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Research Article

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The spatially homogeneous Hopf bifurcation induced jointly by memory and general delays in a diffusive system

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Abstract

In this paper, by incorporating the general delay to the reaction term in the memory-based diffusive system, we propose a diffusive system with memory delay and general delay (e.g., gestation, hunting, migration and maturation delays, etc.). We first derive an algorithm for calculating the normal form of Hopf bifurcation in a diffusive system with memory and general delays. The developed algorithm for calculating the normal form can be used to investigate the direction and stability of Hopf bifurcation. Then, we consider a diffusive predator-prey model with ratio-dependent Holling type-III functional response, which includes with memory and gestation delays. The Hopf bifurcation analysis without considering gestation delay is first studied, then the Hopf bifurcation analysis with memory and gestation delays is studied. Furthermore, by using the developed algorithm for calculating the normal form, the supercritical and stable spatially homogeneous periodic solutions induced jointly by memory and general delays are found theoretically. The stable spatially homogeneous periodic solutions are also found by the numerical simulations which confirms our analytic result.

Keywords: Memory-based diffusion, Memory delay, General delay, Hopf bifurcation, Normal form, Periodic solution

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1. Introduction

In many mathematical modeling of specific disciplines, such as physics, chemistry and biology [1, 2, 3], the reaction-diffusion equations have been widely used. In general, the reaction-diffusion equations are based on the Fick's law, that is the movement flux is in the direction of negative gradient of the density distribution function [4]. The diffusion term based on the Fick's law is usually called as the random diffusion driven by inherent mechanism. Moreover, the diffusion-advection systems have been studied by many scholars, such as the chemotaxis model [5, 6, 7, 8, 9], the predator-prey model with prey-taxis [10, 11, 12, 13, 14], the predator-prey model with indirect prey-taxis [15, 16, 17], the competition-diffusion-advection model in the river environment [18, 19, 20] and the reaction-diffusion-advection population model with delay in reaction term [21]. However, the animal movements are different from the chemical movements, especially for highly developed animals, because they can even remember the historic distribution or clusters of the species in space. Therefore, in order to include the episodic-like spatial memory of animals, Shi et al. [4] proposed a modified Fick's law that in addition to the negative gradient of the density distribution function

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at the present time, there is a directed movement toward the negative or positive gradient of the density distribution function at past time, and they proposed the following diffusive model with spatial memory

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) + d_2 (u(x,t)u_x(x,t-\tau))_x + f(u(x,t)), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}}(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,t) = u_0(x,t), & x \in \Omega, -\tau \leq t \leq 0, \end{cases} \quad (1.1)$$

where $u(x,t)$ is the population density at the spatial location x and at time t , d_1 and d_2 are the Fickian diffusion coefficient and the memory-based diffusion coefficient, respectively, $\Omega \subset \mathbb{R}$ is a smooth and bounded domain, $u_0(x,t)$ is the initial function, $\Delta u(x,t) = \partial^2 u(x,t)/\partial x^2$, $u_x(x,t) = \partial u(x,t)/\partial x$, $u_x(x,t-\tau) = \partial u(x,t-\tau)/\partial x$, $u_{xx}(x,t-\tau) = \partial^2 u(x,t-\tau)/\partial x^2$, and $\mathbf{n}$ is the outward unit normal vector at the smooth boundary $\partial\Omega$. Here, the time delay $\tau > 0$ represents the averaged memory period, which is usually called as the memory delay, and $f(u(x,t))$ describes the chemical reaction or biological birth/death. Notice that such movement is based on the memory (or history) of a particular past time density distribution. However, by the stability analysis, they found that the stability of the positive constant steady state fully depends on the relationship between the diffusion coefficients d_1 and d_2 , but is independent of the memory delay. In order to further investigate the influence of memory delay on the stability of the positive constant steady state, Shi et al. [22] studied the spatial memory diffusion model with memory and maturation delays

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) + d_2 (u(x,t)u_x(x,t-\tau))_x + f(u(x,t), u(x,t-\sigma)), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}}(x,t) = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where $\sigma > 0$ is the maturation delay. They found that memory-based diffusion with memory and maturation delays can induce more complicated spatiotemporal dynamics, such as spatially homogeneous and inhomogeneous periodic solutions.

By introducing the non-local effect to the memory-based diffusive system (1.1), Song et al. [23] proposed the single population model with memory-based diffusion and non-local interaction

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) + d_2 (u(x,t)u_x(x,t-\tau))_x + f(u(x,t), \hat{u}), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}}(x,t) = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where $\Omega = (0, \ell\pi)$ with $\ell \in \mathbb{R}^+$, $\hat{u} = (1/\ell\pi) \int_0^{\ell\pi} u(y,t) dy$. Many complicated spatiotemporal dynamics are found, such as the stable spatially homogeneous or inhomogeneous periodic solutions, homogeneous or inhomogeneous steady states, the transition from one of these solutions to another, and the coexistence of two stable spatially inhomogeneous steady states or two spatially inhomogeneous periodic solutions near the Turing-Hopf bifurcation point. Recently, for the single-species model with spatial memory, Song et al. [24] studied the memory-based movement with spatiotemporal distributed delays in diffusion and reaction terms.

In addition, Song et al. [25] considered the following resource-consumer model with random and memory-

40 based diffusions

$$\left\{ \begin{array}{ll} \frac{\partial u(x, t)}{\partial t} = d_{11} \Delta u(x, t) + f(u(x, t), v(x, t)), & x \in \Omega, t > 0, \\ \frac{\partial v(x, t)}{\partial t} = d_{22} \Delta v(x, t) - d_{21} (v(x, t) u_x(x, t - \tau))_x + g(u(x, t), v(x, t)), & x \in \Omega, t > 0, \\ u_x(0, t) = u_x(\ell\pi, t) = v_x(0, t) = v_x(\ell\pi, t) = 0, & t \geq 0, \\ u(x, t) = u_0(x, t), & x \in \Omega, -\tau \leq t \leq 0, \\ v(x, t) = v_0(x), & x \in \Omega, \end{array} \right. \quad (1.2)$$

where $u(x, t)$ and $v(x, t)$ are the densities of resource and consumer, respectively, $d_{11} \geq 0$ and $d_{22} \geq 0$ are the random diffusion coefficients, $d_{21} \geq 0$ is the memory-based diffusion coefficient, $v_0(x)$ is also the initial function, and $f(u(x, t), v(x, t))$ and $g(u(x, t), v(x, t))$ are the reaction terms. The well-posedness of solutions is studied, and the rich dynamics of the system (1.2) with Holling type-I or type-II functional responses are found. Notice that by comparing with the classical reaction-diffusion systems with delay, the system (1.2) has the two main differences, one is that the memory delay appears in the diffusion term, another is that the diffusion term is nonlinear. Thus, the normal form for Hopf bifurcation in the classical reaction-diffusion systems is not suitable for the system (1.2). Recently, Song et al. [26] developed an algorithm for calculating the normal form of Hopf bifurcation in the system (1.2), and they studied the direction and stability of Hopf bifurcation by using their newly developed algorithm for calculating the normal form. The existences of stable spatially inhomogeneous periodic solutions and the transition from one unstable spatially inhomogeneous periodic solution to another stable spatially inhomogeneous periodic solution are found.

For the chemical reaction models, it is practical to consider the influence of the time delay caused by gene expression. The Brusselator model with gene expression delay has been studied in [27]. For the artificial neural networks, it is practical to consider the influence of the time delay caused by leakage delay. The delay-dependent stability of neutral neural networks with leakage term delays has been studied in [28]. For the biology model, especially for the predator-prey model, the gestation, hunting, migration and maturation delays are usually considered [29, 30, 31], and in this paper, we call these delays as the general delays. By incorporating the general delay to the reaction term in the memory-based diffusive system, we propose the following diffusive system with memory and general delays

$$\left\{ \begin{array}{ll} \frac{\partial u(x, t)}{\partial t} = d_{11} \Delta u(x, t) + f(u(x, t), v(x, t), u(x, t - \tau), v(x, t - \tau)), & x \in (0, \ell\pi), t > 0, \\ \frac{\partial v(x, t)}{\partial t} = d_{22} \Delta v(x, t) - d_{21} (v(x, t) u_x(x, t - \tau))_x + g(u(x, t), v(x, t), u(x, t - \tau), v(x, t - \tau)), & x \in (0, \ell\pi), t > 0, \\ u_x(0, t) = u_x(\ell\pi, t) = v_x(0, t) = v_x(\ell\pi, t) = 0, & t \geq 0, \\ u(x, t) = u_0(x, t), v(x, t) = v_0(x, t), & x \in (0, \ell\pi), -\tau \leq t \leq 0, \end{array} \right. \quad (1.3)$$

where $v_0(x, t)$ is the initial function.

More precisely, the paper is divided into the following five sections. In Section 2, we derive an algorithm for calculating the normal form of Hopf bifurcation induced jointly by memory and general delays. In Section 3, we obtain the normal form of Hopf bifurcation truncated to the third-order term by using the algorithm developed in Sec.2, and we give the detail calculation process of its corresponding coefficients. In Section

4, we consider a diffusive predator-prey model with ratio-dependent Holling type-III functional response, which includes with memory and gestation delays. Then we give the detail Hopf bifurcation analysis for two cases, i.e., with memory delay and without gestation delay, and with memory and gestation delays. Furthermore, we study the direction and stability of Hopf bifurcation corresponding to the above two cases. Finally, we give a brief conclusion and discussion in Section 5.

2. Algorithm for calculating the normal form of Hopf bifurcation induced jointly by memory and general delays

2.1. Characteristic equation at the positive constant steady state

Define the real-valued Sobolev space

$$X := \left\{ (u, v)^T \in (W^{2,2}(0, \ell\pi))^2 : \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \text{ at } x = 0, \ell\pi \right\}$$

with the inner product defined by

$$[U_1, U_2] = \int_0^{\ell\pi} U_1^T U_2 \, dx \text{ for } U_1 = (u_1, v_1)^T \in X \text{ and } U_2 = (u_2, v_2)^T \in X,$$

where the symbol T represents the transpose of vector, and let $\mathcal{C} := C([-1, 0]; X)$ be the Banach space of continuous mappings from $[-1, 0]$ to X with the sup norm. It is well known that the eigenvalue problem

$$\begin{cases} \tilde{\varphi}''(x) = \tilde{\lambda} \tilde{\varphi}(x), & x \in (0, \ell\pi), \\ \tilde{\varphi}'(0) = \tilde{\varphi}'(\ell\pi) = 0 \end{cases}$$

has eigenvalues $\tilde{\lambda}_n = -n^2/\ell^2$ with corresponding normalized eigenfunctions

$$\beta_n^{(j)} = \gamma_n(x) e_j, \quad \gamma_n(x) = \frac{\cos(nx/\ell)}{\|\cos(nx/\ell)\|_{L^2}} = \begin{cases} \frac{1}{\sqrt{\ell\pi}}, & n = 0, \\ \frac{\sqrt{2}}{\sqrt{\ell\pi}} \cos\left(\frac{nx}{\ell}\right), & n \geq 1, \end{cases} \quad (2.1)$$

where e_j , $j = 1, 2$ is the unit coordinate vector of $\mathbb{R}^2$, and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is often called wave number, $\mathbb{N}_0$ is the set of all non-negative integers, $\mathbb{N} = \{1, 2, \dots\}$ represents the set of all positive integers.

Without loss of generality, we assume that $E_*(u_*, v_*)$ is the positive constant steady state of system (1.3). The linearized equation of (1.3) at $E_*(u_*, v_*)$ is

$$\begin{pmatrix} \frac{\partial u(x, t)}{\partial t} \\ \frac{\partial v(x, t)}{\partial t} \end{pmatrix} = D_1 \begin{pmatrix} \Delta u(x, t) \\ \Delta v(x, t) \end{pmatrix} + D_2 \begin{pmatrix} \Delta u(x, t - \tau) \\ \Delta v(x, t - \tau) \end{pmatrix} + A_1 \begin{pmatrix} u(x, t) \\ v(x, t) \end{pmatrix} + A_2 \begin{pmatrix} u(x, t - \tau) \\ v(x, t - \tau) \end{pmatrix}, \quad (2.2)$$

where

$$D_1 = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad (2.3)$$

and

$$\begin{aligned} a_{11} &= \frac{\partial f(u_*, v_*)}{\partial u(x, t)}, \quad a_{12} = \frac{\partial f(u_*, v_*)}{\partial v(x, t)}, \quad a_{21} = \frac{\partial g(u_*, v_*)}{\partial u(x, t)}, \quad a_{22} = \frac{\partial g(u_*, v_*)}{\partial v(x, t)}, \\ b_{11} &= \frac{\partial f(u_*, v_*)}{\partial u(x, t - \tau)}, \quad b_{12} = \frac{\partial f(u_*, v_*)}{\partial v(x, t - \tau)}, \quad b_{21} = \frac{\partial g(u_*, v_*)}{\partial u(x, t - \tau)}, \quad b_{22} = \frac{\partial g(u_*, v_*)}{\partial v(x, t - \tau)}. \end{aligned} \quad (2.4)$$

Therefore, the characteristic equation of system (2.2) is

$$\prod_{n \in \mathbb{N}_0} \Gamma_n(\lambda) = 0,$$

where $\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda))$ with

$$\mathcal{M}_n(\lambda) = \lambda I_2 + \frac{n^2}{\ell^2} D_1 + \frac{n^2}{\ell^2} e^{-\lambda\tau} D_2 - A_1 - A_2 e^{-\lambda\tau}. \quad (2.5)$$

Here, $\det(\cdot)$ represents the determinant of a matrix, I_2 is the identity matrix of 2×2 , and D_1, D_2, A_1, A_2 are defined by (2.3). Then we obtain

$$\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda)) = \lambda^2 - T_n \lambda + \tilde{J}_n(\tau) = 0, \quad (2.6)$$

where

$$\begin{aligned} T_n &= (a_{11} + a_{22}) - (d_{11} + d_{22}) \frac{n^2}{\ell^2}, \\ \tilde{J}_n(\tau) &= d_{11} d_{22} \frac{n^4}{\ell^4} - (d_{11} a_{22} + d_{22} a_{11} + (d_{11} b_{22} + d_{22} b_{11} + d_{21} a_{12} v_*) e^{-\lambda\tau} + d_{21} b_{12} v_* e^{-2\lambda\tau}) \frac{n^2}{\ell^2} \\ &\quad + (a_{11} b_{22} + a_{22} b_{11} - a_{12} b_{21} - a_{21} b_{12}) e^{-\lambda\tau} - (b_{11} + b_{22}) \lambda e^{-\lambda\tau} + (b_{11} b_{22} - b_{12} b_{21}) e^{-2\lambda\tau} \\ &\quad + a_{11} a_{22} - a_{12} a_{21}. \end{aligned} \quad (2.7)$$

2.2. Basic assumption and equation transformation

Assumption 2.1. Assume that at $\tau = \tau_c$, (2.6) has a pair of purely imaginary roots $\pm i\omega_{n_c}$ with $\omega_{n_c} > 0$ for $n = n_c \in \mathbb{N}_0$ and all other eigenvalues have negative real part. Let $\lambda(\tau) = \alpha_1(\tau) \pm i\alpha_2(\tau)$ be a pair of roots of (2.6) near $\tau = \tau_c$ satisfying $\alpha_1(\tau_c) = 0$ and $\alpha_2(\tau_c) = \omega_{n_c}$. In addition, the corresponding transversality condition holds.

Let $\tau = \tau_c + \mu$, $|\mu| \ll 1$ such that $\mu = 0$ corresponds to the Hopf bifurcation value for system (1.3). Moreover, we shift $E_*(u_*, v_*)$ to the origin by setting

$$U(x, t) = (U_1(x, t), U_2(x, t))^T = (u(x, t), v(x, t))^T - (u_*, v_*)^T,$$

and normalize the delay by rescaling the time variable $t \rightarrow t/\tau$. Furthermore, we rewrite $U(t)$ for $U(x, t)$, and $U_t \in \mathcal{C}$ for $U_t(\theta) = U(x, t + \theta)$, $-1 \leq \theta \leq 0$. Then, the system (1.3) becomes the compact form

$$\frac{dU(t)}{dt} = d(\mu)\Delta(U_t) + L(\mu)(U_t) + F(U_t, \mu), \quad (2.8)$$

where for $\varphi = (\varphi^{(1)}, \varphi^{(2)})^T \in \mathcal{C}$, $d(\mu)\Delta$ is given by

$$d(\mu)\Delta(\varphi) = d_0\Delta(\varphi) + F^d(\varphi, \mu)$$

with

$$\begin{aligned} d_0\Delta(\varphi) &= \tau_c D_1 \Delta\varphi(0) + \tau_c D_2 \Delta\varphi(-1), \\ F^d(\varphi, \mu) &= -d_{21}(\tau_c + \mu) \begin{pmatrix} 0 \\ \varphi_x^{(1)}(-1)\varphi_x^{(2)}(0) + \varphi_{xx}^{(1)}(-1)\varphi^{(2)}(0) \end{pmatrix} \\ &\quad + \mu \begin{pmatrix} d_{11}\varphi_{xx}^{(1)}(0) \\ -d_{21}v_*\varphi_{xx}^{(1)}(-1) + d_{22}\varphi_{xx}^{(2)}(0) \end{pmatrix}. \end{aligned} \quad (2.9)$$

Furthermore, $L(\mu) : \mathcal{C} \rightarrow X$ is given by

$$L(\mu)(\varphi) = (\tau_c + \mu) (A_1\varphi(0) + A_2\varphi(-1)), \quad (2.10)$$

and $F : \mathcal{C} \times \mathbb{R}^2 \rightarrow X$ is given by

$$F(\varphi, \mu) = (\tau_c + \mu) \begin{pmatrix} f(\varphi^{(1)}(0) + u_*, \varphi^{(2)}(0) + v_*, \varphi^{(1)}(-1) + u_*, \varphi^{(2)}(-1) + v_*) \\ g(\varphi^{(1)}(0) + u_*, \varphi^{(2)}(0) + v_*, \varphi^{(1)}(-1) + u_*, \varphi^{(2)}(-1) + v_*) \end{pmatrix} - L(\mu)(\varphi). \quad (2.11)$$

In what follows, we assume that $F(\varphi, \mu)$ is $C^k (k \geq 3)$ function, which is smooth with respect to φ and μ . Notice that μ is the perturbation parameter and is treated as a variable in the calculation of normal form. Moreover, from (2.10), if we denote $L_0(\varphi) = \tau_c (A_1\varphi(0) + A_2\varphi(-1))$, then (2.8) can be rewritten as

$$\frac{dU(t)}{dt} = d_0\Delta(U_t) + L_0(U_t) + \tilde{F}(U_t, \mu), \quad (2.12)$$

where the linear and nonlinear terms are separated, and

$$\tilde{F}(\varphi, \mu) = \mu (A_1\varphi(0) + A_2\varphi(-1)) + F(\varphi, \mu) + F^d(\varphi, \mu). \quad (2.13)$$

Thus, the linearized equation of (2.12) can be written as

$$\frac{dU(t)}{dt} = d_0\Delta(U_t) + L_0(U_t). \quad (2.14)$$

Moreover, the characteristic equation for the linearized equation (2.14) is

$$\prod_{n \in \mathbb{N}_0} \tilde{\Gamma}_n(\lambda) = 0, \quad (2.15)$$

where $\tilde{\Gamma}_n(\lambda) = \det(\tilde{\mathcal{M}}_n(\lambda))$ with

$$\tilde{\mathcal{M}}_n(\lambda) = \lambda I_2 + \tau_c \frac{n^2}{\ell^2} D_1 + \tau_c \frac{n^2}{\ell^2} e^{-\lambda} D_2 - \tau_c A_1 - \tau_c A_2 e^{-\lambda}. \quad (2.16)$$

