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A Viscous Velocity Potential/Stream Function

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The motion of fluids presents interesting phenomena including flow separation, wakes, turbulence etc. The physics of these are enshrined in the continuity equation and the NSE. Therefore, their studies are important in mathematics and physics. They also have engineering applications. These studies can either be carried out experimentally, numerically, or theoretically. Theoretical studies using classical potential theory (CPT) have some gaps when compared to experiments. The present publication is part of a series introducing refined potential (RPT) that bridges these gaps. It leverages experimental observations, physical deductions and the match between CPT and experimentally observed flows in the theoretical development. It analytically imitates the numerical source/vortex panel method to describe how wall bounded eddies in a three-dimensional cylinder crossflow are linked to the detached wake eddies. Unlike discrete and arbitrary vortices/sources on the cylinder surface whose strengths are numerically determined in the panel method, the vortices/sources/sinks in RPT are mutually concentric and continuously distributed on the cylinder surface. Their strengths are analytically determined from CPT using physical deductions starting from Reynolds number dependence. This study results in the incompressible Kwasu function which is a Eulerian velocity potential/stream function that captures vorticity, boundary layer, shed wake vortices, three-dimensional effects, and turbulence. This Eulerian Kwasu function also theorizes streaklines. The Lagrangian form of the function is further exploited to obtain flow pathlines.

1. Introduction

The motion of fluids presents interesting phenomena including flow separation, wakes, turbulence etc. The studies of these phenomena are important in mathematics and physics and have engineering applications. Figure 1a presents an overview of the three fundamental approaches to exploring these phenomena (Amoloye 2020; Anderson 2011). Experimentally observed flow patterns or features of incompressible fluid dynamics are governed by the continuity and the Navier-Stokes equations (NSE). The physics of the continuity equation and the NSE are the conservation of mass and momentum respectively (Amoloye 2020; Anderson 2011; Amoloye 2021; Institute 2018; Amoloye 2018). These also are the physics behind computational fluid dynamics (CFD) and the analysis depicted in figure 1b for the classical potential theory (CPT) on a non-lifting cylinder crossflow (Amoloye 2018, 2020, 2021; Anderson 2011). Because the incompressible continuity equation for the flow satisfies Laplace equation, the stream function, ψ , is a superposition of individual stream functions of a uniform and a doublet flows (where κ is the doublet strength) (Amoloye 2018, 2020, 2021; Anderson 2011). With the cylinder

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radius, R , equivalent to $\sqrt{\kappa/2\pi r V_\infty}$, the inviscid stream function for the flow is

$$\psi_{inv}(r, \theta, V_\infty) = V_\infty r \sin \theta \left(1 - \frac{R^2}{r^2}\right) \quad (1.1)$$

(inv: inviscid and V_∞ : freestream velocity). The velocity components are

$$\begin{aligned} V_{r_{inv}} &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} = V_\infty \cos \theta \left(1 - \frac{R^2}{r^2}\right) \\ V_{\theta_{inv}} &= -\frac{\partial \psi}{\partial r} = -V_\infty \sin \theta \left(1 + \frac{R^2}{r^2}\right) \end{aligned} \quad (1.2)$$

and the speed is

$$\begin{aligned} V_{inv} &= V_\infty \sqrt{\left(\cos \theta \left(1 - \frac{R^2}{r^2}\right)\right)^2 + \left(-\sin \theta \left(1 + \frac{R^2}{r^2}\right)\right)^2} \\ &= V_\infty \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}. \end{aligned} \quad (1.3)$$

The pressure, p , follows from the NSE as

$$\begin{aligned} p &= p_\infty + \frac{1}{2} \rho_\infty [V_\infty^2 - V_{inv}^2] \\ &= p_\infty + \frac{1}{2} \rho_\infty V_\infty^2 \left[1 - \left(\cos \theta \left(1 - \frac{R^2}{r^2}\right)\right)^2 - \left(-\sin \theta \left(1 + \frac{R^2}{r^2}\right)\right)^2\right] \end{aligned} \quad (1.4)$$

where p_∞ and ρ_∞ are the freestream static pressure and density respectively.

Equation 1.1 is an inviscid solution since it features no viscosity term. However, actual flow features such as boundary layer, flow separation/reattachment, wake formation, vortex shedding, drag occurrence etc. are viscous-driven and dependent on Reynolds number, Re (based V , the cylinder diameter, D , and the fluid kinematic viscosity, ν , as $Re = VD/\nu$). Also contrary to actual flows that are usually unsteady and three-dimensional, Eqn. 1.1 is steady and two-dimensional because it is not dependent on time, t , and the third cylindrical polar coordinate variable, z , that extends out of the paper.

Table 1 highlights these and other gaps between CPT and experimentally observed flows. These motivated the development of refined potential theory (RPT) (Amoloye 2020, 2021). The items in the lower block of table 1 are the specific features of CPT that RPT addresses. RPT assumes an actual cylinder flow obeys Laplace equation and has a stream function and velocity components described as

$$\psi_{vis}(r, \theta, z, V_\infty, \nu, t, Re) = V_\infty r \sin \theta \left(1 - \frac{R^2}{r^2}\right) + B_0(r, \theta, V_\infty, \nu, t, Re) \ln \left(\frac{r}{R}\right), \quad (1.5)$$

and

$$\begin{aligned} V_{r_{vis}} &= V_\infty \cos \theta \left(1 - \frac{R^2}{r^2}\right) + \frac{1}{r} \frac{\partial B_0}{\partial \theta} \ln \left(\frac{r}{R}\right) \\ V_{\theta_{vis}} &= -V_\infty \sin \theta \left(1 + \frac{R^2}{r^2}\right) - \frac{B_0}{r} - \frac{\partial B_0}{\partial r} \ln \left(\frac{r}{R}\right) \end{aligned} \quad (1.6)$$

respectively (Amoloye 2021, 2020). The double arrows in figure 1a indicate that there are mutual interactions between the three approaches in fluid dynamics (Anderson 2011; Amoloye 2020). Therefore, observations of experimental flow physics and numerical

(b) The superposition of a uniform flow and a doublet into the inviscid over a circular cylinder in CPT (Amoloye 2021).

Figure 1: The three main approaches to fluid dynamics.

solutions are employed in the functional definition of B_0 starting from a basic definition as (Amoloye 2021, 2020)

$$B_0 = \frac{\Gamma}{2\pi} = \frac{\Gamma_0}{2\pi} e^{\left(-\frac{r^2}{4\nu t}\right)} . \tag{1.7}$$

Γ is the circulation (Anderson 2011, p. 174), and Γ_0 is the initial circulation. Equation 1.7 is based on a Burger’s vortex model (Amoloye 2021, 2020). The items in the upper block of table 1 for which there is a match between CPT and experimentally observed flows are leveraged in the development of RPT. The aim is to construct a viscous and time-dependent stream function, ψ_{vis} , from the classical theory (Amoloye 2020, 2021). (Amoloye 2021) discusses how ψ_{vis} satisfies the viscous boundary conditions and conserves mass and momentum. These are also elaborated on in (Amoloye 2020).

Section 2 of the present publication presents the derivation of ψ_{vis} . It is obtained as

$$\psi_{vis} = \psi_{inv} + \psi_{vt} + \psi_{ss} + \psi_p + \psi_{vtsd} \tag{1.8}$$

(vis: viscous, vt: vortex, ss: sink/source, p: perifocal, and vtsd: vortex shedding) using superposition. Usually, the superposition principle does not pertain to non-linear systems

Table 1: An overview of the gap between CPT and experimentally observed flows (Amoloye 2020).

	CPT	Experimentally observed flows	RPT
Satisfies Continuity Eqn., $\nabla \cdot \mathbf{V} = 0$	✓		✓ ?
Bounded at infinity	✓		✓ ?
Smooth and Continuous	✓		✓ ?
Satisfies NSE	✓		✓ ?
Physical streamlines	✓		✓ ?
Three-dimensions	mostly ×		✓ ?
No-slip condition	×	×	✓ ?
Re_∞ dependence	×		✓ ?
Vorticity, $\omega \neq 0$	×		✓ ?
Boundary layer	×		✓ ?
Wake	×		✓ ?
Vortex shedding	×		✓ ?
Unsteadiness, Turbulence	×		✓ ?
Compressible	✓		✓ ?
Drag	×		✓ ?

like a cylinder flow. However, the use of a superposed stream function offers an avenue to resolve the mathematical non-linearity through experimental observations of the flow physics (Amoloye 2020). This is because the stream function has both mathematical description and physical representation as the mass flow rate between streamlines. A streamline is a line drawn in a flow such that the local velocity is always tangential to it (Taneda 1985; Amoloye 2020). The streamlines of an experimentally observed flow are identifiable with flow visualization techniques (Anderson 2011; Merzkirch 2007; Taneda 1985; Van Dyke 1988; Wu *et al.* 2006; White 2006; Zdravkovich 1997; Amoloye 2020). Some other researchers including Talaei and Garrett (Talaei & Garrett 2020) have also used summations of stream functions in their viscous flow analyses (Amoloye 2020). Additionally, the stream function satisfies the linear Laplace equation when the non-linear vorticity (Anderson 2011, pp. 163-174) is identically zero (Amoloye 2020, 2021; Anderson 2011).

Subsequently, ψ_{vis} is integrated to obtain an Eulerian viscous scalar potential/stream function, $\tilde{\kappa}$ (incompressible *Kwasu* function), that solves the NSE problem (Amoloye 2018, 2020, 2021) in Section 4. Section 5 discusses how the Lagrangian *Kwasu* function theorizes flow streaklines (Amoloye 2021, 2020). Section 6 concludes this article.

2. Viscous Incompressible Two-Dimensional Stream Function Without Vortex Shedding

2.1. Surface Sink-Source-Vortex Sheet

The idea is to analytically imitate the numerical source/vortex panel method (Anderson 2011, secs. 3.17 and 4.10). However, unlike discrete and arbitrary vortices/sources on the cylinder surface whose strengths are numerically determined in the panel method, the vortices/sources/sinks in RPT are mutually concentric and continuously distributed

Figure 2: Contour of normalized speed, $V/V_\infty = \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}$, for inviscid case showing the distance $(r - R)$ (Amoloye 2020)

on the cylinder surface. Their strengths are analytically determined from CPT using physical deductions starting from Re -dependence (Amoloye 2018, 2020, 2021).

To introduce Re into the expression for B_0 , the vortices it describes are assumed to be Burgers' vortices that have their characteristic time, t , fixed in Eqn. 1.7 (Amoloye 2021, 2020). The circulation introduced by these vortices is then represented as

$$\Gamma = \Gamma_0 \exp\left(-\frac{(r - R)^2}{r_c^2}\right) \quad (2.1)$$

where the radial distance, r , in Eqn. 1.7 has been replaced with $r - R$. $r - R$ is the distance measured radially from the cylinder surface that is depicted in figure 2. Figure 2 also presents a contour of the normalized inviscid speed (Amoloye 2020).

