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Modified Van der Pol-Helmholtz oscillator equation with exact harmonic solutions

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Abstract. In this paper, we present an exceptional Lienard equation consisting of a modified Van der Pol-Helmholtz oscillator equation. The equation, a frequency-dependent damping oscillator, does not satisfy the classical existence theorems but, nevertheless, has an isochronous centre at the origin. We exhibit the exact and explicit general harmonic and isochronous solutions by using the first integral approach. The numerical results match very well analytical solutions.

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1. Introduction

Many problems in applied mathematics and physics are often described in terms of nonlinear differential equations. One of the most investigated second-order nonlinear differential equations is the Lienard equation [1, 2, 3]

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \quad (1.1)$$

where overdot means differentiation with respect to time, and $f(x)$ and $g(x)$ are functions of x . This general class of equations has been intensively studied in the literature from the viewpoint of theorem for the existence of periodic solutions [1, 2, 3]. A famous example of equations of type (1.1) is the Van der Pol oscillator known to have a unique limit cycle [1, 3]. It is also known since the work performed by Sabatini [2] that equation (1.1) can have an isochronous centre at the origin. However, it was in 2005 that such a property was explicitly proven in [4]. Recently, Monsia and his coworkers presented by

adequately choosing the functions $f(x)$ and $g(x)$, several equations of type (1.1) that have harmonic and isochronous periodic solutions [5, 6, 7]. The equations of type (1.1) are very interesting from a physical point of view since dynamical systems are, in the real world, subject to dissipative terms. Thus, the problem of finding periodic or isochronous solutions to equations of the form (1.1) has become an attractive research field for pure and applied mathematics. In this context, consider the generalized equation

$$\ddot{x} + \beta(\alpha - \gamma x^2)^p \dot{x} + g(x) = 0. \quad (1.2)$$

An interesting family of equations of type (1.2) is the Van der Pol-Helmholtz equation.

$$\ddot{x} + \beta(\alpha - \gamma x^2)^p \dot{x} + a_1 x + a_2 x^2 + a_3 = 0, \quad (1.3)$$

where $\alpha, \beta, \gamma, a_1, a_2, a_3$ and p are arbitrary constants, $f(x) = \beta(\alpha - \gamma x^2)^p$, and $g(x) = a_1 x + a_2 x^2 + a_3$. When $p = 1, a_2 = a_3 = 0$, the equation (1.3) is said to be the generalized Van der Pol equation

$$\ddot{x} + \beta(\alpha - \gamma x^2) \dot{x} + a_1 x = 0. \quad (1.4)$$

When $\beta = 0$, the equation (1.3) reduces to

$$\ddot{x} + a_1 x + a_2 x^2 + a_3 = 0, \quad (1.5)$$

known as the conservative quadratic nonlinear Helmholtz oscillator or quadratic anharmonic oscillator [8, 9, 10]. When $\gamma = 0$ and $p = 1$, equation (1.3) becomes the damped anharmonic Helmholtz oscillator

$$\ddot{x} + \lambda \dot{x} + a_1 x + a_2 x^2 + a_3 = 0, \quad (1.6)$$

where $\lambda = \alpha\beta$, which has been treated in numerous works [8, 11, 12]. With such damping, the general solutions of equation (1.6) are rather nonperiodic [11, 12], contrary to the periodic solutions of equation (1.5). Now, let $p = \frac{1}{2}$. Then, the generalized Van der Pol-Helmholtz equation (1.3) becomes

$$\ddot{x} + \beta\sqrt{\alpha - \gamma x^2} \dot{x} + a_1 x + a_2 x^2 + a_3 = 0. \quad (1.7)$$

Thus, an interesting mathematical problem to investigate is to ask whether equation (1.7) can exhibit periodic solutions. This problem becomes more interesting when the question is to find general harmonic and isochronous periodic solutions such as the solution of the linear harmonic oscillator. To the best of our knowledge, such a question has not been previously solved for the modified Van der Pol-Helmholtz equation of form (1.7) in the literature. In this situation, the objective in this paper is to explicitly prove that equation (1.7) can have an isochronous centre at the origin for an appropriate choice of system parameters, contrary to the existence theorem predictions. To attain this objective, we analyze equation (1.7) in light of usual existence theorems and the phase plane (section 2) and explicitly calculate the general harmonic periodic solutions that prove the existence of an isochronous center at the origin (section 3). A conclusion is sketched finally for the work.

