x, y, z, t) = \delta_1 \cosh(k_1(a_1 x + b_1 y + c_1 z + d_1 t + \gamma_1)) + \delta_2 \cos(k_2(b_2 y + c_2 z + d_2 t + \gamma_2)) \quad (18)$$

An exact one breather-wave solution of Eq.(1) is obtained by combining Eq.(17), (18) and (2), the spatial structure is shown in Fig.5(a).

Similarly, we take $k_2 = r k_1, \delta_1 = \cos(\imath l k_1), \delta_2 = -\cosh(\imath s k_1), \imath$ is imaginary number, and let $k_1 \rightarrow 0$. New type lump solution is obtained (Fig.5(b)).

$$u(x, y, z, t) = \frac{6 l_3}{l_4} \left(\frac{2 a_1^2}{\zeta_1^2 + r^2 \zeta_2^2 + l^2 + s^2} - \frac{(2 a_1 \zeta_1)^2}{(\zeta_1^2 + r^2 \zeta_2^2 + l^2 + s^2)^2} \right). \quad (19)$$

where $\zeta_1 = a_1 x - \frac{l_1 a_1 c_1 (l^2 + s^2)}{3 l_3 a_1^3 + l_2 d_1 (l^2 + s^2)} y + c_1 z + d_1 t + \gamma_1, \zeta_2 = b_2 y - \left(\frac{l_2 b_2 d_1}{l_1 a_1} + \frac{3 l_1 l_3 a_1^4 c_1^2 (l^2 + s^2)}{r^2 b_2 (3 l_3 a_1^3 + l_2 d_1 (l^2 + s^2))} \right) z + \frac{3 l_1 l_3 a_1^4 c_1}{r^2 l_2 b_2 (3 l_3 a_1^3 + l_2 d_1 (l^2 + s^2))} t + \gamma_2$ and $l^2 + s^2 > 0$.

Fig. 5: (a) Spatial structure of breather in Eq.(18) as $l_1 = 1, l_2 = 1, l_3 = 1, l_4 = 12, \delta_1 = \frac{1}{5}, \delta_2 = \frac{1}{10}, k_1 = 1, k_2 = 2, a_1 = \frac{1}{8}, b_2 = 4, c_1 = \frac{1}{2}, c_2 = 1, d_1 = 10, \gamma_i = 0 (i = 1, 2, 3), z = 1, t = 0$; (b) Spatial structure of lump wave in Eq.(19) as $l_1 = 1, l_2 = 1, l_3 = 1, l_4 = 12, a_1 = r = 1, c_2 = -1, d_1 = \frac{1}{6}, l = s = \frac{3\sqrt{2}}{2}, z = 1, t = 0$.

3.2 Full degeneration of N -solitons to M -lump solutions

In this subsection, we study M -lump solutions of Eq.(1) by long wave limit method. In order to obtain M -lump ($N = 2M$) solutions, we need take each $\exp(\eta_i^{(0)}) = -1$, $k_i/k_j = O(1)$, $p_i = O(1)$, $q_i = O(1)$ and $k_i \rightarrow 0$ in Eq.(4) and (5). Then we get

$$f = f_N = \sum_{\mu=0,1} \prod_{i=1}^N (-1)^{\mu_i} (1 + \mu_i k_i \theta_i) \prod_{i < j}^{(N)} (1 + \mu_i \mu_j k_i k_j B_{ij}) + O(k^{N+1}). \quad (20)$$

For the sake of simplification, we omit constant factor $\prod_{i=1}^N k_i$ in f_N and get the simplified form as follows:

$$\begin{aligned} f_l^{[M]} = & \prod_{i=1}^N \theta_i + \frac{1}{2} \sum_{i,j}^N B_{ij} \prod_{l \neq i,j}^N \theta_l + \frac{1}{2!2^2} \sum_{i,j,r,s}^N B_{ij} B_{sr} \prod_{l \neq i,j,s,r}^N \theta_l + \dots \\ & + \frac{1}{M!2^M} \sum_{i,j,\dots,m,n}^N \overbrace{B_{ij} B_{kl} \dots B_{mn}}^M \prod_{p \neq i,j,k,l,\dots,m,n}^N \theta_p + \dots \end{aligned} \quad (21)$$

where

$$\theta_i = x + p_i y + q_i z - \frac{l_1 q_i}{l_2 p_i} t, B_{ij} = \frac{6 l_3 p_i p_j (p_i + p_j)}{l_1 (p_i - p_j) (p_i q_j - p_j q_i)}.$$

If we take $p_{M+i} = p_i^*$, $q_{M+i} = q_i^*$ with the condition $B_{ij} > 0$, a class of M -lump solutions can be constructed.

