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The Flexibility of Home Away Pattern Sets

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Abstract

A popular approach to construct a schedule for a round-robin tournament is known as first-break, then-schedule. Thus, when given a Home Away Pattern (HAP) for each team, which specifies for each round whether the team plays a home game or an away game, the remaining challenge is to find a round for each match that is compatible with both team's patterns. When using such an approach, it matters how many rounds are available for each match: the more rounds are available for a match, the more options exist to accommodate particular constraints. We investigate the notion of flexibility of a set of HAPs, and introduce a number of measures assessing this flexibility. We show how the so-called Canonical Pattern Set (CPS) behaves on these measures, and, by solving integer programs, we give explicit values for all single-break HAP-sets with at most 16 teams.

1 Introduction

Round robin tournaments are abundantly used in all kinds of sport competitions, both in professional leagues, as well as in amateur leagues. The setting where each pair of teams meets once (Single Round Robin (SRR)), or twice (Double Round Robin (DRR)) has proven to be a very popular format to arrive at a ranking of the participating teams [7]. Very often, each team has a venue, and a match between two teams takes place at the venue of one of the two teams, meaning that one team plays home (H), while the other team plays away (A). Deciding upon the matches to be played in each round of a single or double round robin tournament is known to be of great

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practical importance in scheduling professional leagues such as soccer competitions. Indeed, scheduling a round robin tournament has attracted a lot of attention in the scientific literature [9, 16]; a popular approach to do so is known as a first-break, then-schedule approach [14]. The relevance of the first-break, then-schedule approach is strengthened by the observation that, when scheduling multiple leagues consecutively, first-break then-schedule is able to take capacity considerations of clubs that have teams playing in different leagues, into account [2]. We now informally describe this approach (see Section 2 for more precise terminology).

First-break, then-schedule is a hierarchical approach consisting of two phases. In the first phase each team receives a Home Away Pattern (HAP), i.e., it is specified for each round whether the team plays at its home venue, or not. Next, in the second phase, each match that needs to be played is assigned to a round; of course, for such an assignment of matches to be feasible it must hold that each team plays at most one match in each round, and that the assignment is compatible with the patterns from phase 1. In such a hierarchical approach, it is clear that the scheduling decision (which matches to play in which round) crucially depends on the HAP-set that is chosen in the first phase. For instance, it is conceivable that in the second phase, a set of constraints is revealed that are incompatible with the given HAP-set. This would need to be solved by either changing the HAP-set, or by putting energy into mitigating the effects of violating that specific set of constraints.

The theme of this contribution is that not all HAP-sets have the same risk of leading to incompatible constraints. Indeed, some HAP-sets are more flexible than others. In Section 2, we give a number of definitions, and in Section 3 we introduce our measures for the flexibility of a HAP-set. We also show how single-break HAP-sets, in particular the Canonical Pattern Set, behave with respect to these measures. In Section 4, we describe how to compute these measures using integer programming; we conclude in Section 5.

2 Preliminaries

We consider a time-constrained SRR with an even number of teams, denoted by $2n, n \in \mathbb{N}$. To avoid trivialities, we assume $2n \geq 4$. A *match* between two distinct teams i and j is denoted by an unordered pair $\{i, j\}$, $1 \leq i \neq j \leq 2n$. Every team plays every other team exactly once. A *schedule* is a specification of all $\binom{2n}{2}$ matches in $2n-1$ rounds, including the venues. Thus, in a schedule

every match $\{i, j\}$ is assigned to a specific round, the venue of the match is specified, and every team must play exactly one match in every round. For a survey of round robin scheduling, we refer to Rasmussen and Trick [15] and Drexl and Knust [5].

Although we restrict our analysis to SRRs, we claim that many of the ideas can be generalized to DRRs as well (or k-round robin settings with $k \geq 2$). In particular, the definitions of the measures (see Section 3) can be generalized to DRRs. However, we want to point out that a match in a DRR tournament is generally seen as an ordered pair (i, j), rather than an unorderd pair of teams. Indeed, whereas in a single round robin tournament, the home advantage is to be decided in the schedule, a double round robin tournament typically assumes that each team plays at home against each other team exactly once. Moreover, very often it is required that one encounter between a pair of teams occurs in the first half, whereas the other encounter occurs in the second half of the schedule; and these two encounters should be separated by a given number of rounds. Taking such issues into account would impact the corresponding definitions for DRR tournaments. We now proceed with defining our terminology.

Definition 2.1. *[based on [15]] A* Home-Away Pattern (HAP) *is a vector* $h = (h_1, h_2, \ldots, h_{2n-1})$*, where* $h_r \in \{H, A\}$ *specifies whether a team that plays according to pattern* h *plays Home or Away in round* r*, with* $r = 1, 2, \ldots, 2n-1$.

An important property of a HAP is formed by the occurrence of so-called *breaks*: the presence of two consecutive symbols that are identical (see De Werra [4] and Goossens and Spieksma [6]). To simplify notation, we take a circular view of a HAP, i.e., we define $h_0 \equiv h_{2n-1}$.

Definition 2.2 (based on [15])**.** *A Home-Away Pattern* h *has a* break *in round* r *if* $h_{r-1} = h_r$*, with* $r = 1, 2, \ldots, 2n-1$*. In case* $h_{r-1} = h_r = A$ *we call the break an* away-break*, otherwise we call it a* home-break*.*

Notice that, since $2n-1$ is odd, and given our circular view of a HAP, any HAP has at least one break. Indeed, observe that a HAP that consists of alternating H's and A's has a single break in round 1. In fact, motivated by practice, we are interested in the number of breaks present in a HAP, and we define the break-number of a HAP, denoted by $bn(h)$, to capture this number.

Definition 2.3. *The* break-number *of Home-Away Pattern* h *equals* $bn(h) = |\{r : \ h_{r-1} = h_r, r = 1, 2, \ldots, 2n-1\}|$.

	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7	
Team 1 (Lewabo)	A	H	A	H	A	H	H	$P^H(7)$
Team 2 (Spijkenisse)	A	H	A	H	H	A	H	$P^H(5)$
Team 3 (Suthwalda)	H	A	H	A	H	A	H	$P^H(1)$
Team 4 (Nieuwekerk)	H	A	H	A	A	H	A	$P^A(5)$
Team 5 (Arnolduspark)	H	A	H	A	H	A	A	$P^A(7)$
Team 6 (Leimonias)	A	H	A	H	H	A	A	$P^A(5,7)$
Team 7 (Naaldwijk)	A	H	A	H	A	H	A	$P^A(1)$
Team 8 (Kimbria)	H	A	H	A	A	H	H	$P^H(5,7)$

Table 1: HAPs for the 2019-2020 top Dutch male tennis league

Definition 2.4. *A Home-Away Pattern h is called a* single-break *pattern if* $bn(h) = 1$.

