Inhomogeneous spatial patterns in diffusive predator-prey system with spatial memory and predator-taxis

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Research Article

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Abstract In this paper, we consider a diffusive predator-prey system with spatial memory and predator-taxis. Since in this system, the memory delay appears in the diffusion term, and the diffusion term is nonlinear, the classical normal form of Hopf bifurcation in the reaction-diffusion system with delay can't be applied to this system. Thus, in this paper, we first derive an algorithm for calculating the normal form of Hopf bifurcation in this system. Then in order to illustrate the effectiveness of our newly developed algorithm, we consider the diffusive Holling-Tanner model with spatial memory and predator-taxis. The stability and Hopf bifurcation analysis of this model are investigated, and the direction and stability of Hopf bifurcation periodic solution are also researched by using our newly developed algorithm for calculating the normal form of Hopf bifurcation. At last, we carry out some numerical simulations, two stable spatially inhomogeneous periodic solutions corresponding to the mode-1 and mode-2 Hopf bifurcations are found, which verifies our theoretical analysis results.

Keywords Predator-prey system · Memory delay · Predator-taxis · Hopf bifurcation · Normal form · Periodic solutions

Mathematics Subject Classification (2020) 35B10 · 37G05 · 37L10 · 92D25

1 Introduction

The reaction-diffusion equations based on the Fick's law have been widely used in physics, chemistry and biology [1, 2, 3]. More precisely, based on the Fick's law, that is the movement flux is in the direction of negative gradient of the density distribution function, the diffusive Brusselator model with gene expression time delay [4], the diffusive predator-prey model in heterogeneous environment [5, 6], the diffusive predator-prey model with a protection zone [7], the diffusive predator-prey model with prey social behavior [8], the diffusive predator-prey model with protection zone and predator harvesting [9] have been studied by many scholars. Furthermore, in order to include the episodic-like spatial memory of animals, Shi et al. [10] directed movement toward the negative gradient of the density distribution function at past time, and they proposed the following diffusive model with spatial memory

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = d_1 \Delta u(x, t) + d_2 (u(x, t) u_x(x, t - \tau))_x + f(u(x, t)), & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \mathbf{n}}(x, t) = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, t) = u_0(x, t), & x \in \Omega, \ -\tau \leq t \leq 0, \end{cases} \quad (1.1)$$

where $u(x, t)$ is the population density at spatial location x and time t , d_1 and d_2 are the Fickian diffusion coefficient and the memory-based diffusion coefficient, respectively, $\Omega \subset \mathbb{R}$ is a smooth and bounded domain, $u_0(x, t)$ is the initial function, $\Delta u(x, t) = \partial^2 u(x, t) / \partial x^2$, $u_x(x, t) = \partial u(x, t) / \partial x$, $u_x(x, t - \tau) = \partial u(x, t - \tau) / \partial x$, $u_{xx}(x, t - \tau) = \partial^2 u(x, t - \tau) / \partial x^2$, and $\mathbf{n}$ is the outward unit normal vector at the smooth boundary $\partial\Omega$. Here, the time delay $\tau > 0$ represents the averaged memory period, which is usually called as the memory delay, and $f(u(x, t))$ describes the chemical reaction or biological birth/death. Moreover, in order to further investigate the influence of memory delay on the stability of the positive constant steady state, on the basis

of model (1.1), Shi et al. [11] studied the spatial memory diffusion model with memory and maturation delays. Furthermore, Song et al. [12] proposed the following memory-based diffusion system subjects to homogeneous Neumann boundary condition

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_{11}\Delta u(x,t) + f(u(x,t), v(x,t)), & x \in \Omega, t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_{22}\Delta v(x,t) - d_{21}(v(x,t)u_x(x,t-\tau))_x + g(u(x,t), v(x,t)), & x \in \Omega, t > 0, \\ u_x(0,t) = u_x(\ell\pi,t) = v_x(0,t) = v_x(\ell\pi,t) = 0, & t \geq 0, \\ u(x,t) = u_0(x,t), & x \in \Omega, -\tau \leq t \leq 0, \\ v(x,t) = v_0(x), & x \in \Omega, \end{cases} \quad (1.2)$$

where $\Omega = (0, \ell\pi)$ with $\ell \in \mathbb{R}^+$, $u(x,t)$ and $v(x,t)$ represent the densities of prey and predator at the location x and time t , respectively, $d_{11} \geq 0$ and $d_{22} \geq 0$ are the random diffusion coefficients, $d_{21} \geq 0$ is the memory-based diffusion coefficient, $v_0(x)$ is also the initial function, and $f(u(x,t), v(x,t))$ and $g(u(x,t), v(x,t))$ are the reaction terms. Furthermore, they derived an algorithm for calculating the normal form of Hopf bifurcation in the system (1.2). By numerical simulations, the stable spatially inhomogeneous periodic solutions and the transition from the unstable mode-2 spatially inhomogeneous periodic solution to the stable mode-1 spatially inhomogeneous periodic solution are found.

For the general predator-prey models in ecology, in addition to random diffusion of the predator and prey populations, the spatial movement of predator and prey populations also occurs, which is usually shown by the predator pursuing prey and prey escaping from predator [13]. The pursuit and evasion between the predator and prey populations also have a strong impact on the movement pattern of the predator and prey populations [14, 15, 16]. Furthermore, by noticing that such movement is not random but directed, i.e., predator moves toward the gradient direction of prey distribution, which is called prey-taxis, or prey moves opposite to the gradient of predator distribution, which is called predator-taxis. Recently, the predator-prey model with prey-taxis [17, 18, 19, 20, 21], the predator-prey model with indirect prey-taxis [22, 23, 24], the predator-prey model with predator-taxis [25] and the predator-prey model with indirect predator-taxis [26] have been researched. Especially, Wang et al. [13] considered the following diffusive predator-prey model with both predator-taxis and prey-taxis. The corresponding mathematical model is

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \xi(u(x,t)v_x(x,t))_x + f(u(x,t), v(x,t)), & x \in \Omega, t > 0, \\ \frac{\partial v(x,t)}{\partial t} = \Delta v(x,t) - \eta(v(x,t)u_x(x,t))_x + g(u(x,t), v(x,t)), & x \in \Omega, t > 0, \\ \frac{\partial u(x,t)}{\partial \mathbf{n}} = \frac{\partial v(x,t)}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \quad (1.3)$$

where $u_0(x)$ is the initial function, d is the rescaled diffusion coefficient for the prey population, and the diffusion coefficient of the predator population is rescaled as 1. Furthermore, the term $\xi(u(x,t)v_x(x,t))_x$ represents the prey moves away from predator, and $\xi \geq 0$ is the intrinsic predator-taxis rate. The term $-\eta(v(x,t)u_x(x,t))_x$ represents the predator moves towards prey, and $\eta \geq 0$ is the intrinsic prey-taxis rate.

Therefore, based on the models (1.2) and (1.3), and by considering that the spatial memory and predator-taxis, we proposed the following diffusive predator-prey model with spatial memory and predator-taxis subjects to homogeneous Neumann boundary condition

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_{11}\Delta u(x,t) + \xi(u(x,t)v_x(x,t))_x + f(u(x,t), v(x,t)), & x \in (0, \ell\pi), t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_{22}\Delta v(x,t) - d_{21}(v(x,t)u_x(x,t-\tau))_x + g(u(x,t), v(x,t)), & x \in (0, \ell\pi), t > 0, \\ u_x(0,t) = u_x(\ell\pi,t) = v_x(0,t) = v_x(\ell\pi,t) = 0, & t \geq 0, \\ u(x,t) = u_0(x,t), & x \in (0, \ell\pi), -\tau \leq t \leq 0, \\ v(x,t) = v_0(x), & x \in (0, \ell\pi). \end{cases} \quad (1.4)$$

This paper is organized as follows. In Section 2, we derive an algorithm for calculating the normal form of Hopf bifurcation in the model (1.4). In Section 3, we obtain the normal form of Hopf bifurcation truncated to the third-order term by using our newly developed algorithm developed in Sec.2, and the mathematical expressions of the corresponding coefficients are given. In Section 4, we consider the diffusive Holling-Tanner model with spatial memory and predator-taxis. The stability and Hopf bifurcation analysis in this model

are studied, and some numerical simulations are also given. In Section 5, we give a brief conclusion and discussion.

2 Algorithm for calculating the normal form of Hopf bifurcation in the system (1.4)

2.1 Characteristic equation at the positive constant steady state

Define the real-valued Sobolev space

$$X := \left\{ (u, v)^T \in (W^{2,2}(0, \ell\pi))^2 : \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \text{ at } x = 0, \ell\pi \right\}$$

with the inner product defined by

$$[U_1, U_2] = \int_0^{\ell\pi} U_1^T U_2 \, dx \text{ for } U_1 = (u_1, v_1)^T \in X \text{ and } U_2 = (u_2, v_2)^T \in X,$$

where the symbol T represents the transpose of vector, and let $\mathcal{C} := C([-1, 0]; X)$ be the Banach space of continuous mappings from $[-1, 0]$ to X with the sup norm. It is well known that the eigenvalue problem

$$\begin{cases} \tilde{\varphi}''(x) = \tilde{\lambda}\tilde{\varphi}(x), & x \in (0, \ell\pi), \\ \tilde{\varphi}'(0) = \tilde{\varphi}'(\ell\pi) = 0 \end{cases}$$

has eigenvalues $\tilde{\lambda}_n = -n^2/\ell^2$ with corresponding normalized eigenfunctions

$$\beta_n^{(j)} = \gamma_n(x)e_j, \quad \gamma_n(x) = \frac{\cos(nx/\ell)}{\|\cos(nx/\ell)\|_{L^2}} = \begin{cases} \frac{1}{\sqrt{\ell\pi}}, & n = 0, \\ \frac{\sqrt{2}}{\sqrt{\ell\pi}} \cos\left(\frac{nx}{\ell}\right), & n \geq 1, \end{cases} \quad (2.1)$$

where e_j , $j = 1, 2$ is the unit coordinate vector of $\mathbb{R}^2$, and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is often called wave number, $\mathbb{N}_0$ is the set of all non-negative integers, $\mathbb{N} = \{1, 2, \dots\}$ represents the set of all positive integers.

Without loss of generality, we assume that $E_*(u_*, v_*)$ is the positive constant steady state of system (1.4). The linearized equation of (1.4) at $E_*(u_*, v_*)$ is

$$\begin{pmatrix} \frac{\partial u(x, t)}{\partial t} \\ \frac{\partial v(x, t)}{\partial t} \end{pmatrix} = D_1 \begin{pmatrix} \Delta u(x, t) \\ \Delta v(x, t) \end{pmatrix} + D_2 \begin{pmatrix} \Delta u(x, t - \tau) \\ \Delta v(x, t - \tau) \end{pmatrix} + A \begin{pmatrix} u(x, t) \\ v(x, t) \end{pmatrix}, \quad (2.2)$$

where

$$D_1 = \begin{pmatrix} d_{11} & \xi u_* \\ 0 & d_{22} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (2.3)$$

and

$$a_{11} = \frac{\partial f(u_*, v_*)}{\partial u}, \quad a_{12} = \frac{\partial f(u_*, v_*)}{\partial v}, \quad a_{21} = \frac{\partial g(u_*, v_*)}{\partial u}, \quad a_{22} = \frac{\partial g(u_*, v_*)}{\partial v}. \quad (2.4)$$

Therefore, the characteristic equation of (2.2) is

$$\prod_{n \in \mathbb{N}_0} \Gamma_n(\lambda) = 0,$$

where $\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda))$ with

$$\mathcal{M}_n(\lambda) = \lambda I_2 + \frac{n^2}{\ell^2} D_1 + \frac{n^2}{\ell^2} e^{-\lambda\tau} D_2 - A. \quad (2.5)$$

Here, $\det(\cdot)$ represents the determinant of a matrix, I_2 is the identity matrix of 2×2 , and D_1, D_2, A are defined by (2.3). Then we can obtain

$$\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda)) = \lambda^2 - T_n \lambda + \tilde{J}_n(\tau) = 0, \quad (2.6)$$

where

$$\begin{aligned} T_n &= \text{Tr}(A) - \text{Tr}(D_1) \frac{n^2}{\ell^2}, \\ \tilde{J}_n(\tau) &= (d_{11}d_{22} + d_{21}\xi u_* v_* e^{-\lambda\tau}) \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* + d_{21}v_* a_{12} e^{-\lambda\tau}) \frac{n^2}{\ell^2} + \text{Det}(A) \end{aligned} \quad (2.7)$$

with $\text{Tr}(A) = a_{11} + a_{22}$, $\text{Tr}(D_1) = d_{11} + d_{22}$ and $\text{Det}(A) = a_{11}a_{22} - a_{12}a_{21}$.

2.2 Basic assumption and equation transformation

Assumption 1 Assume that at $\tau = \tau_c$, (2.6) has a pair of purely imaginary roots $\pm i\omega_{n_c}$ with $\omega_{n_c} > 0$ for $n = n_c \in \mathbb{N}_0$ and all other eigenvalues have negative real parts. Let $\lambda(\tau) = \alpha_1(\tau) \pm i\alpha_2(\tau)$ be a pair of roots of (2.6) near $\tau = \tau_c$ satisfying $\alpha_1(\tau_c) = 0$ and $\alpha_2(\tau_c) = \omega_{n_c}$. In addition, the corresponding transversality condition holds.