By comparing (2.16) with (2.5), we know that (2.15) has a pair of purely imaginary roots $\pm i\omega_c$ for $n = n_c \in \mathbb{N}_0$, and all other eigenvalues have negative real parts, where $\omega_c = \tau_c \omega_{n_c}$. In order to write (2.12) as an abstract ordinary differential equation in a Banach space, follows by [32], we can take the enlarged space

$$\mathcal{BC} := \left\{ \tilde{\psi} : [-1, 0] \rightarrow X : \tilde{\psi} \text{ is continuous on } [-1, 0), \exists \lim_{\theta \rightarrow 0^-} \tilde{\psi}(\theta) \in X \right\},$$

then the equation (2.12) is equivalent to an abstract ordinary differential equation on $\mathcal{BC}$

$$\frac{dU_t}{dt} = \tilde{A}U_t + X_0\tilde{F}(U_t, \mu).$$

Here, $\tilde{A}$ is a operator from $\mathcal{C}_0^1 = \{\varphi \in \mathcal{C} : \dot{\varphi} \in \mathcal{C}, \varphi(0) \in \text{dom}(\Delta)\}$ to $\mathcal{BC}$, which is defined by

$$\tilde{A}\varphi = \dot{\varphi} + X_0(\tau_c D_1 \Delta\varphi(0) + \tau_c D_2 \Delta\varphi(-1) + L_0(\varphi) - \dot{\varphi}(0)),$$

and $X_0 = X_0(\theta)$ is given by

$$X_0(\theta) = \begin{cases} 0, & -1 \leq \theta < 0, \\ 1, & \theta = 0. \end{cases}$$

118 In the following, the method given in [32] is used to complete the decomposition of $\mathcal{BC}$. Let $C := C([-1, 0], \mathbb{R}^2)$, $C^* := C([0, 1], \mathbb{R}^{2*})$, where $\mathbb{R}^{2*}$ is the two-dimensional space of row vectors, and define the
120 adjoint bilinear form on $C^* \times C$ as follows

$$\langle \Psi(s), \Phi(\theta) \rangle = \Psi(0)\Phi(0) - \int_{-1}^0 \int_0^\theta \Psi(\xi - \theta) dM_n(\theta) \Phi(\xi) d\xi$$

for $\Psi \in C^*$, $\Phi \in C$ and $\xi \in [-1, 0]$, where $M_n(\theta)$ is a bounded variation function from $[-1, 0]$ to $\mathbb{R}^{2 \times 2}$, i.e.,
122 $M_n(\theta) \in BV([-1, 0]; \mathbb{R}^{2 \times 2})$, such that for $\Phi(\theta) \in \mathcal{C}$, one has

$$-\tau_c \frac{n^2}{\ell^2} D_1 \Phi(0) - \tau_c \frac{n^2}{\ell^2} D_2 \Phi(-1) + L_0(\Phi(\theta)) = \int_{-1}^0 dM_n(\theta) \Phi(\theta).$$

By choosing

$$\Phi(\theta) = (\phi(\theta), \bar{\phi}(\theta)), \quad \Psi(s) = \text{col} \left(\psi^T(s), \bar{\psi}^T(s) \right),$$

124 where the $\text{col}(\cdot)$ represents the column vector, $\phi(\theta) = \text{col}(\phi_1(\theta), \phi_2(\theta)) = \phi e^{i\omega_c \theta} \in \mathbb{C}^2$ with $\phi = \text{col}(\phi_1, \phi_2)$ is the eigenvector of (2.14) associated with the eigenvalue $i\omega_c$, and $\psi(s) = \text{col}(\psi_1(s), \psi_2(s)) = \psi e^{-i\omega_c s} \in \mathbb{C}^2$
126 with $\psi = \text{col}(\psi_1, \psi_2)$ is the corresponding adjoint eigenvector such that

$$\langle \Psi(s), \Phi(\theta) \rangle = I_2,$$

where

$$\phi = \begin{pmatrix} 1 \\ \frac{i\omega_{n_c} + d_{11}(n_c/\ell)^2 - a_{11} - b_{11}e^{-i\omega_c}}{a_{12} + b_{12}e^{-i\omega_c}} \end{pmatrix}, \quad \psi = \eta \begin{pmatrix} 1 \\ \frac{a_{12} + b_{12}e^{-i\omega_c}}{i\omega_{n_c} + d_{22}(n_c/\ell)^2 - a_{22} - b_{22}e^{-i\omega_c}} \end{pmatrix}.$$

128 Here,

$$\eta = \frac{1}{1 + k_1 k_2 + e^{-i\omega_c} \tau_c b_{11} + e^{-i\omega_c} k_2 \left(\tau_c b_{21} + \tau_c d_{21} v_* (n_c/\ell)^2 \right)}$$

with

$$k_1 = \frac{i\omega_{n_c} + d_{11}(n_c/\ell)^2 - a_{11} - b_{11}e^{-i\omega_c}}{a_{12} + b_{12}e^{-i\omega_c}}, \quad k_2 = \frac{a_{12} + b_{12}e^{-i\omega_c}}{i\omega_{n_c} + d_{22}(n_c/\ell)^2 - a_{22} - b_{22}e^{-i\omega_c}}.$$

130 According to [32], the phase space $\mathcal{C}$ can be decomposed as

$$\mathcal{C} = \mathcal{P} \oplus \mathcal{Q}, \quad \mathcal{P} = \text{Im } \pi, \quad \mathcal{Q} = \text{Ker } \pi,$$

where for $\tilde{\phi} \in \mathcal{C}$, the projection $\pi : \mathcal{C} \rightarrow \mathcal{P}$ is defined by

$$\pi(\tilde{\phi}) = \left(\Phi \left\langle \Psi, \begin{pmatrix} [\tilde{\phi}(\cdot), \beta_{n_c}^{(1)}] \\ [\tilde{\phi}(\cdot), \beta_{n_c}^{(2)}] \end{pmatrix} \right\rangle \right)^T \beta_{n_c}. \quad (2.17)$$

132 Therefore, according to the method given in [32], $\mathcal{BC}$ can be divided into a direct sum of center subspace and its complementary space, that is

$$\mathcal{BC} = \mathcal{P} \oplus \text{ker } \pi, \quad (2.18)$$

134 where $\dim \mathcal{P} = 2$. It is easy to see that the projection π which is defined by (2.17), is extended to a continuous projection (which is still denoted by π), that is, $\pi : \mathcal{BC} \mapsto \mathcal{P}$. In particular, for $\alpha \in \mathcal{C}$, we have

$$\pi(X_0(\theta)\alpha) = \left(\Phi(\theta)\Psi(0) \begin{pmatrix} [\alpha, \beta_{n_c}^{(1)}] \\ [\alpha, \beta_{n_c}^{(2)}] \end{pmatrix} \right)^T \beta_{n_c}. \quad (2.19)$$

136 By combining with (2.18) and (2.19), $U_t(\theta)$ can be decomposed as

$$\begin{aligned} U_t(\theta) &= \left(\Phi(\theta) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^T \begin{pmatrix} \beta_{n_c}^{(1)} \\ \beta_{n_c}^{(2)} \end{pmatrix} + w = (z_1 \phi e^{i\omega_c \theta} + z_2 \bar{\phi} e^{-i\omega_c \theta}) \gamma_{n_c}(x) + w \\ &= (\phi(\theta) \quad \bar{\phi}(\theta)) \begin{pmatrix} z_1 \gamma_{n_c}(x) \\ z_2 \gamma_{n_c}(x) \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \end{aligned} \quad (2.20)$$

where $w = \text{col}(w_1, w_2)$ and

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left\langle \Psi(0), \begin{pmatrix} [U_t(\theta), \beta_{n_c}^{(1)}] \\ [U_t(\theta), \beta_{n_c}^{(2)}] \end{pmatrix} \right\rangle.$$

138 If we assume that

$$\Phi(\theta) = (\phi(\theta), \bar{\phi}(\theta)), \quad z_x = (z_1 \gamma_{n_c}(x), z_2 \gamma_{n_c}(x))^T,$$

then (2.20) can be rewritten as

$$U_t(\theta) = \Phi(\theta)z_x + w \text{ with } w \in \mathcal{C}_0^1 \cap \text{Ker } \pi := \mathcal{Q}^1. \quad (2.21)$$

140 Then by combining with (2.21), the system (2.12) is decomposed as a system of abstract ordinary differential equations (ODEs) on $\mathbb{R}^2 \times \text{Ker } \pi$, with finite and infinite dimensional variables are separated in
142 the linear term. That is

$$\begin{cases} \dot{z} = Bz + \Psi(0) \begin{pmatrix} [\tilde{F}(\Phi(\theta)z_x + w, \mu), \beta_{n_c}^{(1)}] \\ [\tilde{F}(\Phi(\theta)z_x + w, \mu), \beta_{n_c}^{(2)}] \end{pmatrix}, \\ \dot{w} = A_{\mathcal{Q}^1}w + (I - \pi)X_0(\theta)\tilde{F}(\Phi(\theta)z_x + w, \mu), \end{cases} \quad (2.22)$$

where I is the identity matrix, $z = (z_1, z_2)^T$, $B = \text{diag}\{i\omega_c, -i\omega_c\}$ is the diagonal matrix, and $A_{\mathcal{Q}^1} : \mathcal{Q}^1 \rightarrow$

144 $\text{Ker } \pi$ is defined by

$$A_{\mathcal{Q}^1}w = \dot{w} + X_0(\theta)(\tau_c D_1 \Delta w(0) + \tau_c D_2 \Delta w(-1) + L_0(w) - \dot{w}(0)).$$

Consider the formal Taylor expansion

$$\tilde{F}(\varphi, \mu) = \sum_{j \geq 2} \frac{1}{j!} \tilde{F}_j(\varphi, \mu), \quad F(\varphi, \mu) = \sum_{j \geq 2} \frac{1}{j!} F_j(\varphi, \mu), \quad F^d(\varphi, \mu) = \sum_{j \geq 2} \frac{1}{j!} F_j^d(\varphi, \mu).$$

146 From (2.13), we have

$$\tilde{F}_2(\varphi, \mu) = 2\mu(A_1\varphi(0) + A_2\varphi(-1)) + F_2(\varphi, \mu) + F_2^d(\varphi, \mu) \quad (2.23)$$

and

$$\tilde{F}_j(\varphi, \mu) = F_j(\varphi, \mu) + F_j^d(\varphi, \mu), \quad j = 3, 4, \dots \quad (2.24)$$

By combining with (2.19), the system (2.22) can be rewritten as

$$\begin{cases} \dot{z} = Bz + \sum_{j \geq 2} \frac{1}{j!} f_j^1(z, w, \mu), \\ \dot{w} = A_{\mathcal{Q}^1} w + \sum_{j \geq 2} \frac{1}{j!} f_j^2(z, w, \mu), \end{cases}$$

where

$$\begin{aligned} f_j^1(z, w, \mu) &= \Psi(0) \begin{pmatrix} \tilde{F}_j(\Phi(\theta)z_x + w, \mu), \beta_{n_c}^{(1)} \\ \tilde{F}_j(\Phi(\theta)z_x + w, \mu), \beta_{n_c}^{(2)} \end{pmatrix}, \\ f_j^2(z, w, \mu) &= (I - \pi)X_0(\theta)\tilde{F}_j(\Phi(\theta)z_x + w, \mu). \end{aligned} \quad (2.25)$$

150 In terms of the normal form theory of partial functional differential equations [32], after a recursive transformation of variables of the form

$$(z, w) = (\tilde{z}, \tilde{w}) + \frac{1}{j!} (U_j^1(\tilde{z}, \mu), U_j^2(\tilde{z}, \mu)), \quad j \geq 2, \quad (2.26)$$

152 where $z, \tilde{z} \in \mathbb{R}^2$, $w, \tilde{w} \in \mathcal{Q}^1$ and $U_j^1: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $U_j^2: \mathbb{R}^3 \rightarrow \mathcal{Q}^1$ are homogeneous polynomials of degree j in $\tilde{z}$ and μ , a locally center manifold for (2.12) satisfies $w = 0$ and the flow on it is given by the two-dimensional
154 ODEs

$$\dot{z} = Bz + \sum_{j \geq 2} \frac{1}{j!} g_j^1(z, 0, \mu),$$

which is the normal form as in the usual sense for ODEs.

156 By following [32] and [33], we have

$$g_2^1(z, 0, \mu) = \text{Proj}_{\text{Ker}(M_2^1)} f_2^1(z, 0, \mu) \quad (2.27)$$

and

$$g_3^1(z, 0, \mu) = \text{Proj}_{\text{Ker}(M_3^1)} \tilde{f}_3^1(z, 0, \mu) = \text{Proj}_S \tilde{f}_3^1(z, 0, 0) + O(\mu^2|z|), \quad (2.28)$$

158 where $\text{Proj}_p(q)$ represents the projection of q on p , and $\tilde{f}_3^1(z, 0, \mu)$ is vector and its element is the cubic polynomial of (z, μ) after the variable transformation of (2.26), and it can be determined by (2.38),

$$\begin{aligned} \text{Ker}(M_2^1) &= \text{Span} \left\{ \begin{pmatrix} \mu z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu z_2 \end{pmatrix} \right\}, \\ \text{Ker}(M_3^1) &= \text{Span} \left\{ \begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu^2 z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu^2 z_2 \end{pmatrix} \right\}, \end{aligned} \quad (2.29)$$

160 and

$$S = \text{Span} \left\{ \begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix} \right\}. \quad (2.30)$$

In the following, for notational convenience, we let

$$\mathcal{H}(\alpha z_1^{q_1} z_2^{q_2} \mu) = \begin{pmatrix} \alpha z_1^{q_1} z_2^{q_2} \mu \\ \bar{\alpha} z_1^{q_2} z_2^{q_1} \mu \end{pmatrix}, \quad \alpha \in \mathbb{C}.$$

162 We then calculate $g_j^1(z, 0, \mu)$, $j = 2, 3$ step by step.

2.3. Algorithm for calculating the normal form of Hopf bifurcation

2.3.1. Calculation of $g_2^1(z, 0, \mu)$

From the second mathematical expression in (2.9), we have

$$F_2^d(\varphi, \mu) = F_{20}^d(\varphi) + \mu F_{21}^d(\varphi) \quad (2.31)$$

and

$$F_3^d(\varphi, \mu) = \mu F_{31}^d(\varphi), \quad F_j^d(\varphi, \mu) = (0, 0)^T, \quad j = 4, 5, \dots, \quad (2.32)$$

where

$$\begin{cases} F_{20}^d(\varphi) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \varphi_x^{(1)}(-1)\varphi_x^{(2)}(0) + \varphi_{xx}^{(1)}(-1)\varphi^{(2)}(0) \end{pmatrix}, \\ F_{21}^d(\varphi) = 2D_1\Delta\varphi(0) + 2D_2\Delta\varphi(-1), \\ F_{31}^d(\varphi) = -6d_{21} \begin{pmatrix} 0 \\ \varphi_x^{(1)}(-1)\varphi_x^{(2)}(0) + \varphi_{xx}^{(1)}(-1)\varphi^{(2)}(0) \end{pmatrix}. \end{cases} \quad (2.33)$$

Furthermore, it is easy to verify that

$$\begin{aligned} \begin{pmatrix} \left[2\mu(A_1(\Phi(0)z_x) + A_2(\Phi(-1)z_x)), \beta_{n_c}^{(1)} \right] \\ \left[2\mu(A_1(\Phi(0)z_x) + A_2(\Phi(-1)z_x)), \beta_{n_c}^{(2)} \right] \end{pmatrix} &= 2\mu A_1 \left(\Phi(0) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) + 2\mu A_2 \left(\Phi(-1) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right), \\ \begin{pmatrix} \left[\mu F_{21}^d(\Phi(\theta)z_x), \beta_{n_c}^{(1)} \right] \\ \left[\mu F_{21}^d(\Phi(\theta)z_x), \beta_{n_c}^{(2)} \right] \end{pmatrix} &= -2\frac{n_c^2}{\ell^2} \mu \left(D_1 \left(\Phi(0) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) + D_2 \left(\Phi(-1) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) \right). \end{aligned} \quad (2.34)$$

From (2.11), we have for all $\mu \in \mathbb{R}$, $F_2(\Phi(\theta)z_x, \mu) = F_2(\Phi(\theta)z_x, 0)$. It follows from the first mathematical expression in (2.25) that

$$f_2^1(z, 0, \mu) = \Psi(0) \begin{pmatrix} \left[\tilde{F}_2(\Phi(\theta)z_x, \mu), \beta_{n_c}^{(1)} \right] \\ \left[\tilde{F}_2(\Phi(\theta)z_x, \mu), \beta_{n_c}^{(2)} \right] \end{pmatrix}.$$

This, together with (2.23), (2.27), (2.29), (2.31), (2.32), (2.33) and (2.34), yields to

$$g_2^1(z, 0, \mu) = \text{Proj}_{\text{Ker}(M_2^1)} f_2^1(z, 0, \mu) = \mathcal{H}(B_1\mu z_1), \quad (2.35)$$

where

$$B_1 = 2\psi^T(0) \left(A_1\phi(0) + A_2\phi(-1) - \frac{n_c^2}{\ell^2} (D_1\phi(0) + D_2\phi(-1)) \right). \quad (2.36)$$

2.3.2. Calculation of $g_3^1(z, 0, \mu)$

Notice that the calculation of $g_3^1(z, 0, \mu)$ is very similar to that in [26]. Here, we simply give the results. In this subsection, we calculate the third term $g_3^1(z, 0, 0)$ in terms of (2.28).