Definition 2.5 (based on [15])**.** *Two Home-Away patterns h^1, h^2 are called* complementary *iff $h^1_r \neq h^2_r$ for each $r = 1, 2, \ldots, 2n-1$.*

Notice that an entire HAP h is determined by the rounds in which its breaks occur, and by its value in the final round. Thus, we can write a HAP h as follows: $P^H(r_1, r_2, \ldots, r_{bn(h)})$, where h denotes a HAP that has (i) $bn(h)$ breaks occurring in rounds $r_1, r_2, \ldots, r_{bn(h)}$, and (ii) an H in the final round. In case HAP h has an A in its final round, and has breaks in rounds $r_1, r_2, \ldots, r_{bn(h)}$, we write $P^A(r_1, r_2, \ldots, r_{bn(h)})$. Clearly, $P^H(r_1, r_2, \ldots, r_{bn(h)})$ and $P^A(r_1, r_2, \ldots, r_{bn(h)})$ are complementary patterns.

We sometimes omit the distinction between P^H, P^A and simply use P. When discussing complementary pairs of patterns, we denote by P^c the complement of HAP P.

As a real-life example illustrating this terminology, we consider the HAPs used by the professional Dutch tennis competition for teams in 2019, organized by the Royal Dutch Tennis Association. This is a competition between 8 teams who play a Single Round Robin, according to the HAPs given in Table 1 (for the precise schedule, we refer to the KNLTB [10]). The first column shows the team's names, and the next 7 columns describe the Home-Away Patterns. The final column gives the corresponding description of the particular pattern.

We remark that many other real-life examples of SRR tournaments exist. For instance, the TATA Steel Chess tournament is organized as a single

round robin, where playing with white (black) can be regarded as playing home (away) [12].

We now turn our attention to sets of HAPs; we define a HAP-set as follows:

Definition 2.6.

A set of $2n$ *HAPs of length* $2n-1$*, denoted by* F_{2n}*, is called a* HAP-set*. We call* $2n$ the order *of* F_{2n}*.*

A schedule is called *compatible* with a HAP-set if the set of HAPs implied by the schedule coincides with the given HAP-set. We formally define the following notions:

Definition 2.7. *For each* $2n \geq 4$*:*

- *A HAP-set* F_{2n} *is* feasible *if there exists a schedule compatible with* F_{2n}*.*
- *A HAP-set* F_{2n} *is* complementary*, if for every pattern* $P(r_1, \ldots, r_b) \in F_{2n}$*, also the complementary pattern* $P^c(r_1, \ldots, r_b) \in F_{2n}$*.*
- *A HAP-set* F_{2n} *is called* single-break *if each pattern* $h \in F_{2n}$ *is a single-break pattern.*

The HAP-set given in Table 1 is an example of a complementary HAP-set; 6 of the 8 HAPs are single-break, the remaining two patterns each have two breaks.

Identifying whether a given HAP-set is feasible is known as the pattern set feasibility problem. As far as we are aware, the complexity status of this problem is not settled; Miyashiro et al. [13] describe a necessary condition for feasibility, see also [1, 8]. In addition, observe that when given a schedule compatible with a certain HAP-set for a SRR, one can (i) interchange any two rounds, and (ii) change in a single round each H to an A, and vice versa, and arrive at another HAP-set that must be feasible.

We are primarily interested in feasible HAP-sets that consist of patterns with a limited number of breaks, as witnessed by the following definition.

Definition 2.8. *For each* $2n \geq 4$*,* $k \geq 1$*, we define*

$$\mathcal{H}_{2n,k} = \{F_{2n} : bn(h) \leqslant k \text{ for each } h \in F_{2n},\ F_{2n} \text{ feasible}\}.$$

In particular, $\mathcal{H}_{2n,1}$ is the collection of single-break HAP-sets of order $2n$.

Finally, we point out that Definitions 2.1-2.8 can be generalized to Double Round Robin tournaments by using $4n-2$ (instead of $2n-1$) rounds.

2.1 About single-break HAP-sets

Single-break HAP-sets are of special interest. De Werra [4] showed that a feasible single-break HAP-set exists for any $2n$ teams. Furthermore, for single-break HAP-sets, the conditions of Miyashiro et al. [13] have been (experimentally) shown to be sufficient for $2n \leqslant 26$ [13]. De Werra [4] showed that each feasible single-break HAP-set for $2n$ teams is complementary; as a consequence, in each round either 0 or 2 HAPs will feature a break. Notice that two teams with identical HAPs $h^1, h^2 \in F_{2n}$ are never able to play against each other.

We can describe a feasible, single-break HAP-set F_{2n} quite compactly as follows. It follows from De Werra [4] that out of the $2n-1$ rounds, there are n pairwise distinct rounds in which two patterns have a break, one team having an away break, and the other having a home break. Let us denote these n rounds by r_i, $i = 1, \ldots, n$.

We sort these r_i's in increasing order. Next, we define the difference between consecutive rounds as follows: $d_i = r_{i+1} - r_i$, $i = 1, \ldots, n$, where we use $r_{n+1} \equiv r_1 + 2n - 1$. So the sequence $D = (d_1, d_2, \ldots, d_n)$ gives the *break-gaps*, i.e., the number of rounds between two consecutive breaks (notice that $d_i \geqslant 1$, $i = 1, \ldots, n$). We refer to this representation of a feasible, single-break HAP-set as the *break-gap representation* or D-notation (see also [4] and [11]).

Not only can a feasible, single-break HAP-set F_{2n} be represented by a sequence $(d_1, d_2, \ldots, d_n)$, it is also true that any set of n positive integers that sum to $2n-1$, corresponds to a (set of) single-break HAP-sets (which is not necessarily feasible). Although a HAP-set, when specified in terms of r_i values, uniquely determines a corresponding D sequence, a D sequence can correspond to multiple HAP-sets with different r_i's. Consider for instance, $n = 2$ and the following two distinct HAP-sets: $\{HAA, HHA, AHH, AAH\}$ and $\{HAH, HHA, AHA, AAH\}$. The first one has $r_1 = 2$ and $r_2 = 3$ leading to $D = (1, 2)$ while the second one has $r_1 = 1$ and $r_2 = 2$, also leading to $D = (1, 2)$.