When $N = 2$, the expression f in Eq.(21) is written as

$$f_2 = \theta_1 \theta_2 + B_{12},$$

$$B_{12} = \frac{6l_3 p_1 p_2 (p_1 + p_2)}{l_1 (p_1 - p_2) (p_1 q_2 - p_2 q_1)}. \quad (22)$$

Taking $p_1 = p_2^* = a_1 + b_1 i$, $q_1 = q_2^* = c_1 + d_1 i$, the 1-lump $u_l^{[1]} = \frac{6l_3}{l_4} (\ln f_l^{[1]})_{xx}$ with the $f_l^{[1]}$ as (see Fig.6(a))

$$f_l^{[1]} = (x' + a_1 y' + c_1 z' - \frac{l_1(a_1 c_1 + b_1 d_1)}{l_2(a_1^2 + b_1^2)} t')^2 + (b_1 y' + d_1 z' - \frac{l_1(a_1 d_1 - b_1 c_1)}{l_2(a_1^2 + b_1^2)} t')^2 + \Gamma_3.$$

where

$$\begin{aligned} x' &= x - \frac{l_1(a_1 c_1 + b_1 d_1)}{l_2(a_1^2 + b_1^2)} t, \\ y' &= y + \frac{l_1 c_1}{l_2(a_1^2 + b_1^2)} t, \\ z' &= z - \frac{l_1 a_1}{l_2(a_1^2 + b_1^2)} t, \\ \Gamma_3 &= -\frac{l_1(a_1 c_1 + b_1 d_1)}{l_2(a_1^2 + b_1^2)} > 0. \end{aligned}$$

When $a_1 \neq 0$, the trajectory can be defined along the path $(x(t), y(t))$ as follows

$$\begin{cases} x + a_1 y + c_1 z + \Gamma_1 t = \pm \sqrt{\Gamma_3}, \\ b_1 y + d_1 z + \Gamma_2 t = 0. \end{cases} \quad (23)$$

with

$$\begin{aligned} \Gamma_1 &= -\frac{l_1(a_1 c_1 + b_1 d_1)}{l_2(a_1^2 + b_1^2)}, \\ \Gamma_2 &= -\frac{l_1(a_1 d_1 - b_1 c_1)}{l_2(a_1^2 + b_1^2)}. \end{aligned}$$

On (x, y) -plane, when $z = 0$ the lump reaches the $u_{max} = \frac{4b_1(a_1 d_1 - b_1 c_1)}{3l_3 a_1(a_1^2 + b_1^2)}$ at the point $(-\frac{l_1(a_1^2 d_1 - 2a_1 b_1 c_1 + b_1^2 d_1)t}{l_2 b_1(a_1^2 + b_1^2)}, \frac{l_1(a_1 d_1 - b_1 c_1)t}{l_2 b_1(a_1^2 + b_1^2)})$, and reaches the $u_{min} = \frac{4l_1 b_1(a_1 d_1 - b_1 c_1)(1 + 3a_1 b_1^2 l_3 + 3a_1^3 l_3)}{3l_3 a_1(a_1^2 + b_1^2)(-1 + 3a_1 b_1^2 l_3 + 3a_1^3 l_3)}$ at $(-\frac{l_1(a_1^2 d_1 - 2a_1 b_1 c_1 + b_1^2 d_1)t}{l_2 b_1(a_1^2 + b_1^2)} - \frac{3l_3 a_1(a_1^2 + b_1^2)}{\sqrt{l_1 b_1(b_1 c_1 - a_1 d_1)}}, \frac{l_1(a_1 d_1 - b_1 c_1)t}{l_2 b_1(a_1^2 + b_1^2)})$ and $(-\frac{l_1(a_1^2 d_1 - 2a_1 b_1 c_1 + b_1^2 d_1)t}{l_2 b_1(a_1^2 + b_1^2)} + \frac{3l_3 a_1(a_1^2 + b_1^2)}{\sqrt{l_1 b_1(b_1 c_1 - a_1 d_1)}}, \frac{l_1(a_1 d_1 - b_1 c_1)t}{l_2 b_1(a_1^2 + b_1^2)})$.