The determination of the vortex core radius, r_c , follows an iterative trial and error process. Using the radial distance from the wall and the expression for the inviscid speed, a basic expression is derived as (Amoloye 2020)

$$\begin{aligned} r_c^2 &= 4\nu t \\ &= 4\nu \times \frac{\text{distance}}{\text{speed}} \\ &= \frac{4\nu(r-R)}{V_\infty \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} \\ &= \frac{4\nu R\left(\frac{r}{R} - 1\right)}{V_\infty \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} \\ &= \frac{4\nu R(2R)\left(\frac{r}{R} - 1\right)}{V_\infty (2R) \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} \\ &= \frac{\nu}{V_\infty (2R)} \frac{8R^2\left(\frac{r}{R} - 1\right)}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} \\ &= \frac{1}{Re} \frac{R^2\left(\frac{r}{R} - 1\right)}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} . \end{aligned} \quad (2.2)$$

This is used to define the strength of the surface vortices as

$$\begin{aligned} \Gamma &= \Gamma_0 \exp\left(-\frac{R^2\left(\frac{r}{R} - 1\right)^2}{r_c^2}\right) \\ &= \Gamma_0 \exp\left(-\frac{Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}} \left(\frac{r}{R} - 1\right)}{8}\right) . \end{aligned} \quad (2.3)$$

An analytic expression for the initial circulation is obtained from Eqn. 1.2 as

$$V_\theta = -V_\infty \sin \theta \left(1 + \frac{R^2}{r^2} \right) = \frac{\Gamma_0}{2\pi r},$$

so that the initial circulation of the surface vortices is

$$\Gamma_0 = -2\pi V_\infty \sin \theta r \left(1 + \frac{R^2}{r^2} \right), \quad \pi \geq \theta \geq -\pi. \quad (2.4)$$

This is in consistence with the initialization Fefferman (Feffermann 2006) discusses (Amoloye 2020). The full expression for the strength of the vortices is

$$\Gamma = -2\pi V_\infty \sin \theta r \left(1 + \frac{R^2}{r^2} \right) \exp \left(-\frac{Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}} \left(\frac{r}{R} - 1\right)}{8} \right). \quad (2.5)$$

Then, the surface vortex stream function is expressed as (Amoloye 2020)

$$\psi_{vt} = \frac{\Gamma}{2\pi} \ln \left(\frac{r}{R} \right) = -V_\infty r \sin \theta \left(1 + \frac{R^2}{r^2} \right) \exp \left(-\frac{Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}} \left(\frac{r}{R} - 1\right)}{8} \right) \ln \left(\frac{r}{R} \right). \quad (2.6)$$

The contour of this expression normalized by $V_\infty R$ is illustrated in figure 3 for $Re = 26$. The overall flow is from left to right. Figure 3a reveals two vortices located on each half of the cylinder. The vortices' radial and circumferential extents are bounded. The close-up views of the vortices overlaid with the quiver of the velocity vector in figures 3b and 3c reveal that they are counter-rotating. The quiver arrowheads are longest towards the cylinder surface because Γ_0 is maximum at the surface. The exponential term dissipates Γ_0 and shortens the arrowheads away from the surface. The natural logarithmic term produces the curvilinear path of the vortices' streamlines (Amoloye 2020).

For the surface vortices/sources/sinks to be concentric, the radial velocity field, V_r , of the vortex sheet should be equal to that from the source/sink sheet. To set the source/sink sheet strength, Λ , $V_r = \Lambda/2\pi r$ (Anderson 2011; Amoloye 2020, 2018) is employed. V_r , is obtained from Eqn. 2.6 (Amoloye 2020, 2018). It is

$$V_r = -V_\infty \left(1 + \frac{R^2}{r^2} \right) \left(\cos \theta - \frac{Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} \right) \\ \times \exp \left(-\frac{Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}} \left(\frac{r}{R} - 1\right)}{8} \right) \ln \left(\frac{r}{R} \right).$$

Thus, Λ is

$$\Lambda = 2\pi r V_r \\ = -2\pi V_\infty r \left(1 + \frac{R^2}{r^2} \right) \\ \times \left(\cos \theta - \frac{Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} \right) \exp \left(-\frac{Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}} \left(\frac{r}{R} - 1\right)}{8} \right) \ln \left(\frac{r}{R} \right). \quad (2.7)$$

The strength varies with the cosine of the angle, and the remnant of the initial ψ_{inv}

Figure 3: Contour of $\psi_{vt}/V_\infty R$ at $Re = 26$ (Amoloye 2020).

has been carried over in the $(1 + R^2/r^2)$ term. The source/sink sheet strength has a contribution from the uniform flow and another from the doublet. However, these two components should have alternate directions. Examination of the experimentally observed flow dictates the direction of the sinks/sources. At the front stagnation point, where $\cos(\theta)$ is negative, the reflection of sound waves into the upstream flow indicates that these waves should be treated as if coming from a local source on the base doublet at that point (Amoloye 2018, 2020). Thus, it is more appropriate to employ the equation

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$$\Lambda = 2\pi V_\infty r \left(1 - \frac{R^2}{r^2}\right) \times \left(\cos \theta - \frac{Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} \right) \exp \left(-\frac{Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}} \left(\frac{r}{R} - 1\right)}{8} \right) \ln \left(\frac{r}{R} \right), \quad (2.8)$$

in which $(1 + R^2/r^2)$ has been replaced by $(1 - R^2/r^2)$, and the leading negative sign, dropped (Amoloye 2018, 2020). Hence, the Re -dependent stream function of the source/sink sheet is

$$\begin{aligned} \psi_{ss} &= \frac{\Lambda}{2\pi} \theta \\ &= V_\infty r \left(1 - \frac{R^2}{r^2}\right) \times \left(\cos \theta - \frac{Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} \right) \exp \left(-\frac{Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}} \left(\frac{r}{R} - 1\right)}{8} \right) \theta \ln \left(\frac{r}{R} \right). \end{aligned} \quad (2.9)$$

The normalized sink/source sheet stream function is illustrated in figure 4. Figures 4a and

Figure 5: Contour plots showing the superposition of $\psi_{vis}/V_{\infty}R$ at $Re = 26$ (Amoloye 2020).

4b employ Eqns. 2.7 and 2.8 respectively for the strength of the sheet. Figure 4c presents a close-up view of figure 4b overlaid with a quiver of the velocity field. The overall flow is still from left to right. Two dominant counter-rotating vortices are observable at the front of the cylinder (Amoloye 2018, 2020). Then, using superposition as depicted in figure 5, a viscous stream function is

$$\begin{aligned}
 \psi_{vis} &= \psi_{inv} + \psi_{vt} + \psi_{ss} \\
 &= V_{\infty}r \sin \theta \left(1 - \frac{R^2}{r^2}\right) \\
 &\quad - V_{\infty}r \sin \theta \left(1 + \frac{R^2}{r^2}\right) \exp \left(- \frac{Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}} \left(\frac{r}{R} - 1\right)}{8} \right) \ln \left(\frac{r}{R}\right) \\
 &\quad + V_{\infty}r \left(1 - \frac{R^2}{r^2}\right) \left(\cos \theta - \frac{Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}}} \right) \exp \left(- \frac{Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2}} \left(\frac{r}{R} - 1\right)}{8} \right) \\
 &\quad \times (\theta + \sigma) \ln \left(\frac{r}{R}\right)
 \end{aligned} \tag{2.10}$$

for the cylinder flow. Figure 5 illustrates the superposition of the normalized viscous stream function at $Re = 26$. The flow is from left to right in each of the constituent figures (Amoloye 2020). There is a bias with θ in the third term of Eqn. 2.10 if θ varies from 0 to 2π . This bias is illustrated in figure 5d. Therefore, in the foregoing, θ varies from 0 at the rear stagnation point to $+\pi$ at the front stagnation point on the upper half of the cylinder and from 0 at the rear stagnation point to $-\pi$ at the front stagnation point for the lower half (Amoloye 2018, 2020). Thus, the bias with θ is avoided in figures 5a to 5c. A constant, σ , appropriately places the stagnation point at the front of the cylinder and

Figure 6: Contour of $\psi_{vis}/V_\infty R$ at $Re = 26$ showing the effect of σ (Amoloye 2020).

the region of recirculating eddies at the rear as figure 6 illustrates. σ is $\pm\pi$ depending on the semicircle of the cylinder (Amoloye 2020).

All the variables and constants in Eqn. 2.10 are directly from CPT and Eqn. 2.1. Equation 2.10 is viscous but not time-dependent. It identically satisfies Laplace equation. It also satisfies the no-slip condition. However, the region of the recirculating eddies and the regions of vorticity around the cylinder predicted with Eqn. 2.10 diminishes with increasing Re as in figure 7. Particularly at $Re = 1$, the recirculating eddies are overblown and non-physical. However, the streamlines of an experimentally observed flow at $Re \approx 1$ are symmetrical about the cylinder (Amoloye 2018, 2020). They are un-separated from the cylinder. The current literature further records that for $0 < Re < 4$, the creeping flow, the flow is almost steadily symmetrical about the longitudinal and lateral axes of the cylinder (Anderson 2011; Patel 1978; Van Dyke 1988; Zdravkovich 1997; Amoloye 2018, 2020). A wake-like structure first becomes noticeable for $Re \mathcal{O}(1)$ (Patel 1978; Van Dyke 1988; Zdravkovich 1997; Amoloye 2018, 2020), and a stable pair of recirculating eddies forms in the wake after the flow separates at the cylinder rear for $Re > 6$ (Patel 1978; Van Dyke 1988; Zdravkovich 1997; Amoloye 2018, 2020). Generally, this recirculation region grows in length and width as a monotonic function of Re (Förnberg 1985; Sen *et al.* 2009; White 2006; Amoloye 2018, 2020). However, it is reported to diminish with increasing blockage ratio (Sen *et al.* 2009; Amoloye 2020). Additionally, some reported numerical results display a trend towards reduction of the streamwise extent of the bubble for Re between 10 and 100 (Sen *et al.* 2009; Amoloye 2020). For $Re > 40$, the recirculation region destabilizes, the flow becomes unsteady and asymmetric, and the vortices are periodically and alternately shed from the top and bottom sides of the cylinder to form the von Kármán vortex street (Anderson 2011; White 2006; Patel 1978;

Figure 7: Contour of Eqn. 2.10 normalized with $V_\infty R$ for different Reynolds numbers $c = 0$ (Amoloye 2020).

Zdravkovich 1997; Amoloye 2018, 2020). These observations are contrary to figure 7. Equation 2.10 also features a singularity when $r = R$ and $\theta = 0$ in the square root in the denominator of the Re -dependent term. These all suggest careful modifications of Eqn. 2.10 are necessary (Amoloye 2018, 2020). These modifications are incorporated by introducing some constants, q and c , as (Amoloye 2020)

$$\begin{aligned}
 \psi_{vis} = & V_\infty r \sin \theta \left(1 - \frac{R^2}{r^2} \right) \\
 & - V_\infty r \sin \theta \left(1 + \frac{R^2}{r^2} \right) \exp \left(\frac{-qRe \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r}{R} - 1 \right)}}{8} \right) \ln \left(\frac{r}{R} \right) \\
 & + V_\infty r \left(1 - \frac{R^2}{r^2} \right) \left(\cos \theta - \frac{qRe}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) \\
 & \times \exp \left(- \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r}{R} - 1 \right)}}{8} \right) (\theta + \sigma) \ln \left(\frac{r}{R} \right).
 \end{aligned} \tag{2.11}$$

c can be any small number and is likely not Re - or time-dependent because the experimentally observed flow does not have singularities at all. To preserve the features already captured with Eqn. 2.10 illustrated in figure 8, c should be between zero and the maximum value of the square rooted expression at the surface that is four (4). Therefore, the number two (2) is arbitrarily chosen (Amoloye 2020).