Furthermore, De Werra [4] shows that a sequence D is feasible if and only if all cyclic permutations of D are feasible. Hence, without loss of generality, when dealing with single-break HAP-sets F_{2n}, we will assume that the two (complementary) patterns with a break in round 1 are always present in F_{2n} and we restrict ourselves to the lexicographically largest sequence among this set of equivalent sequences.

A number of observations follow, where we identify a pattern with a team:

Observation 2.1. *If two patterns h^1, h^2 in a single-break HAP-set F_{2n} that start with the same symbol (i.e., $h^1_1 = h^2_1$) have their break in rounds $r, r+1$ respectively, then in any feasible schedule, the corresponding match is played in round r.*

This is explained in the following way. As h^1, h^2 start with the same symbol, until round $r-1$, we have $h^1_j = h^2_j$, $j = 1, \ldots, r-1$. Then, since h^1 has a break in round r, we get $h^1_r \neq h^2_r$. However, as h^2 has a break in round $r+1$, we see that $h^1_{r+1} = h^2_{r+1}$ and as both patterns are single break, $h^1_j = h^2_j$ for $r+1 \leq j \leq 2n-1$. Ergo, h^1, h^2 can only play each other in round r.

Observation 2.2. *If three patterns h^1, h^2, h^3 in a single-break HAP-set F_{2n} that start with the same symbol (i.e., $h^1_1 = h^2_1 = h^3_1$) have their break in rounds $r, r+1, r+2$ respectively (so their breaks are consecutive), there is no feasible schedule. This can be seen because all three teams can only play each other in rounds $r+1, r+2$, while they should play 3 matches in total (this also follows from the condition in Miyashiro et al. [13]).*

Observation 2.3. *If there is no team with a break in round $r+1$, then round $r+1$ is* opposite *to round r, i.e., for each $h \in F_{2n}$, we have $h_r \neq h_{r+1}$. Then, in every feasible schedule, the matches scheduled in round r, can all be played in round $r+1$, and vice versa (provided that we invert the team playing home and the team playing away).*

One of the most popular single-break HAP-sets in practice is the Canonical Pattern Set (CPS) of order $2n$ (see De Werra [3]); it is defined as follows.

Definition 2.9. *The* Canonical Pattern Set *of order $2n$ (CPS_{2n}) is the following single-break HAP-set:*
$CPS_{2n} = \{P^A(2i-1), P^H(2i-1) : i = 1, \ldots, n\}$; its break-gap representation is $D = (2, 2, \ldots, 2, 1)$.

In other words, we can order the teams whose HAPs start with an H (A) such that the break-gap is 2 for any two consecutive teams, except for one pair of teams, for which the break-gap equals 1. This is indeed a possible representation, since $\sum_i d_i = 2n-1$.

3 Measuring the flexibility of a HAP-set

Given a HAP-set F_{2n}, we are interested in the diversity of the schedules that are compatible with F_{2n} ($2n \geq 4$). To make this concrete, we introduce three

distinct measures called the width (Section 3.1), the fixed part (Section 3.2), and the spread (Section 3.3), and we show how the CPS fares on these measures.

3.1 Measure 1: the width

We call two schedules, each compatible with a given HAP-set F_{2n}, *distinct* when there exists a match that is played in one round in schedule 1 and in another round in schedule 2. Let us define here a stronger notion of distinctness.

Definition 3.1. *Two schedules, each compatible with a given HAP-set F_{2n}, are* match-distinct *when each match is played in a different round (when comparing schedule 1 with schedule 2).*

Using Definition 3.1, we define the width of a HAP-set:

Definition 3.2. *The* width *of a HAP-set F_{2n}, denoted by $width(F_{2n})$, is the number of pairwise match-distinct schedules compatible with F_{2n}.*

For instance, if F_{2n} is infeasible, $width(F_{2n}) = 0$, and if a HAP-set has width 2 (or higher), then there exist two schedules such that each match occurs in different rounds in the two schedules. As an example, the width of the HAP-set given in Table 1 equals 1, implying that no pair of match-distinct schedules compatible with that HAP-set exists. Further, we refer to Table 2 (right) for an example of a HAP-set with width 2.

We remark here that, in order to generalize the definition of width to DRR tournaments, it suffices to make a distinction between matches (i, j) and (j, i).

Single-break HAP-sets do not allow large width; in fact, there are always matches that must be played in a particular round, as witnessed by the following theorem.

Theorem 3.1. *For each $2n \geq 4$: $width(F_{2n}) = 1$ for each $F_{2n} \in \mathcal{H}_{2n,1}$.*

Proof. Consider the break-gap representation of any feasible single-break HAP-set F_{2n}; as discussed in Section 2, it follows that the corresponding entries must sum to $2n - 1$. Clearly, when choosing n positive integers that sum up to $2n - 1$, there must be at least one '1' among them. Since F_{2n} is complementary (De Werra [4]), there exist two pairs of patterns which have breaks in consecutive rounds. Then Observation 2.1 shows that, in any schedule compatible with F_{2n}, the corresponding two matches each have

only one round where they can be played. Thus width$(F_{2n}) = 1$, for each $2n \geq 4$. □

In fact, the argument above implies that any feasible HAP-set F_{2n} which contains two single-break patterns of the form $P^H(r)$ and $P^H(s)$ (or $P^A(r)$ and $P^A(s)$) with $|r - s| = 1$ has width 1.

Theorem 3.1 is tight in the following sense: even when all patterns except one are single-break patterns, HAP-sets with width 2 exist. In fact, even for $2n = 4$, it is possible to find a HAP-set with width equal to 2 containing only one pattern that is not a single-break pattern. Such a HAP-set consisting of 4 teams is given by Q in the following example.

Team	R1	R2	R3
1	H	A	H
2	H	A	A
3	A	H	A
4	A	H	H

Team	R1	R2	R3
1	H	H	H
2	H	A	A
3	A	H	A
4	A	A	H

Table 2: On the left CPS_4, on the right HAP-set Q.