Denote

$$\begin{aligned} f_2^{(1,1)}(z, w, 0) &= \Psi(0) \begin{pmatrix} \left[F_2(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(1)} \right] \\ \left[F_2(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(2)} \right] \end{pmatrix}, \\ f_2^{(1,2)}(z, w, 0) &= \Psi(0) \begin{pmatrix} \left[F_2^d(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(1)} \right] \\ \left[F_2^d(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(2)} \right] \end{pmatrix}. \end{aligned} \quad (2.37)$$

It follows from (2.35) that $g_2^1(z, 0, 0) = (0, 0)^T$. Then $\tilde{f}_3^1(z, 0, 0)$ is determined by

$$\begin{aligned} \tilde{f}_3^1(z, 0, 0) = f_3^1(z, 0, 0) + \frac{3}{2} \Big((D_z f_2^1(z, 0, 0)) U_2^1(z, 0) + (D_w f_2^{(1,1)}(z, 0, 0)) U_2^2(z, 0) \\ + (D_{w, w_x, w_{xx}} f_2^{(1,2)}(z, 0, 0)) U_2^{(2,d)}(z, 0)(\theta) \Big), \end{aligned} \quad (2.38)$$

178 where $f_2^1(z, 0, 0) = f_2^{(1,1)}(z, 0, 0) + f_2^{(1,2)}(z, 0, 0)$,

$$\begin{aligned} D_{w, w_x, w_{xx}} f_2^{(1,2)}(z, 0, 0) = \left(D_w f_2^{(1,2)}(z, 0, 0), D_{w_x} f_2^{(1,2)}(z, 0, 0), D_{w_{xx}} f_2^{(1,2)}(z, 0, 0) \right), \\ U_2^1(z, 0) = (M_2^1)^{-1} \text{Proj}_{\text{Im}(M_2^1)} f_2^1(z, 0, 0), \quad U_2^2(z, 0)(\theta) = (M_2^2)^{-1} f_2^2(z, 0, 0), \end{aligned} \quad (2.39)$$

and

$$U_2^{(2,d)}(z, 0)(\theta) = \text{col} \left(U_2^2(z, 0)(\theta), U_{2x}^2(z, 0)(\theta), U_{2xx}^2(z, 0)(\theta) \right). \quad (2.40)$$

180 We calculate $\text{Proj}_S \tilde{f}_3^1(z, 0, 0)$ in the following four steps.

182 **Step 1: Calculation of $\text{Proj}_S f_3^1(z, 0, 0)$**

Writing $F_3(\Phi(\theta)z_x, 0)$ as follows

$$F_3(\Phi(\theta)z_x, 0) = \sum_{q_1+q_2=3} A_{q_1 q_2} z_1^{q_1} z_2^{q_2} \gamma_{n_c}^3(x), \quad (2.41)$$

184 where $A_{q_1 q_2} = \overline{A_{q_2 q_1}}$ with $q_1, q_2 \in \mathbb{N}_0$. From (2.24) and (2.32), we have $\tilde{F}_3(\Phi(\theta)z_x, 0) = F_3(\Phi(\theta)z_x, 0)$, and thus

$$\text{Proj}_S f_3^1(z, 0, 0) = \mathcal{H}(B_{21} z_1^2 z_2),$$

186 where

$$B_{21} = \frac{3}{2\ell\pi} \psi^T A_{21}. \quad (2.42)$$

188 **Step 2: Calculation of $\text{Proj}_S ((D_z f_2^1(z, 0, 0)) U_2^1(z, 0))$**

Form (2.23) and (2.31), we have

$$\tilde{F}_2(\Phi(\theta)z_x, 0) = F_2(\Phi(\theta)z_x, 0) + F_{20}^d(\Phi(\theta)z_x). \quad (2.43)$$

190 By (2.11), we write

$$\begin{aligned} F_2(\Phi(\theta)z_x + w, \mu) &= F_2(\Phi(\theta)z_x + w, 0) \\ &= \sum_{q_1+q_2=2} A_{q_1 q_2} z_1^{q_1} z_2^{q_2} \gamma_{n_c}^2(x) + \mathcal{S}_2(\Phi(\theta)z_x, w) + O(|w|^2), \end{aligned} \quad (2.44)$$

where $\mathcal{S}_2(\Phi(\theta)z_x, w)$ is the product term of $\Phi(\theta)z_x$ and w .

192 By (2.31) and (2.33), we write

$$F_2^d(\Phi(\theta)z_x, 0) = F_{20}^d(\Phi(\theta)z_x) = \frac{n_c^2}{\ell^2} \sum_{q_1+q_2=2} A_{q_1 q_2}^d z_1^{q_1} z_2^{q_2} (\xi_{n_c}^2(x) - \gamma_{n_c}^2(x)), \quad (2.45)$$

where $\xi_{n_c}(x) = (\sqrt{2}/\sqrt{\ell\pi}) \sin((n_c/\ell)x)$, and

$$\begin{cases} A_{20}^d = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_1(-1)\phi_2(0) \end{pmatrix} = \overline{A_{02}^d}, \\ A_{11}^d = -4d_{21}\tau_c \begin{pmatrix} 0 \\ \operatorname{Re}\{\phi_1(-1)\overline{\phi_2(0)}\} \end{pmatrix}. \end{cases} \quad (2.46)$$

194 From (2.1), it is easy to verify that

$$\int_0^{\ell\pi} \gamma_{n_c}^3(x) dx = \int_0^{\ell\pi} \xi_{n_c}^2(x) \gamma_{n_c}(x) dx = 0.$$

Then, from (2.43), (2.44) and (2.45), we have

$$f_2^1(z, 0, 0) = \Psi(0) \begin{pmatrix} [\tilde{F}_2(\Phi(\theta)z_x, 0), \beta_{n_c}^{(1)}] \\ [\tilde{F}_2(\Phi(\theta)z_x, 0), \beta_{n_c}^{(2)}] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.47)$$

196 Hence, by combining with (2.30) and (2.47), we have

$$\operatorname{Proj}_S((D_z f_2^1(z, 0, 0))U_2^1(z, 0)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

198 **Step 3: Calculation of $\operatorname{Proj}_S((D_w f_2^{(1,1)}(z, 0, 0))U_2^2(z, 0)(\theta))$**

Let

$$U_2^2(z, 0)(\theta) \triangleq h(\theta, z) = \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x),$$

200 where $h_n(\theta, z) = \sum_{q_1+q_2=2} h_{n, q_1 q_2}(\theta) z_1^{q_1} z_2^{q_2}$. Then, we have

$$\begin{aligned} & \begin{pmatrix} \left[\mathcal{S}_2 \left(\Phi(\theta)z_x, \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right), \beta_{n_c}^{(1)} \right] \\ \left[\mathcal{S}_2 \left(\Phi(\theta)z_x, \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right), \beta_{n_c}^{(2)} \right] \end{pmatrix} \\ &= \sum_{n \in \mathbb{N}_0} b_n \left(\mathcal{S}_2(\phi(\theta)z_1, h_n(\theta, z)) + \mathcal{S}_2(\overline{\phi}(\theta)z_2, h_n(\theta, z)) \right), \end{aligned}$$

where

$$b_n = \int_0^{\ell\pi} \gamma_{n_c}^2(x) \gamma_n(x) dx = \begin{cases} \frac{1}{\sqrt{\ell\pi}}, & n = 0, \\ \frac{1}{\sqrt{2\ell\pi}}, & n = 2n_c \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

202 Hence, we have

$$\begin{aligned} & (D_w f_2^{(1,1)}(z, 0, 0))U_2^2(z, 0)(\theta) \\ &= \Psi(0) \left(\sum_{n=0, 2n_c} b_n \left(\mathcal{S}_2(\phi(\theta)z_1, h_n(\theta, z)) + \mathcal{S}_2(\overline{\phi}(\theta)z_2, h_n(\theta, z)) \right) \right), \end{aligned}$$

and

$$\operatorname{Proj}_S((D_w f_2^{(1,1)}(z, 0, 0))U_2^2(z, 0)(\theta)) = \mathcal{H}(B_{22}z_1^2 z_2),$$

204 where

$$\begin{aligned}
B_{22} &= \frac{1}{\sqrt{\ell\pi}} \psi^T \left(\mathcal{S}_2(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2(\bar{\phi}(\theta), h_{0,20}(\theta)) \right) \\
&\quad + \frac{1}{\sqrt{2\ell\pi}} \psi^T \left(\mathcal{S}_2(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) \right).
\end{aligned} \tag{2.48}$$

206 **Step 4: Calculation of Proj_S** $\left((D_{w,w_x,w_{xx}} f_2^{(1,2)}(z, 0, 0)) U_2^{(2,d)}(z, 0)(\theta) \right)$

Denote $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))^T = \Phi(\theta)z_x$ and

$$\begin{aligned}
F_2^d(\varphi(\theta), w, w_x, w_{xx}) &= F_2^d(\varphi(\theta) + w, 0) = F_{20}^d(\varphi(\theta) + w) \\
&= -2d_{21}\tau_c \begin{pmatrix} 0 \\ \left(\varphi_{xx}^{(1)}(-1) + (w_1)_{xx}(-1) \right) \left(\varphi^{(2)}(0) + w_2(0) \right) \end{pmatrix} \\
&\quad - 2d_{21}\tau_c \begin{pmatrix} 0 \\ \left(\varphi_x^{(1)}(-1) + (w_1)_x(-1) \right) \left(\varphi_x^{(2)}(0) + (w_2)_x(0) \right) \end{pmatrix}.
\end{aligned}$$

208 Furthermore, from (2.37), (2.39) and (2.40), we have

$$\begin{aligned}
&(D_{w,w_x,w_{xx}} f_2^{(1,2)}(z, 0, 0)) U_2^{(2,d)}(z, 0)(\theta) \\
&= \Psi(0) \begin{pmatrix} \left[D_{w,w_x,w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx}) U_2^{(2,d)}(z, 0)(\theta), \beta_{n_c}^{(1)} \right] \\ \left[D_{w,w_x,w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx}) U_2^{(2,d)}(z, 0)(\theta), \beta_{n_c}^{(2)} \right] \end{pmatrix},
\end{aligned}$$

and then we obtain

$$\text{Proj}_S \left((D_{w,w_x,w_{xx}} f_2^{(1,2)}(z, 0, 0)) U_2^{(2,d)}(z, 0)(\theta) \right) = \mathcal{H}(B_{23} z_1^2 z_2),$$

210 where

$$\begin{aligned}
B_{23} &= -\frac{1}{\sqrt{\ell\pi}} (n_c/\ell)^2 \psi^T \left(\mathcal{S}_2^{(d,1)}(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2^{(d,1)}(\bar{\phi}(\theta), h_{0,20}(\theta)) \right) \\
&\quad + \frac{1}{\sqrt{2\ell\pi}} \psi^T \sum_{j=1,2,3} b_{2n_c}^{(j)} \left(\mathcal{S}_2^{(d,j)}(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2^{(d,j)}(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) \right)
\end{aligned} \tag{2.49}$$

with

$$b_{2n_c}^{(1)} = -\frac{n_c^2}{\ell^2}, \quad b_{2n_c}^{(2)} = 2\frac{n_c^2}{\ell^2}, \quad b_{2n_c}^{(3)} = -\frac{(2n_c)^2}{\ell^2}.$$

212 Furthermore, for $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T$, $y(\theta) = (y_1(\theta), y_2(\theta))^T \in C([-1, 0], \mathbb{R}^2)$, we have

$$\begin{cases} \mathcal{S}_2^{(d,1)}(\phi(\theta), y(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_1(-1)y_2(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,2)}(\phi(\theta), y(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_2(0)y_1(-1) + \phi_1(-1)y_2(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,3)}(\phi(\theta), y(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_2(0)y_1(-1) \end{pmatrix}. \end{cases}$$

3. Normal form of the Hopf bifurcation and the corresponding coefficients

According to the algorithm developed in Section 2, we obtain the normal form of the Hopf bifurcation truncated to the third-order term

$$\dot{z} = Bz + \frac{1}{2} \begin{pmatrix} B_1 z_1 \mu \\ \bar{B}_1 z_2 \mu \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} B_2 z_1^2 z_2 \\ \bar{B}_2 z_1 z_2^2 \end{pmatrix} + O(|z|\mu^2 + |z|^4), \quad (3.1)$$

where

$$B_1 = 2\psi^T(0) \left(A_1 \phi(0) + A_2 \phi(-1) - \frac{n_c^2}{\ell^2} (D_1 \phi(0) + D_2 \phi(-1)) \right),$$

$$B_2 = B_{21} + \frac{3}{2} (B_{22} + B_{23}).$$

Here, B_1 is determined by (2.36), B_{21} , B_{22} and B_{23} are determined by (2.42), (2.48), (2.49), respectively, and they can be calculated by using the MATLAB software. The normal form (3.1) can be written in real coordinates through the change of variables $z_1 = v_1 - iv_2$, $z_2 = v_1 + iv_2$, and then changing to polar coordinates by $v_1 = \rho \cos \Theta$, $v_2 = \rho \sin \Theta$, where Θ is the azimuthal angle. Therefore, by the above transformation and removing the azimuthal term Θ , (3.1) can be rewritten as

$$\dot{\rho} = K_1 \mu \rho + K_2 \rho^3 + O(\mu^2 \rho + |(\rho, \mu)|^4),$$

where

$$K_1 = \frac{1}{2} \operatorname{Re}(B_1), \quad K_2 = \frac{1}{3!} \operatorname{Re}(B_2).$$

According to [34], the sign of $K_1 K_2$ determines the direction of the Hopf bifurcation, and the sign of K_2 determines the stability of the Hopf bifurcation periodic solution. More precisely, we have the following results

- (i) when $K_1 K_2 < 0$, the Hopf bifurcation is supercritical, and the Hopf bifurcation periodic solution is stable for $K_2 < 0$ and unstable for $K_2 > 0$;
- (ii) when $K_1 K_2 > 0$, the Hopf bifurcation is subcritical, and the Hopf bifurcation periodic solution is stable for $K_2 < 0$ and unstable for $K_2 > 0$.

From (2.42), (2.48) and (2.49), it is obvious that in order to obtain the value of K_2 , we still need to calculate $h_{0,20}(\theta)$, $h_{0,11}(\theta)$, $h_{2n_c,20}(\theta)$, $h_{2n_c,11}(\theta)$ and A_{ij} .

3.1. Calculations of $h_{0,20}(\theta)$, $h_{0,11}(\theta)$, $h_{2n_c,20}(\theta)$ and $h_{2n_c,11}(\theta)$

From [32], we have

$$M_2^2(h_n(\theta, z)\gamma_n(x)) = D_z(h_n(\theta, z)\gamma_n(x))Bz - A_{\mathcal{Q}^1}(h_n(\theta, z)\gamma_n(x)),$$

which leads to

$$\begin{pmatrix} \left[M_2^2(h_n(\theta, z)\gamma_n(x)), \beta_n^{(1)} \right] \\ \left[M_2^2(h_n(\theta, z)\gamma_n(x)), \beta_n^{(2)} \right] \end{pmatrix} = 2i\omega_c (h_{n,20}(\theta)z_1^2 - h_{n,02}(\theta)z_2^2) - \left(\dot{h}_n(\theta, z) + X_0(\theta) \left(\mathcal{L}_0(h_n(\theta, z)) - \dot{h}_n(0, z) \right) \right), \quad (3.2)$$

where

$$\mathcal{L}_0(h_n(\theta, z)) = -\tau_c(n/\ell)^2 (D_1 h_n(0, z) + D_2 h_n(-1, z)) + \tau_c(A_1 h_n(0, z) + A_2 h_n(-1, z)).$$

236 By (2.19) and the second mathematical expression in (2.25), we have

$$\begin{aligned} f_2^2(z, 0, 0) &= X_0(\theta) \tilde{F}_2(\Phi(\theta) z_x, 0) - \pi \left(X_0(\theta) \tilde{F}_2(\Phi(\theta) z_x, 0) \right) \\ &= X_0(\theta) \tilde{F}_2(\Phi(\theta) z_x, 0) - \Phi(\theta) \Psi(0) \begin{pmatrix} \tilde{F}_2(\Phi(\theta) z_x, 0), \beta_{n_c}^{(1)} \\ \tilde{F}_2(\Phi(\theta) z_x, 0), \beta_{n_c}^{(2)} \end{pmatrix} \gamma_{n_c}(x). \end{aligned} \quad (3.3)$$

Furthermore, by (2.43), (2.44) and (2.45), we have

$$\begin{pmatrix} f_2^2(z, 0, 0), \beta_n^{(1)} \\ f_2^2(z, 0, 0), \beta_n^{(2)} \end{pmatrix} = \begin{cases} \frac{1}{\sqrt{\ell\pi}} X_0(\theta) (A_{20} z_1^2 + A_{02} z_2^2 + A_{11} z_1 z_2), & n = 0, \\ \frac{1}{\sqrt{2\ell\pi}} X_0(\theta) (\tilde{A}_{20} z_1^2 + \tilde{A}_{02} z_2^2 + \tilde{A}_{11} z_1 z_2), & n = 2n_c, \end{cases} \quad (3.4)$$

238 where $\tilde{A}_{j_1 j_2}$ is defined as follows

$$\begin{cases} \tilde{A}_{j_1 j_2} = A_{j_1 j_2} - 2(n_c/\ell)^2 A_{j_1 j_2}^d, \\ j_1, j_2 = 0, 1, 2, \quad j_1 + j_2 = 2, \end{cases} \quad (3.5)$$

where $A_{j_1 j_2}^d$ is determined by (2.46), and $A_{j_1 j_2}$ will be calculated in the following section. Therefore, from

240 (2.39), (3.2), (3.3), (3.4), and by matching the coefficients of z_1^2 and $z_1 z_2$, we have

$$n = 0, \quad \begin{cases} z_1^2 : \begin{cases} \dot{h}_{0,20}(\theta) - 2i\omega_c h_{0,20}(\theta) = (0, 0)^T, \\ \dot{h}_{0,20}(0) - L_0(h_{0,20}(\theta)) = \frac{1}{\sqrt{\ell\pi}} A_{20}, \end{cases} \\ z_1 z_2 : \begin{cases} \dot{h}_{0,11}(\theta) = (0, 0)^T, \\ \dot{h}_{0,11}(0) - L_0(h_{0,11}(\theta)) = \frac{1}{\sqrt{\ell\pi}} A_{11} \end{cases} \end{cases} \quad (3.6)$$

and

$$n = 2n_c, \quad \begin{cases} z_1^2 : \begin{cases} \dot{h}_{2n_c,20}(\theta) - 2i\omega_c h_{2n_c,20}(\theta) = (0, 0)^T, \\ \dot{h}_{2n_c,20}(0) - \mathcal{L}_0(h_{2n_c,20}(\theta)) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{20}, \end{cases} \\ z_1 z_2 : \begin{cases} \dot{h}_{2n_c,11}(\theta) = (0, 0)^T, \\ \dot{h}_{2n_c,11}(0) - \mathcal{L}_0(h_{2n_c,11}(\theta)) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11}. \end{cases} \end{cases} \quad (3.7)$$

242 Next, by combining with (3.6) and (3.7), we will give the mathematical expressions of $h_{0,20}(\theta)$, $h_{0,11}(\theta)$, $h_{2n_c,20}(\theta)$ and $h_{2n_c,11}(\theta)$.