Let $\tau = \tau_c + \mu$, $|\mu| \ll 1$ such that $\mu = 0$ corresponds to the Hopf bifurcation value for system (1.4). Moreover, we shift $E_*(u_*, v_*)$ to the origin by setting

$$U(x, t) = (U_1(x, t), U_2(x, t))^T = (u(x, t), v(x, t))^T - (u_*, v_*)^T,$$

and normalize the delay by rescaling the time variable $t \rightarrow t/\tau$. Furthermore, we rewrite $U(t)$ for $U(x, t)$, and $U_t \in \mathcal{C}$ for $U_t(\theta) = U(x, t + \theta)$, $-1 \leq \theta \leq 0$. Then the system (1.4) becomes the compact form

$$\frac{dU(t)}{dt} = d(\mu)\Delta(U_t) + L(\mu)(U_t) + F(U_t, \mu), \quad (2.8)$$

where for $\varphi = (\varphi^{(1)}, \varphi^{(2)})^T \in \mathcal{C}$, $d(\mu)\Delta$ is given by

$$d(\mu)\Delta(\varphi) = d_0\Delta(\varphi) + F^d(\varphi, \mu)$$

with

$$\begin{aligned} d_0\Delta(\varphi) &= \tau_c D_1 \Delta\varphi(0) + \tau_c D_2 \Delta\varphi(-1), \\ F^d(\varphi, \mu) &= \begin{pmatrix} \xi(\tau_c + \mu)(\varphi_x^{(1)}(0)\varphi_x^{(2)}(0) + \varphi^{(1)}(0)\varphi_{xx}^{(2)}(0)) \\ -d_{21}(\tau_c + \mu)(\varphi_x^{(1)}(-1)\varphi_x^{(2)}(0) + \varphi_{xx}^{(1)}(-1)\varphi^{(2)}(0)) \end{pmatrix} \\ &\quad + \mu \begin{pmatrix} d_{11}\varphi_{xx}^{(1)}(0) + \xi u_* \varphi_{xx}^{(2)}(0) \\ -d_{21}v_* \varphi_{xx}^{(1)}(-1) + d_{22}\varphi_{xx}^{(2)}(0) \end{pmatrix}. \end{aligned} \quad (2.9)$$

Furthermore, $L(\mu) : \mathcal{C} \rightarrow X$ is given by

$$L(\mu)(\varphi) = (\tau_c + \mu) A\varphi(0), \quad (2.10)$$

and $F : \mathcal{C} \times \mathbb{R}^2 \rightarrow X$ is given by

$$F(\varphi, \mu) = (\tau_c + \mu) \begin{pmatrix} f(\varphi^{(1)}(0) + u_*, \varphi^{(2)}(0) + v_*) \\ g(\varphi^{(1)}(0) + u_*, \varphi^{(2)}(0) + v_*) \end{pmatrix} - L(\mu)(\varphi). \quad (2.11)$$

In what follows, we assume that $F(\varphi, \mu)$ is C^k ($k \geq 3$) function, which is smooth with respect to φ and μ . Notice that μ is the perturbation parameter and is treated as a variable in the calculation of normal form. Moreover, by (2.10), if we denote $L_0(\varphi) = \tau_c A\varphi(0)$, then (2.8) can be rewritten as

$$\frac{dU(t)}{dt} = d_0\Delta(U_t) + L_0(U_t) + \tilde{F}(U_t, \mu), \quad (2.12)$$

where the linear and nonlinear terms are separated, and

$$\tilde{F}(\varphi, \mu) = \mu A\varphi(0) + F(\varphi, \mu) + F^d(\varphi, \mu). \quad (2.13)$$

Thus, the linearized equation of (2.12) can be written as

$$\frac{dU(t)}{dt} = d_0\Delta(U_t) + L_0(U_t). \quad (2.14)$$

Moreover, the characteristic equation for the linearized equation (2.14) is

$$\prod_{n \in \mathbb{N}_0} \tilde{I}_n(\lambda) = 0, \quad (2.15)$$

where $\tilde{I}_n(\lambda) = \det(\tilde{\mathcal{M}}_n(\lambda))$ with

$$\tilde{\mathcal{M}}_n(\lambda) = \lambda I_2 + \tau_c \frac{n^2}{\ell^2} D_1 + \tau_c \frac{n^2}{\ell^2} e^{-\lambda} D_2 - \tau_c A. \quad (2.16)$$

By comparing (2.16) with (2.5), we know that (2.15) has a pair of purely imaginary roots $\pm i\omega_c$ for $n = n_c \in \mathbb{N}_0$, and all other eigenvalues have negative real parts, where $\omega_c = \tau_c \omega_{n_c}$. In order to write (2.12) as an abstract ordinary differential equation in a Banach space, follows by [27], we can take the enlarged space

$$\mathcal{BC} := \left\{ \tilde{\psi} : [-1, 0] \rightarrow X : \tilde{\psi} \text{ is continuous on } [-1, 0), \exists \lim_{\theta \rightarrow 0^-} \tilde{\psi}(\theta) \in X \right\},$$

then the equation (2.12) is equivalent to an abstract ordinary differential equation on $\mathcal{BC}$

$$\frac{dU_t}{dt} = \tilde{A}U_t + X_0 \tilde{F}(U_t, \mu).$$

Here, $\tilde{A}$ is a operator from $\mathcal{C}_0^1 = \{\varphi \in \mathcal{C} : \dot{\varphi} \in \mathcal{C}, \varphi(0) \in \text{dom}(\Delta)\}$ to $\mathcal{BC}$, which is defined by

$$\tilde{A}\varphi = \dot{\varphi} + X_0 (\tau_c D_1 \Delta \varphi(0) + \tau_c D_2 \Delta \varphi(-1) + L_0(\varphi) - \dot{\varphi}(0)),$$

and $X_0 = X_0(\theta)$ is given by

$$X_0(\theta) = \begin{cases} 0, & -1 \leq \theta < 0, \\ 1, & \theta = 0. \end{cases}$$

In the following, the method given in [27] is used to complete the decomposition of $\mathcal{BC}$. Let $C := C([-1, 0], \mathbb{R}^2)$, $C^* := C([0, 1], \mathbb{R}^{2*})$, where $\mathbb{R}^{2*}$ is the two-dimensional space of row vectors, and define the adjoint bilinear form on $C^* \times C$ as follows

$$\langle \Psi(s), \Phi(\theta) \rangle = \Psi(0)\Phi(0) - \int_{-1}^0 \int_0^\theta \Psi(\xi - \theta) dM_n(\theta) \Phi(\xi) d\xi$$

for $\Psi \in C^*$, $\Phi \in C$ and $\xi \in [-1, 0]$, where $M_n(\theta)$ is a bounded variation function from $[-1, 0]$ to $\mathbb{R}^{2 \times 2}$, i.e., $M_n(\theta) \in BV([-1, 0]; \mathbb{R}^{2 \times 2})$, such that for $\Phi(\theta) \in C$, one has

$$-\tau_c \frac{n^2}{\ell^2} D_1 \Phi(0) - \tau_c \frac{n^2}{\ell^2} D_2 \Phi(-1) + L_0(\Phi(\theta)) = \int_{-1}^0 dM_n(\theta) \Phi(\theta).$$

By choosing

$$\Phi(\theta) = (\phi(\theta), \bar{\phi}(\theta)), \quad \Psi(s) = \text{col} \left(\psi^T(s), \bar{\psi}^T(s) \right),$$

where the $\text{col}(\cdot)$ represents the column vector, $\phi(\theta) = \text{col}(\phi_1(\theta), \phi_2(\theta)) = \phi e^{i\omega_c \theta} \in \mathbb{C}^2$ with $\phi = \text{col}(\phi_1, \phi_2)$ is the eigenvector of (2.14) associated with the eigenvalue $i\omega_c$, and $\psi(s) = \text{col}(\psi_1(s), \psi_2(s)) = \psi e^{-i\omega_c s} \in \mathbb{C}^2$ with $\psi = \text{col}(\psi_1, \psi_2)$ is the corresponding adjoint eigenvector such that

$$\langle \Psi(s), \Phi(\theta) \rangle = I_2,$$

where

$$\phi = \left(\frac{1}{\frac{a_{11} - i\omega_{n_c} - d_{11}(n_c^2/\ell^2)}{\xi u_*(n_c^2/\ell^2) - a_{12}}} \right), \quad \psi = \eta \left(\frac{1}{\frac{a_{12} - \xi u_*(n_c^2/\ell^2)}{i\omega_{n_c} + d_{22}(n_c^2/\ell^2) - a_{22}}} \right)$$

and

$$\eta = \frac{i\omega_{n_c} + (n_c/\ell)^2 d_{22} - a_{22}}{2i\omega_{n_c} + (n_c/\ell)^2 d_{11} - a_{11} + (n_c/\ell)^2 d_{22} - a_{22} + \tau_c a_{12} d_{21} v_*(n_c/\ell)^2 e^{-i\omega_c}}.$$

According to [27], the phase space $\mathcal{C}$ can be decomposed as

$$\mathcal{C} = \mathcal{P} \oplus \mathcal{Q}, \quad \mathcal{P} = \text{Im } \pi, \quad \mathcal{Q} = \text{Ker } \pi,$$

where for $\tilde{\phi} \in \mathcal{C}$, the projection $\pi : \mathcal{C} \rightarrow \mathcal{P}$ is defined by

$$\pi(\tilde{\phi}) = \left(\Phi \left\langle \Psi, \begin{pmatrix} \left[\tilde{\phi}(\cdot), \beta_{n_c}^{(1)} \right] \\ \left[\tilde{\phi}(\cdot), \beta_{n_c}^{(2)} \right] \end{pmatrix} \right\rangle \right)^T \beta_{n_c}. \quad (2.17)$$

Therefore, by following the method in [27], $\mathcal{BC}$ can be divided into a direct sum of center subspace and its complementary space, that is

$$\mathcal{BC} = \mathcal{P} \oplus \text{ker } \pi, \quad (2.18)$$

where $\dim \mathcal{P} = 2$. It is easy to see that the projection π which is defined by (2.17), is extended to a continuous projection (which is still denoted by π), that is, $\pi : \mathcal{BC} \mapsto \mathcal{P}$. In particular, for $\alpha \in \mathcal{C}$, we have

$$\pi(X_0(\theta)\alpha) = \left(\Phi(\theta)\Psi(0) \begin{pmatrix} \left[\alpha, \beta_{n_c}^{(1)} \right] \\ \left[\alpha, \beta_{n_c}^{(2)} \right] \end{pmatrix} \right)^T \beta_{n_c}. \quad (2.19)$$

By combining with (2.18) and (2.19), $U_t(\theta)$ can be decomposed as

$$\begin{aligned} U_t(\theta) &= \left(\Phi(\theta) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^T \begin{pmatrix} \beta_{n_c}^{(1)} \\ \beta_{n_c}^{(2)} \end{pmatrix} + w = (z_1 \phi e^{i\omega_c \theta} + z_2 \bar{\phi} e^{-i\omega_c \theta}) \gamma_{n_c}(x) + w \\ &= (\phi(\theta) \quad \bar{\phi}(\theta)) \begin{pmatrix} z_1 \gamma_{n_c}(x) \\ z_2 \gamma_{n_c}(x) \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \end{aligned} \quad (2.20)$$

where $w = \text{col}(w_1, w_2)$ and

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left\langle \Psi(0), \begin{pmatrix} [U_t(\theta), \beta_{n_c}^{(1)}] \\ [U_t(\theta), \beta_{n_c}^{(2)}] \end{pmatrix} \right\rangle.$$

If we assume that

$$\Phi(\theta) = (\phi(\theta), \bar{\phi}(\theta)), \quad z_x = (z_1 \gamma_{n_c}(x), z_2 \gamma_{n_c}(x))^T,$$

then (2.20) can be rewritten as

$$U_t(\theta) = \Phi(\theta)z_x + w \text{ with } w \in \mathcal{C}_0^1 \cap \text{Ker } \pi := \mathcal{Q}^1. \quad (2.21)$$

Then by combining with (2.21), the system (2.12) is decomposed as a system of abstract ordinary differential equations (ODEs) on $\mathbb{R}^2 \times \text{Ker } \pi$, with finite and infinite dimensional variables are separated in the linear term. That is

$$\begin{cases} \dot{z} = Bz + \Psi(0) \begin{pmatrix} [\tilde{F}(\Phi(\theta)z_x + w, \mu), \beta_{n_c}^{(1)}] \\ [\tilde{F}(\Phi(\theta)z_x + w, \mu), \beta_{n_c}^{(2)}] \end{pmatrix}, \\ \dot{w} = A_{\mathcal{Q}^1}w + (I - \pi)X_0(\theta)\tilde{F}(\Phi(\theta)z_x + w, \mu), \end{cases} \quad (2.22)$$

where I is the identity matrix, $z = (z_1, z_2)^T$, $B = \text{diag}\{\dot{\omega}_c, -i\dot{\omega}_c\}$ is the diagonal matrix, and $A_{\mathcal{Q}^1} : \mathcal{Q}^1 \rightarrow \text{Ker } \pi$ is defined by

$$A_{\mathcal{Q}^1}w = \dot{w} + X_0(\theta)(\tau_c D_1 \Delta w(0) + \tau_c D_2 \Delta w(-1) + L_0(w) - \dot{w}(0)).$$

Consider the formal Taylor expansion

$$\tilde{F}(\varphi, \mu) = \sum_{j \geq 2} \frac{1}{j!} \tilde{F}_j(\varphi, \mu), \quad F(\varphi, \mu) = \sum_{j \geq 2} \frac{1}{j!} F_j(\varphi, \mu), \quad F^d(\varphi, \mu) = \sum_{j \geq 2} \frac{1}{j!} F_j^d(\varphi, \mu).$$

From (2.13), we have

$$\tilde{F}_2(\varphi, \mu) = 2\mu A\varphi(0) + F_2(\varphi, \mu) + F_2^d(\varphi, \mu) \quad (2.23)$$

and

$$\tilde{F}_j(\varphi, \mu) = F_j(\varphi, \mu) + F_j^d(\varphi, \mu), \quad j = 3, 4, \dots \quad (2.24)$$

By combining with (2.19), the system (2.22) can be rewritten as

$$\begin{cases} \dot{z} = Bz + \sum_{j \geq 2} \frac{1}{j!} f_j^1(z, w, \mu), \\ \dot{w} = A_{\mathcal{Q}^1}w + \sum_{j \geq 2} \frac{1}{j!} f_j^2(z, w, \mu), \end{cases}$$

where

$$\begin{aligned} f_j^1(z, w, \mu) &= \Psi(0) \begin{pmatrix} [\tilde{F}_j(\Phi(\theta)z_x + w, \mu), \beta_{n_c}^{(1)}] \\ [\tilde{F}_j(\Phi(\theta)z_x + w, \mu), \beta_{n_c}^{(2)}] \end{pmatrix}, \\ f_j^2(z, w, \mu) &= (I - \pi)X_0(\theta)\tilde{F}_j(\Phi(\theta)z_x + w, \mu). \end{aligned} \quad (2.25)$$

In terms of the normal form theory of partial functional differential equations [27], after a recursive transformation of variables of the form

$$(z, w) = (\tilde{z}, \tilde{w}) + \frac{1}{j!} (U_j^1(\tilde{z}, \mu), U_j^2(\tilde{z}, \mu)), \quad j \geq 2, \quad (2.26)$$

where $z, \tilde{z} \in \mathbb{R}^2$, $w, \tilde{w} \in \mathcal{Q}^1$ and $U_j^1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $U_j^2 : \mathbb{R}^2 \rightarrow \mathcal{Q}^1$ are homogeneous polynomials of degree j in $\tilde{z}$ and μ , a locally center manifold for (2.12) satisfies $w = 0$ and the flow on it is given by the two-dimensional ODEs

$$\dot{z} = Bz + \sum_{j \geq 2} \frac{1}{j!} g_j^1(z, 0, \mu),$$

which is the normal form as in the usual sense for ODEs.