The HAP-set on the left is CPS_4, the HAP-set on the right differs from CPS_4 in two entries, and is given by $Q = \{P^H(1,2,3), P^A(3), P^A(1), P^H(2)\}$. Both HAP-sets are compatible with two distinct schedules, however, there are two schedules compatible with HAP-set Q that are match-distinct, while this is not the case for CPS_4. Table 3 displays two schedules compatible with CPS_4, and two that are compatible with Q.

Thus, while CPS_4 has width 1, the HAP-set Q has width 2.

3.2 Measure 2: the fixed part

As can be deduced from Theorem 3.1, the width is a measure that does not differentiate between feasible single-break HAP-sets. Still, different feasible, single-break HAP-sets are not necessarily equal when it comes to the *number*

Round	S1	S2
R1	(1,3),(2,4)	(1,4),(2,3)
R2	(4,1),(3,2)	(3,1),(4,2)
R3	(1,2),(4,3)	(1,2),(4,3)

Round	S3	S4
1	(1,3),(2,4)	(1,4),(2,3)
2	(1,4),(3,2)	(1,2),(3,4)
3	(1,2),(4,3)	(1,3),(4,2)

Table 3: S1, S2 are compatible with CPS_4; S3, S4 are compatible with Q.

of matches that must take place in the same round. Therefore, we introduce here a more refined measure that is able to do so.

Definition 3.3. *The* fixed part *of a feasible HAP-set F_{2n}, denoted by $FP(F_{2n})$, is the number of matches that are played in the same round in every schedule compatible with F_{2n}, for each $2n \geq 4$.*

Notice that if there is just a single schedule compatible with F_{2n}, the value of this measure equals $\binom{2n}{2}$. Notice also that if the width of a HAP-set F_{2n} equals 2, it follows that $FP(F_{2n}) = 0$. For example, the fixed part of the HAP-set given in Table 1 is 4. More in particular, it consists of the matches Team 1 - Team 7, Team 2 - Team 6, Team 3 - Team 5, and Team 4 - Team 8, which all have to be played in round 7.

The following lower and upper bounds for $\mathrm{FP}(F_{2n})$ are readily obtained. Given a feasible, single-break HAP-set, let $(d_1, d_2, \ldots, d_n)$ be its break-gap representation. We use $I = |i : d_i = 1|$.

Lemma 3.1. *For any feasible, single-break HAP-set F_{2n} ($2n \geq 4$), we have: $2I \leq FP(F_{2n}) \leq nI$.*

Proof. The lower bound follows from Observation 2.1 and the fact that a feasible, single-break HAP-set is complementary. To see the upper bound, consider a round in which no team has a break, say round r is such a round in a feasible HAP-set F_{2n}. Since the HAP-set is feasible, there exists a schedule compatible with it. Since no team has a break in round r, we can interchange all matches in round $r-1$ with all matches in round r (while inverting the home-away assignment), and arrive at another feasible schedule. Each occurrence of $d_i = 1$ reveals the presence of a round r in which two teams have a break, followed by a round $r+1$ in which two other teams have a break. The interchange argument does not hold for the n matches in round r. □

3.2.1 The FP of the CPS

We now proceed by investigating the FP of the CPS. We start by recalling the following definition and insights that come from Miyashiro et al. [13].

Definition 3.4. *Let $U \subseteq F_{2n}$ be a subset of patterns of order $2n$. We define, for each $r \in \{1, \ldots, 2n-1\}$:*
$H_r(U) = |\{h \in U|\ h_r = H\}|$,
$A_r(U) = |\{h \in U|\ h_r = A\}|$,
$m_r^-(U) = \min\{H_r(U), A_r(U)\}$,

$m_r^+(U) = \max\{H_r(U), A_r(U)\}$,
$M_r^-(U) = \arg\min\{H_r(U), A_r(U)\}$.

Notice that the teams of a set U all have to play each other once in a feasible schedule, thus resulting in a total amount of $\frac{|U|(|U|-1)}{2}$ matches that have to be scheduled between the teams of U. We introduce $A(U) = \sum_r m_r^-(U)$, summing the maximum number of the matches within U that can be scheduled per round. Next, we define $\alpha(U) = A(U) - \frac{|U|(|U|-1)}{2}$ and as observed in Miyashiro [13], if $\alpha(U) < 0$, the HAP-set is infeasible, since not all matches within U can be scheduled.

If $\alpha(U) = 0$, we say that U is *tight*, meaning that in every round r, exactly $m_r^-(U)$ matches between teams in U have to be scheduled, and that all teams $t \in M_r^-(U)$ have to be scheduled against other teams of $U \setminus M_r^-(U)$.

Now, to facilitate our analysis, we first identify subsets of patterns (teams) from the CPS that are tight.

Definition 3.5. *For* $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$*:*

$$U_i = \{P^A(2n-(2i-1)), \ldots, P^A(2n-1), P^H(1), P^H(3), \ldots, P^H(2i-1)\},$$
$$U_i^c = \{P^H(2n-(2i-1)), \ldots, P^H(2n-1), P^A(1), P^A(3), \ldots, P^A(2i-1)\}.$$

Notice that $\{P^A(2n-1), P^H(1)\} = U_1 \subset U_2 \subset \ldots \subset U_{\lfloor \frac{n}{2} \rfloor}$, and that $|U_i| = |U_i^c| = 2i$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. We also identify a set of rounds as follows:

Definition 3.6. *For* $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$*:*
$R_i = \{2n-2i+1, 2n-2i+2, \ldots, 2n-2\} \cup \{2n-1\} \cup \{1, 2, \ldots, 2i-3, 2i-2\}$.

Observe that $\{2n-1\} = R_1 \subset R_2 \subset \ldots \subset R_{\lfloor \frac{n}{2} \rfloor}$, and that $|R_i| = 4i-3$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Indeed, notice that round $2n-1 \in R_i$ for all $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, and that with each unit increase of index i four rounds are added, two to the "left" of round $2n-1$ and two to the "right" of round $2n-1$.