244

(1) Calculations of $h_{0,20}(\theta)$ and $h_{0,11}(\theta)$

246

(i) Notice that

$$\begin{cases} \dot{h}_{0,20}(\theta) - 2i\omega_c h_{0,20}(\theta) = (0, 0)^T, \\ \dot{h}_{0,20}(0) - L_0(h_{0,20}(\theta)) = \frac{1}{\sqrt{\ell\pi}} A_{20}, \end{cases} \quad (3.8)$$

then from (3.8), we have $h_{0,20}(\theta) = e^{2iw_c\theta}h_{0,20}(0)$, and hence $h_{0,20}(-1) = e^{-2iw_c}h_{0,20}(0)$. Furthermore,
 248 from (3.8) and $L_0(h_{0,20}(\theta)) = \tau_c(A_1h_{0,20}(0) + A_2h_{0,20}(-1))$, we have

$$2i\omega_ch_{0,20}(0) = \frac{1}{\sqrt{\ell\pi}}A_{20} + \tau_c(A_1h_{0,20}(0) + A_2h_{0,20}(-1)). \quad (3.9)$$

Therefore, by combining with $h_{0,20}(-1) = e^{-2iw_c}h_{0,20}(0)$ and (3.9), we can obtain

$$(2i\omega_cI_2 - \tau_cA_1 - \tau_cA_2e^{-2iw_c})h_{0,20}(0) = \frac{1}{\sqrt{\ell\pi}}A_{20},$$

250 and hence $h_{0,20}(\theta) = e^{2iw_c\theta}C_1$ with

$$C_1 = (2i\omega_cI_2 - \tau_cA_1 - \tau_cA_2e^{-2iw_c})^{-1} \frac{1}{\sqrt{\ell\pi}}A_{20}.$$

(ii) Notice that

$$\begin{cases} \dot{h}_{0,11}(\theta) = (0,0)^T, \\ \dot{h}_{0,11}(0) - L_0(h_{0,11}(\theta)) = \frac{1}{\sqrt{\ell\pi}}A_{11}. \end{cases} \quad (3.10)$$

252 then from (3.10), we have $h_{0,11}(\theta) = h_{0,11}(0)$, and hence $h_{0,11}(-1) = h_{0,11}(0)$. Furthermore, from (3.10) and $L_0(h_{0,11}(\theta)) = \tau_c(A_1h_{0,11}(0) + A_2h_{0,11}(-1))$, we have

$$(0,0)^T = \tau_c(A_1h_{0,11}(0) + A_2h_{0,11}(-1)) + \frac{1}{\sqrt{\ell\pi}}A_{11}. \quad (3.11)$$

254 Therefore, by combining with $h_{0,11}(-1) = h_{0,11}(0)$ and (3.11), we can obtain

$$(-\tau_cA_1 - \tau_cA_2)h_{0,11}(0) = \frac{1}{\sqrt{\ell\pi}}A_{11},$$

and hence $h_{0,11}(\theta) = C_2$ with

$$C_2 = (-\tau_cA_1 - \tau_cA_2)^{-1} \frac{1}{\sqrt{\ell\pi}}A_{11}.$$

256

(2) Calculations of $h_{2n_c,20}(\theta)$ and $h_{2n_c,11}(\theta)$

258 (i) Notice that

$$\begin{cases} \dot{h}_{2n_c,20}(\theta) - 2i\omega_ch_{2n_c,20}(\theta) = (0,0)^T, \\ \dot{h}_{2n_c,20}(0) - \mathcal{L}_0(h_{2n_c,20}(\theta)) = \frac{1}{\sqrt{2\ell\pi}}\tilde{A}_{20}, \end{cases} \quad (3.12)$$

then from (3.12), we have $h_{2n_c,20}(\theta) = e^{2iw_c\theta}h_{2n_c,20}(0)$, and hence $h_{2n_c,20}(-1) = e^{-2iw_c}h_{2n_c,20}(0)$. Further-
 260 more, from (3.12) and

$$\mathcal{L}_0(h_{2n_c,20}(\theta)) = -\tau_c \frac{4n_c^2}{\ell^2} (D_1h_{2n_c,20}(0) + D_2h_{2n_c,20}(-1)) + \tau_cA_1h_{2n_c,20}(0) + \tau_cA_2h_{2n_c,20}(-1),$$

we have

$$2i\omega_ch_{2n_c,20}(0) = \frac{1}{\sqrt{2\ell\pi}}\tilde{A}_{20} - \tau_c \frac{4n_c^2}{\ell^2} (D_1h_{2n_c,20}(0) + D_2h_{2n_c,20}(-1)) + \tau_cA_1h_{2n_c,20}(0) + \tau_cA_2h_{2n_c,20}(-1). \quad (3.13)$$

262 Therefore, by combining with $h_{2n_c,20}(-1) = e^{-2iw_c}h_{2n_c,20}(0)$ and (3.13), we can obtain

$$(2i\omega_cI_2 + \tau_c \frac{4n_c^2}{\ell^2}D_1 + \tau_c \frac{4n_c^2}{\ell^2}D_2e^{-2iw_c} - \tau_cA_1 - \tau_cA_2e^{-2iw_c})h_{0200}(0) = \frac{1}{\sqrt{2\ell\pi}}\tilde{A}_{20},$$

and hence $h_{2n_c,20}(\theta) = e^{2iw_c\theta}C_3$ with

$$C_3 = (2iw_c I_2 + \tau_c \frac{4n_c^2}{\ell^2} D_1 + \tau_c \frac{4n_c^2}{\ell^2} D_2 e^{-2iw_c} - \tau_c A_1 - \tau_c A_2 e^{-2iw_c})^{-1} \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{20}.$$

Here, A_{20}^d and $\tilde{A}_{20}$ are defined by (2.46) and (3.5), respectively.

(ii) Notice that

$$\begin{cases} \dot{h}_{2n_c,11}(\theta) = (0,0)^T, \\ \dot{h}_{2n_c,11}(0) - \mathcal{L}_0(h_{2n_c,11}(\theta)) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11}, \end{cases} \quad (3.14)$$

then from (3.14), we have $h_{2n_c,11}(\theta) = h_{2n_c,11}(0)$, and hence $h_{2n_c,11}(-1) = h_{2n_c,11}(0)$. Furthermore, from (3.14) and

$$\mathcal{L}_0(h_{2n_c,11}(\theta)) = -\tau_c \frac{4n_c^2}{\ell^2} (D_1 h_{2n_c,11}(0) + D_2 h_{2n_c,11}(-1)) + \tau_c A_1 h_{2n_c,11}(0) + \tau_c A_2 h_{2n_c,11}(-1),$$

we have

$$(0,0)^T = -\tau_c \frac{4n_c^2}{\ell^2} (D_1 h_{2n_c,11}(0) + D_2 h_{2n_c,11}(-1)) + \tau_c A_1 h_{2n_c,11}(0) + \tau_c A_2 h_{2n_c,11}(-1) + \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11}. \quad (3.15)$$

Therefore, by combining with $h_{2n_c,11}(-1) = h_{2n_c,11}(0)$ and (3.15), we can obtain

$$\left(\tau_c \frac{4n_c^2}{\ell^2} D_1 + \tau_c \frac{4n_c^2}{\ell^2} D_2 - \tau_c A_1 - \tau_c A_2 \right) h_{2n_c,11}(0) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11},$$

and hence $h_{2n_c,11}(\theta) = C_4$ with

$$C_4 = \left(\tau_c \frac{4n_c^2}{\ell^2} D_1 + \tau_c \frac{4n_c^2}{\ell^2} D_2 - \tau_c A_1 - \tau_c A_2 \right)^{-1} \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11}.$$

Here, A_{11}^d and $\tilde{A}_{11}$ are defined by (2.46) and (3.5), respectively.

3.2. Calculations of $A_{i,j}$ and $\mathcal{S}_2(\Phi(\theta)z_x, w)$

In this subsection, let $F(\varphi, \mu) = (F^{(1)}(\varphi, \mu), F^{(2)}(\varphi, \mu))^T$ and $\varphi = (\varphi_1, \varphi_2)^T \in \mathcal{C}$, and we write

$$\frac{1}{j!} F_j(\varphi, \mu) = \sum_{j_1+j_2+j_3+j_4=j} \frac{1}{j_1!j_2!j_3!j_4!} f_{j_1j_2j_3j_4} \varphi_1^{j_1}(0) \varphi_2^{j_2}(0) \varphi_1^{j_3}(-1) \mu^{j_4}, \quad (3.16)$$

where

$$f_{j_1j_2j_3j_4} = \left(f_{j_1j_2j_3j_4}^{(1)}, f_{j_1j_2j_3j_4}^{(2)} \right)^T$$

with

$$f_{j_1j_2j_3j_4}^{(k)} = \frac{\partial^{j_1+j_2+j_3+j_4} F^{(k)}(0,0,0,0)}{\partial \varphi_1^{j_1}(0) \partial \varphi_2^{j_2}(0) \partial \varphi_1^{j_3}(-1) \partial \mu^{j_4}}, \quad k = 1, 2.$$

Then, from (3.16), we have

$$\begin{aligned} F_2(\varphi, \mu) &= F_2(\varphi, 0) \\ &= 2 \sum_{j_1+j_2+j_3+j_4=2} \frac{1}{j_1!j_2!j_3!j_4!} f_{j_1j_2j_3j_4} \varphi_1^{j_1}(0) \varphi_2^{j_2}(0) \varphi_1^{j_3}(-1) \mu^{j_4} \\ &= f_{0020} \varphi_1^2(-1) + 2f_{0110} \varphi_2(0) \varphi_1(-1) + f_{0200} \varphi_2^2(0) \\ &\quad + 2f_{1010} \varphi_1(0) \varphi_1(-1) + 2f_{1100} \varphi_1(0) \varphi_2(0) + f_{2000} \varphi_1^2(0) \end{aligned} \quad (3.17)$$

and

$$\begin{aligned}
F_3(\varphi, 0) &= 6 \sum_{j_1+j_2+j_3+j_4=3} \frac{1}{j_1!j_2!j_3!j_4!} f_{j_1j_2j_3j_4} \varphi_1^{j_1}(0) \varphi_2^{j_2}(0) \varphi_1^{j_3}(-1) \mu^{j_4} \\
&= f_{0030} \varphi_1^3(-1) + 3f_{0120} \varphi_2(0) \varphi_1^2(-1) + 3f_{0210} \varphi_2^2(0) \varphi_1(-1) \\
&\quad + f_{0300} \varphi_2^3(0) + 3f_{1020} \varphi_1(0) \varphi_1^2(-1) + 6f_{1110} \varphi_1(0) \varphi_2(0) \varphi_1(-1) \\
&\quad + 3f_{1200} \varphi_1(0) \varphi_2^2(0) + 3f_{2010} \varphi_1^2(0) \varphi_1(-1) + 3f_{2100} \varphi_1^2(0) \varphi_2(0) \\
&\quad + f_{3000} \varphi_1^3(0).
\end{aligned} \tag{3.18}$$

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Notice that

$$\begin{aligned}
\varphi(\theta) &= \Phi(\theta) z_x = \phi(\theta) z_1(t) \gamma_{n_c}(x) + \bar{\phi}(\theta) z_2(t) \gamma_{n_c}(x) \\
&= \begin{pmatrix} \phi_1(\theta) z_1(t) \gamma_{n_c}(x) + \bar{\phi}_1(\theta) z_2(t) \gamma_{n_c}(x) \\ \phi_2(\theta) z_1(t) \gamma_{n_c}(x) + \bar{\phi}_2(\theta) z_2(t) \gamma_{n_c}(x) \end{pmatrix} \\
&= \begin{pmatrix} \varphi_1(\theta) \\ \varphi_2(\theta) \end{pmatrix},
\end{aligned} \tag{3.19}$$

and similar to (2.41), we have

$$F_2(\Phi(\theta) z_x, 0) = \sum_{q_1+q_2=2} A_{q_1q_2} \gamma_{n_c}^{q_1+q_2}(x) z_1^{q_1} z_2^{q_2}, \tag{3.20}$$

280

then by combining with (3.17), (3.19) and (3.20), we have

$$\begin{aligned}
A_{20} &= f_{0020} \phi_1^2(-1) + 2f_{0110} \phi_2(0) \phi_1(-1) + f_{0200} \phi_2^2(0) + 2f_{1010} \phi_1(0) \phi_1(-1) \\
&\quad + 2f_{1100} \phi_1(0) \phi_2(0) + f_{2000} \phi_1^2(0), \\
A_{02} &= f_{0020} \bar{\phi}_1^2(-1) + 2f_{0110} \bar{\phi}_2(0) \bar{\phi}_1(-1) + f_{0200} \bar{\phi}_2^2(0) + 2f_{1010} \bar{\phi}_1(0) \bar{\phi}_1(-1) \\
&\quad + 2f_{1100} \bar{\phi}_1(0) \bar{\phi}_2(0) + f_{2000} \bar{\phi}_1^2(0), \\
A_{11} &= 2f_{0020} \phi_1(-1) \bar{\phi}_1(-1) + 2f_{0110} (\phi_2(0) \bar{\phi}_1(-1) + \bar{\phi}_2(0) \phi_1(-1)) + 2f_{0200} \phi_2(0) \bar{\phi}_2(0) \\
&\quad + 2f_{1010} (\phi_1(0) \bar{\phi}_1(-1) + \bar{\phi}_1(0) \phi_1(-1)) + 2f_{1100} (\phi_1(0) \bar{\phi}_2(0) + \bar{\phi}_1(0) \phi_2(0)) \\
&\quad + 2f_{2000} \phi_1(0) \bar{\phi}_1(0).
\end{aligned}$$

Furthermore, from (2.41), (3.18) and (3.19), we have

$$\begin{aligned}
A_{30} &= f_{0030} \phi_1^3(-1) + 3f_{0120} \phi_2(0) \phi_1^2(-1) + 3f_{0210} \phi_2^2(0) \phi_1(-1) + f_{0300} \phi_2^3(0) \\
&\quad + 3f_{1020} \phi_1(0) \phi_1^2(-1) + 6f_{1110} \phi_1(0) \phi_2(0) \phi_1(-1) \\
&\quad + 3f_{1200} \phi_1(0) \phi_2^2(0) + 3f_{2010} \phi_1^2(0) \phi_1(-1) \\
&\quad + 3f_{2100} \phi_1^2(0) \phi_2(0) + f_{3000} \phi_1^3(0), \\
A_{03} &= f_{0030} \bar{\phi}_1^3(-1) + 3f_{0120} \bar{\phi}_2(0) \bar{\phi}_1^2(-1) + 3f_{0210} \bar{\phi}_2^2(0) \bar{\phi}_1(-1) + f_{0300} \bar{\phi}_2^3(0) \\
&\quad + 3f_{1020} \bar{\phi}_1(0) \bar{\phi}_1^2(-1) + 6f_{1110} \bar{\phi}_1(0) \bar{\phi}_2(0) \bar{\phi}_1(-1) \\
&\quad + 3f_{1200} \bar{\phi}_1(0) \bar{\phi}_2^2(0) + 3f_{2010} \bar{\phi}_1^2(0) \bar{\phi}_1(-1) \\
&\quad + 3f_{2100} \bar{\phi}_1^2(0) \bar{\phi}_2(0) + f_{3000} \bar{\phi}_1^3(0)
\end{aligned}$$

282 and

$$\begin{aligned}
A_{21} &= 3f_{0030}\phi_1^2(-1)\bar{\phi}_1(-1) + 3f_{0120}(\phi_2(0)2\phi_1(-1)\bar{\phi}_1(-1) + \bar{\phi}_2(0)\phi_1^2(-1)) \\
&\quad + 3f_{0210}(2\phi_2(0)\bar{\phi}_2(0)\bar{\phi}_1(-1) + \phi_2^2(0)\bar{\phi}_1(-1)) + 3f_{0300}\phi_2^2(0)\bar{\phi}_2(0), \\
&\quad + 3f_{1020}(2\phi_1(0)\phi_1(-1)\bar{\phi}_1(-1) + \bar{\phi}_1(0)\phi_1^2(-1)) \\
&\quad + 6f_{1110}(\phi_1(0)\phi_2(0)\bar{\phi}_1(-1) + \phi_1(0)\bar{\phi}_2(0)\phi_1(-1) + \bar{\phi}_1(0)\phi_2(0)\phi_1(-1)) \\
&\quad + 3f_{1200}(2\phi_1(0)\phi_2(0)\bar{\phi}_2(0) + \bar{\phi}_1(0)\phi_2^2(0)) + 3f_{2100}(\phi_1^2(0)\bar{\phi}_2(0) + 2\phi_1(0)\bar{\phi}_1(0)\phi_2(0)) \\
&\quad + 3f_{2010}(2\phi_1(0)\bar{\phi}_1(0)\phi_1(-1) + \phi_1^2(0)\bar{\phi}_1(-1)) \\
&\quad + 3f_{3000}\phi_1^2(0)\bar{\phi}_1(0), \\
A_{12} &= 3f_{0030}\phi_1(-1)\bar{\phi}_1^2(-1) + 3f_{0120}(\phi_2(0)\bar{\phi}_1^2(-1) + 2\bar{\phi}_2(0)\phi_1(-1)\bar{\phi}_1(-1)) \\
&\quad + 3f_{0210}(2\phi_2(0)\bar{\phi}_2(0)\bar{\phi}_1(-1) + \bar{\phi}_2^2(0)\phi_1(-1)) + 3f_{0300}\phi_2(0)\bar{\phi}_2^2(0), \\
&\quad + 3f_{1020}(\phi_1(0)\bar{\phi}_1^2(-1) + 2\bar{\phi}_1(0)\phi_1(-1)\bar{\phi}_1(-1)) \\
&\quad + 6f_{1110}(\phi_1(0)\bar{\phi}_2(0)\bar{\phi}_1(-1) + \bar{\phi}_1(0)\phi_2(0)\bar{\phi}_1(-1) + \bar{\phi}_1(0)\bar{\phi}_2(0)\phi_1(-1)) \\
&\quad + 3f_{1200}(\phi_1(0)\bar{\phi}_2^2(0) + 2\bar{\phi}_1(0)\phi_2(0)\bar{\phi}_2(0)) + 3f_{2100}(2\phi_1(0)\bar{\phi}_1(0)\bar{\phi}_2(0) + \bar{\phi}_1^2(0)\phi_2(0)) \\
&\quad + 3f_{2010}(\bar{\phi}_1^2(0)\phi_1(-1) + 2\phi_1(0)\bar{\phi}_1(0)\bar{\phi}_1(-1)) \\
&\quad + 3f_{3000}\phi_1(0)\bar{\phi}_1^2(0).
\end{aligned}$$

Moreover, from (3.16), we have

$$\begin{aligned}
F_2(\varphi(\theta) + w, \mu) &= 2 \sum_{j_1+j_2+j_3+j_4=2} \frac{1}{j_1!j_2!j_3!j_4!} f_{j_1j_2j_3j_4}(\varphi_1(0) + w_1(0))^{j_1}(\varphi_2(0) + w_2(0))^{j_2}(\varphi_1(-1) + w_1(-1))^{j_3}\mu^{j_4} \\
&= f_{0020}(\varphi_1(-1) + w_1(-1))^2 + 2f_{0110}(\varphi_2(0) + w_2(0))(\varphi_1(-1) + w_1(-1)) + f_{0200}(\varphi_2(0) + w_2(0))^2 \\
&\quad + 2f_{1010}(\varphi_1(0) + w_1(0))(\varphi_1(-1) + w_1(-1)) \\
&\quad + 2f_{1100}(\varphi_1(0) + w_1(0))(\varphi_2(0) + w_2(0)) + f_{2000}(\varphi_1(0) + w_1(0))^2.
\end{aligned} \tag{3.21}$$

284 Notice that

$$\begin{aligned}
\varphi(\theta) + w(\theta) &= \Phi(\theta)z_x + w(\theta) = \phi(\theta)z_1(t)\gamma_{n_c}(x) + \bar{\phi}(\theta)z_2(t)\gamma_{n_c}(x) + w(\theta) \\
&= \begin{pmatrix} \phi_1(\theta)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(\theta)z_2(t)\gamma_{n_c}(x) + w_1(\theta) \\ \phi_2(\theta)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_2(\theta)z_2(t)\gamma_{n_c}(x) + w_2(\theta) \end{pmatrix} \\
&= \begin{pmatrix} \varphi_1(\theta) + w_1(\theta) \\ \varphi_2(\theta) + w_2(\theta) \end{pmatrix}
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
F_2(\Phi(\theta)z_x + w, \mu) &= F_2(\Phi(\theta)z_x + w, 0) \\
&= \sum_{q_1+q_2=2} A_{q_1q_2}\gamma_{n_c}^{q_1+q_2}(x)z_1^{q_1}z_2^{q_2} + \mathcal{S}_2(\Phi(\theta)z_x, w) + O(|w|^2),
\end{aligned} \tag{3.23}$$

286 then by combining with (3.21), (3.22) and (3.23), we have

$$\begin{aligned}
& \mathcal{S}_2(\Phi(\theta)z_x, w) \\
&= 2f_{0020}(\phi_1(-1)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(-1)z_2(t)\gamma_{n_c}(x))w_1(-1) \\
&+ 2f_{0110}((\phi_2(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_2(0)z_2(t)\gamma_{n_c}(x))w_1(-1) + (\phi_1(-1)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(-1)z_2(t)\gamma_{n_c}(x))w_2(0)) \\
&+ 2f_{0200}(\phi_2(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_2(0)z_2(t)\gamma_{n_c}(x))w_2(0) \\
&+ 2f_{1010}((\phi_1(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(0)z_2(t)\gamma_{n_c}(x))w_1(-1) + (\phi_1(-1)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(-1)z_2(t)\gamma_{n_c}(x))w_1(0)) \\
&+ 2f_{1100}((\phi_1(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(0)z_2(t)\gamma_{n_c}(x))w_2(0) + (\phi_2(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_2(0)z_2(t)\gamma_{n_c}(x))w_1(0)) \\
&+ 2f_{2000}(\phi_1(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(0)z_2(t)\gamma_{n_c}(x))w_1(0).
\end{aligned}$$

4. Application to a predator-prey model with memory and gestation delays

288 In this section, we consider the following diffusive predator-prey model with ratio-dependent Holling type-III functional response, which includes with memory and gestation delays

$$\begin{cases}
\frac{\partial u(x, t)}{\partial t} = d_{11}\Delta u(x, t) + u(x, t)(1 - u(x, t)) - \frac{\beta u^2(x, t)v(x, t)}{u^2(x, t) + mv^2(x, t)}, & x \in (0, \ell\pi), t > 0, \\
\frac{\partial v(x, t)}{\partial t} = d_{22}\Delta v(x, t) - d_{21}(v(x, t)u_x(x, t - \tau))_x + \gamma v(x, t)\left(1 - \frac{v(x, t)}{u(x, t - \tau)}\right), & x \in (0, \ell\pi), t > 0, \\
u_x(0, t) = u_x(\ell\pi, t) = v_x(0, t) = v_x(\ell\pi, t) = 0, & t \geq 0, \\
u(x, t) = u_0(x, t), v(x, t) = v_0(x, t), & x \in (0, \ell\pi), -\tau \leq t \leq 0,
\end{cases} \quad (4.1)$$

290 where $u(x, t)$ and $v(x, t)$ stand for the densities of the prey and predators at location x and time t , respectively, $\beta > 0$, $m > 0$ and $\gamma > 0$.