By following [27] and [28], we have

$$g_2^1(z, 0, \mu) = \text{Proj}_{\text{Ker}(M_2^1)} f_2^1(z, 0, \mu) \quad (2.27)$$

and

$$g_3^1(z, 0, \mu) = \text{Proj}_{\text{Ker}(M_3^1)} \tilde{f}_3^1(z, 0, \mu) = \text{Proj}_S \tilde{f}_3^1(z, 0, 0) + O(\mu^2 |z|), \quad (2.28)$$

where $\text{Proj}_p(q)$ represents the projection of q on p , and $\tilde{f}_3^1(z, 0, \mu)$ is vector and its element is the cubic polynomial of (z, μ) after the variable transformation of (2.26), and it is determined by (2.38),

$$\begin{aligned} \text{Ker}(M_2^1) &= \text{Span} \left\{ \begin{pmatrix} \mu z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu z_2 \end{pmatrix} \right\}, \\ \text{Ker}(M_3^1) &= \text{Span} \left\{ \begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu^2 z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu^2 z_2 \end{pmatrix} \right\}, \end{aligned} \quad (2.29)$$

and

$$S = \text{Span} \left\{ \begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix} \right\}. \quad (2.30)$$

In the following, for notational convenience, we let

$$\mathcal{H}(\alpha z_1^{q_1} z_2^{q_2} \mu) = \begin{pmatrix} \alpha z_1^{q_1} z_2^{q_2} \mu \\ \bar{\alpha} z_1^{q_2} z_2^{q_1} \mu \end{pmatrix}, \quad \alpha \in \mathbb{C}.$$

We then calculate $g_j^1(z, 0, \mu)$, $j = 2, 3$ step by step.

2.3 Algorithm for calculating the normal form of Hopf bifurcation

2.3.1 Calculation of $g_2^1(z, 0, \mu)$

From the second mathematical expression in (2.9), we have

$$F_2^d(\varphi, \mu) = F_{20}^d(\varphi) + \mu F_{21}^d(\varphi) \quad (2.31)$$

and

$$F_3^d(\varphi, \mu) = \mu F_{31}^d(\varphi), \quad F_j^d(\varphi, \mu) = (0, 0)^T, \quad j = 4, 5, \dots, \quad (2.32)$$

where

$$\begin{cases} F_{20}^d(\varphi) = 2 \begin{pmatrix} \xi \tau_c \left(\varphi_x^{(1)}(0) \varphi_x^{(2)}(0) + \varphi^{(1)}(0) \varphi_{xx}^{(2)}(0) \right) \\ -d_{21} \tau_c \left(\varphi_x^{(1)}(-1) \varphi_x^{(2)}(0) + \varphi_{xx}^{(1)}(-1) \varphi^{(2)}(0) \right) \end{pmatrix}, \\ F_{21}^d(\varphi) = 2D_1 \Delta \varphi(0) + 2D_2 \Delta \varphi(-1), \\ F_{31}^d(\varphi) = 6 \begin{pmatrix} \xi \mu \left(\varphi_x^{(1)}(0) \varphi_x^{(2)}(0) + \varphi^{(1)}(0) \varphi_{xx}^{(2)}(0) \right) \\ -d_{21} \mu \left(\varphi_x^{(1)}(-1) \varphi_x^{(2)}(0) + \varphi_{xx}^{(1)}(-1) \varphi^{(2)}(0) \right) \end{pmatrix}. \end{cases} \quad (2.33)$$

Furthermore, it is easy to verify that

$$\begin{aligned} \begin{pmatrix} \left[2\mu A(\Phi(0)z_x), \beta_{n_c}^{(1)} \right] \\ \left[2\mu A(\Phi(0)z_x), \beta_{n_c}^{(2)} \right] \end{pmatrix} &= 2\mu A \left(\Phi(0) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right), \\ \begin{pmatrix} \left[\mu F_{21}^d(\Phi(\theta)z_x), \beta_{n_c}^{(1)} \right] \\ \left[\mu F_{21}^d(\Phi(\theta)z_x), \beta_{n_c}^{(2)} \right] \end{pmatrix} &= -\frac{2n_c^2}{\ell^2} \mu \left(D_1 \left(\Phi(0) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) + D_2 \left(\Phi(-1) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) \right). \end{aligned} \quad (2.34)$$

From (2.11), we have $F_2(\Phi(\theta)z_x, \mu) = F_2(\Phi(\theta)z_x, 0)$ for all $\mu \in \mathbb{R}$. It follows from the first mathematical expression in (2.25) that

$$f_2^1(z, 0, \mu) = \Psi(0) \begin{pmatrix} \left[\tilde{F}_2(\Phi(\theta)z_x, \mu), \beta_{n_c}^{(1)} \right] \\ \left[\tilde{F}_2(\Phi(\theta)z_x, \mu), \beta_{n_c}^{(2)} \right] \end{pmatrix}.$$

This, together with (2.23), (2.27), (2.29), (2.31), (2.32), (2.33) and (2.34), yields to

$$g_2^1(z, 0, \mu) = \text{Proj}_{\text{Ker}(M_2^1)} f_2^1(z, 0, \mu) = \mathcal{H}(B_1 \mu z_1), \quad (2.35)$$

where

$$B_1 = 2\psi^T(0) \left(A\phi(0) - \frac{n_c^2}{\ell^2} (D_1\phi(0) + D_2\phi(-1)) \right). \quad (2.36)$$

2.3.2 Calculation of $g_3^1(z, 0, \mu)$

In this subsection, we calculate the third term $g_3^1(z, 0, \mu)$ in terms of (2.28). Denote

$$\begin{aligned} f_2^{(1,1)}(z, w, 0) &= \Psi(0) \begin{pmatrix} \left[F_2(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(1)} \right] \\ \left[F_2(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(2)} \right] \end{pmatrix}, \\ f_2^{(1,2)}(z, w, 0) &= \Psi(0) \begin{pmatrix} \left[F_2^d(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(1)} \right] \\ \left[F_2^d(\Phi(\theta)z_x + w, 0), \beta_{n_c}^{(2)} \right] \end{pmatrix}. \end{aligned} \quad (2.37)$$

It follows from (2.35) that $g_2^1(z, 0, 0) = (0, 0)^T$. Then $\tilde{f}_3^1(z, 0, 0)$ is determined by

$$\begin{aligned} \tilde{f}_3^1(z, 0, 0) &= f_3^1(z, 0, 0) + \frac{3}{2} \left((D_z f_2^1(z, 0, 0)) U_2^1(z, 0) + (D_w f_2^{(1,1)}(z, 0, 0)) U_2^2(z, 0) \right. \\ &\quad \left. + (D_{w, w_x, w_{xx}} f_2^{(1,2)}(z, 0, 0)) U_2^{(2,d)}(z, 0)(\theta) \right), \end{aligned} \quad (2.38)$$

where $f_2^1(z, 0, 0) = f_2^{(1,1)}(z, 0, 0) + f_2^{(1,2)}(z, 0, 0)$,

$$\begin{aligned} D_{w, w_x, w_{xx}} f_2^{(1,2)}(z, 0, 0) &= \left(D_w f_2^{(1,2)}(z, 0, 0), D_{w_x} f_2^{(1,2)}(z, 0, 0), D_{w_{xx}} f_2^{(1,2)}(z, 0, 0) \right), \\ U_2^1(z, 0) &= (M_2^1)^{-1} \text{Proj}_{\text{Im}(M_2^1)} f_2^1(z, 0, 0), \quad U_2^2(z, 0)(\theta) = (M_2^2)^{-1} f_2^2(z, 0, 0), \end{aligned} \quad (2.39)$$

and

$$U_2^{(2,d)}(z, 0)(\theta) = \text{col} \left(U_2^2(z, 0)(\theta), U_{2x}^2(z, 0)(\theta), U_{2xx}^2(z, 0)(\theta) \right). \quad (2.40)$$

We calculate $\text{Proj}_S \tilde{f}_3^1(z, 0, 0)$ in the following four steps.

Step 1: Calculation of $\text{Proj}_S f_3^1(z, 0, 0)$

Writing $F_3(\Phi(\theta)z_x, 0)$ as follows

$$F_3(\Phi(\theta)z_x, 0) = \sum_{q_1+q_2=3} A_{q_1 q_2} z_1^{q_1} z_2^{q_2} \gamma_{n_c}^3(x), \quad (2.41)$$

where $A_{q_1 q_2} = \overline{A_{q_2 q_1}}$ with $q_1, q_2 \in \mathbb{N}_0$. From (2.24) and (2.32), we have $\tilde{F}_3(\Phi(\theta)z_x, 0) = F_3(\Phi(\theta)z_x, 0)$, and thus

$$\text{Proj}_S f_3^1(z, 0, 0) = \mathcal{H}(B_{21} z_1^2 z_2),$$

where

$$B_{21} = \frac{3}{2\ell\pi} \psi^T A_{21}. \quad (2.42)$$

Step 2: Calculation of $\text{Proj}_S ((D_z f_2^1(z, 0, 0))U_2^1(z, 0))$

Form (2.23) and (2.31), we have

$$\tilde{F}_2(\Phi(\theta)z_x, 0) = F_2(\Phi(\theta)z_x, 0) + F_{20}^d(\Phi(\theta)z_x). \quad (2.43)$$

By (2.11), we write

$$\begin{aligned} F_2(\Phi(\theta)z_x + w, \mu) &= F_2(\Phi(\theta)z_x + w, 0) \\ &= \sum_{q_1+q_2=2} A_{q_1 q_2} z_1^{q_1} z_2^{q_2} \gamma_{n_c}^2(x) + \mathcal{S}_2(\Phi(\theta)z_x, w) + O(|w|^2), \end{aligned} \quad (2.44)$$

where $\mathcal{S}_2(\Phi(\theta)z_x, w)$ is the product term of $\Phi(\theta)z_x$ and w .

By (2.31) and (2.33), we write

$$F_2^d(\Phi(\theta)z_x, 0) = F_{20}^d(\Phi(\theta)z_x) = \frac{n_c^2}{\ell^2} \sum_{q_1+q_2=2} A_{q_1 q_2}^d z_1^{q_1} z_2^{q_2} (\xi_{n_c}^2(x) - \gamma_{n_c}^2(x)), \quad (2.45)$$

where $\xi_{n_c}(x) = (\sqrt{2}/\sqrt{\ell\pi}) \sin((n_c/\ell)x)$, and

$$\begin{cases} A_{20}^d = \begin{pmatrix} 2\xi\tau_c\phi_1(0)\phi_2(0) \\ -2d_{21}\tau_c\phi_1(-1)\phi_2(0) \end{pmatrix} = \overline{A_{02}^d}, \\ A_{11}^d = \begin{pmatrix} 4\xi\tau_c \text{Re}\{\phi_1(0)\overline{\phi_2(0)}\} \\ -4d_{21}\tau_c \text{Re}\{\phi_1(-1)\overline{\phi_2(0)}\} \end{pmatrix}. \end{cases} \quad (2.46)$$

From (2.1), it is easy to verify that

$$\int_0^{\ell\pi} \gamma_{n_c}^3(x) dx = \int_0^{\ell\pi} \xi_{n_2}^2(x) \gamma_{n_c}(x) dx = 0.$$

Then, from (2.43), (2.44) and (2.45), we have

$$f_2^1(z, 0, 0) = \Psi(0) \begin{pmatrix} \left[\tilde{F}_2(\Phi(\theta)z_x, 0), \beta_{n_c}^{(1)} \right] \\ \left[\tilde{F}_2(\Phi(\theta)z_x, 0), \beta_{n_c}^{(2)} \right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.47)$$

Thus, by combining with (2.30) and (2.47), we have

$$\text{Proj}_S ((D_z f_2^1(z, 0, 0))U_2^1(z, 0)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Step 3: Calculation of $\text{Proj}_S ((D_w f_2^{(1,1)}(z, 0, 0))U_2^2(z, 0)(\theta))$

Let

$$U_2^2(z, 0)(\theta) \triangleq h(\theta, z) = \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x), \quad (2.48)$$

where $h_n(\theta, z) = \sum_{q_1+q_2=2} h_{n, q_1 q_2}(\theta) z_1^{q_1} z_2^{q_2}$. Then, we have

$$\begin{aligned} & \begin{pmatrix} \left[\mathcal{S}_2 \left(\Phi(\theta)z_x, \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right), \beta_{n_c}^{(1)} \right] \\ \left[\mathcal{S}_2 \left(\Phi(\theta)z_x, \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right), \beta_{n_c}^{(2)} \right] \end{pmatrix} \\ &= \sum_{n \in \mathbb{N}_0} b_n (\mathcal{S}_2(\phi(\theta)z_1, h_n(\theta, z)) + \mathcal{S}_2(\overline{\phi}(\theta)z_2, h_n(\theta, z))), \end{aligned}$$

where

$$b_n = \int_0^{\ell\pi} \gamma_{n_c}^2(x) \gamma_n(x) dx = \begin{cases} \frac{1}{\sqrt{\ell\pi}}, & n = 0, \\ \frac{1}{\sqrt{2\ell\pi}}, & n = 2n_c \neq 0, \\ 0, & \text{otherwise} . \end{cases} \quad (2.49)$$

Hence, we have

$$\begin{aligned} & (D_w f_2^{(1,1)}(z, 0, 0)) U_2^2(z, 0)(\theta) \\ &= \Psi(0) \left(\sum_{n=0, 2n_c} b_n (\mathcal{S}_2(\phi(\theta)z_1, h_n(\theta, z)) + \mathcal{S}_2(\bar{\phi}(\theta)z_2, h_n(\theta, z))) \right), \end{aligned}$$

and

$$\text{Proj}_S \left((D_w f_2^{(1,1)}(z, 0, 0)) U_2^2(z, 0)(\theta) \right) = \mathcal{H}(B_{22} z_1^2 z_2),$$

where

$$\begin{aligned} B_{22} &= \frac{1}{\sqrt{\ell\pi}} \psi^T (\mathcal{S}_2(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2(\bar{\phi}(\theta), h_{0,20}(\theta))) \\ &+ \frac{1}{\sqrt{2\ell\pi}} \psi^T (\mathcal{S}_2(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2(\bar{\phi}(\theta), h_{2n_c,20}(\theta))). \end{aligned} \quad (2.50)$$