Lemma 3.2. *U_i is tight for each $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$.*

Proof. Consider any fixed i with $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Observe that for any round $r \notin R_i$ all patterns in U_i are identical, i.e., $h_r^1 = h_r^2$ for all $h^1, h^2 \in U_i, r \notin R_i$. It follows that any match between a pair of teams within U_i takes place in a round from R_i. Consider the first two rounds in some set R_i, i.e., rounds $2n-2i+1$ and $2n-2i+2$. There is only one team in U_i which has a symbol in $\{H, A\}$ that is different from the symbols of the other teams from U_i in these rounds, namely team $P^A(2n-2i+1)$: the team that features a break

in round $2n-2i+1$. Thus, $m^-_{2n-2i+1}(U_i) = m^-_{2n-2i+2}(U_i) = 1$. Next, for the third and fourth round in R_i, i.e., rounds $2n-2i+3$ and $2n-2i+4$, there are two teams that have a symbol in $\{H, A\}$ that is different from the symbols of the other teams from U_i in these rounds, namely team $P^A(2n-2i+1)$ and $P^A(2n-2i+3)$. This implies that $m_{2n-2i+3}(U_i) = m_{2n-2i+4}(U_i) = 2$. In fact, continuing in this way, we see $m^-_{2n-2i+j} = \lceil \frac{j}{2} \rceil$ and we arrive at $m_{2n-1}(U_i) = \frac{|U_i|}{2}$. By symmetry, we claim that $m^-_{2n-1-j} = m^-_j$ for $j = 1, 2, \ldots, 2i-2$. Thus $A(U_i) = \sum_{r=2n-2i+1}^{2i-2} m^-_r(U_i) = 1+1+2+2+\ldots+\frac{|U_i|-2}{2}+\frac{|U_i|-2}{2}+\frac{|U_i|}{2}+\frac{|U_i|-2}{2}+\frac{|U_i|-2}{2}+\ldots+2+2+1+1 = \frac{|U_i|(|U_i|-1)}{2}$. It follows that U_i is tight. □

We are now in a position to prove that, for schedules that are compatible with CPS_{2n}, exactly n matches have precisely one round in which they can be played.

Theorem 3.2. *For each $2n \geq 4$: $FP(CPS_{2n}) = n$.*

Proof. To prove this theorem, we (i) identify n matches, each of which, in any schedule compatible with CPS_{2n}, can be played in one round only, and (ii) show that, for each other match, solutions exist where this match is played in different rounds.

Claim: In any schedule compatible with CPS_{2n}, all matches of the form $(P^A(2n-(2i-1)), P^H(2i-1))$ are played in round $2n-1$, for $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

Proof. We prove this claim by induction. Clearly, the statement is true for $i = 1$ as round $2n-1$ is the only round in which the match between the two teams $P^A(2n-1)$ and $P^A(1)$ can be played.

Suppose now the claim is true for $i = 1, 2, \ldots, k-1$ ($k-1 < \lfloor \frac{n}{2} \rfloor$). As U_k is tight by Lemma 3.2, we know that $t = P^A(2n-(2k-1))$ has to be scheduled against a team in U_k in every round where $t \in M^-_r(U_k)$. This is true for the rounds $2(k-1)-1, \ldots, 2n-1$. Notice that in round $2n-1$, by the induction hypothesis, all teams in U_{k-1} are already scheduled against teams from U_{k-1}; thus t has to be scheduled against a team in $U_k \setminus U_{k-1} = \{t, P^H(2k-1)\}$. Therefore, the match $(P^A(2n-(2k-1)), P^H(2k-1))$ has to be scheduled in round $2n-1$. □

By symmetry, it follows that a similar analysis for the teams from the sets U^c_i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ implies that all teams from these sets play a match that can only be played in round $2n-1$. Together with the claim this means that we have shown that $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ matches can only be played in round

$2n-1$. If n is odd, this leaves out two teams: $P^A(n)$ and $P^H(n)$. However, since these two teams have to play some match in round $2n-1$, and as all other teams are already paired, they have to play each other. Thus, in any feasible schedule compatible with the CPS, the entire round $2n-1$ consists of matches that can only be played in this round.

Finally, we point out that, due to Observation 2.3, in any feasible schedule compatible with CPS_{2n}, the matches in round r can be interchanged with the matches in round $r+1$ ($r-1$) if r is odd (even), for $r = 1, \ldots, 2n-2$. This proves the theorem. □

Observe that the proof of Theorem 3.2 implies that all matches in the last round of a schedule compatible with a CPS are *fixed*, i.e., these matches cannot be played in any other round.

3.3 Measure 3: the spread

Given a HAP-set, we call the *spread* of a match $\{i, j\}$ the number of distinct rounds in which it can be played, i.e., for which there is a schedule compatible with the given HAP-set that has this match in this round. Notice that this is not necessarily equal to the number of rounds r where h_r^i and h_r^j differ. Next, by simply summing the spread of a match over all matches, we arrive at the spread of a HAP-set. Formally:

Definition 3.7. *The* spread *of match $\{i, j\}$ with respect to feasible HAP-set F_{2n} is denoted by $spr(i, j) =$*

$$|\{r : \exists \text{ a schedule } S \text{ compatible with } F_{2n} \text{ where match } \{i, j\} \text{ is in round } r\}|.$$

Definition 3.8. *The* spread *of a HAP-set F_{2n} is the sum, over the matches, of the spread of a match, i.e., $spread(F_{2n}) = \sum_{\{i,j\}} spr(i, j)$.*

A higher value of the spread for a feasible HAP-set, indicates more flexibility in scheduling individual matches. As an example, the spread of the HAP-set given in Table 1 is 84 (16 matches can be played in 4 rounds, 8 matches have 2 possible rounds, and the fixed part consists of 4 matches). We now state an upper bound on the spread of CPS_{2n}.

Theorem 3.3.

$$spread\,(CPS_{2n}) \leq \frac{n}{6}\left(10n^2 - 9n + 11\right) - \lceil \frac{n}{2} \rceil.$$

Proof. First, we derive an upper bound for the spread of each individual match; then, we sum these upper bounds to obtain the result. As we identify each team with a pattern $P^H(2i+1)$ or $P^A(2i+1)$ $0 \le i, j \le 2n-1$, a match is either of the form $(P^H(2i+1), P^H(2j+1))$ or $(P^A(2i+1), P^A(2j+1))$ (with $i < j$), or it is of the form $(P^H(2i+1), P^A(2j+1))$ (with $i, j = 0, \ldots, n-1$). We refer to the former set of matches as matches of Type 1, and to the latter set of matches as matches of Type 2. Obviously:

$$spread\,(CPS_{2n}) = \sum_{i<j}(spr(P^H(2i+1), P^H(2j+1)) + spr(P^A(2i+1), P^A(2j+1))) + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} spr(P^H(2i+1), P^A(2j+1)). \quad (1)$$