When $N = 4$, Eq.(21) can be reduced to the following expression:

$$f_l^{[2]} = \theta_1 \theta_2 \theta_3 \theta_4 + B_{12} \theta_3 \theta_4 + B_{13} \theta_2 \theta_4 + B_{14} \theta_2 \theta_3 + B_{23} \theta_1 \theta_4 + B_{24} \theta_1 \theta_3 + B_{34} \theta_1 \theta_2 + B_{12} B_{34} + B_{13} B_{24} + B_{14} B_{23}, \quad (24)$$

where

$$\theta_i = x + p_i y + q_i z - \frac{l_1 q_i}{l_2 p_i} t, B_{ij} = \frac{6l_3 p_i p_j (p_i + p_j)}{l_1 (p_i - p_j) (p_i q_j - p_j q_i)}.$$

2-lump (Fig.6(b)) will be obtained by taking $p_1 = p_3^* = a_1 + b_1 i$, $p_2 = p_4^* = a_2 + b_2 i$, $q_1 = q_3^* = c_1 + d_1 i$, $q_2 = q_4^* = c_2 + d_2 i$.

When $N = 6$, taking $p_1 = p_4^* = a_1 + b_1 i$, $p_2 = p_5^* = a_2 + b_2 i$, $p_3 = p_6^* = a_3 + b_3 i$, $q_1 = q_4^* = c_1 + d_1 i$, $q_2 = q_5^* = c_2 + d_2 i$, $q_3 = q_6^* = c_3 + d_3 i$ into Eq.(21), we obtain 3-lump solution (Fig.6(c)).

Therefore, we find that a M -lump can be degenerated from a N -order soliton. From Fig.6, the lump waves include both bright waves and dark waves, and the shapes will change obviously with energy transformation between them as no elastic interaction occurs, and the different parameter values are

$l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0$, (a) $p_1 = -\frac{1}{2} + i, p_2 = -\frac{1}{2} - i, q_1 = -1 + 3i, q_2 = -1 - 3i$. (b) $p_1 = -\frac{1}{2} + i, p_2 = -\frac{1}{10} - i, p_3 = -\frac{1}{2} - i, p_4 = -\frac{1}{10} + i, q_1 = -1 + 3i, q_2 = -5 - 8i, q_3 = -1 - 3i, q_4 = -5 + 8i$. (c) $p_1 = -\frac{1}{2} + i, p_2 = -\frac{1}{10} - i, p_3 = \frac{1}{6} + \frac{1}{6}i, p_4 = -\frac{1}{2} + i, p_5 = -\frac{1}{10} + i, p_6 = \frac{1}{6} - \frac{1}{6}i, q_1 = -1 + 3i, q_2 = -5 - 8i, q_3 = \frac{1}{3} + \frac{1}{2}i, q_4 = -1 - 3i, q_5 = -5 + 8i, q_6 = \frac{1}{3} - \frac{1}{2}i$.

Fig. 6: Spatial evolution behavior of M -lump $u_t^{[M]}$ with the increasing number (a) $M = 1$, (b) $M = 2$, (c) $M = 3$.

4 The hybrid solutions

In this section, abundant hybrid solutions and dynamics behaviours can be obtained during partial degeneration of N -soliton. Several kinds of interaction solutions consisting of breathers and solitons, lump waves and solitons are discussed based on above sections.

4.1 $(2H + K)$ -order hybrid solutions

Interaction solutions composed of breathers and solitons, we take the parameters in Eq.(4) as

$$N = 2H + K, k_j = k_{H+j}^* = k_{j1} + ik_{j2}, p_j = p_{H+j}^* = p_{j1} + ip_{j2}, j = 1, 2, \dots, H;$$

$$k_t = k_t, p_t = p_t, t = 2H + 1, 2H + 2, \dots, 2H + K, q_i = q_i, \eta_i^{(0)} = 0, i = 1, 2, \dots, 2H + K.$$

k_t, p_t and q_i are real numbers. Interaction solutions for Eq.(1) between H -breathers and K -soliton are constructed.

When $N = 3$, interaction solutions between 1-breather and 1-soliton are constructed with

$$k_1 = k_2^* = k_{11} + ik_{12}, k_3 = k_3, p_1 = p_2^* = p_{11} + ip_{12}, p_3 = p_3, q_i = q_i, i = 1, 2, 3. \quad (25)$$

As it is shown in Fig.7(a), the 1-breather and 1-soliton are intersected and keep colliding with each other on the (x, y) -plane with $l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0, k_1 = \frac{1}{2} + i, k_2 = \frac{1}{2} - i, k_3 = 1, p_1 = 1 + \frac{1}{2}i, p_2 = 1 - \frac{1}{2}i, p_3 = 1, q_1 = q_2 = q_3 = -1$.