292 4.1. The case of with memory delay and without gestation delay

When system (4.1) includes memory delay and doesn't include gestation delay, that is to say, in the
294 model (1.3), we let

$$\begin{aligned}
f(u(x, t), v(x, t)) &= u(x, t)(1 - u(x, t)) - \frac{\beta u^2(x, t)v(x, t)}{u^2(x, t) + mv^2(x, t)}, \\
g(u(x, t), v(x, t)) &= \gamma v(x, t)\left(1 - \frac{v(x, t)}{u(x, t)}\right).
\end{aligned}$$

Then, the model (1.3) can be written as

$$\begin{cases}
\frac{\partial u(x, t)}{\partial t} = d_{11}\Delta u(x, t) + u(x, t)(1 - u(x, t)) - \frac{\beta u^2(x, t)v(x, t)}{u^2(x, t) + mv^2(x, t)}, & x \in (0, \ell\pi), t > 0, \\
\frac{\partial v(x, t)}{\partial t} = d_{22}\Delta v(x, t) - d_{21}(v(x, t)u_x(x, t - \tau))_x + \gamma v(x, t)\left(1 - \frac{v(x, t)}{u(x, t)}\right), & x \in (0, \ell\pi), t > 0, \\
u_x(0, t) = u_x(\ell\pi, t) = v_x(0, t) = v_x(\ell\pi, t) = 0, & t \geq 0, \\
u(x, t) = u_0(x, t), & x \in (0, \ell\pi), -\tau \leq t \leq 0, \\
v(x, t) = v_0(x, t), & x \in (0, \ell\pi).
\end{cases} \quad (4.2)$$

296 Notice that for the system (4.2), when $d_{21} = 0$, the global asymptotic stability of the positive constant steady state in this system has been investigated by Shi et al. in [35]. Furthermore, the normal form for Hopf
298 bifurcation can be calculated by using the developed algorithm in [26], and the detail calculation procedures

are give in Appendix A. In the following, we first give the stability and Hopf bifurcation analysis for the model (4.2), then by employing the developed procedure in [26] for calculating the normal form for Hopf bifurcation, the direction and stability of the Hopf bifurcation are determined.

4.1.1. Stability and Hopf bifurcation analysis

The system (4.2) has the positive constant steady state $E_*(u_*, v_*)$, where

$$u_* = v_* = 1 - \frac{\beta}{m+1} \quad (4.3)$$

with $0 < \beta < m+1$. For $E_*(u_*, v_*)$, form (2.4), when $m > 1$, we have

$$a_{11} = \frac{2\beta}{(m+1)^2} - 1 \begin{cases} \leq 0, & 0 < \beta \leq \frac{(m+1)^2}{2}, \\ > 0, & \beta > \frac{(m+1)^2}{2}. \end{cases}$$

Notice that when $m > 1$, if $a_{11} > 0$, then we have $\beta > \frac{(m+1)^2}{2} > m+1$, which is contradict to the condition $0 < \beta < m+1$. Thus, when $m > 1$, $a_{11} \leq 0$ under the condition $0 < \beta < m+1$. When $0 < m < 1$, we have

$$a_{11} = \frac{2\beta}{(m+1)^2} - 1 \begin{cases} \leq 0, & 0 < \beta \leq \frac{(m+1)^2}{2}, \\ > 0, & \frac{(m+1)^2}{2} < \beta < m+1. \end{cases} \quad (4.4)$$

Furthermore, we have

$$\begin{aligned} a_{12} &= \frac{\beta(m-1)}{(m+1)^2} \begin{cases} \leq 0, & 0 < m \leq 1, \\ > 0, & m > 1, \end{cases} \\ a_{21} &= \gamma > 0, \quad a_{22} = -\gamma < 0, \\ b_{11} &= 0, \quad b_{12} = 0, \quad b_{21} = 0, \quad b_{22} = 0. \end{aligned} \quad (4.5)$$

Moreover, by combining with (4.4), (4.5),

$$D_1 = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

and

$$\mathcal{M}_n(\lambda) = \lambda I_2 + \frac{n^2}{\ell^2} D_1 + \frac{n^2}{\ell^2} e^{-\lambda\tau} D_2 - A_1 - A_2 e^{-\lambda\tau},$$

or according to (2.7), the characteristic equation of system (4.2) can be written as

$$\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda)) = \lambda^2 - T_n \lambda + \tilde{J}_n(\tau) = 0, \quad (4.6)$$

where

$$\begin{aligned} T_n &= (a_{11} + a_{22}) - (d_{11} + d_{22}) \frac{n^2}{\ell^2}, \\ \tilde{J}_n(\tau) &= d_{11}d_{22} \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} + d_{21}a_{12}v_*e^{-\lambda\tau}) \frac{n^2}{\ell^2} + \text{Det}(A_1) \end{aligned} \quad (4.7)$$

with $\text{Det}(A_1) = a_{11}a_{22} - a_{12}a_{21}$.

When $d_{21} = 0$, from the second mathematical expression in (4.7), we denote

$$J_n := d_{11}d_{22} \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11}) \frac{n^2}{\ell^2} + \text{Det}(A_1), \quad (4.8)$$

314 then from (4.4), (4.5), (4.7) and (4.8), it is easy to verify that $T_n < 0$ and $J_n > 0$ provided that

$$(C_0) : 0 < \beta \leq \frac{(m+1)^2}{2}, \quad 0 < m \leq 1.$$

This implies that when $d_{21} = 0$ and the condition (C_0) holds, the positive constant steady state $E_*(u_*, v_*)$ 316 is asymptotically stable for $d_{11} \geq 0$ and $d_{22} \geq 0$. In this subsection, we always assume that the condition (C_0) holds.

318 Since $J_n > 0$ and $a_{12} < 0$ under the condition (C_0) , then according to (4.6), we have

$$\Gamma_n(0) = J_n - d_{21}a_{12}v_* \frac{n^2}{\ell^2} > 0.$$

This implies that $\lambda = 0$ is not a root of (4.6). Let $\lambda = i\omega_n (\omega_n > 0)$ be a root of (4.6). From (4.4), (4.5) and 320 by substituting $\lambda = i\omega_n (\omega_n > 0)$ into (4.6), and separating the real from the imaginary parts, we have

$$\begin{cases} J_n - \omega_n^2 = \frac{n^2}{\ell^2} d_{21}a_{12}v_* \cos(\omega_n \tau), \\ T_n \omega_n = \frac{n^2}{\ell^2} d_{21}a_{12}v_* \sin(\omega_n \tau), \end{cases} \quad (4.9)$$

which yields

$$\omega^4 + P_n \omega^2 + Q_n = 0, \quad (4.10)$$

322 where

$$P_n = T_n^2 - 2J_n = (d_{11}^2 + d_{22}^2) \frac{n^4}{\ell^4} - 2(d_{11}a_{11} + d_{22}a_{22}) \frac{n^2}{\ell^2} + a_{11}^2 + a_{22}^2 + 2a_{12}a_{21}, \quad (4.11)$$

and

$$Q_n = \left(J_n + d_{21}a_{12}v_* \frac{n^2}{\ell^2} \right) \left(J_n - d_{21}a_{12}v_* \frac{n^2}{\ell^2} \right). \quad (4.12)$$

324 Here,

$$a_{11}^2 + a_{22}^2 + 2a_{12}a_{21} \begin{cases} \leq 0, & c_* \leq 0, \\ > 0, & c_* > 0 \end{cases} \quad (4.13)$$

with

$$c_* = \frac{4\beta^2 - 4\beta(m+1)^2 + (m+1)^4 + \gamma^2(m+1)^4 + 2\beta\gamma(m-1)(m+1)^2}{(m+1)^4}. \quad (4.14)$$

326 Notice that from (4.10), we can define

$$\omega_n^\pm := \sqrt{\frac{-P_n \pm \sqrt{P_n^2 - 4Q_n}}{2}}.$$

Moreover, by combining with (4.4), (4.5), (4.11) and (4.13), if we assume that $c_* > 0$, then $P_n > 0$ for any 328 $n \in \mathbb{N}_0$. Furthermore, by defining

$$d_{21}^{(n)} = -\frac{J_n}{a_{12}v_*(n/\ell)^2} = -\frac{1}{a_{12}v_*} \left(d_{11}d_{22}(n/\ell)^2 + \frac{\text{Det}(A_1)}{(n/\ell)^2} - (d_{11}a_{22} + d_{22}a_{11}) \right) > 0, \quad (4.15)$$

then for fixed n , by (4.12) we have

$$Q_n \begin{cases} > 0, & 0 < d_{21} < d_{21}^{(n)}, \\ = 0, & d_{21} = d_{21}^{(n)}, \\ < 0, & d_{21} > d_{21}^{(n)}. \end{cases} \quad (4.16)$$

Thus, when $d_{21} > d_{21}^{(n)}$, (4.10) has one positive root ω_n , where

$$\omega_n = \sqrt{\frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2}}. \quad (4.17)$$

Notice that $T_n < 0$ for any $n \in \mathbb{N}_0$ and $a_{12} < 0$ under the condition (C_0) , then from the second mathematical expression in (4.9), we have

$$\sin(\omega_n \tau) = \frac{T_n \omega_n}{(n/\ell)^2 a_{12} d_{21} v_*} > 0.$$

Thus, from the first mathematical expression in (4.9), we can set

$$\tau_{n,j} = \frac{1}{\omega_n} \left\{ \arccos \left\{ \frac{J_n - \omega_n^2}{d_{21} a_{12} v_* (n/\ell)^2} \right\} + 2j\pi \right\}, \quad n \in \mathbb{N}, \quad j \in \mathbb{N}_0. \quad (4.18)$$

Furthermore, it is easy to verify that the transversality condition satisfies

$$\left. \frac{d \operatorname{Re}(\lambda(\tau))}{d\tau} \right|_{\tau=\tau_{n,j}} > 0.$$

Furthermore, if we let

$$d_{21}^* = \min_{n \in \mathbb{N}} \{d_{21}^{(n)}\} > 0, \quad (4.19)$$

then from (4.15), it is easy to verify that $d_{21}^{(n)}$ is decreasing for $n < \ell \sqrt[4]{\frac{\operatorname{Det}(A_1)}{d_{11} d_{22}}}$, is increasing for $n > \ell \sqrt[4]{\frac{\operatorname{Det}(A_1)}{d_{11} d_{22}}}$ and $d_{21}^{(n)} \rightarrow \infty$ as $n \rightarrow \infty$. This implies that d_{21}^* exists. For fixed $d_{21} > d_{21}^*$, define an index set

$$U(d_{21}) = \{n \in \mathbb{N} : d_{21}^{(n)} < d_{21}\}.$$

Moreover, according to the above analysis, we have the following results.

Theorem 4.1. *If the condition (C_0) holds and $c_* > 0$, then we have the following conclusions:*

- (a) when $0 < d_{21} \leq d_{21}^*$, the positive constant steady state $E_*(u_*, v_*)$ of system (4.2) is locally asymptotically stable for any $\tau \geq 0$;
- (b) when $d_{21} > d_{21}^*$, if denote

$$\tau_*(d_{21}) = \min_{n \in U(d_{21})} \{\tau_{n,0}\},$$

then the positive constant steady state $E_*(u_*, v_*)$ of system (4.2) is asymptotically stable for $0 \leq \tau < \tau_*(d_{21})$ and unstable for $\tau > \tau_*(d_{21})$. Furthermore, system (4.2) undergoes Hopf bifurcations at $\tau = \tau_{n,0}$ for $n \in U(d_{21})$.

4.1.2. Direction and stability of the Hopf bifurcation

We now investigate the direction and stability of the Hopf bifurcation by some numerical simulations.

In this section, we use the following initial conditions for the system (4.2)

$$u(x, t) = u_0(x), \quad v(x, t) = v_0(x), \quad t \in [-\tau, 0],$$

and we set the parameters as follows

$$d_{11} = 0.6, \quad d_{22} = 0.8, \quad m = 0.5, \quad \gamma = 0.5, \quad \beta = 1, \quad \ell = 2.$$

Figure 1: Stable region (below the blue line and below the red line) and Hopf bifurcation curves $\tau = \tau_{n,0}$, $n = 1, 2, 3$ in $d_{21} - \tau$ plane for the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$. Hopf bifurcation curves $\tau = \tau_{2,0}$ and $\tau = \tau_{3,0}$ intersect at the point $P_1(42.87, 0.817)$.

Figure 2: For the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$, and (d_{21}, τ) is chosen as the point $P_2(2, 4.2)$ which satisfies $0 < d_{21} < d_{21}^{(2)} = 13.98$. The positive constant steady state $E_*(u_*, v_*) = (0.3333, 0.3333)$ is locally asymptotically stable for any $\tau \geq 0$, see (a)-(d) for detail. The initial values are $u_0(x) = 0.3333 - 0.1 \cos(x)$, $v_0(x) = 0.3333 + 0.1 \cos(x)$.

Then, according to (4.3), (4.4) and (4.5), we have $E_*(u_*, v_*) = (0.3333, 0.3333)$,

$$a_{11} = -0.1111, a_{12} = -0.2222, a_{21} = 0.5, a_{22} = -0.5.$$

It follows from (4.11) and (4.15) that

$$P_n = 0.0625n^4 + 0.2333n^2 + 0.0401 > 0$$

and

$$d_{21}^{(n)} = 1.62n^2 + \frac{9}{n^2} + 5.25. \quad (4.20)$$

Notice that $P_n > 0$ for any $n \in \mathbb{N}_0$, which together with (4.16), implies that for a fixed n , (4.10) has no positive root for $d_{21} < d_{21}^{(n)}$ and has only one positive root for $d_{21} \geq d_{21}^{(n)}$. From (4.20), it is easy to verify that $d_{21}^{(n)} < d_{21}^{(n+1)}$ for any $n \in \mathbb{N}_0$, and

$$d_{21}^{(1)} = 15.87, d_{21}^{(2)} = 13.98 < d_{21}^{(3)} = 20.83. \quad (4.21)$$

Therefore, by combining with (4.19) and (4.21), we have $d_{21}^* = d_{21}^{(2)} = 13.98$. It follows from (4.14) that $c_* = 0.0401 > 0$. By Theorem 4.1, we have the following Propositions 4.1 and 4.2.

Proposition 4.2. *For system (4.2) with the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$, when $0 \leq d_{21} < d_{21}^{(2)} = 13.98$, the positive constant steady state $E_*(u_*, v_*) = (0.3333, 0.3333)$ is locally asymptotically stable for any $\tau \geq 0$.*

Figure 1 illustrates the stability region, and the Hopf bifurcation curves are plotted in the $d_{21} - \tau$ plane for $20 \leq d_{21} \leq 150$. The Hopf bifurcation curves $\tau = \tau_{2,0}$ and $\tau = \tau_{3,0}$ intersect at the point $P_1(42.87, 0.817)$, which is the Hopf-Hopf bifurcation point. Furthermore, when $d_{21} = 2$ and $\tau = 4.2$, according to (4.21), we can see that the point $P_2(2, 4.2)$ satisfies $0 \leq d_{21} < d_{21}^{(2)} = 13.98$. According to Proposition 1, we know that under the above parameter settings, as long as $0 \leq d_{21} < d_{21}^{(2)} = 13.98$, the positive constant steady state $E_*(u_*, v_*)$ of system (4.2) is locally asymptotically stable for any $\tau \geq 0$. Especially, by taking the point $P_2(2, 4.2)$ which satisfies $0 \leq d_{21} < d_{21}^{(2)} = 13.98$, we illustrate this result in Fig.2 with the initial values $u_0(x) = 0.3333 - 0.1 \cos(x)$, $v_0(x) = 0.3333 + 0.1 \cos(x)$.

Proposition 4.3. *For system (4.2) with the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$, and for fixed $d_{21} > d_{21}^{(2)} = 13.98$, the positive constant steady state $E_*(u_*, v_*)$ is asymptotically stable for $\tau < \tau_*(d_{21})$ and unstable for $\tau > \tau_*(d_{21})$.*

From Fig.1, it is obvious to see that

$$\tau_*(d_{21}) = \begin{cases} \tau_{2,0}, & d_{21}^{(2)} < d_{21} < 42.87, \\ \tau_{3,0}, & 42.87 < d_{21} < 150, \end{cases}$$

and when $d_{21} = 21$, it follows from (4.17) and (4.18) that

$$\tau_{2,0} = 2.5896 < \tau_{3,0} = 17.9261. \quad (4.22)$$

For $d_{21} = 21$ which satisfies $d_{21}^{(2)} = 13.98 < d_{21} < 42.87$, according to (4.22), we know that system (4.2) undergoes Hopf bifurcation at $\tau_{2,0} = 2.5896$. Furthermore, the direction and stability of the Hopf

bifurcation can be determined by calculating K_1K_2 and K_2 using the procedures listed in Appendix A. By a direct calculation, we obtain

$$K_1 = 0.1092 > 0, \quad K_2 = 103.5071 > 0, \quad K_1K_2 = 11.2997 > 0,$$

which implies that the spatially inhomogeneous Hopf bifurcation at $\tau_{2,0}$ is subcritical and unstable. When $d_{21} = 21$ and $\tau = 1.5$, by combining with (4.21) and (4.22), we can see that the point $P_3(21, 1.5)$ satisfies $d_{21}^{(2)} = 13.98 < d_{21} < 42.87$ and $\tau = 1.5 < \tau_{2,0} = 2.5896$. There exists an unstable spatially inhomogeneous periodic solution, and its amplitude is decreasing, see Fig.3 (a)-(d) for detail. The initial values are $u_0(x) = 0.3333 + 0.02 \cos(x)$, $v_0(x) = 0.3333 + 0.02 \cos(x)$.

Figure 3: For the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$, and (d_{21}, τ) is chosen as the point $P_3(21, 1.5)$ which satisfies $d_{21}^{(2)} < d_{21} < 42.87$ and $\tau < \tau_*(d_{21}^{(2)}) = 2.5896$. There exists an unstable spatially inhomogeneous periodic solution, and its amplitude is decreasing. The initial values are $u_0(x) = 0.3333 + 0.02 \cos(x)$, $v_0(x) = 0.3333 + 0.02 \cos(x)$.