Consider a match of Type 1. Given two teams $P^H(2i+1)$, $P^H(2j+1)$ with $0 \le i < j \le n-1$, their HAPs differ in the rounds $R_{i,j} := [2i+1, 2j]$. As two teams can only be scheduled in rounds where they have a different HAPs, it follows immediately that, for each $0 \le i < j \le n-1$:

$$spr(P^H(2i+1), P^H(2j+1)) \le |R_{i,j}| = 2(j-i). \quad (2)$$

Using (2), we are able to bound the sum of the spreads of the corresponding matches; we find for $2n \ge 4$:

$$\sum_{i<j} spr(P^H(2i+1), P^H(2j+1)) \le \sum_{j=1}^{n}\sum_{i=0}^{j-1} 2(j-i) = \sum_{j=1}^{n-1} j(j+1) =$$

$$= \frac{1}{6}(n-1)n(2n-1) + \frac{n(n-1)}{2} = \frac{1}{6}((n-1)n(2n-1) + 3n(n-1)) =: Z_{1,2n}.$$

Notice that we also need to count the spreads of the matches $(P^A(2i+1), P^A(2j+1))$; by symmetry, their sum will also be upper bounded by $Z_{1,2n}$, which leads to:

$$\sum_{i<j}(spr(P^H(2i+1), P^H(2j+1)) + spr(P^A(2i+1), P^A(2j+1))) \le 2Z_{1,2n}. \quad (3)$$

Consider now a match of Type 2. Two teams $P^H(2i+1), P^A(2j+1)$ have different HAPs in rounds $R^c_{i,j} := [1, 2\min(i,j)] \cup [2\max(i,j)+1, 2n-1]$. It

follows that the spread of a match between two such teams obeys, for each $i, j = 0, \ldots, n-1$:

$$spr(P^H(2i+1), P^A(2j+1)) \leq |R^c_{i,j}|. \tag{4}$$

However, here we can prove better bounds. We distinguish three cases when looking at matches of Type 2. We assume $i \leq j$, as the analysis is similar if $i \geq j$.

Case 1: $i = n - j - 1$.

From the proof of Theorem 3.2, we get immediately that the match $(P^H(2i+1), P^A(2j+1))$ can be scheduled in round $2n-1$ if and only if $i = n - j - 1$; obviously, in this case, the spread of this match is equal to 1. Hence, for $i, j = 0, \ldots, n-1$ with $i = n - j - 1$:

$$spr(P^H(2i+1), P^A(2j+1)) = 1. \tag{5}$$

Case 2: $i < n - j - 1$.

From Definition 3.5 it follows that $P^H(2i+1) \in U_i$, which is a tight set by Lemma 3.2. This means that in rounds $[1, 2i]$, $P^H(2i+1)$ has to be scheduled against $2i$ teams from U_i. As $P^A(2j'+1) \in U_i$ if and only if $n - j' - 1 \leq i$, $P^A(2j+1) \notin U_i$. Therefore, $(P^H(2i+1), P^A(2j+1))$ cannot be scheduled in $[1, 2i]$. Since, in addition, round $2n-1$ is not available, it follows that this match can only be played in one of the rounds in $[2j+1, 2n-2]$. We conclude, for $i, j = 0, \ldots, n-1$ with $i < n - j - 1$:

$$spr(P^H(2i+1), P^A(2j+1)) \leq 2(n - j - 1). \tag{6}$$

Case 3: $i > n - j - 1$.

We apply a similar argument as in Case 2. In this case, we observe that U_{n-j-1} is tight, which means that $P^A(2j+1)$ can only play teams from U_{n-j-1} in rounds $[2j+1, 2n-2]$, and $P^H(2i+1) \notin U_{n-j-1}$ as $i > n - j - 1$. Therefore, $(P^H(2i+1), P^A(2j+1))$ cannot be scheduled in $[2j+1, 2n-1]$, and must be scheduled in $[1, 2i]$. It follows that, for $i, j = 0, \ldots, n-1$ with $i > n - j - 1$:

$$spr(P^H(2i+1), P^A(2j+1)) \leq 2i. \tag{7}$$

We now derive an upper bound for the sum of the spreads of a subset of the matches of Type 2. Let $\lfloor \frac{n}{2} \rfloor = k$, so $n = 2k$ if n is even, and $n = 2k+1$

if n is odd. We use the right-hand sides of the expressions (5), (6), and (7), to obtain an upper bound for the sum of the spreads of the subset of the matches of Type 2, where we let i vary from 0 till $k-1$; we find for each $2n \geq 4$:

$$\sum_{i=0}^{k-1}\sum_{j=0}^{n-1} spr(P^H(2i+1), P^A(2j+1)) =$$

$$= \sum_{i=0}^{k-1}\sum_{j=i}^{n-1} spr(P^H(2i+1), P^A(2j+1)) + \sum_{i=0}^{k-1}\sum_{j=0}^{i-1} spr(P^H(2i+1), P^A(2j+1)) =$$

$$= \sum_{i=0}^{k-1}\sum_{j:i=n-j-1} spr(P^H(2i+1), P^A(2j+1)) + \sum_{i=0}^{k-1}\sum_{j:j<n-i-1} spr(P^H(2i+1), P^A(2j+1)) +$$

$$\sum_{i=0}^{k-1}\sum_{j:j>n-i-1} spr(P^H(2i+1), P^A(2j+1)) + \sum_{i=0}^{k-1}\sum_{j=0}^{i-1} spr(P^H(2i+1), P^A(2j+1)) \leq$$

$$\leq \sum_{i=0}^{k-1}[1 + \sum_{j=i}^{n-i-2} 2(n-j-1) + \left(\sum_{j=n-i}^{n-1} 2i\right) + \sum_{j=0}^{i-1} 2(n-i-1)] =$$

$$= \sum_{i=0}^{k-1}[1 + n(n-2i-1) + 2i \cdot i + 2i(n-i-1)] =$$

$$= \sum_{i=0}^{k-1}[1 + n^2 - n - 2i]$$

$$= k(n^2 - n + 1) - k(k-1) =: Z_{2,2n}.$$

In the above derivations, validity of the inequality follows from the inequalities (5), (6), and (7). Notice that $Z_{2,2n}$ is the result of the sum of the upper bounds of the spreads corresponding to the matches $(P^H(2i+1), P^A(2j+1))$, where $i < k$. By symmetry, $Z_{2,2n}$ is also equal to the sum of the spreads corresponding to matches $(P^H(2i+1), P^A(2j+1))$ where $i \geq n-k$ (recall that $k = \lfloor \frac{n}{2} \rfloor$). If n is even, this would mean $k = n-k$ and the sum over all upper bounds is equal to $Z_{2,2n}$, leading to:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} spr(P^H(2i+1), P^A(2j+1)) \leq 2Z_{2,2n}. \tag{8}$$