However, q is likely Re -dependent because the experimentally observed flow ranges from a symmetrical un-separated flow at $Re = 1$ to a separated flow with a distinct wake at $Re \gg \gg 1$ (Anderson 2011; White 2006; Amoloye 2020). It is observable that Eqn. 2.10 features an overblown re-circulation region at $Re = 1$ that gradually closes up

Figure 8: Contour of Eqn. 2.11 normalized with $V_\infty R$ for different values of parameter c at $Re = 26$ (Amoloye 2020).

into a symmetrically un-separated flow at $Re \geq 130$. Thus, to reverse this non-physical trend, q should have the form

$$q \approx \frac{130}{Re^2} \quad (2.12)$$

such that at $Re \leq 1$, an un-separated and symmetric flow is predicted as in figure 9 (Amoloye 2020). As Re progresses from figure 9a to 9c, the re-circulation region also grows progressively in consistence with experimentally observed flows. When $Re \gg 1$ in figure 9d, the region does not form an unsteady wake but becomes non-physically overblown. However, the width of the separated region in an experimentally observed flow ceases to grow at $Re \approx 50$ when the wake sheds into a vortex street (Anderson 2011; White 2006; Patel 1978; Amoloye 2020). Therefore, q is modified with some constants, b and l , into

$$q \approx \frac{130}{Re^2} \left(\frac{bRe}{130} \right)^{\exp\left(-\frac{l}{Re}\right)} \quad (2.13)$$

in which $\exp(-l/Re)$ varies the parenthetical term between one (1) and $bRe/130$ as itself varies between zero for $Re = 1$ and one (1) for $Re \gg 1$ respectively (Amoloye 2020). l and b are likely independent of flow conditions. For $l \geq 10$, this behavior is obtainable. For $b \approx 4$, the predicted width of the re-circulation region at $Re \approx 50$ is observed to be qualitatively consistent with experimentally observed flows. Thus, after trials and errors,

Figure 9: Contour of Eqn. 2.11 normalized with $V_\infty R$ for varying Re (Amoloye 2020).

l and b were chosen to be 10 and 3.9792 respectively. So, q becomes (Amoloye 2020)

$$q = \frac{130}{Re^2} \left(\frac{3.9792 Re}{130} \right)^{\exp\left(-\frac{10}{Re}\right)} \quad (2.14)$$

$$= \frac{130}{Re^2} (0.03061 Re)^{\exp\left(-\frac{10}{Re}\right)}.$$

Figure 10 presents a consistence of Eqn.2.11 with experimentally observed streamlines at $Re = 26$. Table 2 also presents a consistence in the theoretical prediction of the separation points, θ_s (measured from the rear stagnation point), with experimental and numerical trends for varying Re (Amoloye 2020). However, the core of the vortex is smeared and much closer to the centerline in the theoretical result than it is in the experimentally observed flow. Figure 11 also shows a disparity between the surface vorticity distribution predicted from Eqn. 2.11 and some computational result by Dennis and Shimshoni (Dennis & Shimshoni 1965) for a cylinder at $Re = 20$. The theoretical result from Eqn. 2.11 does not approach zero at the cylinder rear where $\theta = 0$ as the numerical result predicts. This is non-physical. Therefore, ψ_{ss} which produces the recirculating eddies in Eqn. 2.11 is modified into (Amoloye 2020)

$$\psi_{ss} = V_\infty r \left(1 - \frac{R^2}{r^2} \right) \left(\cos \theta - \frac{q Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) \times \exp \left(- \frac{q Re \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r}{R} - 1 \right)}{8} \right) (\theta + \sigma) \underbrace{(\chi \sin \theta)^{\exp\left(-\frac{V_\infty t}{r} \left(1 - \frac{R}{r} \right) \right)}}_{new} \ln \left(\frac{r}{R} \right), \quad (2.15)$$

Figure 10: Comparison between an experimentally observed flow (upper image - Source: Photograph by Sadatoshi Taneda (Anderson 2011; Van Dyke 1988)) and contour of Eqn. 2.11 (lower image) at $Re = 26$ (Amoloye 2020).

Table 2: Comparison of separation point predictions between Eqn.2.17, CFD results (N-K,K-T,D-C and D-D) and experimental results (Thom, Homann and Grove et al). N-K: Nieuwstadt and Keller, K-T: Keller and Takami, D-C: Dennis and Chang, D-D: D'Alessio and Dennis (Amoloye 2020)

θ_s (deg)								
Re	Eqn.2.11	N-K [†]	K-T [‡]	D-C [‡]	D-D [‡]	Patel [¶]	Thom	Homann
10	30.41	27.96	29.22	29.79	—	—	≈ 35	—
20	41.76	43.37	43.54	43.54	43.1	43.78	≈ 45	—
30	49.24	49.39	49.85	—	—	—	—	≈ 50
40	54.70	53.34	53.29	53.86	53.0	53.6	≈ 51	—

[†] Source: (Nieuwstadt & Keller 1973)

[‡] Source: (D'Alessio & Dennis 1994)

[¶] Source: (Patel 1987)

^{||} Source: (Wu *et al.* 2004)

where

$$\sigma = \begin{cases} -\pi, & \theta \geq 0 \\ \pi, & \theta < 0 \end{cases}$$

and

$$\chi = \begin{cases} 1, & \theta \geq 0 \\ -1, & \theta < 0. \end{cases}$$

In this equation, a product of the new term $(\chi \sin \theta)^{\exp(-\frac{V_{\infty} t}{r}(1-\frac{R}{r}))}$ with Eqn. 2.9

Figure 11: Present $\omega_z R / (V_\infty \sqrt{Re})$ distribution from Eqn. 2.11 compared with computations of Dennis and Shimshoni (Dennis & Shimshoni 1965) at $Re = 20$ (Amoloye 2020).

enforces the vanishing of the wall vorticity with the sine function when $\theta = 0$. The planar vorticity, ω_z , distribution obtained from a superposed stream function using Eqn. 2.15 at a non-dimensional time, $V_\infty t / R \approx 0$ achieves this in figure 11. Figures 12 to 14 illustrate the effects of the constituent functions in the present modification. In figure 12a, the sinusoidal term further affects the displacement of the wake vortex core from the centreline. However, it disrupts the flow symmetry. χ restores the flow symmetry as figures 12b, 12c and 13 display. Figure 13 illustrates the velocity field showing the no-slip condition. In figure 14a, the sinusoidal term also shortens and rounds the tail of the wake (Amoloye 2020). The flow pattern is qualitatively consistent with figure 14b that presents a transient stage in an experimentally observed cylinder flow development from an impulsive start (Prandtl & Tietjens 1934, figure 6, p. 279); (Schlichting 1979, figure 15.5d, p. 426); (Van Dyke 1988, figure 59, p. 36); (White 2006, figure 4-50, p. 319); (Amoloye 2020).

At the cylinder surface where $r = R$, the modification is

$$\begin{aligned}
 & (\chi \sin \theta)^{\exp(-\frac{V_\infty t}{R}(1-\frac{R}{R}))} \\
 &= (\chi \sin \theta)^{\exp(-0)} \\
 &= (\chi \sin \theta)^1 \\
 &= (\chi \sin \theta),
 \end{aligned}$$

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and as $r \rightarrow \infty$, it is

$$\begin{aligned} & (\chi \sin \theta)^{\exp(-\frac{V_\infty t}{r})} \\ &= (\chi \sin \theta)^{\exp(-0)} \\ &= (\chi \sin \theta)^1 \\ &= (\chi \sin \theta) . \end{aligned}$$

Thus, the effect of the sine function is maintained for a constant time (Amoloye 2020). Similarly, for a specific r when $t = 0$, the modification is

$$\begin{aligned} & (\chi \sin \theta)^{\exp(-\frac{V_\infty \times 0}{r}(1-\frac{R}{r}))} \\ &= (\chi \sin \theta)^{\exp(-0)} \\ &= (\chi \sin \theta)^1 \\ &= (\chi \sin \theta) , \end{aligned}$$

and the effect of the sine function is maintained (Amoloye 2020). However, as $t \rightarrow \infty$, it is

$$\begin{aligned} & (\chi \sin \theta)^{\exp(-\frac{V_\infty \times \infty}{r}(1-\frac{R}{r}))} \\ &= (\chi \sin \theta)^{\exp(-\infty)} \\ &= (\chi \sin \theta)^0 \\ &= 1 , \end{aligned}$$

and the effect of the sine function vanishes (Amoloye 2020). Then, the tail of the wake elongates as illustrated in figure 14c. Hence, $(\chi \sin \theta)^{\exp(-\frac{V_\infty t}{r}(1-\frac{R}{r}))}$ incorporates a temporal and spatial variation of the wake dimension depending on the freestream velocity (Amoloye 2020).

The impulsive start of an experimentally observed cylinder flow begins with a nearly inviscid flow that is similar in pattern to the classical potential flow (White 2006; Prandtl & Tietjens 1934; Schlichting 1979; Eldredge 2019; Amoloye 2020). Then, the boundary layer starts to develop. At $V_\infty t/R \approx 0.35$, separation begins from the rear stagnation point (White 2006; Amoloye 2020). The resulting region of reverse flow at the rear travels upstream and thickens the boundary layer. This process gives rise to the wake vortices that continue to grow (Schlichting 1979; White 2006; Amoloye 2020). By $V_\infty t/R \approx 1.5$, a large pair of counter-rotating vortices will have emerged (White 2006; Amoloye 2020). The present theoretical flow pattern in figure 14a is not consistent with these experimental observations illustrated in figure 14b, but the one in figure 14c is consistent with the observations (Amoloye 2020). Therefore, the basic functional form of B_0 (Eqn. 1.7) is modified as

$$\begin{aligned} B_0 &= \frac{\Gamma_0}{2\pi} \exp \left(\underbrace{-\frac{r^2}{4(V_\infty^2 t) t}}_{new} \right) \exp \left(-\frac{r^2}{4\nu t} \right) \\ &= \frac{\Gamma_0}{2\pi} \exp \left[-\left(\frac{r^2}{4V_\infty^2 t^2} + \frac{(r-R)^2}{r_c^2} \right) \right] \\ &= \frac{\Gamma_0}{2\pi} \exp \left[-\left(\frac{r^2}{4V_\infty^2 t^2} + \frac{qRe \sqrt{\left(1-\frac{R^2}{r^2}\right)^2 + 4\frac{R^2 \sin^2 \theta}{r^2} + c\left(\frac{r}{R}-1\right)}}{8} \right) \right] . \end{aligned} \quad (2.16)$$

The new exponential term in Eqn. 2.16 is modeled on the previous exponential term in the same equation. The kinematic viscosity, ν that has a dimension of $[\frac{Length \times Length}{Time}]$ is replaced with $V_\infty^2 t$ that has the same dimension (Amoloye 2020). This is because the

Figure 12: Contour of the superposed stream function with Eqn. 2.15 showing the effect of the $(\theta + \sigma) \chi \sin \theta$ term (figures (a) and (b)) and contour of $\omega_z R / (V_\infty \sqrt{Re})$ from the same superposition (figure (c)) at $Re = 26$ and $V_\infty t / R \approx 0$ (Amoloye 2020).

Figure 13: Contour of the velocity field of Eqn. 2.15 at $Re = 26$ and $V_\infty t / R \approx 0$ showing the no-slip condition (Amoloye 2020).