2. Existence theorem and phase plane analysis

The Lienard equation (1.1) was deeply investigated in [1] from the perspective of the theorem for the existence of a centre at the origin (see **Theorem 11.3**, page 390 of [1]). This theorem is in agreement with the theorems for the existence of a centre at the origin formulated in [2, 3]. Then, according to [1, 2, 3], the Lienard equation (1.1) has a centre at the origin when $f(x)$ and $g(x)$ are continuous functions, $f(x)$ and $g(x)$ are odd, and $g(x) > 0$ for $x > 0$, that is $g(0) = 0$. In the case of equation (1.7), $f(x) = \beta\sqrt{\alpha - \gamma x^2}$, and $g(x) = a_1x + a_2x^2 + a_3$. As observed, $f(x)$ is neither odd nor even. $g(x)$ is not also odd and $g(0) \neq 0$, since $a_3 \neq 0$. These results are sufficient to conclude that, according to [1, 2, 3], the isochronous center at the origin is excluded from equation (1.7) for any arbitrary nonzero value of $a_1, a_2, a_3, \alpha, \beta$ and γ . Now, consider the dynamical system

$$\dot{x} = y, \quad \dot{y} = -\beta\sqrt{\alpha - \gamma x^2}y - a_1x - a_2x^2 - a_3, \quad (2.1)$$

equivalent to equation (1.7). The equilibrium points are given by $y = 0$, and the quadratic equation

$$a_2x^2 + a_1x + a_3 = 0. \quad (2.2)$$

The discriminant of equation (2.2) reads

$$\Delta = a_1^2 - 4a_2a_3, \quad (2.3)$$

and the solution are written as

$$x_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_3}}{2a_2}, \quad (2.4)$$

and

$$x_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2a_3}}{2a_2}. \quad (2.5)$$

The origin is a single equilibrium point for equation (1.7) when $x_1 = x_2 = 0$, that is, $a_2a_3 = 0$. This involves that $a_2 = 0$, or $a_3 = 0$. As these coefficients must be different from zero, equation (1.7) cannot have a centre at the origin. However, in the next section, we exhibit harmonic and isochronous periodic solutions to equation (1.7) to explicitly prove the existence of an isochronous centre at the origin.

3. Harmonic and isochronous solutions

We state the conditions of integrability in terms of exact and explicit general harmonic and isochronous solutions of equation (1.7) and the existence of a centre in this section.

3.1. Conditions of integrability of equation (1.7) and existence of a centre

To explicitly integrate equation (1.7), consider the general class of velocity-dependent Lienard equations

$$\ddot{x} \pm \frac{1}{2} \frac{u'(x)}{u^2(x)} \sqrt{u(x)(b - av(x))} \dot{x} + \frac{a}{2} \frac{v'(x)}{u(x)} = 0, \quad (3.1)$$

which can be formulated from the theory performed in [9, 13], where $u(x)$ and $v(x)$ are functions of x and a and b are constants. Putting $u(x) = e^{2\beta x}$, and $v(x) = (cx^2 - q)e^{2\beta x}$, leads to the equations

$$\ddot{x} \pm \beta \sqrt{be^{-2\beta x} - a(cx^2 - q)} \dot{x} + acx + ac\beta x^2 - aq\beta = 0, \quad (3.2)$$

where c and q are free constants, which can reduce to

$$\ddot{x} \pm \beta \sqrt{aq - acx^2} \dot{x} + acx + ac\beta x^2 - aq\beta = 0, \quad (3.3)$$

when $b = 0$. Equation (3.3) is a frequency-dependent damping system. The first integral of equation (13) can be expressed as

$$b = u(x)\dot{x}^2 + av(x). \quad (3.4)$$

Substituting the previous expressions of $u(x)$ and $v(x)$ into equation (3.4) yields a first integral of equation (3.3) in the form

$$aq = \dot{x}^2 + acx^2. \quad (3.5)$$

From this perspective, equation (3.3) becomes completely integrable. Since equation (1.7) is identical to equation (3.3) for $+\beta, \alpha = aq, \gamma = ac, a_1 = ac, a_2 = ac\beta$ and $a_3 = aq\beta$, it also becomes completely integrable under these conditions. As such, equation (3.3) of interest does not satisfy the existence theorem for an isochronous centre at the origin mentioned above. However, the time-independent first integral (3.5) can be interpreted as the Hamiltonian of system (3.3) under the form

$$H(x, y) = \frac{1}{2}aq = \frac{1}{2}y^2 + \frac{1}{2}acx^2, \quad (3.6)$$

where $aq > 0, ac > 0$ and $y = \dot{x}$. This Hamiltonian (3.6), as well known, corresponds to closed trajectories in the (x, y) phase plane with a centre at the origin. This shows that equation (3.3) of interest has a centre at the origin, contrary to the predictions of classical existence theorems. Now, we can formulate the exact and isochronous solutions of equation (3.3).