If n is odd, we still need to bound the spread of the matches $(P^H(2k+1), P^A(2j+1)$ where $j \geq k$, and $(P^A(2k+1), P^H(2j+1))$ where $j > k$. It is

not difficult to see that the sum of the spread of these matches is bounded from above by $Z_{odd} = 2k \cdot k + 1 + 2k \cdot k = 4k^2 + 1$, leading to:

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} spr(P^H(2i+1), P^A(2j+1)) \leq 2Z_{2,2n} + Z_{odd}. \tag{9}$$

Combining (1), (3) and (8), for even n we get an upper bound of:

$$\begin{aligned}
&spread\,(CPS_{2n}) \leq 2Z_{1,2n} + 2Z_{2,2n} = \\
&= \frac{1}{6}(2(n-1)n(2n-1) + 6n(n-1)) + 2k(n^2 - n + 1) - 2k(k-1) = \\
&= \frac{1}{6}(4n^3 - 6n^2 + 2n + 6n^2 - 6n + 6n^3 - 6n^2 + 6n - 6n(k-1)) = \\
&= \frac{1}{6}(10n^3 - 6n^2 + 8n - 6nk) = \\
&= \frac{1}{6}(10n^3 - 9n^2 + 8n).
\end{aligned}$$

Combining (1), (3) and (9), for odd n we get an upper bound of:

$$\begin{aligned}
&spread\,(CPS_{2n}) \leq 2Z_{1,2n} + 2Z_{2,2n} + 4k^2 + 1 = \\
&= \frac{1}{6}(2(n-1)n(2n-1) + 6n(n-1)) + 2k(n^2 - n + 1) - 2k(k-1) + 4k^2 + 1 = \\
&= \frac{1}{6}(2(n-1)n(2n-1) + 6n(n-1) + 6(n-1)(n^2 - n + 1) + 12k(k+1) + 6) = \\
&= \frac{1}{6}(10n^3 - 12n^2 + 8n + 6(n-1)(k+1) + 6) = \\
&= \frac{1}{6}(10n^3 - 6n^2 + 8n - 6(nk - k + n)) = \\
&= \frac{1}{6}(10n^3 - 9n^2 + 8n + 3).
\end{aligned}$$

When n is even, $3n = 6\lceil \frac{n}{2} \rceil$, and when n is odd, $3n = 6\lceil \frac{n}{2} \rceil - 3$. Therefore, we can write, for any n, that:

$$spread\,(CPS_{2n}) \leq \frac{1}{6}(10n^3 - 9n^2 + 11n) - \lceil \frac{n}{2} \rceil.$$

This finishes the proof. □

We can calculate the actual value of the spread of CPS_{2n} using integer programming (see Section 4). In Table 4, we explicitly list the value of the spread of CPS_{2n} and its upper bound for values of $2n$ up to 30, and note that this bound is tight. In fact, we conjecture that this bound is tight for any even number of teams.

$2n$	4	6	8	10	12	14	16	18	20	22	24	26	28	30
upper bound	10	35	88	177	314	507	768	1105	1530	2051	2680	3425	4298	5307
spread(CPS)	10	35	88	177	314	507	768	1105	1530	2051	2680	3425	4298	5307

Table 4: Values for the spread and the upper bound for CPS with $2n$ teams

Conjecture 3.1.

$$spread\,(CPS_{2n}) = \frac{n}{6}\left(10n^2 - 9n + 11\right) - \lceil\frac{n}{2}\rceil.$$

4 Computing the measures

In Section 4.1, we introduce two mathematical formulations that are instrumental in computing the flexibility measures proposed earlier. In Section 4.2 we present computational results for single-break HAP-sets.

4.1 Mathematical formulations

In order to check whether a given HAP-set F_{2n} has a width of at least W, we develop a feasibility integer programming (IP) formulation, such that if this formulation has a feasible solution, W pairwise match-distinct schedules can be constructed that are compatible with F_{2n}. The given HAP-set consist of $2n$ HAPs h^i, represented by the parameter h^i_r which equals 1 if h^i has a home game on round r and 0 otherwise. We refer to t_i as the team that plays according to HAP h^i. The binary decision variable $x^w_{i,j,r}$ equals 1 if in schedule w, t_i is playing a home game against team t_j on round r, and 0 otherwise. We use R to denote the set of rounds $\{1, 2, ..., 2n-1\}$.

$$\sum_{r\in R}(x^w_{i,j,r} + x^w_{j,i,r}) = 1 \quad \forall i,j \in F_{2n} : i \neq j, w \in \{1,\ldots,W\} \quad (10)$$

$$\sum_{j\in F_{2n}}(x^w_{i,j,r} + x^w_{j,i,r}) = 1 \quad \forall i \in F_{2n}, r \in R, w \in \{1,\ldots,W\} \quad (11)$$

$$x^w_{i,j,r} \leqslant h^i_r \quad \forall i,j \in F_{2n}, r \in R, w \in \{1,\ldots,W\} \quad (12)$$

$$x^w_{i,j,r} \leqslant 1 - h^j_r \quad \forall i,j \in F_{2n}, r \in R, w \in \{1,\ldots,W\} \quad (13)$$

$$\sum_{w=1}^{W} x^w_{i,j,r} \leqslant 1 \quad \forall i,j \in F_{2n}, r \in R \quad (14)$$

$$x^w_{i,j,r} \in \{0,1\} \quad \forall i,j \in F_{2n}, r \in R, w \in \{1,\ldots,W\} \quad (15)$$

Constraints (10) enforce that each match is played exactly once in each schedule, while constraints (11) make sure that each team plays exactly one

match per round in each schedule. Constraints (12) and (13) accomplish that a home match for t_i against t_j can only be scheduled on round r if h^i features a home match and h^j specifies an away match on that round. The fact that all W schedules need to be pairwise match-distinct is enforced by constraints (14).

The width of a HAP-set F_{2n} can be determined by solving formulation (10)-(15) by a binary search on potential values for W. Moreover, it is possible to adapt this formulation to study properties of patterns that allow the existence of a feasible HAP-set with width at least W. This can be done by treating the h^i_r parameters as variables, and then adding constraints on these to reflect the properties one wants the patterns to have. Furthermore, an objective function, e.g., minimizing the total number of breaks, could be added.