(a) $Re = 26$ ($V_\infty t/R \approx 0$ for present theory (lower image) but unknown for the experimentally observed flow (upper image) Photograph by Madeleine Coutanceau and Roger Bouard (Van Dyke 1988)).
 (b) An experimentally observed flow at $Re = 500$ and $V_\infty t/R = 6$ (Source: Photograph by Sadatoshi Taneda (Anderson 2011; Van Dyke 1988)))

(c) $Re = 26$ ($V_\infty t/R \approx 1.3$ for present theory (lower image) but unknown for the experimentally observed flow (upper image) - Source: Photograph by Sadatoshi Taneda (Anderson 2011; Van Dyke 1988)))

Figure 14: The effect of the $(\theta + \sigma) \left(\chi \sin \theta \right)^e \frac{-V_\infty t}{r} \left(1 - \frac{R}{r} \right)$ term in the superposed stream function with Eqn. 2.15 (Amoloye 2020).

impulsive start of the flow is a relatively rapid process that is observed to be dependent on the latter (from $V_\infty t/R$) (White 2006, p. 319);(Amoloye 2020). For a specific r , when $t \approx 0$, B_0 is

$$\begin{aligned}
 B_0 &= \frac{\Gamma_0}{2\pi} \exp \left[- \left(\frac{r^2}{4V_\infty^2 \times 0} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r}{R} - 1 \right)}}{8} \right) \right] \\
 &= \frac{\Gamma_0}{2\pi} \exp(-\infty) \\
 &= 0,
 \end{aligned}$$

and the stream function is

$$\begin{aligned}\psi_{vis} &= V_{\infty} r \sin \theta \left(1 - \frac{R^2}{r^2}\right) + B_0 \ln \left(\frac{r}{R}\right) \\ &= V_{\infty} r \sin \theta \left(1 - \frac{R^2}{r^2}\right) .\end{aligned}$$

As $t \rightarrow \infty$, B_0 tends to

$$\begin{aligned}B_0 &\rightarrow \frac{\Gamma_0}{2\pi} \exp \left[- \left(\frac{r^2}{4V_{\infty}^2 \times \infty} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r}{R} - 1\right)}}{8} \right) \right] \\ &= \frac{\Gamma_0}{2\pi} \exp \left[- \left(0 + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r}{R} - 1\right)}}{8} \right) \right] ,\end{aligned}$$

and Eqn. 2.11 is recovered (Amoloye 2020). However, a singularity occurs at the spatio-temporal origin where $r = t = 0$, and B_0 tends to

$$B_0 \rightarrow \frac{\Gamma_0}{2\pi} \exp \left(-\frac{0}{0} \right) .$$

Then, the superposed stream function is undefined. $r^2/(4V_{\infty}^2 t^2)$ is not replaced with $(r - R)^2/(4V_{\infty}^2 t^2)$ in the second row of Eqn. 2.16 so that the singularity does not occur at the cylinder surface (Amoloye 2020). Therefore, the modified viscous stream function is

$$\begin{aligned}\psi_{vis} &= V_{\infty} r \sin \theta \left(1 - \frac{R^2}{r^2}\right) \\ &- V_{\infty} r \sin \theta \left(1 + \frac{R^2}{r^2}\right) \exp \left[- \left(\frac{r^2}{4V_{\infty}^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r}{R} - 1\right)}}{8} \right) \right] \ln \left(\frac{r}{R}\right) \\ &+ V_{\infty} r \left(1 - \frac{R^2}{r^2}\right) (\theta + \sigma) (\chi \sin \theta)^{\exp \left[-\frac{V_{\infty} t}{r} \left(1 - \frac{R}{r}\right) \right]} \left(\cos \theta - \frac{qRe}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) \\ &\times \exp \left[- \left(\frac{r^2}{4V_{\infty}^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r}{R} - 1\right)}}{8} \right) \right] \ln \left(\frac{r}{R}\right)\end{aligned}\tag{2.17}$$

where

$$\sigma = \begin{cases} -\pi, & \theta \geq 0 \\ \pi, & \theta < 0 \end{cases}$$

and

$$\chi = \begin{cases} 1, & \theta \geq 0 \\ -1, & \theta < 0 . \end{cases}$$

This is illustrated in figure 15 for different values of $V_{\infty} t/R$ and compared to an experimentally observed flow at $Re = 26$. The experimentally observed flow in these figures are the same, and the time is not reported from the source (Van Dyke 1988, p. 28). In figure 15a, the theoretical flow is starting up with an unseparated symmetrical flow pattern. At $V_{\infty} t/R \approx 0.86$ in figure 15b, the flow has separated. The vortices form and continue to grow. The present theoretical flow patterns in figures 15d and 15e are the most consistent with the photograph of the experimentally observed flow. At a much later time in figure 15f, a steady state has been attained. Thus, this figure is similar to figure 10 that is obtained with a time-independent Eqn. 2.11 (Amoloye 2020).

(a) $V_\infty t/R \approx 0$ (b) $V_\infty t/R \approx 0.86$ (c) $V_\infty t/R \approx 1.29$ (d) $V_\infty t/R \approx 2.15$ (e) $V_\infty t/R \approx 4.30$ (f) $V_\infty t/R \approx 43$

Figure 15: Contour of Eqn. 2.17 normalized with $V_\infty R$ (lower images (Amoloye 2020)) for different values of $V_\infty t/R$ at $Re = 26$ (upper images - Source: Photograph by Sadatoshi Taneda (Van Dyke 1988)).

The Incompressible Kwasu Function

(a) Orbits of various eccentricities having a common focus, F, and a periapsis, P (Curtis 2014).

(b) The streamlines of an experimentally observed flow at $Re = 26$ (Source: Photograph by Sadatoshi Taneda (Van Dyke 1988)).

Figure 16: The reasoning behind the gravity analogy (Amoloye 2020).

2.2. Perifocal Stream Function

A gravity analogy is used to describe the affinity between a fluid element and the cylinder wall in RPT. This is to predict the points of separation, transition, and reattachment, and the thickness of the boundary layer. The analogy introduces the perifocal frame of fluid motion and stream function (Amoloye 2018, 2020).

Figure 16a presents different orbits with corresponding eccentricity, ε (e in figure 16a). A body in motion can occupy any of these orbits under mutual gravitation with a bigger dominant body at the focus, F (Amoloye 2020; Curtis 2014). The eccentricity is a measure of the circularity of an orbit (Curtis 2014). The orbits are solutions to the two-body problem in astrodynamics. Depending on the sum of its kinetic energy and potential energy with respect to the focus, an orbiting body launched from the periapsis, P, may occupy a circular or an elliptic orbit ($0 < \varepsilon < 1$) and remain bound under the mutual gravitational interaction with the dominant body at F. It may also occupy either a parabolic ($\varepsilon = 1$) or a hyperbolic ($\varepsilon > 1$) orbit and coast to infinity away from the dominant body. These orbits are pathlines of the orbiting body (Amoloye 2020; Curtis 2014). Similarly, figure 16b presents the fluid element pathline in an experimentally observed flow at $Re = 26$. A fluid element arriving at the stagnation point remains bound to the influence of the cylinder as it flows around it ($0 < \varepsilon < 1$) before it eventually

separates and advects to infinity ($\varepsilon \geq 1$). This behavior is assumed to be dependent on the sum of the kinetic energy of the fluid element and its potential energy with respect to the cylinder (Amoloye 2020).

The gravitational field, Φ , of a body (the cylinder in the present case) is

$$\begin{aligned} \Phi(r, \theta, \varphi) &= \frac{\mu_g}{r} \left[1 + \sum_{n=1}^{\infty} \left(\frac{R_B}{r} \right)^n \sum_{m=1}^n \frac{1}{2^n n!} \frac{d^{n+m}}{d(\sin \varphi)^{n+m}} (\sin^2 \varphi - 1)^n \right. \\ &\quad \left. (C_{n,m} (\cos(m\theta) \cos(m\varphi)) + S_{n,m} (\sin(m\theta) \cos(m\varphi))) \right], \end{aligned} \quad (2.18)$$

where φ is the azimuthal spherical coordinate variable, the gravitational parameter, $\mu_g \approx GM_B$, G is the gravitational constant, and M_B is the body mass (Fitzpatrick 2012; Amoloye 2020). μ_g is referred to as the wake-analogy factor in RPT (Amoloye 2020). Equation 2.18 is a separated variables solution of Laplace equation in the spherical polar coordinate system (Fitzpatrick 2012; Pines 1973; Amoloye 2020). $C_{n,m}$ and $S_{n,m}$ are unique mass coefficients that are determined empirically (Pines 1973; Amoloye 2020). However, for a spherically symmetric mass distribution, the gravitational potential is simply

$$\Phi(r) = -\frac{\mu_g}{r}. \quad (2.19)$$

This is employed as a suitable potential for the cylinder center plane (Amoloye 2020). The recommended value of the gravitational constant, G , is $6.67408(31) \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$ (Mohr *et al.* 2016; Amoloye 2020). This means that the wake-analogy factor is a very small number for all practical purposes of the cylinder flow. Thus, an arbitrary value of the cylinder mass, M_B , and its actual distribution will not affect the theoretical flow (Amoloye 2020). This is consistent with dimensional analyses that are in agreement with experimental and numerical observations (Anderson 2011; White 2006; Amoloye 2020). Therefore, using Eqn. 2.19 for the analogy is justified (Amoloye 2020). Since the differentials of the gravitational potential give the components of gravitational acceleration, the gravitational velocity potential is derived by integrating the gravitational potential in the perifocal time, t_ϱ . The perifocal frame is a planar frame that is centered at the focus, F, of an orbit as illustrated in figure 17 (refer to Curtis (Curtis 2014, p. 108) for a discussion on perifocal frames). The orbit is depicted with the thick curvilinear red arrowhead in this figure (Amoloye 2020).

For a steady and uniform mass distribution in the cylinder, the gravitational velocity potential, ϕ_p , is thus obtained as

$$\phi_p(r, t_\varrho) = \int_0^{\mathcal{T}} \Phi(r) dt_\varrho = -\frac{\mu_g \mathcal{T}}{r} \quad (2.20)$$

where $\mathcal{T}$ is the orbital period. The path of two bodies moving solely under the influence of mutual gravity is a Keplerian orbit (conic section) defined by

$$\varrho = \frac{h^2}{\mu_g} \frac{1}{1 + \varepsilon \cos \Theta} \quad (2.21)$$

where ϱ is the radial coordinate or orbital path, h is the specific angular momentum of the orbiting body, ε is the eccentricity of the orbit, and Θ is the circumferential coordinate or true anomaly of the orbit (Amoloye 2020; Curtis 2014). The origin in Eqn. 2.21 coincides with the center of mass of the two-body system that is located at one of the foci of the orbit (Curtis 2014; Fitzpatrick 2012). Equation 2.21 is also a solution of Newton's second law of motion. It has an inherent assumption that there are no influences from other

Figure 17: The perifocal frame of fluid motion overlaid on an experimentally observed flow at $Re = 26$ (Source of adapted image: Photograph by Sadatoshi Taneda (Van Dyke 1988)).

bodies on the two-body system (Curtis 2014; Fitzpatrick 2012; Amoloye 2020). Therefore, the foregoing analogy is most consistent with the conservation of the momentum of a perfect gas flowing about a cylinder (Amoloye 2020). A perfect gas is one for which negligible or no mutual interactions are assumed between the molecules (Anderson 2006; Amoloye 2020). For a fluid particle and the cylinder, the origin of the two-body system is essentially the cylinder's center of mass. This is assumed to be coincident with the geometric center of the cylinder in figure 17. The stagnation point and the periapsis are also coincident (Amoloye 2020).