When $N = 4$, interaction solutions between 1-breather and 2-soliton are constructed with

$$k_1 = k_2^* = k_{11} + ik_{12}, k_3 = k_3, k_4 = k_4, p_1 = p_2^* = p_{11} + ip_{12}, p_3 = p_3, p_4 = p_4, q_i = q_i, i = 1, 2, 3, 4. \quad (26)$$

As it is shown in Fig.8(a), the 1-breather and 2-soliton are interacting with each other on the (x, y) -plane with $l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0, k_1 = \frac{1}{2} + i, k_2 = \frac{1}{2} - i, k_3 = 1, k_4 = 1, p_1 = 1 + \frac{1}{2}i, p_2 = 1 - \frac{1}{2}i, p_3 = 2, p_4 = \frac{3}{4}, q_1 = q_2 = q_3 = q_4 = -1$.

Similarly, taking $N = 5$:

Case I. The 1-breather and 3-soliton are interacting with each other on the (x, y) -plane (Fig.9(a)) with

$$\begin{aligned} k_1 = k_2^* = k_{11} + ik_{12}, k_3 = k_3, k_4 = k_4, k_5 = k_5, p_1 = p_2^* = p_{11} + ip_{12}, \\ p_3 = p_3, p_4 = p_4, p_5 = p_5, q_i = q_i, i = 1, 2, 3, 4, 5. \end{aligned} \quad (27)$$

specific parameters taking

$$\begin{aligned} l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0, k_1 = \frac{1}{2} + i, k_2 = \frac{1}{2} - i, k_3 = k_4 = k_5 = 1, \\ p_1 = 1 + \frac{1}{2}i, p_2 = 1 - \frac{1}{2}i, p_3 = 2, p_4 = \frac{3}{4}, p_5 = \frac{1}{4}, q_1 = q_2 = q_3 = q_4 = q_5 = -1. \end{aligned}$$

Case II. The interaction between 2-breather and 1-soliton on the (x, y) -plane is shown in Fig.10(a) with

$$\begin{aligned} k_1 = k_2^* = k_{11} + ik_{12}, k_3 = k_4^* = k_{21} + ik_{22}, k_5 = k_5, p_1 = p_2^* = p_{11} + ip_{12}, \\ p_3 = p_4^* = p_{21} + ip_{22}, p_5 = p_5, q_1 = q_2^* = q_{11} + iq_{12}, q_3 = q_4^* = q_{21} + iq_{22}, q_5 = q_5. \end{aligned} \quad (28)$$

and

$$\begin{aligned} l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0, k_1 = \frac{1}{2} + i, k_2 = \frac{1}{2} - i, k_3 = \frac{1}{2} + i, k_4 = \frac{1}{2} - i, k_5 = 1, p_1 = 1 + \frac{1}{2}i, \\ p_2 = 1 - \frac{1}{2}i, p_3 = 1 + i, p_4 = 1 - i, p_5 = -\frac{1}{2}, q_1 = -\frac{1}{2} + i, q_2 = -\frac{1}{2} - i, q_3 = -\frac{1}{2} + i, q_4 = -\frac{1}{2} - i, q_5 = -\frac{1}{2}. \end{aligned}$$

According to the characteristic of the figures and propagating mechanism, above interactions are elastic.

4.2 $(2H + L)$ -order hybrid solutions

Interaction solutions composed of H -lump and L -solitons for Eq.(1) by taking the parameters in Eq.(4) as follows:

$$N = 2H + L, k_i = k_i, e^{\eta_i^{(0)}} = -1, k_i \rightarrow 0, i = 1, 2, \dots, 2H. \quad (29)$$

Firstly, setting $N = 3, H = 1, L = 1$, elastic interaction solution between 1-lump and 1-soliton is displayed in Fig.7(b) with

$$f_3 = \theta_1 \theta_2 + B_{12} + (\theta_1 \theta_2 + B_{12} + B_{23} \theta_1 + B_{13} \theta_2 + B_{13} B_{23}) e^{\eta_3}. \quad (30)$$

and

$$l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0, k_1 = \frac{1}{2} + i, k_2 = \frac{1}{2} - i, k_3 = 1,$$

$$p_1 = 1 + \frac{1}{2}i, p_2 = 1 - \frac{1}{2}i, p_3 = \frac{1}{20}, q_1 = q_2 = q_3 = -3.$$