Step 4: Calculation of $\text{Proj}_S \left((D_{w, w_x, w_{xx}} f_2^{(1,2)}(z, 0, 0)) U_2^{(2,d)}(z, 0)(\theta) \right)$

Denote $\varphi(\theta) = (\varphi^{(1)}(\theta), \varphi^{(2)}(\theta))^T = \Phi(\theta)z_x$,

$$\begin{aligned} F_2^d(\varphi(\theta), w, w_x, w_{xx}) &= F_2^d(\varphi(\theta) + w, 0) = F_{20}^d(\varphi(\theta) + w) \\ &= 2 \begin{pmatrix} \xi \tau_c \left((\varphi_x^{(1)}(0) + w_x^{(1)}(0))(\varphi_x^{(2)}(0) + w_x^{(2)}(0)) + (\varphi^{(1)}(0) + w^{(1)}(0))(\varphi_{xx}^{(2)}(0) + w_{xx}^{(2)}(0)) \right) \\ -d_{21} \tau_c \left((\varphi_x^{(1)}(-1) + w_x^{(1)}(-1))(\varphi_x^{(2)}(0) + w_x^{(2)}(0)) + (\varphi_{xx}^{(1)}(-1) + w_{xx}^{(1)}(-1))(\varphi^{(2)}(0) + w^{(2)}(0)) \right) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{S}}_2^{(d,1)}(\varphi(\theta), w) &= 2 \begin{pmatrix} \xi \tau_c \varphi_{xx}^{(2)}(0) w^{(1)}(0) \\ -d_{21} \tau_c \varphi_{xx}^{(1)}(-1) w^{(2)}(0) \end{pmatrix}, \\ \tilde{\mathcal{S}}_2^{(d,2)}(\varphi(\theta), w_x) &= 2 \begin{pmatrix} \xi \tau_c (\varphi_x^{(2)}(0) w_x^{(1)}(0) + \varphi_x^{(1)}(0) w_x^{(2)}(0)) \\ -d_{21} \tau_c (\varphi_x^{(2)}(0) w_x^{(1)}(-1) + \varphi_x^{(1)}(-1) w_x^{(2)}(0)) \end{pmatrix}, \\ \tilde{\mathcal{S}}_2^{(d,3)}(\varphi(\theta), w_{xx}) &= 2 \begin{pmatrix} \xi \tau_c \varphi^{(1)}(0) w_{xx}^{(2)}(0) \\ -d_{21} \tau_c \varphi^{(2)}(0) w_{xx}^{(1)}(-1) \end{pmatrix}. \end{aligned}$$

By combining with (2.1) and (2.48), we have

$$\begin{cases} U_{2x}^2(z, 0)(\theta) = h_x(\theta, z) = -\frac{n}{\ell} \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \xi_n(x), \\ U_{2xx}^2(z, 0)(\theta) = h_{xx}(\theta, z) = -\frac{n^2}{\ell^2} \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x). \end{cases}$$

Then we have

$$\begin{aligned} & (D_{w, w_x, w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx})) U_2^{(2,d)}(z, 0)(\theta) \\ &= \tilde{\mathcal{S}}_2^{(d,1)}(\varphi(\theta), h(\theta, z)) + \tilde{\mathcal{S}}_2^{(d,2)}(\varphi(\theta), h_x(\theta, z)) + \tilde{\mathcal{S}}_2^{(d,3)}(\varphi(\theta), h_{xx}(\theta, z)) \end{aligned}$$

and

$$\begin{aligned}
& \begin{pmatrix} \left[\tilde{S}_2^{(d,1)}(\varphi(\theta), h(\theta, z)), \beta_{n_c}^{(1)} \right] \\ \left[\tilde{S}_2^{(d,1)}(\varphi(\theta), h(\theta, z)), \beta_{n_c}^{(2)} \right] \end{pmatrix} \\
&= - (n_c/\ell)^2 \sum_{n \in \mathbb{N}_0} b_n \left(\mathcal{S}_2^{(d,1)}(\phi(\theta)z_1, h_n(\theta, z)) + \mathcal{S}_2^{(d,1)}(\bar{\phi}(\theta)z_2, h_n(\theta, z)) \right), \\
& \begin{pmatrix} \left[\tilde{S}_2^{(d,2)}(\varphi(\theta), h_x(\theta, z)), \beta_{n_c}^{(1)} \right] \\ \left[\tilde{S}_2^{(d,2)}(\varphi(\theta), h_x(\theta, z)), \beta_{n_c}^{(2)} \right] \end{pmatrix} \\
&= (n_c/\ell) \sum_{n \in \mathbb{N}_0} (n/\ell) c_n \left(\mathcal{S}_2^{(d,2)}(\phi(\theta)z_1, h_n(\theta, z)) + \mathcal{S}_2^{(d,2)}(\bar{\phi}(\theta)z_2, h_n(\theta, z)) \right), \\
& \begin{pmatrix} \left[\tilde{S}_2^{(d,3)}(\varphi(\theta), h_{xx}(\theta, z)), \beta_{n_c}^{(1)} \right] \\ \left[\tilde{S}_2^{(d,3)}(\varphi(\theta), h_{xx}(\theta, z)), \beta_{n_c}^{(2)} \right] \end{pmatrix} \\
&= - \sum_{n \in \mathbb{N}_0} (n/\ell)^2 b_n \left(\mathcal{S}_2^{(d,3)}(\phi(\theta)z_1, h_n(\theta, z)) + \mathcal{S}_2^{(d,3)}(\bar{\phi}(\theta)z_2, h_n(\theta, z)) \right),
\end{aligned}$$

where b_n is defined by (2.49) and

$$c_n = \int_0^{\ell\pi} \xi_{n_c}(x) \xi_n(x) \gamma_{n_c}(x) dx = \begin{cases} \frac{1}{\sqrt{2\ell\pi}}, & n = 2n_c \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and for $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta))^T$, $y(\theta) = (y_1(\theta), y_2(\theta))^T \in C([-1, 0], \mathbb{R}^2)$, we have

$$\begin{cases} \mathcal{S}_2^{(d,1)}(\phi(\theta), y(\theta)) = 2 \begin{pmatrix} \xi \tau_c \phi_2(0) y_1(0) \\ -d_{21} \tau_c \phi_1(-1) y_2(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,2)}(\phi(\theta), y(\theta)) = 2 \begin{pmatrix} \xi \tau_c (\phi_2(0) y_1(0) + \phi_1(0) y_2(0)) \\ -d_{21} \tau_c (\phi_2(0) y_1(-1) + \phi_1(-1) y_2(0)) \end{pmatrix}, \\ \mathcal{S}_2^{(d,3)}(\phi(\theta), y(\theta)) = 2 \begin{pmatrix} \xi \tau_c \phi_1(0) y_2(0) \\ -d_{21} \tau_c \phi_2(0) y_1(-1) \end{pmatrix}. \end{cases}$$

Furthermore, from (2.37), (2.39) and (2.40), we have

$$\begin{aligned}
& (D_{w, w_x, w_{xx}} f_2^{(1,2)}(z, 0, 0)) U_2^{(2,d)}(z, 0)(\theta) \\
&= \Psi(0) \begin{pmatrix} \left[D_{w, w_x, w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx}) U_2^{(2,d)}(z, 0)(\theta), \beta_{n_c}^{(1)} \right] \\ \left[D_{w, w_x, w_{xx}} F_2^d(\varphi(\theta), w, w_x, w_{xx}) U_2^{(2,d)}(z, 0)(\theta), \beta_{n_c}^{(2)} \right] \end{pmatrix},
\end{aligned}$$

and then we obtain

$$\text{Proj}_S \left((D_{w, w_x, w_{xx}} f_2^{(1,2)}(z, 0, 0)) U_2^{(2,d)}(z, 0)(\theta) \right) = \mathcal{H}(B_{23} z_1^2 z_2),$$

where

$$\begin{aligned}
B_{23} &= - \frac{1}{\sqrt{\ell\pi}} (n_c/\ell)^2 \psi^T \left(\mathcal{S}_2^{(d,1)}(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2^{(d,1)}(\bar{\phi}(\theta), h_{0,20}(\theta)) \right) \\
&+ \frac{1}{\sqrt{2\ell\pi}} \psi^T \sum_{j=1,2,3} b_{2n_c}^{(j)} \left(\mathcal{S}_2^{(d,j)}(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2^{(d,j)}(\bar{\phi}(\theta), h_{2n_c,20}(\theta)) \right)
\end{aligned} \tag{2.51}$$

with

$$b_{2n_c}^{(1)} = -\frac{n_c^2}{\ell^2}, \quad b_{2n_c}^{(2)} = \frac{2n_c^2}{\ell^2}, \quad b_{2n_c}^{(3)} = -\frac{(2n_c)^2}{\ell^2}.$$

3 Normal form of the Hopf bifurcation and the corresponding coefficients

According to the algorithm developed in Section 2, we can obtain the normal form of the Hopf bifurcation truncated to the third-order term

$$\dot{z} = Bz + \frac{1}{2} \left(\frac{B_1 z_1 \mu}{B_1 z_2 \mu} \right) + \frac{1}{3!} \left(\frac{B_2 z_1^2 z_2}{B_2 z_1 z_2^2} \right) + O(|z|\mu^2 + |z|^4), \quad (3.1)$$

where

$$B_1 = 2\psi^T(0) \left(A\phi(0) - \frac{n_c^2}{\ell^2} (D_1\phi(0) + D_2\phi(-1)) \right),$$

$$B_2 = B_{21} + \frac{3}{2} (B_{22} + B_{23}).$$

Here, B_1 is determined by (2.36), B_{21} , B_{22} and B_{23} are determined by (2.42), (2.50), (2.51), respectively, and they can be calculated by using the MATLAB software. The normal form (3.1) can be written in real coordinates through the change of variables $z_1 = v_1 - iv_2$, $z_2 = v_1 + iv_2$, and then changing to polar coordinates by $v_1 = \rho \cos \Theta$, $v_2 = \rho \sin \Theta$, where Θ is the azimuthal angle. Therefore, by the above transformation and removing the azimuthal term Θ , (3.1) can be rewritten as

$$\dot{\rho} = K_1 \mu \rho + K_2 \rho^3 + O(\mu^2 \rho + |(\rho, \mu)|^4),$$

where

$$K_1 = \frac{1}{2} \operatorname{Re}(B_1), \quad K_2 = \frac{1}{3!} \operatorname{Re}(B_2).$$

According to [29], the sign of $K_1 K_2$ determines the direction of the Hopf bifurcation, and the sign of K_2 determines the stability of the Hopf bifurcation periodic solution. More precisely, we have the following results

- (i) when $K_1 K_2 < 0$, the Hopf bifurcation is supercritical, and the Hopf bifurcation periodic solution is stable for $K_2 < 0$ and unstable for $K_2 > 0$;
- (ii) when $K_1 K_2 > 0$, the Hopf bifurcation is subcritical, and the Hopf bifurcation periodic solution is stable for $K_2 < 0$ and unstable for $K_2 > 0$.

From (2.42), (2.50) and (2.51), it is obvious that in order to obtain the value of K_2 , we still need to calculate $h_{0,20}(\theta)$, $h_{0,11}(\theta)$, $h_{2n_c,20}(\theta)$, $h_{2n_c,11}(\theta)$ and A_{ij} .

3.1 Calculations of $h_{0,20}(\theta)$, $h_{0,11}(\theta)$, $h_{2n_c,20}(\theta)$ and $h_{2n_c,11}(\theta)$

From [27], we have

$$M_2^2(h_n(\theta, z)\gamma_n(x)) = D_z(h_n(\theta, z)\gamma_n(x))Bz - A_{Q^1}(h_n(\theta, z)\gamma_n(x)),$$

which leads to

$$\begin{aligned} & \begin{pmatrix} \left[M_2^2(h_n(\theta, z)\gamma_n(x)), \beta_n^{(1)} \right] \\ \left[M_2^2(h_n(\theta, z)\gamma_n(x)), \beta_n^{(2)} \right] \end{pmatrix} \\ &= 2i\omega_c(h_{n,20}(\theta)z_1^2 - h_{n,02}(\theta)z_2^2) - \left(\dot{h}_n(\theta, z) + X_0(\theta) \left(\mathcal{L}_0(h_n(\theta, z)) - \dot{h}_n(0, z) \right) \right), \end{aligned} \quad (3.2)$$

where

$$\mathcal{L}_0(h_n(\theta, z)) = -\tau_c(n/\ell)^2 (D_1 h_n(0, z) + D_2 h_n(-1, z)) + \tau_c A h_n(0, z).$$

By (2.19) and the second mathematical expression in (2.25), we have

$$\begin{aligned} f_2^2(z, 0, 0) &= X_0(\theta) \tilde{F}_2(\Phi(\theta)z_x, 0) - \pi \left(X_0(\theta) \tilde{F}_2(\Phi(\theta)z_x, 0) \right) \\ &= X_0(\theta) \tilde{F}_2(\Phi(\theta)z_x, 0) - \Phi(\theta) \Psi(0) \begin{pmatrix} \left[\tilde{F}_2(\Phi(\theta)z_x, 0), \beta_{n_c}^{(1)} \right] \\ \left[\tilde{F}_2(\Phi(\theta)z_x, 0), \beta_{n_c}^{(2)} \right] \end{pmatrix} \gamma_{n_c}(x). \end{aligned} \quad (3.3)$$

Furthermore, by (2.43), (2.44) and (2.45), we have

$$\left(\begin{array}{c} \left[f_2^2(z, 0, 0), \beta_n^{(1)} \right] \\ \left[f_2^2(z, 0, 0), \beta_n^{(2)} \right] \end{array} \right) = \begin{cases} \frac{1}{\sqrt{\ell\pi}} X_0(\theta) (A_{20}z_1^2 + A_{02}z_2^2 + A_{11}z_1z_2), & n = 0, \\ \frac{1}{\sqrt{2\ell\pi}} X_0(\theta) (\tilde{A}_{20}z_1^2 + \tilde{A}_{02}z_2^2 + \tilde{A}_{11}z_1z_2), & n = 2n_c, \end{cases} \quad (3.4)$$

where $\tilde{A}_{j_1j_2}$ is defined as follows

$$\begin{cases} \tilde{A}_{j_1j_2} = A_{j_1j_2} - 2(n_c/\ell)^2 A_{j_1j_2}^d, \\ j_1, j_2 = 0, 1, 2, \quad j_1 + j_2 = 2, \end{cases} \quad (3.5)$$

where $A_{j_1j_2}^d$ is determined by (2.46), and $A_{j_1j_2}$ will be calculated in the following section. Therefore, from (2.39), (3.2), (3.3), (3.4), and by matching the coefficients of z_1^2 and z_1z_2 , we have

$$n = 0, \quad \begin{cases} z_1^2 : \begin{cases} \dot{h}_{0,20}(\theta) - 2i\omega_c h_{0,20}(\theta) = (0, 0)^T, \\ \dot{h}_{0,20}(0) - L_0(h_{0,20}(\theta)) = \frac{1}{\sqrt{\ell\pi}} A_{20}, \end{cases} \\ z_1z_2 : \begin{cases} \dot{h}_{0,11}(\theta) = (0, 0)^T, \\ \dot{h}_{0,11}(0) - L_0(h_{0,11}(\theta)) = \frac{1}{\sqrt{\ell\pi}} A_{11} \end{cases} \end{cases} \quad (3.6)$$

and

$$n = 2n_c, \quad \begin{cases} z_1^2 : \begin{cases} \dot{h}_{2n_c,20}(\theta) - 2i\omega_c h_{2n_c,20}(\theta) = (0, 0)^T, \\ \dot{h}_{2n_c,20}(0) - \mathcal{L}_0(h_{2n_c,20}(\theta)) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{20}, \end{cases} \\ z_1z_2 : \begin{cases} \dot{h}_{2n_c,11}(\theta) = (0, 0)^T, \\ \dot{h}_{2n_c,11}(0) - \mathcal{L}_0(h_{2n_c,11}(\theta)) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11}. \end{cases} \end{cases} \quad (3.7)$$

Next, by combining with (3.6) and (3.7), we will give the mathematical expressions of $h_{0,20}(\theta)$, $h_{0,11}(\theta)$, $h_{2n_c,20}(\theta)$ and $h_{2n_c,11}(\theta)$.