In the perifocal frame, the particle's velocity components are defined as (Curtis 2014; Amoloye 2020, 2018)

$$\begin{aligned} V_{\Theta} &= \frac{\mu_g}{h} (1 + \varepsilon \cos \Theta) = -\frac{\partial \Psi}{\partial \varrho} \\ V_{\varrho} &= \frac{\mu_g}{h} \varepsilon \sin \Theta = \frac{1}{\varrho} \frac{\partial \Psi}{\partial \Theta} . \end{aligned} \quad (2.22)$$

The positive directions of these velocity components are illustrated in figure 17. Equation 2.22 indicates a stream function that identically satisfies the steady continuity equation in the perifocal frame of reference as

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \frac{1}{\varrho} \frac{\partial(\varrho V_{\varrho})}{\partial \varrho} + \frac{1}{\varrho} \frac{\partial V_{\Theta}}{\partial \Theta} \\ &= \frac{V_{\varrho}}{\varrho} + \frac{\partial V_{\varrho}}{\partial \varrho} + \frac{1}{\varrho} \frac{\partial V_{\Theta}}{\partial \Theta} \\ &= \frac{1}{\varrho^2} \frac{\partial \Psi}{\partial \Theta} - \frac{1}{\varrho^2} \frac{\partial \Psi}{\partial \Theta} + \frac{1}{\varrho} \frac{\partial^2 \Psi}{\partial \varrho \partial \Theta} - \frac{1}{\varrho} \frac{\partial^2 \Psi}{\partial \Theta \partial \varrho} \\ &= 0 \quad (t \geq 0). \end{aligned} \quad (2.23)$$

This perifocal stream function, Ψ , is integrated to be

$$\Psi = -\frac{\mu_g \varrho}{h} (1 + \varepsilon \cos \Theta) . \quad (2.24)$$

Applying the Keplerian orbit eccentricity (Curtis 2014; Amoloye 2020),

$$\varepsilon = \sqrt{1 + \frac{h^2}{\mu_g^2} (V^2 - \frac{2\mu_g}{\varrho})}, \quad (2.25)$$

and the orbital circumferential coordinate variable or true anomaly (Curtis 2014; Amoloye 2020),

$$\Theta = \cos^{-1} \left[\frac{1}{\varepsilon} \left(\frac{h^2}{\mu_g \varrho} - 1 \right) \right], \quad (2.26)$$

Eqn. 2.24 simplifies to

$$\Psi = -h. \quad (2.27)$$

The orbital period expression is given as (Curtis 2014)

$$\mathcal{T} = \frac{2\pi}{\mu_g^2} \left(\frac{h}{\sqrt{1 - \varepsilon^2}} \right)^3.$$

This simplifies to

$$\mathcal{T} = 2\pi\mu_g \left(\frac{2\mu_g}{\varrho} - V^2 \right)^{-3/2} \quad (2.28)$$

using Eqn. 2.25. The velocity in this expression is

$$V = \sqrt{\left(\frac{1}{r} \left[\frac{\partial\psi_{inv}}{\partial\theta} + \frac{\partial\psi_{vt}}{\partial\theta} + \frac{\partial\psi_{ss}}{\partial\theta} \right] \right)^2 + \left(- \left[\frac{\partial\psi_{inv}}{\partial r} + \frac{\partial\psi_{vt}}{\partial r} + \frac{\partial\psi_{ss}}{\partial r} \right] \right)^2}. \quad (2.29)$$

The origins of the body axis and the perifocal frame are coincident as illustrated in figure 17, and the radial distance around the cylinder, r , is equivalent to the orbital radial coordinate, ϱ . Thus, the specific angular momentum of the fluid particle in the body axis is

$$\mathbf{h} = \mathbf{r} \times \mathbf{V},$$

and the magnitude is

$$h = rV_\theta. \quad (2.30)$$

Differentiating Eqn. 2.20 gives the gravitational circumferential velocity component around the cylinder as

$$V_\theta = \frac{1}{r} \frac{\partial\phi_p}{\partial\theta}. \quad (2.31)$$

Hence, the following expression

$$\psi_p = \Psi = -h = -\frac{\partial\phi_p}{\partial\theta} \quad (2.32)$$

is derived for the perifocal stream function (Amoloye 2018, 2020).

2.3. Viscous Two-Dimensional Stream Function

The summary of the forgoing superposition is the viscous and time-dependent stream function of the uniform flow on a circular cylinder that is

$$\begin{aligned}
 \psi_{vis} &= \psi_{inv} + \psi_{vt} + \psi_{ss} + \psi_p \\
 &= V_\infty r \sin \theta \left(1 - \frac{R^2}{r^2} \right) \\
 &\quad - V_\infty r \sin \theta \left(1 + \frac{R^2}{r^2} \right) \exp \left[- \left(\frac{r^2}{4V_\infty^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r}{R} - 1 \right)}{8} \right) \right] \ln \left(\frac{r}{R} \right) \\
 &\quad + V_\infty r \left(1 - \frac{R^2}{r^2} \right) (\theta + \sigma) (\chi \sin \theta) \exp \left[- \frac{V_\infty t}{r} \left(1 - \frac{R}{r} \right) \right] \left(\cos \theta - \frac{qRe}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) \\
 &\quad \times \exp \left[- \left(\frac{r^2}{4V_\infty^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r}{R} - 1 \right)}{8} \right) \right] \ln \left(\frac{r}{R} \right) - \frac{\partial \phi_p}{\partial \theta},
 \end{aligned} \tag{2.33}$$

where

$$\sigma = \begin{cases} -\pi, & \theta \geq 0 \\ \pi, & \theta < 0 \end{cases}$$

and

$$\chi = \begin{cases} 1, & \theta \geq 0 \\ -1, & \theta < 0. \end{cases}$$

This function resolves the boundary layer and produces a stationary pair of counter-rotating vortices that is dependent on time and/or Re .

3. Viscous Incompressible Two-Dimensional Stream Function With Vortex Shedding

As Re increases to about 50, the experimentally observed flow around the circular cylinder undergoes a Hopf bifurcation, and the stationary counter-rotating vortices begin to alternately shed into the von Kármán vortex street (Anderson 2011; Van Dyke 1988; Wu *et al.* 2006; White 2006; Zdravkovich 1997; Amoloye 2018, 2020). This is shown in figure 18 that presents an experimentally observed flow of water past a cylinder at $Re = 140$. The integrated streaklines of the flow were revealed by electrolytic precipitation of a white colloidal smoke that was illuminated by a sheet of light (Van Dyke 1988; Amoloye 2020). The growth of the street width downstream of the cylinder is highlighted with the red arrowheads. Streaklines are traces of all fluid particles that have flowed continuously through a specific spatial location. They are only coincident with streamlines in a steady flow or a steady region of a flow as illustrated in figure 19a (Anderson 2011; White 2006; Taneda 1985; Wu *et al.* 2006; Amoloye 2020). The photograph was taken with a simultaneous usage of the aluminum flake and electrolytic precipitation methods that show streamlines and streaklines respectively (Taneda 1985; Amoloye 2020). The brighter streakline patterns are observed to match the darker background streamlines in figure 19a. This is not the case in the unsteady vortex shedding phenomenon whose streamlines and streaklines in figure 19b were revealed with the same methods in figure 19a (Taneda 1985; Wu *et al.* 2006; Amoloye 2020). It can be seen that some of the vortex features illuminated by the streamlines are not found on the streaklines (Taneda 1985; Amoloye 2020).

Figure 18: A Kármán vortex street behind a circular cylinder at $Re = 140$ (Source of adapted image: Photograph by Sadatoshi Taneda (Van Dyke 1988)).

However, both the streamline and the streakline patterns in figure 19c are correlated and provide a good visualization of the alternately shed vortices and their staggered location in the street (Amoloye 2020). The streamlines in figure 19c were also revealed with aluminum flakes, but the streaklines were revealed with condensed milk (Taneda 1985; Amoloye 2020).

Theoretical simulations of the shedding process are difficult when the street is from a bluff body. Föppl attempted to theorize the stable separated flow before the bifurcation by placing two vortices behind the cylinder (Zdravkovich 1997; Wu *et al.* 2006; Amoloye 2020) as illustrated in figure 20a. However, the ensuing surface pressure distribution in figure 20b is nonphysical (Zdravkovich 1997; Amoloye 2020). Talaei's and Garrett's (Talaei & Garrett 2020) analytical solution of the asymmetric flow around a fast moving sphere exhibits some eddies in the wake region, but these eddies do not alternately shed (Amoloye 2020). Others have also treated the vortex street as staggered double row point vortices or vortex patch arrays (Wu *et al.* 2006; Amoloye 2020). In the present study, Eqn. 2.33 is modified to model the vortex street. The exponential term of ψ_{ss} (Eqn. 2.15) which produces the stationary bubble in the flow is exploited to introduce spatial and temporal periodicity in the wake (Amoloye 2020). This is achieved by introducing constants ζ and ι in ψ_{ss} as

$$\begin{aligned} \psi_{vtsd} = & V_{\infty} r \left(1 - \frac{R^2}{r^2} \right) \\ & \times \left(\cos \theta - \frac{\overset{\text{new}}{\zeta} Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) (\theta + \sigma) (\chi \sin \theta)^{\exp \left[-\frac{V_{\infty} t}{r} \left(1 - \frac{R}{r} \right) \right]} \\ & \times \exp \left[- \left(\underset{\text{new}}{\iota} + \frac{r^2}{4V_{\infty}^2 t^2} \right) - \left(\frac{\overset{\text{new}}{\zeta} Re \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r}{R} - 1 \right)}{8} \right) \right] \ln \left(\frac{r}{R} \right). \end{aligned} \quad (3.1)$$

ι triggers the vortex shedding at $Re \sim 50$. It is likely only dependent on Re . Its function as a bifurcator in the exponential term suggests that it has an inverse relationship with Re as

$$\iota \propto \frac{1}{Re}.$$

(a) A stable pair of counter-rotating vortices behind a circular cylinder at $Re = 22$ (Taneda 1985).

(b) A Kármán vortex street behind a circular cylinder at $Re = 100$ (Taneda 1985).

(c) A Kármán vortex street behind a flat plate at zero inclination to the freestream (Taneda 1985).

Figure 19: The differences between streamlines and streaklines.

The constant of proportionality is chosen such that at a specific t and r in Eqn. 3.1, ι becomes a big number that dominates the exponential argument and makes ψ_{vtsd} vanish as Re approaches one (1) (Amoloye 2020). However, the influence of ι should diminish at about $Re = 50$ when the shedding begins. This mode of operation in the exponential term is also dependent on the relative value of ζ that controls the divergence of the vortex street width downstream of the cylinder. ζ is similar to q in Eqn. 2.11. When $q = 130/Re^2$ in Eqn. 2.11, it is observed that it grows the width of the stationary vortices with progressing Re . This is shown in figure 9 (Amoloye 2020). Therefore, ζ is set to be

$$\zeta = \frac{130}{Re^2} .$$

Figure 20: The Föppl stationary vortex pair behind a circular cylinder (Zdravkovich 1997).

Thus, ι is $O(130/Re)$. It was observed that setting ι as

$$\iota = \frac{125}{Re}$$

ensures that the bifurcation begins at about $Re = 50$ (Amoloye 2020).

Once the bifurcation starts, r^2 in the exponent dissipates ψ_{vtsd} as $r \rightarrow \infty$, and $4V_\infty^2 t^2$ advances its extent away from the cylinder at the freestream velocity. Figure 21 presents the contour of Eqn. 3.1 for two functional relationships between ζ and Re (Amoloye 2020).