For $d_{21} = 43$ which satisfies $42.87 < d_{21} < 150$, it follows from (4.17) and (4.18) that

$$\tau_{3,0} = 0.813 < \tau_{2,0} = 0.8138. \quad (4.23)$$

According to (4.23), we know that system (4.2) undergoes Hopf bifurcation at $\tau_{3,0} = 0.813$. Furthermore, the direction and stability of the Hopf bifurcation can be determined by calculating K_1K_2 and K_2 using the procedures listed in Appendix A. By a direct calculation, we obtain

$$K_1 = 0.4024 > 0, \quad K_2 = 326.1951 > 0, \quad K_1K_2 = 131.2501 > 0,$$

which implies that the spatially inhomogeneous Hopf bifurcation at $\tau_{3,0}$ is subcritical and unstable. When $d_{21} = 43$ and $\tau = 0.4$, by combining with (4.21) and (4.22), we can see that the point $P_4(43, 0.4)$ satisfies $42.87 < d_{21} < 150$ and $\tau = 0.4 < \tau_{3,0} = 0.813$. There exists an unstable spatially inhomogeneous periodic

Figure 4: For the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$, and (d_{21}, τ) is chosen as the point $P_4(43, 0.4)$ which satisfies $42.87 < d_{21} < 150$ and $\tau < \tau_*(d_{21}^{(3)}) = 0.813$. There exists an unstable spatially inhomogeneous periodic solution, and its amplitude is decreasing. The initial values are $u_0(x) = 0.3333 + 0.02 \cos(3x/2)$, $v_0(x) = 0.3333 + 0.02 \cos(3x/2)$.

solution, and its amplitude is decreasing, see Fig.4 (a)-(d) for detail. The initial values are $u_0(x) = 0.3333 + 0.02 \cos(3x/2)$, $v_0(x) = 0.3333 + 0.02 \cos(3x/2)$.

4.2. The case of with memory and gestation delays

When system (4.1) includes memory and gestation delays, that is to say, in the model (1.3), we let

$$\begin{aligned} f(u(x, t), v(x, t)) &= u(x, t) (1 - u(x, t)) - \frac{\beta u^2(x, t) v(x, t)}{u^2(x, t) + m v^2(x, t)}, \\ g(u(x, t), v(x, t)) &= \gamma v(x, t) \left(1 - \frac{v(x, t)}{u(x, t - \tau)} \right). \end{aligned}$$

Then, the model (1.3) can be written as

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = d_{11} \Delta u(x, t) + u(x, t) (1 - u(x, t)) - \frac{\beta u^2(x, t) v(x, t)}{u^2(x, t) + m v^2(x, t)}, & x \in (0, \ell\pi), t > 0, \\ \frac{\partial v(x, t)}{\partial t} = d_{22} \Delta v(x, t) - d_{21} (v(x, t) u_x(x, t - \tau))_x + \gamma v(x, t) \left(1 - \frac{v(x, t)}{u(x, t - \tau)} \right), & x \in (0, \ell\pi), t > 0, \\ u_x(0, t) = u_x(\ell\pi, t) = v_x(0, t) = v_x(\ell\pi, t) = 0, & t \geq 0, \\ u(x, t) = u_0(x, t), v(x, t) = v_0(x, t), & x \in (0, \ell\pi), -\tau \leq t \leq 0. \end{cases} \quad (4.24)$$

Notice that for the system (4.24), the normal form for Hopf bifurcation can be calculated by using our developed algorithm in Section 2. In the following, we first give the stability and Hopf bifurcation analysis for the system (4.24), then by employing our developed procedure in Section 2 for calculating the normal form of Hopf bifurcation, the direction and stability of the Hopf bifurcation are determined.

4.2.1. Stability and Hopf bifurcation analysis

400 The system (4.24) has the positive constant steady state $E_*(u_*, v_*)$, where

$$u_* = v_* = 1 - \frac{\beta}{m+1} \quad (4.25)$$

with $0 < \beta < m+1$. For $E_*(u_*, v_*)$, from (2.4), when $m > 1$, we have

$$a_{11} = \frac{2\beta}{(m+1)^2} - 1 \begin{cases} \leq 0, & 0 < \beta \leq \frac{(m+1)^2}{2}, \\ > 0, & \beta > \frac{(m+1)^2}{2}. \end{cases}$$

402 Notice that when $m > 1$, if $a_{11} > 0$, then we have $\beta > \frac{(m+1)^2}{2} > m+1$, which is contradict to the condition $0 < \beta < m+1$. Thus, when $m > 1$, $a_{11} \leq 0$ under the condition $0 < \beta < m+1$. When $0 < m < 1$, we have

$$a_{11} = \frac{2\beta}{(m+1)^2} - 1 \begin{cases} \leq 0, & 0 < \beta \leq \frac{(m+1)^2}{2}, \\ > 0, & \frac{(m+1)^2}{2} < \beta < m+1. \end{cases} \quad (4.26)$$

404 Figure 5 shows the curves $f_1 = m+1$ and $f_2 = (m+1)^2/2$ for $0 \leq m \leq 3$, and they intersect at the point $P(1, 2)$.

Figure 5: The curves $f_1 = m+1$ and $f_2 = (m+1)^2/2$ for $0 \leq m \leq 3$. The intersection point is $P(1, 2)$.

406 Furthermore, we have

$$\begin{aligned} a_{12} &= \frac{\beta(m-1)}{(m+1)^2} \begin{cases} \leq 0, & 0 < m \leq 1, \\ > 0, & m > 1, \end{cases} \\ a_{21} &= 0, \quad a_{22} = -\gamma < 0, \\ b_{11} &= 0, \quad b_{12} = 0, \\ b_{21} &= \gamma > 0, \quad b_{22} = 0. \end{aligned} \quad (4.27)$$

Moreover, by combining with (4.26), (4.27),

$$D_1 = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{2\beta}{(m+1)^2} - 1 & \frac{\beta(m-1)}{(m+1)^2} \\ 0 & -\gamma \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$$

408 and

$$\mathcal{M}_n(\lambda) = \lambda I_2 + \frac{n^2}{\ell^2} D_1 + \frac{n^2}{\ell^2} e^{-\lambda\tau} D_2 - A_1 - A_2 e^{-\lambda\tau},$$

or according to (2.7), the characteristic equation of system (4.24) can be written as

$$\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda)) = \lambda^2 - T_n \lambda + \tilde{J}_n(\tau) = 0, \quad (4.28)$$

410 where

$$\begin{aligned} T_n &= (a_{11} + a_{22}) - (d_{11} + d_{22}) \frac{n^2}{\ell^2}, \\ \tilde{J}_n(\tau) &= d_{11}d_{22} \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} + d_{21}a_{12}v_*e^{-\lambda\tau}) \frac{n^2}{\ell^2} + a_{11}a_{22} - a_{12}b_{21}e^{-\lambda\tau}. \end{aligned} \quad (4.29)$$

Notice that when $\tau = 0$, the characteristic equation (4.28) becomes

$$\lambda^2 - T_n\lambda + \tilde{J}_n(0) = 0, \quad (4.30)$$

412 where $\tilde{J}_n(0)$ is defined by

$$\tilde{J}_n(0) = d_{11}d_{22} \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} + d_{21}a_{12}v_*) \frac{n^2}{\ell^2} + a_{11}a_{22} - a_{12}b_{21}.$$

A set of sufficient and necessary condition that all roots of (4.30) have a negative real part is $T_n < 0$, $\tilde{J}_n(0) >$
414 0 , which is always holds provided that $a_{11} < 0$ and $a_{12} < 0$, i.e.,

$$(C_0) : 0 < \beta < \frac{(m+1)^2}{2}, \quad 0 < m < 1.$$

This implies that when $\tau = 0$ and the condition (C_0) holds, the positive steady state $E_*(u_*, v_*)$ is asymp-
416 totically stable for $d_{11} \geq 0$, $d_{21} \geq 0$ and $d_{22} \geq 0$. Meanwhile, if we let $d_{21} = 0$, then we have

$$\tilde{J}_n := d_{11}d_{22} \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11}) \frac{n^2}{\ell^2} + a_{11}a_{22} - a_{12}b_{21}.$$

It is easy to verify that $T_n < 0$ and $\tilde{J}_n > 0$ provided that the condition (C_0) holds. This implies that when
418 $d_{21} = 0$, $\tau = 0$ and the condition (C_0) holds, the positive steady state $E_*(u_*, v_*)$ is asymptotically stable
for $d_{11} \geq 0$ and $d_{22} \geq 0$. Furthermore, since $\Gamma_n(0) = \tilde{J}_n(0) > 0$ under the condition (C_0) , this implies that
420 $\lambda = 0$ is not a root of (4.28).

In the following, we let

$$J_n = d_{11}d_{22} \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11}) \frac{n^2}{\ell^2} + a_{11}a_{22}. \quad (4.31)$$

422 Furthermore, let $\lambda = i\omega_n$ ($\omega_n > 0$) be a root of (4.28). By substituting it along with expressions in (4.26)
and (4.27) into (4.28), and separating the real part from the imaginary part, we have

$$\begin{cases} J_n - \omega_n^2 = \left(d_{21}a_{12}v_* \frac{n^2}{\ell^2} + a_{12}b_{21} \right) \cos(\omega_n\tau), \\ T_n\omega_n = \left(d_{21}a_{12}v_* \frac{n^2}{\ell^2} + a_{12}b_{21} \right) \sin(\omega_n\tau), \end{cases} \quad (4.32)$$

424 which yields

$$\omega_n^4 + P_n\omega_n^2 + Q_n = 0, \quad (4.33)$$

where

$$\begin{aligned} P_n &= T_n^2 - 2J_n \\ &= (d_{11}^2 + d_{22}^2) \frac{n^4}{\ell^4} - 2(d_{11}a_{11} + d_{22}a_{22}) \frac{n^2}{\ell^2} + a_{11}^2 + a_{22}^2, \end{aligned} \quad (4.34)$$

426 and

$$Q_n = \left(J_n + \left(d_{21}a_{12}v_* \frac{n^2}{\ell^2} + a_{12}b_{21} \right) \right) \left(J_n - \left(d_{21}a_{12}v_* \frac{n^2}{\ell^2} + a_{12}b_{21} \right) \right). \quad (4.35)$$

It is easy to verify that $P_n > 0$ for any $n \in \mathbb{N}_0$. Thus, (4.33) has one positive root when $Q_n < 0$. In the following, we will discuss several cases under the condition (C_0) , which are used to guarantee $Q_n < 0$.

When $\tau > 0$, according to (4.31) and (4.35), we can define $Q_n = \Gamma_n(0)\tilde{Q}_n$ with

$$\Gamma_n(0) = \tilde{J}_n(0) = d_{11}d_{22}\frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} + d_{21}a_{12}v_*)\frac{n^2}{\ell^2} + a_{11}a_{22} - a_{12}b_{21}$$

and

$$\tilde{Q}_n = d_{11}d_{22}\frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - d_{21}a_{12}v_*)\frac{n^2}{\ell^2} + a_{11}a_{22} + a_{12}b_{21} = 0, \quad (4.36)$$

and then by a simple analysis, we have $\Gamma_n(0) = \tilde{J}_n(0) > 0$ for any $n \in \mathbb{N}_0$. Therefore, the sign of Q_n coincides with that of $\tilde{Q}_n$, and in order to guaranteeing $Q_n < 0$, we only need to study the case of $\tilde{Q}_n < 0$.

Case 4.4. *It is easy to see that if the conditions (C_0) and*

$$(C_1) : d_{11}a_{22} + d_{22}a_{11} - d_{21}a_{12}v_* < 0, \quad a_{11}a_{22} + a_{12}b_{21} > 0$$

or

$$(C_{11}) : (d_{11}a_{22} + d_{22}a_{11} - d_{21}a_{12}v_*)^2 - 4d_{11}d_{22}(a_{11}a_{22} + a_{12}b_{21}) < 0$$

holds, then (4.36) has no positive roots. Hence, all roots of (4.28) have negative real parts when $\tau \in [0, +\infty)$ under the conditions (C_0) and (C_1) or (C_{11}) .

Case 4.5. *If the conditions (C_0) and*

$$(C_2) : a_{11}a_{22} + a_{12}b_{21} < 0$$

or

$$(C_{21}) : d_{11}a_{22} + d_{22}a_{11} - d_{21}a_{12}v_* > 0,$$

$$(d_{11}a_{22} + d_{22}a_{11} - d_{21}a_{12}v_*)^2 - 4d_{11}d_{22}(a_{11}a_{22} + a_{12}b_{21}) = 0$$

hold, then (4.36) has a positive root. Notice that when $d_{11}a_{22} + d_{22}a_{11} - d_{21}a_{12}v_* > 0$, $\tilde{Q}_n \geq 0$ for all $n \in \mathbb{N}_0$ and we only need to study the case of $\tilde{Q}_n < 0$, thus for the case 4.5, we only consider the condition (C_2) . Moreover, if we let $\tilde{x} = n^2/\ell^2$, then the mathematical expression of $\tilde{Q}_n$ can be rewritten as

$$\tilde{f}(\tilde{x}) = d_{11}d_{22}\tilde{x}^2 - (d_{11}a_{22} + d_{22}a_{11} - d_{21}a_{12}v_*)\tilde{x} + a_{11}a_{22} + a_{12}b_{21}, \quad (4.37)$$

and the unique positive root of this equation is

$$\tilde{x}_* = \frac{d_{11}a_{22} + d_{22}a_{11} - d_{21}a_{12}v_* + \sqrt{(d_{11}a_{22} + d_{22}a_{11} - d_{21}a_{12}v_*)^2 - 4d_{11}d_{22}(a_{11}a_{22} + a_{12}b_{21})}}{2d_{11}d_{22}} \quad (4.38)$$

under the conditions (C_0) and (C_2) or (C_{21}) . Since $\tilde{x}_* = n^2/\ell^2$, then $n_0 = \ell\sqrt{\tilde{x}_*}$, and notice that $\tilde{Q}_n$ is a quadratic polynomial with respect to n^2/ℓ^2 and $\tilde{Q}_0 \leq 0$ under the condition (C_2) . Thus, when the condition (C_2) holds, we can conclude that there exists $n_0 > 0$ such that $\tilde{Q}_{n_0} = 0$ and

$$Q_n = \Gamma_n(0)\tilde{Q}_n \begin{cases} < 0, & 0 \leq n \leq n_*, \\ \geq 0, & n \geq n_* + 1, \end{cases} \quad (4.39)$$

446 where $n \in \mathbb{N}_0$, and n_* is defined by

$$n_* = \begin{cases} n_0 - 1, & n_0 \in \mathbb{N}, \\ [n_0], & n_0 \notin \mathbb{N}. \end{cases} \quad (4.40)$$

Here, $[\cdot]$ stands for the integer part function. Therefore, (4.33) has one positive root ω_n for $0 \leq n \leq n_*$ with
448 $n \in \mathbb{N}_0$, where

$$\omega_n = \sqrt{\frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2}}. \quad (4.41)$$

By combining with (4.32), and notice that $a_{12} < 0$, $T_n < 0$ under the condition (C_0) , then we have

$$\sin(\omega_n \tau) = \frac{T_n \omega_n}{d_{21} a_{12} v_*(n^2/\ell^2) + a_{12} b_{21}} > 0.$$

450 Thus, from the first mathematical expression in (4.32), we can set

$$\tau_{n,j} = \frac{1}{\omega_n} \left\{ \arccos \left\{ \frac{J_n - \omega_n^2}{d_{21} a_{12} v_*(n^2/\ell^2) + a_{12} b_{21}} \right\} + 2j\pi \right\}, \quad n \in \mathbb{N}_0, \quad j \in \mathbb{N}_0. \quad (4.42)$$

Case 4.6. If the conditions (C_0) and

$$(C_3): \quad d_{11} a_{22} + d_{22} a_{11} - d_{21} a_{12} v_* > 0, \quad a_{11} a_{22} + a_{12} b_{21} > 0, \\ (d_{11} a_{22} + d_{22} a_{11} - d_{21} a_{12} v_*)^2 - 4d_{11} d_{22} (a_{11} a_{22} + a_{12} b_{21}) > 0$$

452 hold, then the (4.36) has two positive roots. Without loss of generality, we assume that the two positive roots of (4.37) are $\tilde{x}_1$ and $\tilde{x}_2$, i.e.,

$$\tilde{x}_{1,2} = \frac{d_{11} a_{22} + d_{22} a_{11} - d_{21} v_* a_{12} \mp \sqrt{(d_{11} a_{22} + d_{22} a_{11} - d_{21} v_* a_{12})^2 - 4d_{11} d_{22} (a_{11} a_{22} + a_{12} b_{21})}}{2d_{11} d_{22}}$$

454 under the conditions (C_0) and (C_3) . Since $\tilde{x}_1 = n_1^2/\ell^2$ and $\tilde{x}_2 = n_2^2/\ell^2$, then $n_1 = \ell\sqrt{\tilde{x}_1}$ and $n_2 = \ell\sqrt{\tilde{x}_2}$. By using a geometric argument, we can conclude that

$$Q_n = \Gamma_n(0) \tilde{Q}_n \begin{cases} < 0, & n_1 < n < n_2, \\ \geq 0, & n \leq n_1 \text{ or } n \geq n_2, \end{cases}$$

456 where $n \in \mathbb{N}_0$. Therefore, (4.33) has one positive root ω_n^+ for $n_1 < n < n_2$ with $n \in \mathbb{N}_0$, where

$$\omega_n^+ = \sqrt{\frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2}}.$$

Furthermore, by combining with the second mathematical expression in (4.32), and notice that $a_{12} < 0$,
458 $T_n < 0$ under the condition (C_0) , then we have $\sin(\omega_n^+ \tau) > 0$. Thus, from the first mathematical expression in (4.32), we can set

$$\tau_{n,j}^+ = \frac{1}{\omega_n^+} \left\{ \arccos \left\{ \frac{J_n - (\omega_n^+)^2}{d_{21} a_{12} v_*(n^2/\ell^2) + a_{12} b_{21}} \right\} + 2j\pi \right\}, \quad n \in \mathbb{N}_0, \quad j \in \mathbb{N}_0.$$

460 Next, we continue to verify the transversality conditions for the Cases 4.5 and 4.6.

Lemma 4.7. Suppose that the conditions (C_0) and (C_2) hold, and $0 \leq n \leq n_*$ with $n \in \mathbb{N}_0$, then we have

$$\left. \frac{d \operatorname{Re}(\lambda(\tau))}{d\tau} \right|_{\tau=\tau_{n,j}} > 0,$$

462 where $\operatorname{Re}(\lambda(\tau))$ represents the real part of $\lambda(\tau)$.