(1) Calculations of $h_{0,20}(\theta)$ and $h_{0,11}(\theta)$

(i) Notice that

$$\begin{cases} \dot{h}_{0,20}(\theta) - 2i\omega_c h_{0,20}(\theta) = (0, 0)^T, \\ \dot{h}_{0,20}(0) - L_0(h_{0,20}(\theta)) = \frac{1}{\sqrt{\ell\pi}} A_{20}, \end{cases} \quad (3.8)$$

then from (3.8), we have $h_{0,20}(\theta) = e^{2i\omega_c\theta} h_{0,20}(0)$ and $\dot{h}_{0,20}(0) - 2i\omega_c h_{0,20}(0) = (0, 0)^T$. Notice that $L_0(h_{0,20}(\theta)) = \tau_c A h_{0,20}(0)$, then we have

$$(2i\omega_c I - \tau_c A) h_{0,20}(0) = \frac{1}{\sqrt{\ell\pi}} A_{20},$$

and hence $h_{0,20}(\theta) = e^{2i\omega_c\theta} C_1$ with

$$C_1 = (2i\omega_c I - \tau_c A)^{-1} \frac{1}{\sqrt{\ell\pi}} A_{20}.$$

(ii) Notice that

$$\begin{cases} \dot{h}_{0,11}(\theta) = (0, 0)^T, \\ \dot{h}_{0,11}(0) - L_0(h_{0,11}(\theta)) = \frac{1}{\sqrt{\ell\pi}} A_{11}, \end{cases} \quad (3.9)$$

then from (3.9), we have $h_{0,11}(\theta) = h_{0,11}(0)$ and $\dot{h}_{0,11}(0) = (0, 0)^T$. Notice that $L_0(h_{0,11}(\theta)) = \tau_c A h_{0,11}(0)$, then we have

$$-\tau_c A h_{0,11}(0) = \frac{1}{\sqrt{\ell\pi}} A_{11},$$

and hence $h_{0,11}(\theta) = C_2$ with

$$C_2 = (-\tau_c A)^{-1} \frac{1}{\sqrt{\ell\pi}} A_{11}.$$

(2) Calculations of $h_{2n_c,20}(\theta)$ and $h_{2n_c,11}(\theta)$

(i) Notice that

$$\begin{cases} \dot{h}_{2n_c,20}(\theta) - 2i\omega_c h_{2n_c,20}(\theta) = (0, 0)^T, \\ \dot{h}_{2n_c,20}(0) - \mathcal{L}_0(h_{2n_c,20}(\theta)) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{20}, \end{cases} \quad (3.10)$$

then from (3.10), we have $h_{2n_c,20}(\theta) = e^{2i\omega_c\theta} h_{2n_c,20}(0)$, and hence $h_{2n_c,20}(-1) = e^{-2i\omega_c} h_{2n_c,20}(0)$. Furthermore, from (3.10) and

$$\mathcal{L}_0(h_{2n_c,20}(\theta)) = -\tau_c \frac{4n_c^2}{\ell^2} (D_1 h_{2n_c,20}(0) + D_2 h_{2n_c,20}(-1)) + \tau_c A h_{2n_c,20}(0),$$

we have

$$2i\omega_c h_{2n_c,20}(0) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{20} - \tau_c \frac{4n_c^2}{\ell^2} (D_1 h_{2n_c,20}(0) + D_2 h_{2n_c,20}(-1)) + \tau_c A h_{2n_c,20}(0). \quad (3.11)$$

Therefore, by combining with $h_{2n_c,20}(-1) = e^{-2i\omega_c} h_{2n_c,20}(0)$ and (3.11), we can obtain

$$(2i\omega_c I + \tau_c \frac{4n_c^2}{\ell^2} D_1 + \tau_c \frac{4n_c^2}{\ell^2} D_2 e^{-2i\omega_c} - \tau_c A) h_{2n_c,20}(0) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{20},$$

and hence

$$h_{2n_c,20}(\theta) = e^{2i\omega_c\theta} C_3$$

with

$$C_3 = (2i\omega_c I + \tau_c \frac{4n_c^2}{\ell^2} D_1 + \tau_c \frac{4n_c^2}{\ell^2} D_2 e^{-2i\omega_c} - \tau_c A)^{-1} \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{20}.$$

Here, $\tilde{A}_{20}$ and A_{20}^d are defined by (3.5) and (2.46), respectively.

(ii) Notice that

$$\begin{cases} \dot{h}_{2n_c,11}(\theta) = (0, 0)^T, \\ \dot{h}_{2n_c,11}(0) - \mathcal{L}_0(h_{2n_c,11}(\theta)) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11}, \end{cases} \quad (3.12)$$

then from (3.12), we have $h_{2n_c,11}(\theta) = h_{2n_c,11}(0)$, and hence $h_{2n_c,11}(-1) = h_{2n_c,11}(0)$. Furthermore, from (3.12) and

$$\mathcal{L}_0(h_{2n_c,11}(\theta)) = -\tau_c \frac{4n_c^2}{\ell^2} (D_1 h_{2n_c,11}(0) + D_2 h_{2n_c,11}(-1)) + \tau_c A h_{2n_c,11}(0),$$

we have

$$(0, 0)^T = -\tau_c \frac{4n_c^2}{\ell^2} (D_1 h_{2n_c,11}(0) + D_2 h_{2n_c,11}(-1)) + \tau_c A h_{2n_c,11}(0) + \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11}. \quad (3.13)$$

Therefore, by combining with $h_{2n_c,11}(-1) = h_{2n_c,11}(0)$ and (3.13), we can obtain

$$\left(\tau_c \frac{4n_c^2}{\ell^2} D_1 + \tau_c \frac{4n_c^2}{\ell^2} D_2 - \tau_c A \right) h_{2n_c,11}(0) = \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11},$$

and hence

$$h_{2n_c,11}(\theta) = C_4$$

with

$$C_4 = \left(\tau_c \frac{4n_c^2}{\ell^2} D_1 + \tau_c \frac{4n_c^2}{\ell^2} D_2 - \tau_c A \right)^{-1} \frac{1}{\sqrt{2\ell\pi}} \tilde{A}_{11}.$$

Here, $\tilde{A}_{11}$ and A_{11}^d are defined by (3.5) and (2.46), respectively.

3.2 Calculations of $A_{i,j}$ and $\mathcal{S}_2(\Phi(\theta)z_x, w)$

In this subsection, let $F(\varphi, \mu) = (F^{(1)}(\varphi, \mu), F^{(2)}(\varphi, \mu))^T$ and $\varphi = (\varphi_1, \varphi_2)^T \in \mathcal{C}$, and we write

$$\frac{1}{j!} F_j(\varphi, \mu) = \sum_{j_1+j_2=j} \frac{1}{j_1!j_2!} f_{j_1j_2} \varphi_1^{j_1}(0) \varphi_2^{j_2}(0), \quad (3.14)$$

where $f_{j_1j_2} = \text{col} \left(f_{j_1j_2}^{(1)}, f_{j_1j_2}^{(2)} \right)$ with

$$f_{j_1j_2}^{(k)} = \frac{\partial^{j_1+j_2} F^{(k)}(0, 0)}{\partial \varphi_1^{j_1}(0) \partial \varphi_2^{j_2}(0)}, \quad k = 1, 2.$$

Then from (3.14), we have

$$\begin{aligned} F_2(\varphi, \mu) &= F_2(\varphi, 0) = 2 \sum_{j_1+j_2=2} \frac{1}{j_1!j_2!} f_{j_1j_2} \varphi_1^{j_1}(0) \varphi_2^{j_2}(0) \\ &= f_{20} \varphi_1^2(0) + f_{02} \varphi_2^2(0) + 2f_{11} \varphi_1(0) \varphi_2(0) \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} F_3(\varphi, 0) &= 6 \sum_{j_1+j_2=3} \frac{1}{j_1!j_2!} f_{j_1j_2} \varphi_1^{j_1}(0) \varphi_2^{j_2}(0) \\ &= f_{30} \varphi_1^3(0) + f_{03} \varphi_2^3(0) + 3f_{21} \varphi_1^2(0) \varphi_2(0) + 3f_{12} \varphi_1(0) \varphi_2^2(0). \end{aligned} \quad (3.16)$$

Notice that

$$\begin{aligned} \varphi(\theta) &= \Phi(\theta)z_x = \phi(\theta)z_1(t)\gamma_{n_c}(x) + \bar{\phi}(\theta)z_2(t)\gamma_{n_c}(x) \\ &= \begin{pmatrix} \phi_1(\theta)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(\theta)z_2(t)\gamma_{n_c}(x) \\ \phi_2(\theta)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_2(\theta)z_2(t)\gamma_{n_c}(x) \end{pmatrix} \\ &= \begin{pmatrix} \varphi_1(\theta) \\ \varphi_2(\theta) \end{pmatrix}, \end{aligned} \quad (3.17)$$

and similar to (2.41), we have

$$F_2(\Phi(\theta)z_x, 0) = \sum_{q_1+q_2=2} A_{q_1q_2} \gamma_{n_c}^{q_1+q_2}(x) z_1^{q_1} z_2^{q_2}, \quad (3.18)$$

then by combining with (3.15), (3.17) and (3.18), we have

$$\begin{aligned} A_{20} &= f_{20} \phi_1^2(0) + f_{02} \phi_2^2(0) + 2f_{11} \phi_1(0) \phi_2(0), \\ A_{02} &= f_{20} \bar{\phi}_1^2(0) + f_{02} \bar{\phi}_2^2(0) + 2f_{11} \bar{\phi}_1(0) \bar{\phi}_2(0), \\ A_{11} &= 2f_{20} \phi_1(0) \bar{\phi}_1(0) + 2f_{02} \phi_2(0) \bar{\phi}_2(0) + 2f_{11} (\phi_1(0) \bar{\phi}_2(0) + \bar{\phi}_1(0) \phi_2(0)). \end{aligned}$$

Furthermore, from (2.41), (3.16) and (3.17), we have

$$\begin{aligned} A_{30} &= f_{30} \phi_1^3(0) + f_{03} \phi_2^3(0) + 3f_{21} \phi_1^2(0) \phi_2(0) + 3f_{12} \phi_1(0) \phi_2^2(0), \\ A_{03} &= f_{30} \bar{\phi}_1^3(0) + f_{03} \bar{\phi}_2^3(0) + 3f_{21} \bar{\phi}_1^2(0) \bar{\phi}_2(0) + 3f_{12} \bar{\phi}_1(0) \bar{\phi}_2^2(0), \\ A_{21} &= 3f_{30} \phi_1^2(0) \bar{\phi}_1(0) + 3f_{03} \phi_2^2(0) \bar{\phi}_2(0) + 3f_{21} (\phi_1^2(0) \bar{\phi}_2(0) + 2\phi_1(0) \bar{\phi}_1(0) \phi_2(0) \\ &\quad + 3f_{12} (2\phi_1(0) \phi_2(0) \bar{\phi}_2(0) + \bar{\phi}_1(0) \phi_2^2(0)), \\ A_{12} &= 3f_{30} \phi_1(0) \bar{\phi}_1^2(0) + 3f_{03} \phi_2(0) \bar{\phi}_2^2(0) + 3f_{21} (2\phi_1(0) \bar{\phi}_1(0) \bar{\phi}_2(0) + \bar{\phi}_1^2(0) \phi_2(0)) \\ &\quad + 3f_{12} (\phi_1(0) \bar{\phi}_2^2(0) + 2\bar{\phi}_1(0) \phi_2(0) \bar{\phi}_2(0)). \end{aligned}$$

Moreover, from (3.14), we have

$$\begin{aligned} F_2(\varphi(\theta) + w, \mu) &= F_2(\varphi(\theta) + w, 0) = 2 \sum_{j_1+j_2=2} \frac{1}{j_1!j_2!} f_{j_1j_2} (\varphi_1(0) + w_1(0))^{j_1} (\varphi_2(0) + w_2(0))^{j_2} \\ &= f_{20} (\varphi_1(0) + w_1(0))^2 + f_{02} (\varphi_2(0) + w_2(0))^2 + 2f_{11} (\varphi_1(0) + w_1(0)) (\varphi_2(0) + w_2(0)). \end{aligned} \quad (3.19)$$

Notice that

$$\begin{aligned}
 \varphi(\theta) + w &= \Phi(\theta)z_x + w = \phi(\theta)z_1(t)\gamma_{n_c}(x) + \bar{\phi}(\theta)z_2(t)\gamma_{n_c}(x) + w \\
 &= \begin{pmatrix} \phi_1(\theta)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(\theta)z_2(t)\gamma_{n_c}(x) + w_1 \\ \phi_2(\theta)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_2(\theta)z_2(t)\gamma_{n_c}(x) + w_2 \end{pmatrix} \\
 &= \begin{pmatrix} \varphi_1(\theta) + w_1 \\ \varphi_2(\theta) + w_2 \end{pmatrix}
 \end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
 F_2(\Phi(\theta)z_x + w, \mu) &= F_2(\Phi(\theta)z_x + w, 0) \\
 &= \sum_{q_1+q_2=2} A_{q_1 q_2} \gamma_{n_c}^{q_1+q_2}(x) z_1^{q_1} z_2^{q_2} + \mathcal{S}_2(\Phi(\theta)z_x, w) + O(|w|^2),
 \end{aligned} \tag{3.21}$$

then by combining with (3.19), (3.20) and (3.21), we have

$$\begin{aligned}
 \mathcal{S}_2(\Phi(\theta)z_x, w) &= 2f_{20}(\phi_1(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(0)z_2(t)\gamma_{n_c}(x))w_1(0) \\
 &+ 2f_{02}(\phi_2(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_2(0)z_2(t)\gamma_{n_c}(x))w_2(0) \\
 &+ 2f_{11}((\phi_1(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_1(0)z_2(t)\gamma_{n_c}(x))w_2(0) + (\phi_2(0)z_1(t)\gamma_{n_c}(x) + \bar{\phi}_2(0)z_2(t)\gamma_{n_c}(x))w_1(0)).
 \end{aligned}$$

4 Application to Holling-Tanner model with spatial memory and predator-taxis

In this section, we apply our newly developed algorithm in Section 2 to the Holling-Tanner model with spatial memory and predator-taxis, i.e., in the model (1.4), we let

$$\begin{aligned}
 f(u(x, t), v(x, t)) &= u(x, t)(1 - \beta u(x, t)) - \frac{mu(x, t)v(x, t)}{1 + u(x, t)}, \\
 g(u(x, t), v(x, t)) &= sv(x, t) \left(1 - \frac{v(x, t)}{u(x, t)}\right),
 \end{aligned} \tag{4.1}$$

where $\beta > 0$, $m > 0$ and $s > 0$.