The absolute flow domain in figure 21 is

$$x = r \cos \theta \quad y = r \sin \theta$$

which originates at the center of the cylinder. The radial coordinate of a time-dependent relative domain, $r(t)$, that advances from left to right with velocity components

$$\begin{aligned} V_x &= \frac{\partial \psi_{inv}}{\partial y} \\ &= \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \psi}{r \partial \theta} \frac{r \partial \theta}{\partial y} \\ &= -V_\theta \sin \theta + V_r \cos \theta \\ &= -\left(-V_\infty \sin \theta \left(1 + \frac{R^2}{r^2}\right)\right) \sin \theta + \left(V_\infty \cos \theta \left(1 - \frac{R^2}{r^2}\right)\right) \cos \theta \\ &= V_\infty \left(1 + \frac{R^2}{r^2} (\sin^2 \theta - \cos^2 \theta)\right), \end{aligned}$$

and

$$\begin{aligned}
 V_y &= -\frac{\partial \psi_{inv}}{\partial x} \\
 &= -\left(\frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \psi}{r \partial \theta} \frac{r \partial \theta}{\partial x}\right) \\
 &= V_\theta \cos \theta + V_r \sin \theta \\
 &= \left(-V_\infty \sin \theta \left(1 + \frac{R^2}{r^2}\right)\right) \cos \theta + \left(V_\infty \cos \theta \left(1 - \frac{R^2}{r^2}\right)\right) \sin \theta \\
 &= V_\infty \left(\frac{R^2}{r^2} \sin \theta \cos \theta\right),
 \end{aligned}$$

is constructed as

$$\begin{aligned}
 r(t) &= \sqrt{(x - V_x t)^2 + (y - V_y t)^2} \\
 &= r \sqrt{\left(\sin \theta - \frac{V_\infty t}{r} \left(\frac{R^2}{r^2} \sin \theta \cos \theta\right)\right)^2 + \left(\cos \theta - \frac{V_\infty t}{r} \left(1 + \frac{R^2}{r^2} (\sin^2 \theta - \cos^2 \theta)\right)\right)^2}.
 \end{aligned} \tag{3.2}$$

This is used to advect the vortices downstream (Amoloye 2020). So Eqn. 3.1 is modified with $r(t)$ into

$$\begin{aligned}
 \psi_{vtsd} &= V_\infty r \left(1 - \frac{R^2}{r^2}\right) \\
 &\times \left(\cos \theta - \frac{\zeta Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}}\right) (\theta + \sigma) (\chi \sin \theta)^{\exp\left[-\frac{V_\infty t}{r} \left(1 - \frac{R}{r}\right)\right]} \\
 &\times \exp \left[- \left(\iota + \frac{r^2}{4V_\infty^2 t^2} \right) - \underbrace{\left(\frac{\zeta Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{\overset{\text{new}}{\downarrow} \frac{r(t)}{R} - 1 \right)}{8} \right)}_{\text{new}} \right] \ln \left(\frac{r}{R} \right).
 \end{aligned} \tag{3.3}$$

Then, the under-braced argument of the exponential term in Eqn. 3.3 is multiplied with $\eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2}$ to give

$$\begin{aligned}
 \psi_{vtsd} &= V_\infty r \left(1 - \frac{R^2}{r^2}\right) \\
 &\times \left(\cos \theta - \frac{\zeta Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}}\right) (\theta + \sigma) (\chi \sin \theta)^{\exp\left[-\frac{V_\infty t}{r} \left(1 - \frac{R}{r}\right)\right]} \\
 &\times \exp \left[- \left(\iota + \frac{r^2}{4V_\infty^2 t^2} \right) - \underbrace{\left(\frac{\zeta Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r(t)}{R} - 1 \right)}{8} \right)}_{\text{new}} \right] \left(\eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} \right) \\
 &\times \ln \left(\frac{r}{R} \right),
 \end{aligned} \tag{3.4}$$

Figure 21: Contour of Eqn. 3.1 normalized by $V_\infty R$ at $Re = 140$ and $V_\infty t/R \approx 165.3$ for different values of ζ (Amoloye 2020).

where

$$\eta = \begin{cases} 1, & \theta > \frac{\pi}{2} \parallel \theta < -\frac{\pi}{2} \\ -1, & \frac{\pi}{2} \geq \theta \geq -\frac{\pi}{2}. \end{cases}$$

This is to introduce spatio-temporal periodicity into ψ_{vtsd} when $\eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2}$ becomes a complex number (Amoloye 2020). It takes advantage of the fact that for a generic time-varying function, $F(t)$, with a complex signal frequency, $f_1 + if_2$, Euler's formula (Zill & Wright 2014, p. 119) gives

$$\exp(-(f_1 + if_2)F(t)) = \exp(-f_1 F(t)) (\cos(f_2 F(t)) - i \sin(f_2 F(t))). \quad (3.5)$$

In the windward half of the flow domain $\eta = 1$, and

$$\eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} = (1) \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} = 1$$

always (Amoloye 2020). This agrees with the actual physics of the flow since no vortices are shed in this flow region. However, the value of $\eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2}$ is conditional in the leeward half of the domain where $\eta = -1$ (Amoloye 2020). When $t \approx 0$ and $R \leq r \leq \infty$,

$$\eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} = (-1) \frac{\exp\left(-\frac{r^2}{4V_\infty^2(0)^2}\right)}{2} = 1.$$

This corresponds to the start-up of the flow prior to the vortex-shedding. After the vortex-shedding begins, the vortices advect into a uniform freestream flow far downstream of the cylinder (Amoloye 2020). Thus, when $0 < t < \infty$ as $r \rightarrow \infty$

$$\eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} \rightarrow (-1) \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} = 1,$$

and the freestream flow is recovered (Amoloye 2020). As t progressively grows to infinity, the ratio $r^2/(4V_\infty^2 t^2)$ at any specific location, r , continues to approach zero, and

$$\eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} = (-1) \frac{\exp\left(-\frac{r^2}{4V_\infty^2 (\infty)^2}\right)}{2} = \sqrt{-1} = i.$$

Then, the under-braced exponential argument in Eqn. 3.4 becomes a complex function that is periodic in both space and time because of $r(t)$ (Amoloye 2020). By comparing the under-braced exponential argument of Eqn. 3.4 to the left hand side of Eqn. 3.5, it is seen that

$$\begin{aligned} F(t) &\equiv \left(\frac{\zeta Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r(t)}{R} - 1\right)}}{8} \right) \\ i &\equiv \eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} \\ f_2 &\equiv 1. \end{aligned} \quad (3.6)$$

f_1 is a damping function that is to be determined. These functions are dimensionless. Figure 22a, in which the flow is still from left to right, illustrates some of the effects of η in the flow domain. However, the experimentally observed flow in figure 18 is attached on the windward face of the cylinder until it separates and starts to shed around the crests of the cylinder. The shedding and periodicity of the flow start from the crests and get pronounced (less damped) towards the rear stagnation point. These suggest that the flow periodicity in the immediate vicinity of the cylinder surface is controlled by a damping function that is dependent on the circumferential coordinate, θ . It is assumed that the value of the function ranges from 0 (no damping) at the rear stagnation point (where $\theta = 0$) to 1 (full damping) at either of the crests (where $\theta = \pm\pi/2$). The $\sin^2 \theta$ is a function with this behavior (Amoloye 2020). Thus, f_1 is assumed to be equivalent to $\sin^2 \theta$, and Eqn. 3.4 is modified into

$$\begin{aligned} \psi_{vtsd} &= V_\infty r \left(1 - \frac{R^2}{r^2}\right) \\ &\times \left(\cos \theta - \frac{\zeta Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) (\theta + \sigma) (\chi \sin \theta) \exp \left[-\frac{V_\infty t}{r} \left(1 - \frac{R}{r}\right) \right] \\ &\times \exp \left[-\left(\iota + \frac{r^2}{4V_\infty^2 t^2} \right) - \left(\frac{\zeta Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r(t)}{R} - 1\right)}}{8} \right) \left(\underset{\text{new}}{\sin^2 \theta} + \eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} \right) \right] \\ &\times \ln \left(\frac{r}{R} \right). \end{aligned} \quad (3.7)$$

Therefore, the $\sin^2 \theta$ term damps the extent of the shedding away from the crests towards the cylinder center line as illustrated in figure 22b. The contour of individual vortices are observable in this figure as they are advected away from an absolute reference frame that originates at the cylinder center. With $r(t)$ defined as in Eqn. 3.2, the vortical axes have only relative translation motion, and the vortices are *free* (Amoloye 2020).

Figure 22: Contour of Eqns. 3.4 and 3.7 normalized by $V_\infty R$ at $Re = 140$ and $V_\infty t/R \approx 165.3$ (Amoloye 2020).

If the experimentally observed vortices are assumed to be *forced*, their axes can also spin relative to the cylinder axis (Amoloye 2020). Therefore, a relative circumferential coordinate variable, $\theta(t)$, that varies in a clockwise direction in figure 21 is defined as

$$\begin{aligned}
 \theta(t) &= \theta + \chi \int_0^t \overset{\text{angular velocity}}{\uparrow} \frac{V_{\theta_{inv}}}{r} dt \\
 &= \theta - \frac{\chi V_\infty t \sin \theta}{r} \left(1 + \frac{R^2}{r^2} \right),
 \end{aligned} \tag{3.8}$$

where χ is still

$$\chi = \begin{cases} 1, & \theta \geq 0 \\ -1, & \theta < 0 \end{cases}.$$

Then $r(t)$ is modified into

$$r(t) = r \sqrt{\left(\sin \theta(t) - \frac{V_\infty t}{r} \left(\frac{R^2}{r^2} \sin \theta(t) \cos \theta(t) \right) \right)^2 + \left(\cos \theta(t) - \frac{V_\infty t}{r} \left(1 + \frac{R^2}{r^2} (\sin^2 \theta(t) - \cos^2 \theta(t)) \right) \right)^2} \tag{3.9}$$

so that $\theta(t)$ spins the vortices about the convected vortical axes (Amoloye 2020). The resulting flow pattern of ψ_{vtsd} is depicted in figure 22c. Then, using superposition, the

updated viscous stream function for the cylinder flow is

$$\begin{aligned}
 \psi_{vis} &= \psi_{inv} + \psi_{vt} + \psi_{ss} + \psi_p + \psi_{vtsd} \\
 &= V_{\infty} r \sin \theta \left(1 - \frac{R^2}{r^2} \right) \\
 &\quad - V_{\infty} r \sin \theta \left(1 + \frac{R^2}{r^2} \right) \exp \left[- \left(\frac{r^2}{4V_{\infty}^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r}{R} - 1 \right)}}{8} \right) \right] \ln \left(\frac{r}{R} \right) \\
 &\quad + V_{\infty} r \left(1 - \frac{R^2}{r^2} \right) (\theta + \sigma) (\chi \sin \theta)^{\exp \left[- \frac{V_{\infty} t}{r} \left(1 - \frac{R}{r} \right) \right]} \left(\cos \theta - \frac{qRe}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) \\
 &\quad \times \exp \left[- \left(\frac{r^2}{4V_{\infty}^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r}{R} - 1 \right)}}{8} \right) \right] \ln \left(\frac{r}{R} \right) \\
 &\quad + \underset{\substack{\uparrow \\ \text{new}}}{j} V_{\infty} r \left(1 - \frac{R^2}{r^2} \right) \left(\cos \theta - \frac{\zeta Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) \\
 &\quad \times (\theta + \sigma) (\chi \sin \theta)^{\exp \left[- \frac{V_{\infty} t}{r} \left(1 - \frac{R}{r} \right) \right]} \\
 &\quad \times \exp \left[- \left(\iota + \frac{r^2}{4V_{\infty}^2 t^2} \right) - \left(\frac{\zeta Re \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c \left(\frac{r(t)}{R} - 1 \right)}}{8} \right) \left(\sin^2 \theta + \eta^{\frac{e \left(-\frac{r^2}{4V_{\infty}^2 t^2} \right)}{2}} \right) \right] \\
 &\quad \times \ln \left(\frac{r}{R} \right) - \frac{\partial \phi_p}{\partial \theta}
 \end{aligned} \tag{3.10}$$

where the new constant, j , that represents the ratio of the vortex street velocity, V_{kvs} , relative to V_{∞} , is to be determined (Amoloye 2020).