3.2. General harmonic and isochronous solutions

From the equation (3.5) or (3.6), one can write

$$\frac{dx}{\sqrt{aq - acx^2}} = \pm dt, \quad (3.7)$$

which can be integrated to obtain the harmonic solution of equation (3.3) in the form

$$x(t) = \sqrt{\frac{q}{c}} \sin[\pm\sqrt{ac}(t + K)]. \quad (3.8)$$

Equation (3.8), which characterizes the amplitude-dependent frequency oscillator, becomes isochronous when $c = 1$, that is,

$$x(t) = \sqrt{q} \sin[\sqrt{a}(t + K)], \quad (3.9)$$

where K is a constant of integration. It is worth noting that solution (3.9) is also the solution of the linear harmonic oscillator

$$\ddot{x} + ax = 0, \quad (3.10)$$

where the amplitude of oscillations is kept at $\sqrt{q}$, $a > 0$, and $q > 0$. Equation (3.10) is equivalent to equation (3.3) when $c = 1$. The solution (3.8) or (3.9) does not depend on β . Thus, for $\pm\beta > 0$, corresponding to positive damping, or for $\pm\beta < 0$, corresponding to negative damping, the solution remains harmonic. In this regard, numerical examples can be given to illustrate the theory.

4. Numerical applications

An illustration of the analytical theory is shown in this part by comparison with numerical results using the fourth-order Runge-Kutta (*RK4*) algorithm. In this way, consider the general initial conditions $x(0) = x_0$, and $\dot{x}(0) = v_0$. Thus, substituting these conditions into the general solution (3.9), yields the system of algebraic equations

$$\begin{cases} x(0) = x_0 = \sqrt{q} \sin(\sqrt{q}K) \\ \dot{x}(0) = v_0 = \sqrt{aq} \cos(\sqrt{q}K) \end{cases}, \quad (4.1)$$

from which, one can obtain

$$K = \frac{\sqrt{a}}{a} \operatorname{arccotan}\left(\frac{v_0 \sqrt{a}}{x_0 a}\right). \quad (4.2)$$

Therefore, the general isochronous solution (3.9) takes the form

$$x(t) = \sqrt{q} \sin[\sqrt{a}t + \operatorname{arccotan}\left(\frac{v_0 \sqrt{a}}{x_0 a}\right)]. \quad (4.3)$$

Figures 1, 2, 3 and 4 show the comparison of solution (4.3) to numerical results for different values of model parameters and initial conditions.

FIGURE 1. Comparison of solution (4.3) in the solid line with the numerical solution of equation (3.3) in the circle line. Typical values are $+\beta = 0.0025$, $a = 0.5$, $c = 1$, $q = 0.25$, $x_0 = 0.5$, and $v_0 = 0.001$.

FIGURE 2. Comparison of solution (4.3) in the solid line with the numerical solution of equation (3.3) in the circle line. Typical values are $+\beta = -0.0025$, $a = 0.5$, $c = 1$, $q = 0.25$, $x_0 = 0.5$, and $v_0 = 0.001$.

FIGURE 3. Comparison of solution (4.3) in the solid line with the numerical solution of equation (3.3) in the circle line. Typical values are $-\beta = 0.0025$, $a = 0.5$, $c = 1$, $q = 0.25$, $x_0 = 0.5$, and $v_0 = 0.01$.

FIGURE 4. Comparison of solution (4.3) in the solid line with the numerical solution of equation (3.3) in the circle line. Typical values are $-\beta = -0.0025$, $a = 0.5$, $c = 1$, $q = 0.25$, $x_0 = 0.5$, and $v_0 = 0.01$.

There is excellent agreement between the numerical and exact analytical solutions. Now, we can address a conclusion for the work.

Conclusion

In this paper, we have presented a velocity-dependent Lienard equation consisting of a generalized Van der Pol-Helmholtz equation. The conditions of integrability in terms of periodic solutions are established. Thus, we have been able to exhibit the exact and general harmonic and isochronous solutions of the equation of interest, contrary to the predictions of usual existence theorems.

Declarations

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Data availability

This manuscript has no associated data.

Conflict of interest

The authors declare that they have no conflict of interest.

Ethical approval

The authors state that this article complies with ethical standards.

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