Secondly, letting $N = 4$, $H = 1$, $L = 2$, elastic interaction solution between 1-lump and 2-soliton is displayed in Fig.8(b) with

$$f_4 = \theta_1\theta_2 + B_{12} + (\theta_1\theta_2 + B_{12} + B_{23}\theta_1 + B_{13}\theta_2 + B_{13}B_{23})e^{\eta_3} + (\theta_1\theta_2 + B_{12} + B_{24}\theta_1 + B_{14}\theta_2 + B_{14}B_{24})e^{\eta_4} \\ + (B_{12} + (B_{13} + B_{14} + \theta_1)(B_{23} + B_{24} + \theta_2))e^{\eta_3 + \eta_4 + A_{34}}. \quad (31)$$

and

$$l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0, k_1 = \frac{1}{2} + i, k_2 = \frac{1}{2} - i, k_3 = k_4 = 1, \\ p_1 = 1 + \frac{1}{2}i, p_2 = 1 - \frac{1}{2}i, p_3 = \frac{1}{20}, p_4 = 1, q_1 = q_2 = -3, q_3 = q_4 = -1.$$

Fig.8(c) depicts the elastic collision between 1-lump and 1-breather with

$$l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0, k_1 = i, k_2 = -i, k_3 = i, k_4 = -i, \\ p_1 = \frac{1}{2} + 2i, p_2 = \frac{1}{2} - 2i, p_3 = 1 + i, p_4 = 1 - i, q_1 = q_2 = q_3 = q_4 = -3.$$

When $N = 5$, $H = 1$, $L = 3$, with $l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0, k_1 = \frac{1}{2} + i, k_2 = \frac{1}{2} - i, k_3 = k_4 = k_5 = 1, p_1 = 1 + \frac{1}{2}i, p_2 = 1 - \frac{1}{2}i, p_3 = 3, p_4 = \frac{3}{4}, p_5 = \frac{1}{8}, q_1 = q_2 = -1, q_3 = 3, q_4 = q_5 = -1$, elastic interaction solution between 1-lump and 3-soliton is displayed in Fig.9(b).

Finally, elastic interaction solution between 2-lump and 1-soliton is displayed in Fig.10(b) with $l_1 = 2, l_2 = 1, l_3 = 1, l_4 = 3, z = 0, t = 0, p_1 = \frac{1}{2} + i, p_2 = -\frac{1}{10} - i, p_3 = \frac{1}{2} - i, p_4 = -\frac{1}{10} + i, p_5 = -\frac{1}{2}, q_1 = -1 + 3i, q_2 = -5 - 8i, q_3 = -1 - 3i, q_4 = -5 + 8i, q_5 = -\frac{1}{2}$.

Fig. 7: (a) Interaction phenomenon between 1-breather and 1-soliton, (b) Collision between 1-lump and 1-soliton.

Fig. 8: Different interaction phenomenon including (a) 1-breather and 2-soliton, (b) 1-lump and 2-soliton, (c) 1-lump and 1-breather.

Fig. 9: Two kinds of interaction phenomenon including (a) 1-breather and 3-soliton, (b) 1-lump and 3-soliton.

5 Conclusion

As the high-dimensional NLEEs can describe many real-world phenomenon more accurately, so the study of high-dimensional NLEEs has attracted great interest of mathematical physics researchers. In this manuscript, N -soliton solutions are constructed based on symbolic computation and Hirota bilinear form. In the process of degeneration of N -soliton solutions, T -breathers ($T = 1, 2, 3, \dots$) are derived by taking complexification method, Fig.1,2,3 depict the specific shape and dynamic behaviors, and the collision of breathers is elastic. Then rogue waves expressed in Eq.(16) and (19) are found out during the degeneration of breathers by taking the parameter limit method, Fig.4 and 5 show the process of the evolution. Through full degeneration of N -soliton, M -lump ($M = \frac{N}{2} = 1, 2, 3, \dots$) solutions are derived

Fig. 10: Interaction solutions composed of (a) 2-breather and 1-soliton, (b) 2-lump and 1-soliton.

based on long wave limit approach as shown in Fig.6. If the N -soliton degenerates partially, the abundant hybrid solutions composed of soliton, breather and lump can be constructed, such as interaction between breather and soliton shown in the first plot of Fig.7,8,9,10, interaction between lump and soliton shown in the second plot of Fig.7,8,9,10, interaction between lump and breather shown in Fig.8(c). The dynamic properties of the solutions have been analyzed by taking appropriate parameters. This work greatly enrich the dynamic behavior of Eq.(1), and the methods used to obtain new exact solutions also can be extended to solve other nonlinear partial differential equations. For some important classical mathematical and physical models, the methods of studying solutions are very significant.

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Declarations

Conflict of interest The authors declare that there are no conflicts of interests with publication of this work.

Ethical standards The authors ensure the compliance with ethical standards for this work.

Data availability statement

The authors declare that data supporting the findings of this study are available within the article, the figures are concrete expression.

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