Thus, (1.4) becomes the following model

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} = d_{11}\Delta u(x, t) + \xi(u(x, t)v_x(x, t))_x + u(x, t)(1 - \beta u(x, t)) - \frac{mu(x, t)v(x, t)}{1 + u(x, t)}, & x \in (0, \ell\pi), t > 0, \\ \frac{\partial v(x, t)}{\partial t} = d_{22}\Delta v(x, t) - d_{21}(v(x, t)u_x(x, t - \tau))_x + sv(x, t) \left(1 - \frac{v(x, t)}{u(x, t)}\right), & x \in (0, \ell\pi), t > 0, \\ u_x(0, t) = u_x(\ell\pi, t) = v_x(0, t) = v_x(\ell\pi, t) = 0, & t \geq 0. \end{cases} \tag{4.2}$$

The Holling-tanner model is one of the typical predator-prey models. For the ordinary differential equation (4.2) with $d_{11} = \xi = d_{21} = d_{22} = 0$, it has been completely analyzed in [30]. For the diffusive model (4.2) with $\xi = d_{21} = 0$, the global stability of the positive constant steady state was proved in [31, 32], and the Hopf bifurcation and Turing instability have been studied in [33].

4.1 Stability and Hopf bifurcation analysis

The system (4.2) has the positive constant steady state $E_*(u_*, v_*)$, where

$$u_* = v_* = \frac{1}{2\beta} \left(\sqrt{R^2 + 4\beta} - R \right) \tag{4.3}$$

with $R = \beta + m - 1$. By combining with $E_*(u_*, v_*)$, (2.4) and (4.1), we have

$$\begin{aligned}
 a_{11} &= 1 - 2\beta u_* - \frac{mu_*}{(1 + u_*)^2}, \quad a_{12} = -\frac{mu_*}{1 + u_*} < 0, \\
 a_{21} &= s > 0, \quad a_{22} = -s < 0.
 \end{aligned} \tag{4.4}$$

Moreover, by combining with (2.3), (2.5), (2.6) and (4.4), the characteristic equation of system (4.2) can be written as

$$\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda)) = \lambda^2 - T_n\lambda + \tilde{J}_n(\tau) = 0, \tag{4.5}$$

where

$$T_n = \text{Tr}(A) - \text{Tr}(D_1) \frac{n^2}{\ell^2},$$

$$\tilde{J}_n(\tau) = (d_{11}d_{22} + d_{21}\xi u_* v_* e^{-\lambda\tau}) \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* + d_{21}v_* a_{12} e^{-\lambda\tau}) \frac{n^2}{\ell^2} + \text{Det}(A). \quad (4.6)$$

Notice that the mathematical expression in (4.6) the same as in (2.7).

Furthermore, when $\tau = 0$, the characteristic equation (4.5) becomes

$$\lambda^2 - T_n \lambda + \tilde{J}_n(0) = 0, \quad (4.7)$$

where

$$\tilde{J}_n(0) = (d_{11}d_{22} + d_{21}\xi u_* v_*) \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* + d_{21}v_* a_{12}) \frac{n^2}{\ell^2} + \text{Det}(A). \quad (4.8)$$

A set of sufficient and necessary condition that all roots of (4.7) have negative real parts is $T_n < 0$, $\tilde{J}_n(0) > 0$, which is always holds provided that $a_{11} < 0$, i.e.,

$$(C_0) : 1 - 2\beta u_* - \frac{mu_*}{(1 + u_*)^2} < 0.$$

This implies that when $\tau = 0$ and the condition (C_0) holds, the positive steady state $E_*(u_*, v_*)$ is asymptotically stable for $d_{11} \geq 0$, $\xi \geq 0$, $d_{21} \geq 0$ and $d_{22} \geq 0$. Meanwhile, if we let $d_{21} = 0$, then by (4.8), we have

$$\tilde{J}_n := d_{11}d_{22} \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_*) \frac{n^2}{\ell^2} + \text{Det}(A).$$

It is easy to verify that $T_n < 0$ and $\tilde{J}_n > 0$ provided that the condition (C_0) holds. This implies that when $\tau = 0$, $d_{21} = 0$ and the condition (C_0) holds, the positive steady state $E_*(u_*, v_*)$ is asymptotically stable for $d_{11} \geq 0$, $\xi \geq 0$ and $d_{22} \geq 0$. Furthermore, since $\Gamma_n(0) = \tilde{J}_n(0) > 0$ under the condition (C_0) , this implies that $\lambda = 0$ is not a root of (4.5).

In the following, we let

$$J_n = d_{11}d_{22} \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_*) \frac{n^2}{\ell^2} + \text{Det}(A).$$

Furthermore, let $\lambda = i\omega_n$ ($\omega_n > 0$) be a root of (4.5). By substituting it along with expressions in (4.6) into (4.5), and separating the real part from the imaginary part, we have

$$\begin{cases} \omega_n^2 - J_n = \left(d_{21}\xi u_* v_* \frac{n^4}{\ell^4} - d_{21}v_* a_{12} \frac{n^2}{\ell^2} \right) \cos(\omega_n \tau), \\ -T_n \omega_n = \left(d_{21}\xi u_* v_* \frac{n^4}{\ell^4} - d_{21}v_* a_{12} \frac{n^2}{\ell^2} \right) \sin(\omega_n \tau), \end{cases} \quad (4.9)$$

which yields

$$\omega_n^4 + P_n \omega_n^2 + Q_n = 0, \quad (4.10)$$

where

$$P_n = T_n^2 - 2J_n = (d_{11}^2 + d_{22}^2) \frac{n^4}{\ell^4} - 2(d_{11}a_{11} + d_{22}a_{22} + a_{21}\xi u_*) \frac{n^2}{\ell^2} + a_{11}^2 + a_{22}^2 + 2a_{12}a_{21},$$

and

$$Q_n = \left(J_n + \left(d_{21}\xi u_* v_* \frac{n^4}{\ell^4} - d_{21}v_* a_{12} \frac{n^2}{\ell^2} \right) \right) \left(J_n - \left(d_{21}\xi u_* v_* \frac{n^4}{\ell^4} - d_{21}v_* a_{12} \frac{n^2}{\ell^2} \right) \right). \quad (4.11)$$

Notice that for (4.10), it is easy to see that if

$$\text{either } P_n > 0 \text{ and } Q_n > 0 \text{ or } P_n^2 - 4Q_n < 0,$$

then (4.10) has no positive root. Suppose that

$$Q_n > 0, P_n < 0 \text{ and } P_n^2 - 4Q_n > 0,$$

then (4.10) has two positive roots. In addition, if

$$\text{either } Q_n < 0 \text{ or } P_n < 0 \text{ and } P_n^2 - 4Q_n = 0,$$

then (4.10) has only one positive root.

Case 1 It is easy to see that if the conditions (C_0) and

$$(C_1) : P_n > 0 \text{ and } Q_n > 0 \text{ or } P_n^2 - 4Q_n < 0,$$

hold, then (4.10) has no positive roots. Hence, by combining with the assumption 2.1, we know that all roots of (4.5) have negative real parts when $\tau \in [0, +\infty)$ under the conditions (C_0) and (C_1) .

In the following, we mainly consider the case of $Q_n < 0$, that is (4.10) has only one positive root ω_n . In the following, we will discuss the case which is used to guarantee $Q_n < 0$ under the condition (C_0) .

When $\tau > 0$, according to (4.5) and (4.11), we can define $Q_n = \Gamma_n(0)\tilde{Q}_n$ with

$$\Gamma_n(0) = \tilde{J}_n(0) = (d_{11}d_{22} + d_{21}\xi u_* v_*) \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* + d_{21}v_* a_{12}) \frac{n^2}{\ell^2} + \text{Det}(A)$$

and

$$\tilde{Q}_n = (d_{11}d_{22} - d_{21}\xi u_* v_*) \frac{n^4}{\ell^4} - (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_* a_{12}) \frac{n^2}{\ell^2} + \text{Det}(A), \quad (4.12)$$

and then by a simple analysis, we have $\Gamma_n(0) = \tilde{J}_n(0) > 0$ for any $n \in \mathbb{N}_0$. Therefore, the sign of Q_n coincides with that of $\tilde{Q}_n$, and in order to guaranteeing $Q_n < 0$, we only need to study the case of $\tilde{Q}_n < 0$.

Case 2 If $(d_{11}d_{22} - d_{21}\xi u_* v_*) > 0$ and the conditions (C_0) ,

$$(C_2) : \text{Det}(A) > 0, \quad d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_* a_{12} > 0 \\ (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_* a_{12})^2 - 4(d_{11}d_{22} - d_{21}\xi u_* v_*) \text{Det}(A) > 0$$

hold, then (4.12) has two positive roots. Without loss of generality, we assume that the two positive roots of (4.12) are $\tilde{x}_1$ and $\tilde{x}_2$, i.e.,

$$\tilde{x}_{1,2} = \frac{\tilde{A}_1 \mp \sqrt{\tilde{A}_3}}{2\tilde{A}_2}, \quad (4.13)$$

where

$$\tilde{A}_1 = d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_* a_{12}, \quad \tilde{A}_2 = d_{11}d_{22} - d_{21}\xi u_* v_*, \\ \tilde{A}_3 = \tilde{A}_1^2 - 4\tilde{A}_2 \text{Det}(A). \quad (4.14)$$

Since $\tilde{x}_1 = n_1^2/\ell^2$ and $\tilde{x}_2 = n_2^2/\ell^2$, then $n_1 = \ell\sqrt{\tilde{x}_1}$ and $n_2 = \ell\sqrt{\tilde{x}_2}$. By using a geometric argument, we can conclude that

$$Q_n = \Gamma_n(0)\tilde{Q}_n \begin{cases} < 0, & n_1 < n < n_2, \\ \geq 0, & n \leq n_1 \text{ or } n \geq n_2, \end{cases}$$

where $n \in \mathbb{N}$. Therefore, (4.10) has one positive root ω_n for $n_1 < n < n_2$ with $n \in \mathbb{N}$, where

$$\omega_n = \sqrt{\frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2}}. \quad (4.15)$$

Furthermore, by combining with the second mathematical expression in (4.9), and notice that $a_{12} < 0$, $T_n < 0$ under the condition (C_0) , then we have $\sin(\omega_n \tau) > 0$. Thus, from the first mathematical expression in (4.9), we can set

$$\tau_{n,j} = \frac{1}{\omega_n} \left\{ \arccos \left\{ \frac{\omega_n^2 - J_n}{d_{21}\xi u_* v_* (n^4/\ell^4) - d_{21}v_* a_{12} (n^2/\ell^2)} \right\} + 2j\pi \right\}, \quad n \in \mathbb{N}, \quad j \in \mathbb{N}_0. \quad (4.16)$$

Next, we continue to verify the transversality conditions for the Case 4.2.

Lemma 1 Suppose that $(d_{11}d_{22} - d_{21}\xi u_* v_*) > 0$, the conditions (C_0) , (C_2) hold, and $n_1 < n < n_2$ with $n \in \mathbb{N}$, then we have

$$\left. \frac{d \text{Re}(\lambda(\tau))}{d\tau} \right|_{\tau=\tau_{n,j}} > 0,$$

where $\text{Re}(\lambda(\tau))$ represents the real part of $\lambda(\tau)$.