Proof. By differentiating the two sides of

$$\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda)) = \lambda^2 - T_n\lambda + \tilde{J}_n(\tau) = 0$$

464 with respect to τ , where T_n and $\tilde{J}_n(\tau)$ are defined by (4.29), we have

$$\left(\frac{d\lambda(\tau)}{d\tau}\right)^{-1} = \frac{(2\lambda - T_n)e^{\lambda\tau}}{-\lambda(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})} - \frac{\tau}{\lambda}. \quad (4.43)$$

Therefore, by (4.43), we have

$$\begin{aligned} \operatorname{Re}\left(\frac{d\lambda(\tau)}{d\tau}\Big|_{\tau=\tau_{n,j}}\right)^{-1} &= \operatorname{Re}\left(\frac{(2i\omega_n - T_n)e^{i\omega_n\tau_{n,j}}}{-i\omega_n(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})}\right) \\ &= \operatorname{Re}\left(\frac{(2i\omega_n - T_n)(\cos(\omega_n\tau_{n,j}) + i\sin(\omega_n\tau_{n,j}))}{-i\omega_n(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})}\right) \\ &= \operatorname{Re}\left(\frac{(2i\omega_n - T_n)\cos(\omega_n\tau_{n,j})}{-i\omega_n(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})} + \frac{i(2i\omega_n - T_n)\sin(\omega_n\tau_{n,j})}{-i\omega_n(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})}\right) \\ &= \frac{T_n\sin(\omega_n\tau_{n,j})}{\omega_n(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})} - \frac{2\cos(\omega_n\tau_{n,j})}{(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})}. \end{aligned} \quad (4.44)$$

466 Furthermore, according to (4.32), we have

$$\sin(\omega_n\tau_{n,j}) = \frac{T_n\omega_n}{(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})}, \quad \cos(\omega_n\tau_{n,j}) = \frac{J_n - \omega_n^2}{(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})}. \quad (4.45)$$

Moreover, by combining with (4.44), (4.45) and

$$\omega_n = \sqrt{\frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2}} > 0, \quad a_{12} < 0, \quad P_n = T_n^2 - 2J_n > 0, \quad Q_n < 0,$$

468 we have

$$\begin{aligned} \operatorname{Re}\left(\frac{d\lambda(\tau)}{d\tau}\Big|_{\tau=\tau_{n,j}}\right)^{-1} &= \frac{T_n\sin(\omega_n\tau_{n,j})}{\omega_n(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})} - \frac{2\cos(\omega_n\tau_{n,j})}{(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})} \\ &= \frac{T_n^2 - 2(J_n - \omega_n^2)}{(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})^2} = \frac{\sqrt{P_n^2 - 4Q_n}}{(d_{21}a_{12}v_*(n^2/\ell^2) + a_{12}b_{21})^2} > 0. \end{aligned}$$

This, together with the fact that

$$\operatorname{sign}\left\{\frac{d\operatorname{Re}(\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_{n,j}}\right\} = \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda(\tau)}{d\tau}\Big|_{\tau=\tau_{n,j}}\right)^{-1}\right\}$$

470 completes the proof, where $\operatorname{sign}(\cdot)$ represents the sign function. ■

Remark 4.8. Similarly, if we suppose that the conditions (C_0) and (C_3) hold, and $n_1 < n < n_2$ with

472 $n \in \mathbb{N}_0$, then we have

$$\frac{d\operatorname{Re}(\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_{n,j}^+} > 0.$$

Notice that the transversality condition for $\tau = \tau_{n,j}^+$ can be verified by a similar argument in Lemma 4.7, we

474 hence omit here.

Moreover, according to the above analysis, we have the following results.

Lemma 4.9. *If the condition (C_0) is satisfied, then we have the following conclusions:*

(i) *if the condition (C_1) or (C_{11}) holds, then the positive constant steady state $E_*(u_*, v_*)$ of system (4.24) is asymptotically stable for all $\tau \geq 0$;*

(ii) *if the condition (C_2) holds, and denote $\tau_* = \min \{\tau_{n,0} : 0 \leq n \leq n_*, n \in \mathbb{N}_0\}$, then the positive constant steady state $E_*(u_*, v_*)$ of system (4.24) is asymptotically stable for $0 \leq \tau < \tau_*$ and unstable for $\tau > \tau_*$. Furthermore, system (4.24) undergoes Hopf bifurcations at $\tau = \tau_{n,0}$ for $n \in \mathbb{N}_0$. If $n = 0$, then the bifurcating periodic solutions are all spatially homogeneous, and when $n \geq 1$ and $n \in \mathbb{N}$, these bifurcating periodic solutions are spatially inhomogeneous;*

(iii) *if the condition (C_3) holds, and denote $\tau_* = \min \{\tau_{n,0}^+ : n_1 < n < n_2, n \in \mathbb{N}_0\}$, then the positive constant steady state $E_*(u_*, v_*)$ of system (4.24) is asymptotically stable for $0 \leq \tau < \tau_*$ and unstable for $\tau > \tau_*$. Furthermore, system (4.24) undergoes Hopf bifurcations at $\tau = \tau_{n,0}^+$ for $n \in \mathbb{N}_0$. If $n = 0$, then the bifurcating periodic solutions are all spatially homogeneous, and when $n \geq 1$ and $n \in \mathbb{N}$, these bifurcating periodic solutions are spatially inhomogeneous.*

4.2.2. Direction and stability of the Hopf bifurcation

Figure 6: For the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$ and $\tau = 6 < \tau_{0,0} = 10.078$, the positive constant steady state $E_*(u_*, v_*) = (0.3333, 0.3333)$ is locally asymptotically stable. The initial values are $u_0(x) = 0.3333 + 0.01$, $v_0(x) = 0.3333 + 0.01$.

In this section, we verify the analytical results given in the previous sections by some numerical simulations and investigate the direction and stability of the Hopf bifurcation. We use the following initial conditions for the system (4.24)

$$u(x, t) = u_0(x), \quad v(x, t) = v_0(x), \quad t \in [-\tau, 0],$$

Figure 7: For the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$ and $\tau = 6 < \tau_{0,0} = 10.078$, the behavior and phase portrait of system (4.24) is shown. The initial values are $u_0(x) = 0.3333 + 0.01$, $v_0(x) = 0.3333 + 0.01$.

Figure 8: For the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$ and $\tau = 13 > \tau_{0,0} = 10.078$, there exists a stable spatially homogeneous periodic solution. The initial values are $u_0(x) = 0.3333 - 0.01$, $v_0(x) = 0.3333 + 0.01$.

Figure 9: For the parameters $d_{11} = 0.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$ and $\tau = 13 > \tau_{0,0} = 10.078$, the behavior and phase portrait of system (4.24) is shown. The initial values are $u_0(x) = 0.3333 - 0.01$, $v_0(x) = 0.3333 + 0.01$.

and we set the parameters as follows

$$d_{11} = 0.6, d_{21} = 3.6, d_{22} = 0.8, m = 0.5, \gamma = 0.5, \beta = 1, \ell = 2,$$

we can easily obtain that

$$0 < \beta = 1 < m + 1 = 1.5, 0 < m = 0.5 < 1, a_{11}a_{22} + a_{12}b_{21} = -0.0556 < 0.$$

Therefore, the conditions (C_0) and (C_2) are satisfied under the above parameters settings. In the following, we mainly verify the conclusion in Lemma 4.9 (ii). According to (4.25), (4.26) and (4.27), we have $E_*(u_*, v_*) = (0.3333, 0.3333)$,

$$a_{11} = -0.1111, a_{12} = -0.2222, a_{21} = 0, a_{22} = -0.5,$$

$$b_{11} = 0, b_{12} = 0, b_{21} = 0.5, b_{22} = 0.$$

It follows from (4.34) that

$$P_n = 0.0625n^4 + 0.2333n^2 + 0.2623 > 0.$$

Notice that $P_n > 0$ for any $n \in \mathbb{N}_0$, which together with (4.39) and Lemma 4.9 (ii), implies that for a fixed n , (4.33) has only one positive root for $0 \leq n \leq n_*$. Furthermore, by combining with (4.38), (4.39), (4.40), (4.41) and (4.42), we have $n_* = 0$, $\omega_c = \omega_0 = 0.1775$ and $\tau_c = \tau_{0,0} = 10.078$.

Moreover, by Lemma 4.9 (ii), we have the following proposition.

Proposition 4.10. *For system (4.24) with the parameters $d_{11} = 0.6$, $d_{21} = 3.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$, the positive constant steady state $E_*(u_*, v_*)$ of system (4.24) is asymptotically stable for $0 \leq \tau < \tau_{0,0} = 10.078$ and unstable for $\tau > \tau_{0,0} = 10.078$. Furthermore, system (4.24) undergoes a Hopf bifurcation at the positive constant steady state $E_*(u_*, v_*)$ when $\tau = \tau_{0,0} = 10.078$.*

For the parameters $d_{11} = 0.6$, $d_{21} = 3.6$, $d_{22} = 0.8$, $m = 0.5$, $\gamma = 0.5$, $\beta = 1$, $\ell = 2$, according to Proposition 4.11, we know that system (4.24) undergoes Hopf bifurcation at $\tau_{0,0} = 10.078$. Furthermore, the direction and stability of the Hopf bifurcation can be determined by calculating K_1K_2 and K_2 using the procedures developed in Section 2. After a direct calculation using MATLAB software, we obtain

$$K_1 = 0.0366 > 0, K_2 = -18.8010 < 0, K_1K_2 = -0.6874 < 0,$$

which implies that the Hopf bifurcation at $\tau_{0,0} = 10.078$ is supercritical and stable.

When $\tau = 6 < \tau_{0,0} = 10.078$, Fig.6 (a)-(d) illustrate the evolution of the solution of system (4.24) starting from the initial values $u_0(x) = 0.3333 + 0.01$, $v_0(x) = 0.3333 + 0.01$, finally converging to the positive constant steady state $E_*(u_*, v_*)$. When $\tau = 6 < \tau_{0,0} = 10.078$, Fig.7 shows the behavior and phase portrait of system (4.24). Furthermore, when $\tau = 13 > \tau_{0,0} = 10.078$, Fig.8 (a)-(d) illustrate the existence of the spatially homogeneous periodic solution with the initial values $u_0(x) = 0.3333 - 0.01$, $v_0(x) = 0.3333 + 0.01$. When $\tau = 13 > \tau_{0,0} = 10.078$, Fig.9 shows the behavior and phase portrait of system (4.24).

5. Conclusion and discussion

In this paper, we have developed an algorithm for calculating the normal form of Hopf bifurcation in a diffusive system with memory and general delays. By considering that apart from the memory delay

appears in the diffusion term, the general delay also occurs in the reaction term, the traditional algorithm for calculating the normal form of Hopf bifurcation in the memory-based system which only the memory delay appears in the diffusion term is not applied to this system. To bridge the gap, we derive an algorithm for calculating the normal form associated with the Hopf bifurcation in a diffusive system with memory and general delays, which can be seen a generalization of the existing algorithm for the reaction-diffusion system where only the memory delay appears in the diffusion term.

In order to show the effectiveness of our developed algorithm, and as an application, we consider a diffusive predator-prey model with ratio-dependent Holling type-III functional response, which includes with memory and gestation delays. The memory and gestation delays induced spatially homogeneous Hopf bifurcation is observed.

It is also worth mentioning that in this paper, we assume that the memory delay is equal to the general delay, but different memory delay and general delay are more real. By considering this point, a more reasonable revision of (1.3) would be

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = d_{11} \Delta u(x, t) + f(u(x, t), v(x, t), u(x, t - \sigma), v(x, t - \sigma)), \\ \frac{\partial v(x, t)}{\partial t} = d_{22} \Delta v(x, t) - d_{21} (v(x, t) u_x(x, t - \tau))_x + g(u(x, t), v(x, t), u(x, t - \sigma), v(x, t - \sigma)), \end{cases}$$

which needs further research, where $\sigma > 0$ is the general delay appears in the reaction term, and $\tau = \sigma$ or $\tau \neq \sigma$.

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Declarations

This research did not involve human participants and animals.

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Availability of data and material: This research didn't involve the private data, and the involving data and material are all available.

Code availability: The numerical simulations in this paper are carried by using the MATLAB software.

Authors' contributions: This manuscript is investigated and written by Yehu Lv.

Appendix A

Remark 5.1. Assume that at $\tau = \tau_c$, (4.6) has a pair of purely imaginary roots $\pm i\omega_{n_c}$ with $\omega_{n_c} > 0$ for $n = n_c \in \mathbb{N}$ and all other eigenvalues have negative real part. Let $\lambda(\tau) = \alpha_1(\tau) \pm i\alpha_2(\tau)$ be a pair of roots of (4.6) near $\tau = \tau_c$ satisfying $\alpha_1(\tau_c) = 0$ and $\alpha_2(\tau_c) = \omega_{n_c}$. In addition, the corresponding transversality condition holds.

554 The normal form of Hopf bifurcation for the system (4.2) can be calculated by using the developed algorithm in [26]. Here, we give the detail calculation procedures of $B_1, B_{21}, B_{22}, B_{23}$ steps by steps.

Step 1:

$$B_1 = 2\psi^T(0) \left(A_1\phi(0) - \frac{n_c^2}{\ell^2} (D_1\phi(0) + D_2\phi(-1)) \right)$$

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$$D_1 = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{2\beta}{(m+1)^2} - 1 & \frac{\beta(m-1)}{(m+1)^2} \\ \gamma & -\gamma \end{pmatrix}.$$

Here,

$$\phi = \begin{pmatrix} 1 \\ \frac{i\omega_{n_c} + (n_c/\ell)^2 d_{11} - a_{11}}{a_{12}} \end{pmatrix}, \quad \psi = \eta \begin{pmatrix} 1 \\ \frac{a_{12}}{i\omega_{n_c} + (n_c/\ell)^2 d_{22} - a_{22}} \end{pmatrix}$$

558 with

$$\eta = \frac{i\omega_{n_c} + (n_c/\ell)^2 d_{22} - a_{22}}{2i\omega_{n_c} + (n_c/\ell)^2 d_{11} - a_{11} + (n_c/\ell)^2 d_{22} - a_{22} + \tau_c a_{12} d_{21} v_* (n_c/\ell)^2 e^{-i\omega_c}}.$$

Step 2:

$$B_{21} = \frac{3}{2\ell\pi} \psi^T A_{21}$$

with

$$A_{21} = 3f_{30}\phi_1^2(0)\overline{\phi_1}(0) + 3f_{03}\phi_2^2(0)\overline{\phi_2}(0) + 3f_{21}(\phi_1^2(0)\overline{\phi_2}(0) + 2\phi_1(0)\overline{\phi_1}(0)\phi_2(0)) \\ + 3f_{12}(2\phi_1(0)\phi_2(0)\overline{\phi_2}(0) + \overline{\phi_1}(0)\phi_2^2(0)).$$

560 Here,

$$f_{03}^{(1)} = 6\tau_c\beta m u_*^2 (u_*^2 + m v_*^2)^{-2} - 48\tau_c\beta m^2 u_*^2 v_*^2 (u_*^2 + m v_*^2)^{-3} + 48\tau_c\beta m^3 u_*^2 v_*^4 (u_*^2 + m v_*^2)^{-4}, \\ f_{03}^{(2)} = 0, \\ f_{12}^{(1)} = 12\tau_c\beta m u_* v_* (u_*^2 + m v_*^2)^{-2} - 16\tau_c\beta m^2 u_* v_*^3 (u_*^2 + m v_*^2)^{-3}, \quad f_{12}^{(2)} = 2\tau_c\gamma u_*^{-2}, \\ f_{21}^{(1)} = -2\tau_c\beta (u_*^2 + m v_*^2)^{-1} + 4\tau_c\beta m v_*^2 (u_*^2 + m v_*^2)^{-2} + 10\tau_c\beta u_*^2 (u_*^2 + m v_*^2)^{-2} \\ - 40\tau_c\beta m u_*^2 v_*^2 (u_*^2 + m v_*^2)^{-3} - 8\tau_c\beta u_*^4 (u_*^2 + m v_*^2)^{-3} + 48\tau_c\beta m u_*^4 v_*^2 (u_*^2 + m v_*^2)^{-4}, \\ f_{21}^{(2)} = -4\tau_c\gamma v_* u_*^{-3}, \\ f_{30}^{(1)} = 24\tau_c\beta u_* v_* (u_*^2 + m v_*^2)^{-2} - 72\tau_c\beta u_*^3 v_* (u_*^2 + m v_*^2)^{-3} + 48\tau_c\beta u_*^5 v_* (u_*^2 + m v_*^2)^{-4}, \\ f_{30}^{(2)} = 6\tau_c\gamma v_*^2 u_*^{-4}.$$

Step 3:

$$B_{22} = \frac{1}{\sqrt{\ell\pi}} \psi^T (\mathcal{S}_2(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2(\overline{\phi}(\theta), h_{0,20}(\theta))) \\ + \frac{1}{\sqrt{2\ell\pi}} \psi^T (\mathcal{S}_2(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2(\overline{\phi}(\theta), h_{2n_c,20}(\theta)))$$

with

$$\begin{aligned}
\mathcal{S}_2(\phi(\theta), h_{0,11}(\theta)) &= 2f_{20}\phi_1(0)h_{0,11}^{(1)}(0) + 2f_{02}\phi_2(0)h_{0,11}^{(2)}(0) \\
&\quad + 2f_{11}\left(\phi_1(0)h_{0,11}^{(2)}(0) + \phi_2(0)h_{0,11}^{(1)}(0)\right), \\
\mathcal{S}_2(\bar{\phi}(\theta), h_{0,20}(\theta)) &= 2f_{20}\bar{\phi}_1(0)h_{0,20}^{(1)}(0) + 2f_{02}\bar{\phi}_2(0)h_{0,20}^{(2)}(0) \\
&\quad + 2f_{11}\left(\bar{\phi}_1(0)h_{0,20}^{(2)}(0) + \bar{\phi}_2(0)h_{0,20}^{(1)}(0)\right), \\
\mathcal{S}_2(\phi(\theta), h_{2n_c,11}(\theta)) &= 2f_{20}\phi_1(0)h_{2n_c,11}^{(1)}(0) + 2f_{02}\phi_2(0)h_{2n_c,11}^{(2)}(0) \\
&\quad + 2f_{11}\left(\phi_1(0)h_{2n_c,11}^{(2)}(0) + \phi_2(0)h_{2n_c,11}^{(1)}(0)\right), \\
\mathcal{S}_2(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) &= 2f_{20}\bar{\phi}_1(0)h_{2n_c,20}^{(1)}(0) + 2f_{02}\bar{\phi}_2(0)h_{2n_c,20}^{(2)}(0) \\
&\quad + 2f_{11}\left(\bar{\phi}_1(0)h_{2n_c,20}^{(2)}(0) + \bar{\phi}_2(0)h_{2n_c,20}^{(1)}(0)\right).
\end{aligned}$$

Here,

$$\begin{aligned}
f_{02}^{(1)} &= 6\tau_c\beta mu_*^2 v_*(u_*^2 + mv_*^2)^{-2} - 8\tau_c\beta m^2 u_*^2 v_*^3 (u_*^2 + mv_*^2)^{-3}, \quad f_{02}^{(2)} = -2\tau_c\gamma u_*^{-1}, \\
f_{11}^{(1)} &= -2\tau_c\beta u_*(u_*^2 + mv_*^2)^{-1} + 4\tau_c\beta mu_* v_*^2 (u_*^2 + mv_*^2)^{-2}, \quad f_{11}^{(2)} = 2\tau_c\gamma u_*^{-2} v_*, \\
f_{20}^{(1)} &= -2\tau_c - 2\tau_c\beta v_*(u_*^2 + mv_*^2)^{-1} + 10\tau_c\beta u_*^2 v_*(u_*^2 + mv_*^2)^{-2} - 8\tau_c\beta u_*^4 v_*(u_*^2 + mv_*^2)^{-3}, \\
f_{20}^{(2)} &= -2\tau_c\gamma u_*^{-3} v_*^2.
\end{aligned}$$

Furthermore, we have

$$\begin{cases} h_{0,20}(\theta) = \frac{1}{\sqrt{\ell\pi}} \left(\widetilde{\mathcal{M}}_0(2i\omega_c) \right)^{-1} A_{20} e^{2i\omega_c\theta}, \\ h_{0,11}(\theta) = \frac{1}{\sqrt{\ell\pi}} \left(\widetilde{\mathcal{M}}_0(0) \right)^{-1} A_{11} \end{cases}$$

and

$$\begin{cases} h_{2n_c,20}(\theta) = \frac{1}{\sqrt{2\ell\pi}} \left(\widetilde{\mathcal{M}}_{2n_c}(2i\omega_c) \right)^{-1} \widetilde{A}_{20} e^{2i\omega_c\theta} \\ h_{2n_c,11}(\theta) = \frac{1}{\sqrt{2\ell\pi}} \left(\widetilde{\mathcal{M}}_{2n_c}(0) \right)^{-1} \widetilde{A}_{11} \end{cases}$$

with

$$\widetilde{\mathcal{M}}_n(\lambda) = \lambda I_2 + \tau_c(n/\ell)^2 D_1 + \tau_c(n/\ell)^2 e^{-\lambda} D_2 - \tau_c A_1.$$