To compute the FP, we consider the model above for $W = 2$, where we, for a given match between $t_{i'}$ and $t_{j'}$, replace constraints (14) with

$$\sum_{w=1}^{W}(x^w_{i',j',r} + x^w_{j',i',r}) \leqslant 1 \qquad \forall r \in R. \tag{16}$$

This yields a feasibility IP that allows to check whether or not the match between $t_{i'}$ and $t_{j'}$ is part of the fixed part of HAP-set F_{2n}. FP(F_{2n}) is then obtained by summing the number of IPs that are infeasible over all matches $\{t_{i'}, t_{j'}\}$.

Probably the most challenging measure, from a computational point of view is the spread. When we consider formulation (10)-(15) for $W = 1$, formulation (17)-(21) arises. Note that we dropped the index w in the x variables and that constraint (14) is now redundant. Consider now a match between $t_{i'}$ and $t_{j'}$, to be played in a particular round ℓ (regardless of the home advantage) as enforced by constraint (22).

$$\textstyle\sum_{r\in R}(x_{i,j,r} + x_{j,i,r}) = 1 \quad \forall i, j \in F_{2n} : i \neq j \tag{17}$$
$$\textstyle\sum_{j\in F_{2n}}(x_{i,j,r} + x_{j,i,r}) = 1 \quad \forall i \in F_{2n}, r \in R \tag{18}$$
$$x_{i,j,r} \leqslant h^i_r \qquad \forall(\{i, j\}, r) \neq (\{i', j'\}, \ell) \tag{19}$$
$$x_{i,j,r} \leqslant 1 - h^j_r \qquad \forall(\{i, j\}, r) \neq (\{i', j'\}, \ell) \tag{20}$$
$$x_{i,j,r} \in \{0, 1\} \qquad \forall i, j \in F_{2n}, r \in R \tag{21}$$
$$x_{i',j',\ell} + x_{j',i',\ell} = 1 \tag{22}$$

The idea is that by solving the resulting feasibility IP for each match between $t_{i'}$ and $t_{j'}$ and each round ℓ which is compatible with the given

HAP-set F_{2n}, we can find the spread by solving at most $n \times \frac{n}{2} \times (2n-1)$ IPs. Indeed, each round has $n \times \frac{n}{2}$ H-A pairs, and for each of these, in each of the $2n-1$ rounds, after solving formulation (17)-(22), we record yes/no whether this match can be played in this round. Next, enumerating gives us the spread of each match, from which we determine the spread of the HAP-set.

Note that this also gives us the fixed part of the given HAP-set, however, not before solving formulation (17)-(22) $O(n^3)$ times (as opposed to solving formulation (10)-(13),(15),(16) $O(n^2)$ times).

4.2 Results

We have used the formulations in Section 4.1 to compute the spread and the fixed part for all D-notations corresponding to feasible single-break HAP-sets of $2n \leq 16$. Recall that each such D-notation represents a family of HAP-sets with the same value for the flexibility measures (see Section 2.1). All computations were done using IBM Ilog Cplex 12.8 on a Dell Latitude 7490 with Intel Core i7-8650 @ 1.9 GHz and 16 GB RAM. IP models (10)-(16) and (17)-(22) were solved in less than one second; all flexibility measures were computed in less than 1 minute for up to 16 teams. Recall that we know that the width equals one for all single-break HAP-sets from Theorem 3.1.

The results are summarized in Table 5. While for 4 and 6 teams, all single-break HAP-sets are equivalent with respect to all flexibility measures, this is no longer the case for 8 teams and more. It turns out that for 8 teams, the popular canonical pattern set is the best choice with respect to the spread and as good as any other single-break HAP-set when we focus on the fixed part. For 10 to 16 teams, however, the canonical pattern set is clearly dominated by another type of schedule, indicated by the D-notation 312...21. Moreover, for all values of $2n$ that we considered, this single-break HAP-set shows the highest spread as well as the lowest fixed part.

We also point out the following: we claim that if some match $(P(r), P(s))$ is in the FP, then its complementary match, defined as $(P^c(r), P^c(s))$ is also in the FP. Thus, the set of matches in the FP can be seen as pairs. Interestingly, for $2n \in \{10, 14\}$, there exist feasible single-break HAP-sets with an odd number of matches in the FP. This can only happen when the complement of a match is the match itself, as in the match $(P^H(r), P^A(r))$. Thus, in these cases, although the match is between two teams that have complementary HAPs (and hence could, seemingly, play in each round), apparently, this match can only be played in one round only.

5 Conclusion

In a first-break-then-schedule approach, it is important to have freedom to schedule the individual matches after the Home-Away Patterns have been chosen. We have proposed three measures that indicate the amount of freedom associated with a particular HAP-set. We have theoretically established how the most popular HAP-set, the Canonical Pattern Set (CPS), fares on these measures. Also, we have computed, using integer programming, for all feasible single-break HAP-sets up to order 16, the numerical values of the proposed measures. It is interesting to see that, when the number of teams exceeds 10, the CPS is not the most flexible HAP-set.

More generally, while we have focussed here on single break HAP-sets, we expect that, when allowing HAPs with break-numbers of value 2 and larger, one can construct HAP-sets that score better on our flexibility measures, i.e., achieve a better value for the width, fixed part, and spread respectively. We leave this as an interesting direction to explore.

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$2n$	D-notation	spread	FP
4	21 (CPS)	10	2
6	221 (CPS)	35	3
8	2221 (CPS)	88	4
	3121	76	4
10	22221 (CPS)	177	5
	31221	161	4
12	222221 (CPS)	314	6
	312212	266	6
	312221	332	4
	321221	266	6
	313121	254	6
14	2222221 (CPS)	507	7
	3122212	471	6
	3122221	557	4
	3212221	471	6
	3123121	423	7
	3131221	439	6
	3213121	423	7
16	22222221 (CPS)	768	8
	31222122	686	8
	31222212	796	6
	31222221	864	4
	32122212	632	8
	32122221	796	6
	32212221	686	8
	31223121	672	8
	31231221	690	6
	31312212	614	8
	31312221	838	6
	32122131	614	8
	32123121	684	8
	32131221	690	6
	32213121	672	8
	31313121	640	8
	41213121	552	8

Table 5: Flexibility measures for single-break HAP-sets for up to 16 teams