Figure 23 illustrates the present theoretical vortex shedding at $Re = 140$ and $V_{\infty} t / R \approx 165.3$ in comparison to the streaklines of an experimentally observed flow. The time stamp for the experimentally observed flow is not reported (Van Dyke 1988, figure 94, p. 56). So the choice of $V_{\infty} t / R \approx 165.3$ for the theoretical evaluation is arbitrary. The theoretical streamlines are contours of Eqn. 3.10 normalized by $V_{\infty} R$ and plotted for two different values of j . The vortices are forced with Eqn. 3.8. Equation 3.10 is a complex function that has both real and imaginary parts. Therefore, its absolute value is plotted. However, figure 23a has the contour of the absolute value in white and overlaid on that of the real part of the function that is in red. Figures 23a and 23c consist of eleven linearly-spaced contour levels from 0 to 10. They are not to scale with the experimentally observed flow in figure 23b, but they display about the same number of shedding cycles. In the vortex street, streamlines and streaklines do not coincide because of flow unsteadiness. However, the qualitative similarity of the theoretical flow oscillation to the experimentally observed one is discernible in these figures. For $j = 1$ in figure 23a, the oscillation amplitude about the x -axis is pronounced, and the shedding frequency is lower compared to the experimentally observed flow. Experimental observations of V_{vks} / V_{∞} in the sub-critical Re regime have suggested it to be about $1/6$ (Hoerner 1965, p. 3-6). Therefore, j is assumed to be equivalent to the proportion of ψ_{vtsd} to ψ_{vis} in Eqn. 3.10, and its value is set to $1/5$ in figure 23c. It is observable that the oscillation amplitude is damped towards the oscillation amplitude of the experimentally observed flow in figure 23b. However, the theoretical frequency remains the same with a variation of j (Amoloye 2020).

As Re increases to very high values in an experimentally observed flow visualized with streaklines in figure 24a, the vortex street persists in the wake with more complexity

(a) Eqn. 3.10 with $j = 1$ (red lines for the real part and white lines for the absolute value) (Amoloye 2020).

(b) Experimentally observed streaklines (unknown time - Source: Photograph by Sadatoshi Taneda (Van Dyke 1988))

(c) Eqn. 3.10 with $j = 1/5$ (Amoloye 2020).

Figure 23: Contour of Eqn. 3.10 normalized by $V_\infty R$ (streamlines) showing the Kármán vortex street and the effect of j at $Re = 140$ and $V_\infty t/R \approx 165.3$ (*forced* vortices) in comparison with an experimentally observed flow.

(a) Experimentally observed streaklines (unknown time - Source: Photograph by Thomas Clarke and Hasan Nagib (Van Dyke 1988)).

(b) Eqn. 3.10 at $V_\infty t/R \approx 16,535$ (*forced* vortices) (Amoloye 2020).

(c) Eqn. 3.15 at $V_\infty t/R \approx 16,535$ (*forced* vortices) (Amoloye 2020).

Figure 24: Experimentally observed and theoretical Kármán vortex street behind a circular cylinder at $Re = 10,000$.

(Anderson 2011; Van Dyke 1988; White 2006; Zdravkovich 1997; Amoloye 2020). At a critical $Re \approx 3 \times 10^5$, the wake width shrinks (Anderson 2011; Amoloye 2020). However, a vortex street is not observable in figure 24b that illustrates the normalized absolute value of Eqn. 3.10 at $Re = 10,000$, $V_\infty t/R \approx 16,535$ and $j = 1/5$. An assessment of the Re -dependence of the governing equations informs the necessary modifications to ψ_{vtsd} in the very high Re regime (Amoloye 2020).

The continuity equation is unaffected by changes in Re (Anderson 2011; White 2006; Schlichting 1979; Prandtl & Tietjens 1934; Amoloye 2020). The behavior of the momentum equation in very high Re regime is assessed as follow. Firstly, non-dimensional quantities and operator are defined as

$$\rho^* = \frac{\rho}{\rho_\infty} \quad V^* = \frac{V}{V_\infty} \quad p^* = \frac{p}{p_\infty} \quad \mu^* = \frac{\mu}{\mu_\infty} \quad \omega^* = \frac{2\omega R}{V_\infty} \quad t^* = \frac{V_\infty t}{R} \quad ,$$

$$\nabla^* = 2R\nabla$$

where the quantities with the ∞ subscripts have freestream reference values. For the foregoing incompressible flow analyses, $\rho^* = \rho/\rho_\infty = 1$ (Amoloye 2020). Then, the momentum equation is rewritten as

$$\begin{aligned} & \frac{V_\infty^2}{R} \frac{\partial \mathbf{V}^*}{\partial t^*} + \frac{\nabla^*}{2R} \left(\frac{p_\infty}{\rho_\infty} \frac{p^*}{\rho^*} + V_\infty^2 \frac{V^{*2}}{2} - 2 \frac{\mu_\infty V_\infty}{\rho_\infty 2R} \frac{\mu^*}{\rho^*} \nabla^* \cdot \mathbf{V}^* \right) \\ &= \frac{V_\infty^2}{2R} \mathbf{V}^* \times \omega^* + \frac{\mu_\infty V_\infty}{\rho_\infty 4R^2} \nabla^* \times \left(\frac{\mu^* \omega^*}{\rho^*} \right) \end{aligned}$$

after neglecting body forces (Amoloye 2020). The non-dimensional momentum equation is normalized with $V_\infty^2/2R$ as

$$\begin{aligned} & 2 \frac{\partial \mathbf{V}^*}{\partial t^*} + \nabla^* \left(\frac{p_\infty}{\rho_\infty V_\infty^2} \frac{p^*}{\rho^*} + \frac{V^{*2}}{2} - 2 \frac{\mu_\infty}{\rho_\infty V_\infty 2R} \frac{\mu^*}{\rho^*} \nabla^* \cdot \mathbf{V}^* \right) = \\ & \mathbf{V}^* \times \omega^* + \frac{\mu_\infty}{\rho_\infty V_\infty 2R} \nabla^* \times \left(\frac{\mu^* \omega^*}{\rho^*} \right) . \end{aligned}$$

When the substitutions

$$\frac{\mu_\infty}{\rho_\infty V_\infty 2R} = \frac{1}{Re} \quad \frac{\mu^*}{\rho^*} = \nu^*$$

are made, the normalized non-dimensional momentum equation becomes

$$2 \frac{\partial \mathbf{V}^*}{\partial t^*} + \nabla^* \left(\frac{p_\infty}{\rho_\infty V_\infty^2} \frac{p^*}{\rho^*} + \frac{V^{*2}}{2} - \frac{2}{Re} \nu^* \nabla^* \cdot \mathbf{V}^* \right) = \mathbf{V}^* \times \omega^* + \frac{1}{Re} \nabla^* \times (\nu^* \omega^*) . \quad (3.11)$$

Thus, it is observable that as $Re \rightarrow \infty$ in Eqn. 3.11, the momentum equation approaches an inviscid limit as in

$$2 \frac{\partial \mathbf{V}^*}{\partial t^*} + \nabla^* \left(\frac{p_\infty}{\rho_\infty V_\infty^2} \frac{p^*}{\rho^*} + \frac{V^{*2}}{2} \right) = \mathbf{V}^* \times \omega^* . \quad (3.12)$$

As Re increases to high values in Eqn. 3.10,

$$qRe = \left(\frac{130}{Re^2} \left(\frac{3.9792Re}{130} \right)^{\exp\left(-\frac{10}{Re}\right)} \right) Re \rightarrow 3.9792$$

and

$$\zeta Re = \left(\frac{130}{Re^2} \right) Re \rightarrow 0 .$$

Then, ψ_{vis} becomes less dependent on Re . This is consistent with Eqn. 3.12. The persistence of the vortex shedding at very high Re and the Re -independence of the momentum equation as the inviscid limit is approached both suggest that the shed vortices in very high Re cylinder flows can be modeled on inviscid vortices (Amoloye 2020). A suitable model for this is the Batchelor vortex (Wu *et al.* 2006, p. 260) that has its exponential term only dependent on the radial distance, r , from its core as

$$V_{\theta} = \frac{q}{r} (1 - \exp(-r^2)) . \quad (3.13)$$

Batchelor vortices are also called q -vortices (Wu *et al.* 2006; Amoloye 2020) after the constant q in Eqn. 3.13. This constant is different from the q that has been introduced and used in earlier discussions in this article (Amoloye 2020).

A material surface (Dusan 1976) that is dependent on time and viscosity is defined from the cylinder bounding surface as illustrated in figure 25. The radial distance, $(r - \sqrt{R^2 + 4\nu t})$ employed in the vortex model originates from the material surface (Amoloye 2020). Exponential arguments are non-dimensional, so this distance is normalized with the radius of the material surface as

$$\frac{(r - \sqrt{R^2 + 4\nu t})}{\sqrt{R^2 + 4\nu t}} = \left(\frac{r}{\sqrt{R^2 + 4\nu t}} - 1 \right) .$$

Then, the exponential argument of ψ_{vtsd} is modified with $\left(\frac{r}{\sqrt{R^2 + 4\nu t}} - 1 \right)$ to obtain

$$\begin{aligned} \psi_{vtsd} &= V_\infty r \left(1 - \frac{R^2}{r^2} \right) \\ &\left(\cos \theta - \frac{\zeta Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) (\theta + \sigma) (\chi \sin \theta)^{\exp \left[-\frac{V_\infty t}{r} \left(1 - \frac{R}{r} \right) \right]} \\ &\times \exp \left[-\left(\iota + \frac{r^2}{4V_\infty^2 t^2} \right) - \left(\exp(-\zeta Re) \left(\frac{r}{\sqrt{R^2 + 4\nu t}} - 1 \right)^2 + \frac{\zeta Re \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r(t)}{R} - 1 \right)}{8} \right) \left(\sin^2 \theta + \eta \frac{\exp \left[-\frac{r^2}{4V_\infty^2 t^2} \right]}{2} \right) \right] \\ &\times \ln \left(\frac{r}{R} \right) \end{aligned} \quad (3.14)$$

where $\exp(-\zeta Re)$ tends to 1 and progressively triggers the modification as Re increases (Amoloye 2020).

Then, the viscous stream function for the cylinder flow is

$$\begin{aligned}
 \psi_{vis} &= \psi_{inv} + \psi_{vt} + \psi_{ss} + \psi_p + \psi_{vtsd} \\
 &= V_{\infty} r \sin \theta \left(1 - \frac{R^2}{r^2} \right) \\
 &\quad - V_{\infty} r \sin \theta \left(1 + \frac{R^2}{r^2} \right) \exp \left[- \left(\frac{r^2}{4V_{\infty}^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r}{R} - 1 \right)}{8} \right) \right] \ln \left(\frac{r}{R} \right) \\
 &\quad + V_{\infty} r \left(1 - \frac{R^2}{r^2} \right) (\theta + \sigma) (\chi \sin \theta) \exp \left[- \frac{V_{\infty} t}{r} \left(1 - \frac{R}{r} \right) \right] \\
 &\quad \times \left(\cos \theta - \frac{qRe}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) \exp \left[- \left(\frac{r^2}{4V_{\infty}^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r}{R} - 1 \right)}{8} \right) \right] \ln \left(\frac{r}{R} \right) \\
 &\quad + jV_{\infty} r \left(1 - \frac{R^2}{r^2} \right) \left(\cos \theta - \frac{\zeta Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1 \right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) (\theta + \sigma) (\chi \sin \theta) \exp \left[- \frac{V_{\infty} t}{r} \left(1 - \frac{R}{r} \right) \right] \\
 &\quad \times \exp \left[- \left(\iota + \frac{r^2}{4V_{\infty}^2 t^2} \right) - \left(\exp(-\zeta Re) \left(\frac{r}{\sqrt{R^2 + 4\nu t}} - 1 \right)^2 + \frac{\zeta Re \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r(t)}{R} - 1 \right)}{8} \right) \left(\sin^2 \theta + \eta \frac{\exp \left[- \frac{r^2}{4V_{\infty}^2 t^2} \right]}{2} \right) \right] \ln \left(\frac{r}{R} \right) \\
 &\quad - \frac{\partial \phi_p}{\partial \theta} .
 \end{aligned} \tag{3.15}$$

Figure 25: Vortex core model for high Re vortex shedding (Amoloye 2020).