Here,

$$\begin{aligned}
A_{20} &= f_{20}\phi_1^2(0) + f_{02}\phi_2^2(0) + 2f_{11}\phi_1(0)\phi_2(0), \\
A_{11} &= 2f_{20}\phi_1(0)\bar{\phi}_1(0) + 2f_{02}\phi_2(0)\bar{\phi}_2(0) + 2f_{11}(\phi_1(0)\bar{\phi}_2(0) + \bar{\phi}_1(0)\phi_2(0))
\end{aligned}$$

and

$$\begin{cases} \widetilde{A}_{20} = A_{20} - 2(n_c/\ell)^2 A_{20}^d, \\ \widetilde{A}_{11} = A_{11} - 2(n_c/\ell)^2 A_{11}^d \end{cases}$$

with

$$\begin{cases} A_{20}^d = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_1(-1)\phi_2(0) \end{pmatrix} = \overline{A_{02}^d}, \\ A_{11}^d = -2d_{21}\tau_c \begin{pmatrix} 0 \\ 2\operatorname{Re}\{\phi_1(-1)\bar{\phi}_2(0)\} \end{pmatrix}. \end{cases}$$

Step 4:

$$B_{23} = -\frac{1}{\sqrt{\ell\pi}} (n_c/\ell)^2 \psi^T \left(\mathcal{S}_2^{(d,1)}(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2^{(d,1)}(\bar{\phi}(\theta), h_{0,20}(\theta)) \right) \\ + \frac{1}{\sqrt{2\ell\pi}} \psi^T \sum_{j=1,2,3} b_{2n_c}^{(j)} \left(\mathcal{S}_2^{(d,j)}(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2^{(d,j)}(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) \right)$$

with

$$b_{2n_c}^{(1)} = -\frac{n_c^2}{\ell^2}, \quad b_{2n_c}^{(2)} = \frac{2n_c^2}{\ell^2}, \quad b_{2n_c}^{(3)} = -\frac{(2n_c)^2}{\ell^2}$$

and

$$\left\{ \begin{array}{l} \mathcal{S}_2^{(d,1)}(\phi(\theta), h_{0,11}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_1(-1)h_{0,11}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,1)}(\bar{\phi}(\theta), h_{0,20}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \bar{\phi}_1(-1)h_{0,20}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,1)}(\phi(\theta), h_{2n_c,11}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_1(-1)h_{2n_c,11}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,2)}(\phi(\theta), h_{2n_c,11}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_1(-1)h_{2n_c,11}^{(2)}(0) + \phi_2(0)h_{2n_c,11}^{(1)}(-1) \end{pmatrix}, \\ \mathcal{S}_2^{(d,3)}(\phi(\theta), h_{2n_c,11}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_2(0)h_{2n_c,11}^{(1)}(-1) \end{pmatrix}, \\ \mathcal{S}_2^{(d,1)}(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \bar{\phi}_1(-1)h_{2n_c,20}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,2)}(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \bar{\phi}_1(-1)h_{2n_c,20}^{(2)}(0) + \bar{\phi}_2(0)h_{2n_c,20}^{(1)}(-1) \end{pmatrix}, \\ \mathcal{S}_2^{(d,3)}(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \bar{\phi}_2(0)h_{2n_c,20}^{(1)}(-1) \end{pmatrix}. \end{array} \right.$$

Appendix B

Remark 5.2. Assume that at $\tau = \tau_c$, (4.28) has a pair of purely imaginary roots $\pm i\omega_{n_c}$ with $\omega_{n_c} > 0$ for $n = n_c \in \mathbb{N}_0$ and all other eigenvalues have negative real part. Let $\lambda(\tau) = \alpha_1(\tau) \pm i\alpha_2(\tau)$ be a pair of roots of (4.28) near $\tau = \tau_c$ satisfying $\alpha_1(\tau_c) = 0$ and $\alpha_2(\tau_c) = \omega_{n_c}$. In addition, the corresponding transversality condition holds.

The normal form of Hopf bifurcation for the system (4.24) can be calculated by using our developed algorithm in Section 2. Here, we give the detail calculation procedures of $B_1, B_{21}, B_{22}, B_{23}$ steps by steps.

Step 1:

$$B_1 = 2\psi^T(0) \left(A_1\phi(0) + A_2\phi(-1) - \frac{n_c^2}{\ell^2} (D_1\phi(0) + D_2\phi(-1)) \right)$$

with

$$D_1 = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{2\beta}{(m+1)^2} - 1 & \frac{\beta(m-1)}{(m+1)^2} \\ 0 & -\gamma \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$$

and

$$\phi = \begin{pmatrix} 1 \\ \frac{i\omega_{n_c} + d_{11}(n_c/\ell)^2 - a_{11} - b_{11}e^{-i\omega_c}}{a_{12} + b_{12}e^{-i\omega_c}} \end{pmatrix}, \quad \psi = \eta \begin{pmatrix} 1 \\ \frac{a_{12} + b_{12}e^{-i\omega_c}}{i\omega_{n_c} + d_{22}(n_c/\ell)^2 - a_{22} - b_{22}e^{-i\omega_c}} \end{pmatrix}.$$

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Here,

$$\eta = \frac{1}{1 + k_1 k_2 + e^{-i\omega_c} \tau_c b_{11} + e^{-i\omega_c} k_2 \left(\tau_c b_{21} + \tau_c d_{21} v_* (n_c/\ell)^2 \right)},$$

with

$$k_1 = \frac{i\omega_{n_c} + d_{11}(n_c/\ell)^2 - a_{11} - b_{11}e^{-i\omega_c}}{a_{12} + b_{12}e^{-i\omega_c}}, \quad k_2 = \frac{a_{12} + b_{12}e^{-i\omega_c}}{i\omega_{n_c} + d_{22}(n_c/\ell)^2 - a_{22} - b_{22}e^{-i\omega_c}}.$$

Step 2:

$$B_{21} = \frac{3}{2\ell\pi} \psi^T A_{21}$$

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with

$$\begin{aligned} A_{21} = & 3f_{0030}\phi_1^2(-1)\bar{\phi}_1(-1) + 3f_{0120} \left(2\phi_2(0)\phi_1(-1)\bar{\phi}_1(-1) + \bar{\phi}_2(0)\phi_1^2(-1) \right) \\ & + 3f_{0210} \left(2\phi_2(0)\bar{\phi}_2(0)\bar{\phi}_1(-1) + \phi_2^2(0)\bar{\phi}_1(-1) \right) + 3f_{0300}\phi_2^2(0)\bar{\phi}_2(0) \\ & + 3f_{1200} \left(2\phi_1(0)\phi_2(0)\bar{\phi}_2(0) + \bar{\phi}_1(0)\phi_2^2(0) \right) + 3f_{2100} \left(\phi_1^2(0)\bar{\phi}_2(0) + 2\phi_1(0)\bar{\phi}_1(0)\phi_2(0) \right) \\ & + 3f_{3000}\phi_1^2(0)\bar{\phi}_1(0). \end{aligned}$$

Here,

$$\begin{aligned} f_{0030}^{(1)} &= 0, \quad f_{0030}^{(2)} = 6\tau_c \gamma u_*^{-4} v_*^2, \quad f_{0120}^{(1)} = 0, \quad f_{0120}^{(2)} = -4\tau_c \gamma u_*^{-3} v_*, \\ f_{0210}^{(1)} &= 0, \quad f_{0210}^{(2)} = 2\tau_c \gamma u_*^{-2}, \\ f_{0300}^{(1)} &= 6\tau_c \beta m u_*^2 (u_*^2 + m v_*^2)^{-2} - 48\tau_c \beta m^2 u_*^2 v_*^2 (u_*^2 + m v_*^2)^{-3} + 48\tau_c \beta m^3 u_*^2 v_*^4 (u_*^2 + m v_*^2)^{-4}, \\ f_{0300}^{(2)} &= 0, \\ f_{1200}^{(1)} &= 12\tau_c \beta m u_* v_* (u_*^2 + m v_*^2)^{-2} - 16\tau_c \beta m^2 u_* v_*^3 (u_*^2 + m v_*^2)^{-3}, \quad f_{1200}^{(2)} = 0, \\ f_{2100}^{(1)} &= -2\tau_c \beta (u_*^2 + m v_*^2)^{-1} + 4\tau_c \beta m v_*^2 (u_*^2 + m v_*^2)^{-2} + 10\tau_c \beta u_*^2 (u_*^2 + m v_*^2)^{-2} \\ &\quad - 40\tau_c \beta m u_*^2 v_*^2 (u_*^2 + m v_*^2)^{-3} - 8\tau_c \beta u_*^4 (u_*^2 + m v_*^2)^{-3} + 48\tau_c \beta m u_*^4 v_*^2 (u_*^2 + m v_*^2)^{-4}, \\ f_{2100}^{(2)} &= 0, \\ f_{3000}^{(1)} &= 24\tau_c \beta u_* v_* (u_*^2 + m v_*^2)^{-2} - 72\tau_c \beta u_*^3 v_* (u_*^2 + m v_*^2)^{-3} + 48\tau_c \beta u_*^5 v_* (u_*^2 + m v_*^2)^{-4}, \\ f_{3000}^{(2)} &= 0. \end{aligned}$$

Step 3:

$$\begin{aligned} B_{22} = & \frac{1}{\sqrt{\ell\pi}} \psi^T \left(\mathcal{S}_2(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2(\bar{\phi}(\theta), h_{0,20}(\theta)) \right) \\ & + \frac{1}{\sqrt{2\ell\pi}} \psi^T \left(\mathcal{S}_2(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) \right) \end{aligned}$$

with

$$\begin{aligned}
\mathcal{S}_2(\phi(\theta), h_{0,11}(\theta)) &= 2f_{0020}\phi_1(-1)h_{0,11}^{(1)}(-1) + 2f_{0110}\left(\phi_2(0)h_{0,11}^{(1)}(-1) + \phi_1(-1)h_{0,11}^{(2)}(0)\right) \\
&\quad + 2f_{0200}\phi_2(0)h_{0,11}^{(2)}(0) + 2f_{1100}\left(\phi_1(0)h_{0,11}^{(2)}(0) + \phi_2(0)h_{0,11}^{(1)}(0)\right) + 2f_{2000}\phi_1(0)h_{0,11}^{(1)}(0), \\
\mathcal{S}_2(\phi(\theta), h_{2n_c,11}(\theta)) &= 2f_{0020}\phi_1(-1)h_{2n_c,11}^{(1)}(-1) + 2f_{0110}\left(\phi_2(0)h_{2n_c,11}^{(1)}(-1) + \phi_1(-1)h_{2n_c,11}^{(2)}(0)\right) \\
&\quad + 2f_{0200}\phi_2(0)h_{2n_c,11}^{(2)}(0) + 2f_{1100}\left(\phi_1(0)h_{2n_c,11}^{(2)}(0) + \phi_2(0)h_{2n_c,11}^{(1)}(0)\right) + 2f_{2000}\phi_1(0)h_{2n_c,11}^{(1)}(0), \\
\mathcal{S}_2(\bar{\phi}(\theta), h_{0,20}(\theta)) &= 2f_{0020}\bar{\phi}_1(-1)h_{0,20}^{(1)}(-1) + 2f_{0110}\left(\bar{\phi}_2(0)h_{0,20}^{(1)}(-1) + \bar{\phi}_1(-1)h_{0,20}^{(2)}(0)\right) \\
&\quad + 2f_{0200}\bar{\phi}_2(0)h_{0,20}^{(2)}(0) + 2f_{1100}\left(\bar{\phi}_1(0)h_{0,20}^{(2)}(0) + \bar{\phi}_2(0)h_{0,20}^{(1)}(0)\right) + 2f_{2000}\bar{\phi}_1(0)h_{0,20}^{(1)}(0), \\
\mathcal{S}_2(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) &= 2f_{0020}\bar{\phi}_1(-1)h_{2n_c,20}^{(1)}(-1) + 2f_{0110}\left(\bar{\phi}_2(0)h_{2n_c,20}^{(1)}(-1) + \bar{\phi}_1(-1)h_{2n_c,20}^{(2)}(0)\right) \\
&\quad + 2f_{0200}\bar{\phi}_2(0)h_{2n_c,20}^{(2)}(0) + 2f_{1100}\left(\bar{\phi}_1(0)h_{2n_c,20}^{(2)}(0) + \bar{\phi}_2(0)h_{2n_c,20}^{(1)}(0)\right) + 2f_{2000}\bar{\phi}_1(0)h_{2n_c,20}^{(1)}(0).
\end{aligned}$$

Here,

$$\begin{aligned}
f_{0020}^{(1)} &= 0, \quad f_{0020}^{(2)} = -2\tau_c\gamma u_*^{-3}v_*^2, \quad f_{0110}^{(1)} = 0, \quad f_{0110}^{(2)} = 2\tau_c\gamma u_*^{-2}v_*, \\
f_{0200}^{(1)} &= 6\tau_c\beta m u_*^2 v_*(u_*^2 + mv_*^2)^{-2} - 8\tau_c\beta m^2 u_*^2 v_*^3(u_*^2 + mv_*^2)^{-3}, \quad f_{0200}^{(2)} = -2\tau_c\gamma u_*^{-1}, \\
f_{1100}^{(1)} &= -2\tau_c\beta u_*(u_*^2 + mv_*^2)^{-1} + 4\tau_c\beta m u_* v_*^2(u_*^2 + mv_*^2)^{-2}, \quad f_{1100}^{(2)} = 0, \\
f_{2000}^{(1)} &= -2\tau_c - 2\tau_c\beta v_*(u_*^2 + mv_*^2)^{-1} + 10\tau_c\beta u_*^2 v_*(u_*^2 + mv_*^2)^{-2} - 8\tau_c\beta u_*^4 v_*(u_*^2 + mv_*^2)^{-3}, \\
f_{2000}^{(2)} &= 0.
\end{aligned}$$

Furthermore, we have

$$\begin{cases} h_{0,20}(\theta) = \frac{1}{\sqrt{\ell\pi}} \left(\widetilde{\mathcal{M}}_0(2i\omega_c) \right)^{-1} A_{20} e^{2i\omega_c\theta}, \\ h_{0,11}(\theta) = \frac{1}{\sqrt{\ell\pi}} \left(\widetilde{\mathcal{M}}_0(0) \right)^{-1} A_{11} \end{cases}$$

and

$$\begin{cases} h_{2n_c,20}(\theta) = \frac{1}{\sqrt{2\ell\pi}} \left(\widetilde{\mathcal{M}}_{2n_c}(2i\omega_c) \right)^{-1} \tilde{A}_{20} e^{2i\omega_c\theta}, \\ h_{2n_c,11}(\theta) = \frac{1}{\sqrt{2\ell\pi}} \left(\widetilde{\mathcal{M}}_{2n_c}(0) \right)^{-1} \tilde{A}_{11} \end{cases}$$

with

$$\widetilde{\mathcal{M}}_n(\lambda) = \lambda I_2 + \tau_c \frac{n^2}{\ell^2} D_1 + \tau_c \frac{n^2}{\ell^2} e^{-\lambda} D_2 - \tau_c A_1 - \tau_c A_2 e^{-\lambda}.$$

Here,

$$\begin{aligned}
A_{20} &= f_{0020}\phi_1^2(-1) + 2f_{0110}\phi_2(0)\phi_1(-1) + f_{0200}\phi_2^2(0) + 2f_{1100}\phi_1(0)\phi_2(0) + f_{2000}\phi_1^2(0), \\
A_{02} &= f_{0020}\bar{\phi}_1^2(-1) + 2f_{0110}\bar{\phi}_2(0)\bar{\phi}_1(-1) + f_{0200}\bar{\phi}_2^2(0) + 2f_{1100}\bar{\phi}_1(0)\bar{\phi}_2(0) + f_{2000}\bar{\phi}_1^2(0), \\
A_{11} &= 2f_{0020}\phi_1(-1)\bar{\phi}_1(-1) + 2f_{0110}(\phi_2(0)\bar{\phi}_1(-1) + \bar{\phi}_2(0)\phi_1(-1)) + 2f_{0200}\phi_2(0)\bar{\phi}_2(0) \\
&\quad + 2f_{1100}(\phi_1(0)\bar{\phi}_2(0) + \bar{\phi}_1(0)\phi_2(0)) + 2f_{2000}\phi_1(0)\bar{\phi}_1(0),
\end{aligned}$$

and

$$\begin{cases} \tilde{A}_{20} = A_{20} - 2(n_c/\ell)^2 A_{20}^d, \\ \tilde{A}_{11} = A_{11} - 2(n_c/\ell)^2 A_{11}^d \end{cases}$$

with

$$\begin{cases} A_{20}^d = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_1(-1)\phi_2(0) \end{pmatrix} = \overline{A_{02}^d}, \\ A_{11}^d = -2d_{21}\tau_c \begin{pmatrix} 0 \\ 2\operatorname{Re}\{\phi_1(-1)\overline{\phi_2(0)}\} \end{pmatrix} = -4d_{21}\tau_c \begin{pmatrix} 0 \\ \operatorname{Re}\{\phi_1(-1)\overline{\phi_2(0)}\} \end{pmatrix}. \end{cases}$$

Step 4:

$$\begin{aligned} B_{23} = & -\frac{1}{\sqrt{\ell\pi}} (n_c/\ell)^2 \psi^T \left(\mathcal{S}_2^{(d,1)}(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2^{(d,1)}(\overline{\phi}(\theta), h_{0,20}(\theta)) \right) \\ & + \frac{1}{\sqrt{2\ell\pi}} \psi^T \sum_{j=1,2,3} b_{2n_c}^{(j)} \left(\mathcal{S}_2^{(d,j)}(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2^{(d,j)}(\overline{\phi}(\theta), h_{2n_c,20}(\theta)) \right) \end{aligned}$$

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with

$$b_{2n_c}^{(1)} = -\frac{n_c^2}{\ell^2}, \quad b_{2n_c}^{(2)} = \frac{2n_c^2}{\ell^2}, \quad b_{2n_c}^{(3)} = -\frac{4n_c^2}{\ell^2}$$

and

$$\begin{cases} \mathcal{S}_2^{(d,1)}(\phi(\theta), h_{0,11}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_1(-1)h_{0,11}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,1)}(\overline{\phi}(\theta), h_{0,20}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \overline{\phi}_1(-1)h_{0,20}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,1)}(\phi(\theta), h_{2n_c,11}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_1(-1)h_{2n_c,11}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,2)}(\phi(\theta), h_{2n_c,11}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_2(0)h_{2n_c,11}^{(1)}(-1) + \phi_1(-1)h_{2n_c,11}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,3)}(\phi(\theta), h_{2n_c,11}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \phi_2(0)h_{2n_c,11}^{(1)}(-1) \end{pmatrix}, \\ \mathcal{S}_2^{(d,1)}(\overline{\phi}(\theta), h_{2n_c,20}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \overline{\phi}_1(-1)h_{2n_c,20}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,2)}(\overline{\phi}(\theta), h_{2n_c,20}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \overline{\phi}_2(0)h_{2n_c,20}^{(1)}(-1) + \overline{\phi}_1(-1)h_{2n_c,20}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,3)}(\overline{\phi}(\theta), h_{2n_c,20}(\theta)) = -2d_{21}\tau_c \begin{pmatrix} 0 \\ \overline{\phi}_2(0)h_{2n_c,20}^{(1)}(-1) \end{pmatrix}. \end{cases}$$

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