Proof By differentiating the two sides of

$$\Gamma_n(\lambda) = \det(\mathcal{M}_n(\lambda)) = \lambda^2 - T_n \lambda + \tilde{J}_n(\tau) = 0$$

with respect to τ , where T_n and $\tilde{J}_n(\tau)$ are defined by (4.6), we have

$$\left(\frac{d\lambda(\tau)}{d\tau} \right)^{-1} = \frac{(2\lambda - T_n) e^{\lambda\tau}}{-\lambda d_{21} v_* a_{12} (n^2/\ell^2) + \lambda d_{21} \xi u_* v_* (n^4/\ell^4)} - \frac{\tau}{\lambda}. \quad (4.17)$$

Therefore, by (4.17), we have

$$\begin{aligned} \operatorname{Re} \left(\frac{d\lambda(\tau)}{d\tau} \Big|_{\tau=\tau_{n,j}} \right)^{-1} &= \operatorname{Re} \left(\frac{(2i\omega_n - T_n) e^{i\omega_n \tau_{n,j}}}{-i\omega_n d_{21} v_* a_{12} (n^2/\ell^2) + i\omega_n d_{21} \xi u_* v_* (n^4/\ell^4)} \right) \\ &= \operatorname{Re} \left(\frac{(2i\omega_n - T_n) (\cos(\omega_n \tau_{n,j}) + i \sin(\omega_n \tau_{n,j}))}{-i\omega_n d_{21} v_* a_{12} (n^2/\ell^2) + i\omega_n d_{21} \xi u_* v_* (n^4/\ell^4)} \right) \\ &= \operatorname{Re} \left(\frac{(2i\omega_n - T_n) (\cos(\omega_n \tau_{n,j}) + i \sin(\omega_n \tau_{n,j}))}{i\omega_n (d_{21} \xi u_* v_* (n^4/\ell^4) - d_{21} v_* a_{12} (n^2/\ell^2))} \right) \\ &= \frac{2 \cos(\omega_n \tau_{n,j})}{d_{21} \xi u_* v_* (n^4/\ell^4) - d_{21} v_* a_{12} (n^2/\ell^2)} - \frac{T_n \sin(\omega_n \tau_{n,j})}{\omega_n (d_{21} \xi u_* v_* (n^4/\ell^4) - d_{21} v_* a_{12} (n^2/\ell^2))}. \end{aligned} \quad (4.18)$$

Furthermore, according to (4.9), we have

$$\begin{aligned} \sin(\omega_n \tau_{n,j}) &= \frac{-T_n \omega_n}{d_{21} \xi u_* v_* (n^4/\ell^4) - d_{21} v_* a_{12} (n^2/\ell^2)}, \\ \cos(\omega_n \tau_{n,j}) &= \frac{\omega_n^2 - J_n}{d_{21} \xi u_* v_* (n^4/\ell^4) - d_{21} v_* a_{12} (n^2/\ell^2)}. \end{aligned} \quad (4.19)$$

Moreover, by combining with (4.18), (4.19) and

$$\omega_n = \sqrt{\frac{-P_n + \sqrt{P_n^2 - 4Q_n}}{2}} > 0, \quad Q_n < 0, \quad a_{12} < 0,$$

we have

$$\begin{aligned} \operatorname{Re} \left(\frac{d\lambda(\tau)}{d\tau} \Big|_{\tau=\tau_{n,j}} \right)^{-1} &= \frac{2 \cos(\omega_n \tau_{n,j})}{d_{21} \xi u_* v_* (n^4/\ell^4) - d_{21} v_* a_{12} (n^2/\ell^2)} - \frac{T_n \sin(\omega_n \tau_{n,j})}{\omega_n (d_{21} \xi u_* v_* (n^4/\ell^4) - d_{21} v_* a_{12} (n^2/\ell^2))} \\ &= \frac{2\omega_n^3 + \omega_n (T_n^2 - 2J_n)}{\omega_n (d_{21} \xi u_* v_* (n^4/\ell^4) - d_{21} v_* a_{12} (n^2/\ell^2))^2} \\ &= \frac{\sqrt{P_n^2 - 4Q_n}}{(d_{21} \xi u_* v_* (n^4/\ell^4) - d_{21} v_* a_{12} (n^2/\ell^2))^2} > 0. \end{aligned}$$

This, together with the fact that

$$\operatorname{sign} \left\{ \frac{d \operatorname{Re}(\lambda(\tau))}{d\tau} \Big|_{\tau=\tau_{n,j}} \right\} = \operatorname{sign} \left\{ \operatorname{Re} \left(\frac{d\lambda(\tau)}{d\tau} \Big|_{\tau=\tau_{n,j}} \right)^{-1} \right\}$$

completes the proof, where $\operatorname{sign}(\cdot)$ represents the sign function.

Moreover, according to the above analysis, we have the following results.

Lemma 2 *If the condition (C_0) is satisfied, then we have the following conclusions*

(i) *if the condition (C_1) holds, then the positive constant steady state $E_*(u_*, v_*)$ of system (4.2) is asymptotically stable for all $\tau \geq 0$;*

(ii) *if $(d_{11}d_{22} - d_{21}\xi u_* v_*) > 0$ and the condition (C_2) holds, by denoting $\tau_* = \min \{\tau_{n,0} : n_1 < n < n_2, n \in \mathbb{N}\}$, then the positive constant steady state $E_*(u_*, v_*)$ of system (4.2) is asymptotically stable for $0 \leq \tau < \tau_*$ and unstable for $\tau > \tau_*$. Furthermore, system (4.2) undergoes mode- n Hopf bifurcations at $\tau = \tau_{n,0}$ for $n \in \mathbb{N}$.*

4.2 Numerical simulations

In this section, we verify the analytical results given in the previous sections by some numerical simulations and investigate the direction and stability of Hopf bifurcation. We use the following initial conditions for the system (4.2)

$$u(x, t) = u_0(x), \quad v(x, t) = v_0(x), \quad t \in [-\tau, 0].$$

4.2.1 Mode-1 Hopf bifurcation

Fig. 1 For the parameters $\ell = 2$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, when $\tau = 3 < \tau_{1,0} = 6.1498$, the positive constant steady state $E_*(u_*, v_*) = (1.4142, 1.4142)$ is locally asymptotically stable. The initial values are $u_0(x) = 1.4142 - 0.1 \cos(x/2)$ and $v_0(x) = 1.4142 + 0.1 \cos(x/2)$.

If we set the parameters as follows

$$\ell = 2, \quad d_{11} = 2, \quad d_{22} = 3, \quad d_{21} = 18, \quad \xi = 0.06, \quad \beta = 0.5, \quad m = 0.5, \quad s = 0.8,$$

then we can easily obtain that

$$a_{11} = 1 - 2\beta u_* - \frac{mu_*}{(1 + u_*)^2} = -0.5355 < 0, \quad d_{11}d_{22} - d_{21}\xi u_* v_* = 3.84 > 0,$$

$$\text{Det}(A) = a_{11}a_{22} - a_{12}a_{21} = 0.6627 > 0, \quad d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_*a_{12} = 4.1814 > 0,$$

$$(d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_*a_{12})^2 - 4(d_{11}d_{22} - d_{21}\xi u_* v_*) \text{Det}(A) = 7.3041 > 0.$$

Therefore, the conditions (C_0) and (C_2) are satisfied under the above parameters settings. In the following, we mainly verify the conclusion in Lemma 4.4 (ii). According to (4.3) and (4.4), we have $E_*(u_*, v_*) = (1.4142, 1.4142)$,

$$a_{11} = -0.5355, \quad a_{12} = -0.2929, \quad a_{21} = 0.8, \quad a_{22} = -0.8.$$

By combining with (4.13), (4.14), (4.15) and (4.16), we have $n_1 = 0.8776$, $n_2 = 1.8935$, and consider that $n \in \mathbb{N}$, we have $\omega_c = \omega_1 = 0.418$ and $\tau_c = \tau_{1,0} = 6.1498$. Moreover, by Lemma 4.4 (ii), we have the following proposition.

Proposition 1 For system (4.2) with the parameters $\ell = 2$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, the positive constant steady state $E_*(u_*, v_*)$ of system (4.2) is asymptotically stable for $0 \leq \tau < \tau_{1,0} = 6.1498$ and unstable for $\tau > \tau_{1,0} = 6.1498$. Furthermore, system (4.2) undergoes mode-1 Hopf bifurcation at $\tau = \tau_{1,0} = 6.1498$.

Fig. 2 For the parameters $\ell = 2$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, when $\tau = 8 > \tau_{1,0} = 6.1498$, there exists a stable spatially inhomogeneous periodic solution. (a) and (b) are the transient behaviours for $u(x, t)$ and $v(x, t)$, respectively, (c) and (d) are long-term behaviours for $u(x, t)$ and $v(x, t)$, respectively. The initial values are $u_0(x) = 1.4142 - 0.1 \cos(x/2)$ and $v_0(x) = 1.4142 + 0.1 \cos(x/2)$.

Fig. 3 For the parameters $\ell = 2$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, when $\tau = 8 > \tau_{1,0} = 6.1498$, there exists a stable spatially inhomogeneous periodic solution. The initial values are $u_0(x) = 1.4142 - 0.1 \cos(x/2)$ and $v_0(x) = 1.4142 + 0.1 \cos(x/2)$.

For the parameters $\ell = 2$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, according to Proposition 4.5, we know that system (4.2) undergoes a Hopf bifurcation at $\tau_{1,0} = 6.1498$. Furthermore, the direction and stability of Hopf bifurcation can be determined by calculating $K_1 K_2$ and K_2 using the procedures developed in Section 2. After a direct calculation using MATLAB software, we obtain

$$K_1 = 0.016 > 0, \quad K_2 = -0.9283 < 0, \quad K_1 K_2 = -0.0148 < 0,$$

which implies that the Hopf bifurcation at $\tau_{1,0} = 6.1498$ is supercritical and stable.

When $\tau = 3 < \tau_{1,0} = 6.1498$, Fig.1 (a)-(d) illustrate the evolution of the solution of system (4.2) starting from the initial values $u_0(x) = 1.4142 - 0.1 \cos(x/2)$ and $v_0(x) = 1.4142 + 0.1 \cos(x/2)$, finally converging to the positive constant steady state $E_*(u_*, v_*)$. Furthermore, when $\tau = 8 > \tau_{1,0} = 6.1498$, Fig.2 (a)-(d) illustrate the existence of the spatially homogeneous periodic solution with the initial values $u_0(x) = 1.4142 - 0.1 \cos(x/2)$ and $v_0(x) = 1.4142 + 0.1 \cos(x/2)$. When $\tau = 8 > \tau_{1,0} = 6.1498$, the space-time diagrams for the prey population $u(x, t)$ and the predator population $v(x, t)$ are given in Fig.3 (a) and (b), respectively.

4.2.2 Mode-2 Hopf bifurcation

Fig. 4 For the parameters $\ell = 3$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, when $\tau = 2 < \tau_{2,0} = 3.5361$, the positive constant steady state $E_*(u_*, v_*) = (1.4142, 1.4142)$ is locally asymptotically stable. The initial values are $u_0(x) = 1.4142 - 0.1 \cos(2x/3)$ and $v_0(x) = 1.4142 + 0.1 \cos(2x/3)$.

If we set the parameters as follows

$$\ell = 3, d_{11} = 2, d_{22} = 3, d_{21} = 18, \xi = 0.06, \beta = 0.5, m = 0.5, s = 0.8,$$

then we can also easily obtain that

$$\begin{aligned} a_{11} &= 1 - 2\beta u_* - \frac{mu_*}{(1+u_*)^2} = -0.5355 < 0, \quad d_{11}d_{22} - d_{21}\xi u_* v_* = 3.84 > 0, \\ \text{Det}(A) &= a_{11}a_{22} - a_{12}a_{21} = 0.6627 > 0, \quad d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_*a_{12} = 4.1814 > 0, \\ (d_{11}a_{22} + d_{22}a_{11} - a_{21}\xi u_* - d_{21}v_*a_{12})^2 - 4(d_{11}d_{22} - d_{21}\xi u_* v_*) \text{Det}(A) &= 7.3041 > 0. \end{aligned}$$

Therefore, the conditions (C_0) and (C_2) are satisfied under the above parameters settings. In the following, we mainly verify the conclusion in Lemma 4.4 (ii). According to (4.3) and (4.4), we have $E_*(u_*, v_*) = (1.4142, 1.4142)$,

$$a_{11} = -0.5355, \quad a_{12} = -0.2929, \quad a_{21} = 0.8, \quad a_{22} = -0.8.$$

By combining with (4.13), (4.14), (4.15) and (4.16), we have $n_1 = 1.3164$, $n_2 = 2.8403$, and consider that $n \in \mathbb{N}$, we have $\omega_c = \omega_2 = 0.6870$ and $\tau_c = \tau_{2,0} = 3.5361$. Moreover, by Lemma 4.4 (ii), we have the following proposition.

Proposition 2 For system (4.2) with the parameters $\ell = 3$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, the positive constant steady state $E_*(u_*, v_*)$ of system (4.2) is asymptotically stable for $0 \leq \tau < \tau_{2,0} = 3.5361$ and unstable for $\tau > \tau_{2,0} = 3.5361$. Furthermore, system (4.2) undergoes mode-2 Hopf bifurcation at $\tau = \tau_{2,0} = 3.5361$.

For the parameters $\ell = 3$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, according to Proposition 4.6, we know that system (4.2) undergoes a Hopf bifurcation at $\tau_{2,0} = 3.5361$. Furthermore, the direction and stability of Hopf bifurcation can be determined by calculating $K_1 K_2$ and K_2 using the procedures developed in Section 2. After a direct calculation using MATLAB software, we obtain

$$K_1 = 0.0410 > 0, \quad K_2 = -1.3669 < 0, \quad K_1 K_2 = -0.0561 < 0,$$

Fig. 5 For the parameters $\ell = 3$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, when $\tau = 6 > \tau_{2,0} = 3.5361$, there exists a stable spatially inhomogeneous periodic solution. (a) and (b) are the transient behaviours for $u(x, t)$ and $v(x, t)$, respectively, (c) and (d) are long-term behaviours for $u(x, t)$ and $v(x, t)$, respectively. The initial values are $u_0(x) = 1.4142 - 0.1 \cos(2x/3)$ and $v_0(x) = 1.4142 + 0.1 \cos(2x/3)$.

Fig. 6 For the parameters $\ell = 3$, $d_{11} = 2$, $d_{22} = 3$, $d_{21} = 18$, $\xi = 0.06$, $\beta = 0.5$, $m = 0.5$, $s = 0.8$, when $\tau = 6 > \tau_{2,0} = 3.5361$, there exists a stable spatially inhomogeneous periodic solution. The initial values are $u_0(x) = 1.4142 - 0.1 \cos(2x/3)$ and $v_0(x) = 1.4142 + 0.1 \cos(2x/3)$.

which implies that the Hopf bifurcation at $\tau_{2,0} = 3.5361$ is supercritical and stable.

When $\tau = 2 < \tau_{2,0} = 3.5361$, Fig.4 (a)-(d) illustrate the evolution of the solution of system (4.2) starting from the initial values $u_0(x) = 1.4142 - 0.1 \cos(2x/3)$ and $v_0(x) = 1.4142 + 0.1 \cos(2x/3)$, finally converging to the positive constant steady state $E_*(u_*, v_*)$. Furthermore, when $\tau = 6 > \tau_{2,0} = 3.5361$, Fig.5 (a)-(d) illustrate the existence of the spatially inhomogeneous periodic solution with the initial values $u_0(x) = 1.4142 - 0.1 \cos(2x/3)$ and $v_0(x) = 1.4142 + 0.1 \cos(2x/3)$. When $\tau = 6 > \tau_{2,0} = 3.5361$, the space-time diagrams for the prey population $u(x, t)$ and the predator population $v(x, t)$ are given in Fig.6 (a) and (b), respectively.

5 Conclusion and discussion

In this paper, the diffusive predator-prey system with spatial memory and predator-taxis is proposed, and we derive an algorithm for calculating the normal form of Hopf bifurcation in this system. As a real application, we consider the Holling-Tanner model with spatial memory and predator-taxis. By carrying out the stability and Hopf bifurcation analysis for the Holling-Tanner model with spatial memory and predator-taxis,

the inhomogeneous spatial patterns, i.e., two stable spatially inhomogeneous periodic solutions are found. Furthermore, the supercritical and stable mode-1 and mode-2 Hopf bifurcation periodic solutions are found by using our newly developed algorithm. Numerical simulations verify the results of our theoretical analysis and give us a more intuitive display.

It is worth mentioning that in this paper, for our proposed diffusive predator-prey system with spatial memory and predator-taxis, the delay only occurs in the diffusion term, not in the reaction form. However, for the general predator-prey models, the gestation, hunting, migration and maturation delays, etc., often occur in the reaction term. By noticing this point, on the basis of system (1.4), the system

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_{11} \Delta u(x,t) + \xi (u(x,t)v_x(x,t))_x + f(u(x,t), v(x,t), u(x,t-\sigma), v(x,t-\sigma)), & x \in (0, \ell\pi), t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_{22} \Delta v(x,t) - d_{21} (v(x,t)u_x(x,t-\tau))_x + g(u(x,t), v(x,t), u(x,t-\sigma), v(x,t-\sigma)), & x \in (0, \ell\pi), t > 0, \end{cases}$$

which needs further research, where $\sigma > 0$ is the delay occurs in the reaction term.

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Availability of data and material: This research didn't involve the private data, and the involving data and material are all available.

Code availability: The numerical simulations in this paper are carried by using the MATLAB software.

Authors' contributions: This manuscript is investigated and written by Yehu Lv.

Appendix A

Remark 1 Assume that at $\tau = \tau_c$, system (4.2) has a pair of purely imaginary roots $\pm i\omega_{n_c}$ with $\omega_{n_c} > 0$ for $n = n_c \in \mathbb{N}$ and all other eigenvalues have negative real parts. Let $\lambda(\tau) = \alpha_1(\tau) \pm i\alpha_2(\tau)$ be a pair of roots of system (4.2) near $\tau = \tau_c$ satisfying $\alpha_1(\tau_c) = 0$ and $\alpha_2(\tau_c) = \omega_{n_c}$. In addition, the corresponding transversality condition holds.

The normal form of Hopf bifurcation for the system (4.2) can be calculated by using our newly developed algorithm in Section 2. Here, we give the detail calculation procedures of $B_1, B_{21}, B_{22}, B_{23}$ steps by steps.

Step 1:

$$B_1 = 2\psi^T(0) \left(A\phi(0) - \frac{n_c^2}{\ell^2} (D_1\phi(0) + D_2\phi(-1)) \right)$$

with

$$D_1 = \begin{pmatrix} d_{11} & \xi u_* \\ 0 & d_{22} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Here,

$$\phi = \begin{pmatrix} 1 \\ \frac{a_{11} - i\omega_{n_c} - d_{11}(n_c^2/\ell^2)}{\xi u_* (n_c^2/\ell^2) - a_{12}} \end{pmatrix}, \quad \psi = \eta \begin{pmatrix} 1 \\ \frac{a_{12} - \xi u_* (n_c^2/\ell^2)}{i\omega_{n_c} + d_{22}(n_c^2/\ell^2) - a_{22}} \end{pmatrix}$$

with

$$\eta = \frac{i\omega_{n_c} + (n_c/\ell)^2 d_{22} - a_{22}}{2i\omega_{n_c} + (n_c/\ell)^2 d_{11} - a_{11} + (n_c/\ell)^2 d_{22} - a_{22} + \tau_c a_{12} d_{21} v_* (n_c/\ell)^2 e^{-i\omega_c}}.$$

Step 2:

$$B_{21} = \frac{3}{2\ell\pi} \psi^T A_{21}$$

with

$$A_{21} = 3f_{30}\phi_1^2(0)\overline{\phi_1}(0) + 3f_{03}\phi_2^2(0)\overline{\phi_2}(0) + 3f_{21}(\phi_1^2(0)\overline{\phi_2}(0) + 2\phi_1(0)\overline{\phi_1}(0)\phi_2(0)) \\ + 3f_{12}(2\phi_1(0)\phi_2(0)\overline{\phi_2}(0) + \overline{\phi_1}(0)\phi_2^2(0)).$$

Here,

$$f_{30}^{(1)} = -6\tau_c m(1+u_*)^{-4}v_*, \quad f_{30}^{(2)} = 6\tau_c s u_*^{-4}v_*^2, \\ f_{21}^{(1)} = 2\tau_c m(1+u_*)^{-3}, \quad f_{21}^{(2)} = -4\tau_c s u_*^{-3}v_*, \\ f_{12}^{(1)} = 0, \quad f_{12}^{(2)} = 2\tau_c s u_*^{-2}, \\ f_{03}^{(1)} = 0, \quad f_{03}^{(2)} = 0.$$

Step 3:

$$B_{22} = \frac{1}{\sqrt{\ell\pi}} \psi^T (\mathcal{S}_2(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2(\overline{\phi}(\theta), h_{0,20}(\theta))) \\ + \frac{1}{\sqrt{2\ell\pi}} \psi^T (\mathcal{S}_2(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2(\overline{\phi}(\theta), h_{2n_c,20}(\theta)))$$

with

$$\mathcal{S}_2(\phi(\theta), h_{0,11}(\theta)) = 2f_{20}\phi_1(0)h_{0,11}^{(1)}(0) + 2f_{02}\phi_2(0)h_{0,11}^{(2)}(0) \\ + 2f_{11}(\phi_1(0)h_{0,11}^{(2)}(0) + \phi_2(0)h_{0,11}^{(1)}(0)), \\ \mathcal{S}_2(\overline{\phi}(\theta), h_{0,20}(\theta)) = 2f_{20}\overline{\phi_1}(0)h_{0,20}^{(1)}(0) + 2f_{02}\overline{\phi_2}(0)h_{0,20}^{(2)}(0) \\ + 2f_{11}(\overline{\phi_1}(0)h_{0,20}^{(2)}(0) + \overline{\phi_2}(0)h_{0,20}^{(1)}(0)), \\ \mathcal{S}_2(\phi(\theta), h_{2n_c,11}(\theta)) = 2f_{20}\phi_1(0)h_{2n_c,11}^{(1)}(0) + 2f_{02}\phi_2(0)h_{2n_c,11}^{(2)}(0) \\ + 2f_{11}(\phi_1(0)h_{2n_c,11}^{(2)}(0) + \phi_2(0)h_{2n_c,11}^{(1)}(0)), \\ \mathcal{S}_2(\overline{\phi}(\theta), h_{2n_c,20}(\theta)) = 2f_{20}\overline{\phi_1}(0)h_{2n_c,20}^{(1)}(0) + 2f_{02}\overline{\phi_2}(0)h_{2n_c,20}^{(2)}(0) \\ + 2f_{11}(\overline{\phi_1}(0)h_{2n_c,20}^{(2)}(0) + \overline{\phi_2}(0)h_{2n_c,20}^{(1)}(0)).$$

Here,

$$f_{20}^{(1)} = -2\tau_c \beta + 2\tau_c m(1+u_*)^{-3}v_*, \quad f_{20}^{(2)} = -2\tau_c s u_*^{-3}v_*^2, \\ f_{11}^{(1)} = -\tau_c m(1+u_*)^{-2}, \quad f_{11}^{(2)} = 2\tau_c s u_*^{-2}v_*, \\ f_{02}^{(1)} = 0, \quad f_{02}^{(2)} = -2\tau_c s u_*^{-1}.$$

Furthermore, we have

$$\begin{cases} h_{0,20}(\theta) = \frac{1}{\sqrt{\ell\pi}} \left(\widetilde{\mathcal{M}}_0(2i\omega_c) \right)^{-1} A_{20} e^{2i\omega_c \theta}, \\ h_{0,11}(\theta) = \frac{1}{\sqrt{\ell\pi}} \left(\widetilde{\mathcal{M}}_0(0) \right)^{-1} A_{11} \end{cases}$$

and

$$\begin{cases} h_{2n_c,20}(\theta) = \frac{1}{\sqrt{2\ell\pi}} \left(\widetilde{\mathcal{M}}_{2n_c}(2i\omega_c) \right)^{-1} \widetilde{A}_{20} e^{2i\omega_c \theta}, \\ h_{2n_c,11}(\theta) = \frac{1}{\sqrt{2\ell\pi}} \left(\widetilde{\mathcal{M}}_{2n_c}(0) \right)^{-1} \widetilde{A}_{11} \end{cases}$$

with

$$\widetilde{\mathcal{M}}_n(\lambda) = \lambda I_2 + \tau_c (n/\ell)^2 D_1 + \tau_c (n/\ell)^2 e^{-\lambda} D_2 - \tau_c A.$$

Here,

$$A_{20} = f_{20}\phi_1^2(0) + f_{02}\phi_2^2(0) + 2f_{11}\phi_1(0)\phi_2(0), \\ A_{11} = 2f_{20}\phi_1(0)\overline{\phi_1}(0) + 2f_{02}\phi_2(0)\overline{\phi_2}(0) + 2f_{11}(\phi_1(0)\overline{\phi_2}(0) + \overline{\phi_1}(0)\phi_2(0))$$

and

$$\begin{cases} \widetilde{A}_{20} = A_{20} - 2(n_c/\ell)^2 A_{20}^d, \\ \widetilde{A}_{11} = A_{11} - 2(n_c/\ell)^2 A_{11}^d \end{cases}$$

with

$$\begin{cases} A_{20}^d = \begin{pmatrix} 2\xi\tau_c\phi_1(0)\phi_2(0) \\ -2d_{21}\tau_c\phi_1(-1)\phi_2(0) \end{pmatrix} = \overline{A_{02}^d}, \\ A_{11}^d = \begin{pmatrix} 4\xi\tau_c \operatorname{Re} \{ \phi_1(0)\overline{\phi_2(0)} \} \\ -4d_{21}\tau_c \operatorname{Re} \{ \phi_1(-1)\overline{\phi_2(0)} \} \end{pmatrix}. \end{cases}$$

Step 4:

$$\begin{aligned} B_{23} = & -\frac{1}{\sqrt{\ell\pi}} (n_c/\ell)^2 \psi^T \left(\mathcal{S}_2^{(d,1)}(\phi(\theta), h_{0,11}(\theta)) + \mathcal{S}_2^{(d,1)}(\overline{\phi}(\theta), h_{0,20}(\theta)) \right) \\ & + \frac{1}{\sqrt{2\ell\pi}} \psi^T \sum_{j=1,2,3} b_{2n_c}^{(j)} \left(\mathcal{S}_2^{(d,j)}(\phi(\theta), h_{2n_c,11}(\theta)) + \mathcal{S}_2^{(d,j)}(\overline{\phi}(\theta), h_{2n_c,20}(\theta)) \right) \end{aligned}$$

with

$$b_{2n_c}^{(1)} = -\frac{n_c^2}{\ell^2}, \quad b_{2n_c}^{(2)} = \frac{2n_c^2}{\ell^2}, \quad b_{2n_c}^{(3)} = -\frac{(2n_c)^2}{\ell^2}$$

and

$$\begin{cases} \mathcal{S}_2^{(d,1)}(\phi(\theta), h_{0,11}(\theta)) = 2 \begin{pmatrix} \xi\tau_c\phi_2(0)h_{0,11}^{(1)}(0) \\ -d_{21}\tau_c\phi_1(-1)h_{0,11}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,1)}(\overline{\phi}(\theta), h_{0,20}(\theta)) = 2 \begin{pmatrix} \xi\tau_c\overline{\phi}_2(0)h_{0,20}^{(1)}(0) \\ -d_{21}\tau_c\overline{\phi}_1(-1)h_{0,20}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,1)}(\phi(\theta), h_{2n_c,11}(\theta)) = 2 \begin{pmatrix} \xi\tau_c\phi_2(0)h_{2n_c,11}^{(1)}(0) \\ -d_{21}\tau_c\phi_1(-1)h_{2n_c,11}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,2)}(\phi(\theta), h_{2n_c,11}(\theta)) = 2 \begin{pmatrix} \xi\tau_c(\phi_2(0)h_{2n_c,11}^{(1)}(0) + \phi_1(0)h_{2n_c,11}^{(2)}(0)) \\ -d_{21}\tau_c(\phi_2(0)h_{2n_c,11}^{(1)}(-1) + \phi_1(-1)h_{2n_c,11}^{(2)}(0)) \end{pmatrix}, \\ \mathcal{S}_2^{(d,3)}(\phi(\theta), h_{2n_c,11}(\theta)) = 2 \begin{pmatrix} \xi\tau_c\phi_1(0)h_{2n_c,11}^{(2)}(0) \\ -d_{21}\tau_c\phi_2(0)h_{2n_c,11}^{(1)}(-1) \end{pmatrix}, \\ \mathcal{S}_2^{(d,1)}(\overline{\phi}(\theta), h_{2n_c,20}(\theta)) = 2 \begin{pmatrix} \xi\tau_c\overline{\phi}_2(0)h_{2n_c,20}^{(1)}(0) \\ -d_{21}\tau_c\overline{\phi}_1(-1)h_{2n_c,20}^{(2)}(0) \end{pmatrix}, \\ \mathcal{S}_2^{(d,2)}(\overline{\phi}(\theta), h_{2n_c,20}(\theta)) = 2 \begin{pmatrix} \xi\tau_c(\overline{\phi}_2(0)h_{2n_c,20}^{(1)}(0) + \overline{\phi}_1(0)h_{2n_c,20}^{(2)}(0)) \\ -d_{21}\tau_c(\overline{\phi}_2(0)h_{2n_c,20}^{(1)}(-1) + \overline{\phi}_1(-1)h_{2n_c,20}^{(2)}(0)) \end{pmatrix}, \\ \mathcal{S}_2^{(d,3)}(\overline{\phi}(\theta), h_{2n_c,20}(\theta)) = 2 \begin{pmatrix} \xi\tau_c\overline{\phi}_1(0)h_{2n_c,20}^{(2)}(0) \\ -d_{21}\tau_c\overline{\phi}_2(0)h_{2n_c,20}^{(1)}(-1) \end{pmatrix}. \end{cases}$$

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