Thus, $\exp(-\zeta Re)(\frac{r}{\sqrt{R^2 + 4\nu t}} - 1)^2$ ensures that the vortex shedding continues at moderate to high Re . This is shown in figure 24c that illustrates the theoretical vortex shedding at $Re = 10,000$ and $V_\infty t/R \approx 16,535$. $\exp(-\zeta Re)(\frac{r}{\sqrt{R^2 + 4\nu t}} - 1)^2$ also augments the deficient theoretical shedding frequency in figure 23c. This is evident in figure 26 which compares the experimentally observed flow to the absolute value of Eqn. 3.15 evaluated at the same conditions as figure 23c. Figure 26 illustrates about 3 shedding cycles covered over approximately the same distance for both the experimentally observed and theoretical flows (Amoloye 2020).

An assessment of figure 27 that presents the experimentally observed vortex street streamlines from a moving cylinder in two different reference frames is suggestive that the real value of Eqn. 3.10 corresponds to the flow in the cylinder reference frame, and its magnitude corresponds to the same flow in the Laboratory reference frame (Amoloye 2020).

Using Euler's formula for complex numbers (Eqn. 3.5) (Zill & Wright 2014), the spatio-temporal expressions for the Strouhal number, St and the vortex shedding frequency, f , are extracted from the abridged ψ_{vtsd} in

$$\begin{aligned}\psi_{vtsd} &= \dots \exp \left[-i \left(\exp \left(-\frac{130}{Re} \right) \left(\frac{r}{\sqrt{R^2+4\nu t}} - 1 \right)^2 + \frac{130}{8Re} \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + 2} \left(\frac{r(t)}{R} - 1 \right) \right) \right] \\ &= \dots \exp \left(-i \left[\frac{10(R^2+4\nu t) \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + 2}}{R^2} \frac{2R}{V_\infty} \left(\frac{\exp \left(-\frac{130}{Re} \right) R V_\infty \left(\frac{r}{\sqrt{R^2+4\nu t}} - 1 \right)^2}{20(R^2+4\nu t) \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + 2}} + \frac{13\nu V_\infty t (r(t) - \frac{R}{V_\infty t})}{32R(R^2+4\nu t)} \right) \right] \right) \end{aligned} \quad (3.16)$$

where

$$r(t) = r \sqrt{\left(\sin \theta(t) - \frac{V_\infty t}{r} \left(\frac{R^2}{r^2} \sin \theta(t) \cos \theta(t) \right) \right)^2 + \left(\cos \theta(t) - \frac{V_\infty t}{r} \left(1 + \frac{R^2}{r^2} (\sin^2 \theta(t) - \cos^2 \theta(t)) \right) \right)^2},$$

and the values for the constants c and ζ have been substituted (Amoloye 2020).

After rearrangement, the abridged vortex-shedding stream function is

$$\psi_{vtsd} = \dots \exp \left[-i \left(\frac{10(R^2+4\nu t) \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + 2}}{R^2} St \right) \right], \quad (3.17)$$

and the Strouhal number is

$$St = \frac{2fR}{V_\infty} = \frac{2R}{V_\infty} \left(\frac{\exp \left(-\frac{130}{Re} \right) R V_\infty \left(\frac{r}{\sqrt{R^2+4\nu t}} - 1 \right)^2}{20(R^2+4\nu t) \sqrt{\left(1 - \frac{R^2}{r^2} \right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + 2}} + \frac{13\nu V_\infty t (r_0(t) - \frac{R}{V_\infty t})}{32R(R^2+4\nu t)} \right) \quad (3.18)$$

where

$$r_0(t) = \sqrt{\left(\frac{r \sin \theta(t)}{V_\infty t} - \left(\frac{R^2}{r^2} \sin \theta(t) \cos \theta(t) \right) \right)^2 + \left(\frac{r \cos \theta(t)}{V_\infty t} - \left(1 + \frac{R^2}{r^2} (\sin^2 \theta(t) - \cos^2 \theta(t)) \right) \right)^2},$$

$$\theta(t) = \theta - \frac{\chi V_\infty t \sin \theta}{r} \left(1 - \frac{R^2}{r^2} \right),$$

and χ and η are still as previously defined (Amoloye 2020).

The comparison of Eqn. 3.15 to Eqn. 1.5 which is re-presented below as

$$\psi_{vis}(r, \theta, V_\infty, \nu, t, Re) = V_\infty r \sin \theta \left(1 - \frac{R^2}{r^2} \right) + B_0(r, \theta, V_\infty, \nu, t, Re) \ln \left(\frac{r}{R} \right)$$

reveals that B_0 is

$$\begin{aligned}
 B_0 \approx & -V_\infty r \sin \theta \left(1 + \frac{R^2}{r^2}\right) \exp \left[- \left(\frac{r^2}{4V_\infty^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r}{R} - 1\right)}{8} \right) \right] \\
 & + V_\infty r \left(1 - \frac{R^2}{r^2}\right) (\theta + \sigma) (\chi \sin \theta) \exp \left[- \frac{V_\infty t}{r} \left(1 - \frac{R}{r}\right) \right] \\
 & \times \left(\cos \theta - \frac{qRe}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) \exp \left[- \left(\frac{r^2}{4V_\infty^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r}{R} - 1\right)}{8} \right) \right] \\
 & + jV_\infty r \left(1 - \frac{R^2}{r^2}\right) \left(\cos \theta - \frac{\zeta Re}{2} \frac{\sin^2 \theta \cos \theta \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c}} \right) (\theta + \sigma) (\chi \sin \theta) \exp \left[- \frac{V_\infty t}{r} \left(1 - \frac{R}{r}\right) \right] \\
 & \times \exp \left[- \left(\iota + \frac{r^2}{4V_\infty^2 t^2} \right) - \left(\exp(-\zeta Re) \left(\frac{r}{\sqrt{R^2 + 4\iota t}} - 1 \right)^2 + \frac{\zeta Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta}{r^2} + c} \left(\frac{r(t)}{R} - 1\right)}{8} \right) \left(\sin^2 \theta + \eta \frac{\exp\left(-\frac{r^2}{4V_\infty^2 t^2}\right)}{2} \right) \right]
 \end{aligned} \tag{3.19}$$

since the contribution from ψ_p is negligible (Amoloye 2020).

The conditions on Eqn. 3.15 satisfy the continuity equation as well as the boundary conditions at infinity and the surface. Verifying the algebra of these would be laborious without the use of a mathematical software like *Maple™* (Maple 2018). Using *Maple™*, it was confirmed that Eqn. 3.15 is divergence-free because (Amoloye 2020)

$$\frac{1}{r} \frac{\partial(rV_r)}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} = 0 .$$

Figure 26: Experimentally observed Kármán vortex street (upper image - Source: Photograph by Sadatoshi Taneda (Van Dyke 1988)) behind a circular cylinder compared to theoretical streamlines from Eqn. 3.15 at $Re = 140$ (lower image, $V_\infty t/R \approx 165.3$, forced vortices) (Amoloye 2020).

4. The Eulerian Kwasu Function

Equation 3.15 (ψ_{vis}) is continuously differentiable. So, physical quantities such as the velocity, the vorticity and the shear stress that are products of differentiation of the stream function are directly obtainable (Amoloye 2020). However, it is a formidable one to analytically integrate. This is primarily due to the non-linear combinations of the natural logarithm term with the exponential terms. It presents a difficulty to obtain the pressure that is an integral of non-linear expressions involving the components of the velocity field in

$$\begin{aligned}
 \frac{1}{\rho} \frac{\partial p}{\partial r} &= - \left(\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} \right) \\
 &+ \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_r}{\partial \theta^2} - \frac{2V_r}{r^2} + \frac{2}{r^2} \frac{\partial V_\theta}{\partial \theta} \right) \\
 \frac{1}{\rho r} \frac{\partial p}{\partial \theta} &= - \left(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} \right) \\
 &+ \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_\theta}{\partial \theta^2} - \frac{V_\theta}{r^2} + \frac{2}{r^2} \frac{\partial V_r}{\partial \theta} \right)
 \end{aligned} \tag{4.1}$$

whose complexity is complicated with the non-linearity of the cylindrical polar coordinate system. For instance, the cylindrical polar coordinate system's non-linearity is evident in the coupling of $1/r$ with operations in the θ direction as $\partial/r\partial\theta$ (Amoloye 2020). Differential operations in the Cartesian coordinate system are decoupled so that it offers

(a) Cylinder reference frame (Prandtl & Tietjens 1934)

(b) Laboratory reference frame (Prandtl & Tietjens 1934)

(c) Real part of Eqn. 3.10 with *free* vortices at $V_\infty t/R = 165.35$ (Amoloye 2020). (d) Absolute value of Eqn. 3.10 with *free* vortices at $V_\infty t/R = 165.35$ (Amoloye 2020).

Figure 27: The experimentally observed vortex street streamlines behind a circular cylinder at $Re = 250$ photographed in two reference frames compared with present theoretical streamlines.

relatively simplified pressure differentials as

$$\begin{aligned} \frac{1}{\rho} \frac{\partial p}{\partial x} &= - \left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} \right) + \nu \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} \right) \\ \frac{1}{\rho} \frac{\partial p}{\partial y} &= - \left(\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} \right) + \nu \left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} \right). \end{aligned} \quad (4.2)$$

Therefore, Eqn. 4.3

$$\begin{aligned} \theta_W &= \arctan[x, y] \\ &= \arctan \left[\cos \theta \sqrt{x^2 + y^2}, \sin \theta \sqrt{x^2 + y^2} \right] \\ \theta &= \arctan(y + y_m, x) \\ y_m &= 7R_{3D}/20 \\ R_{3D} &= \sqrt{R_{2D}^2 + B^2 \cos^2 \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)} \end{aligned} \quad (4.3)$$

is employed in ψ_{vis} to define it on a three-dimensional (3D) principal Cartesian coordinate axis, θ_W , about which ω is identically zero. B is the cylinder span. With this, ψ_{vis} is transformed into a viscous velocity potential/stream function (Amoloye 2018, 2020, 2021). This is the incompressible *Kwasu* function, $\tilde{\kappa}$, that solves the NSE problem (Amoloye 2018, 2020, 2021; Feffermann 2006). The full expression for $\tilde{\kappa}$ is

$$\begin{aligned}
\tilde{\kappa} &= V_{\infty} r \sin \theta_W \left(1 - \frac{R^2}{r^2}\right) \\
&- V_{\infty} r \sin \theta_W \left(1 + \frac{R^2}{r^2}\right) \exp \left[- \left(\frac{r^2}{4V_{\infty}^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta_W}{r^2} + c} \left(\frac{r}{R} - 1\right)}{8} \right) \right] \ln \left(\frac{r}{R}\right) \\
&+ V_{\infty} r \left(1 - \frac{R^2}{r^2}\right) \theta_W (\chi \sin \theta_W)^{\exp \left[- \frac{V_{\infty} t}{r} \left(1 - \frac{R}{r}\right) \right]} \\
&\times \left(\cos \theta_W - \frac{qRe}{2} \frac{\sin^2 \theta_W \cos \theta_W \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta_W}{r^2} + c}} \right) \exp \left[- \left(\frac{r^2}{4V_{\infty}^2 t^2} + \frac{qRe \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta_W}{r^2} + c} \left(\frac{r}{R} - 1\right)}{8} \right) \right] \ln \left(\frac{r}{R}\right) \\
&+ j V_{\infty} r \left(1 - \frac{R^2}{r^2}\right) \left(\cos \theta_W - \frac{\zeta Re}{2} \frac{\sin^2 \theta_W \cos \theta_W \left(\frac{r}{R} - 1\right) \frac{R^2}{r^2}}{\sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta_W}{r^2} + c}} \right) \theta_W (\chi \sin \theta_W)^{\exp \left[- \frac{V_{\infty} t}{r} \left(1 - \frac{R}{r}\right) \right]} \\
&\times \exp \left[- \left(\iota + \frac{r^2}{4V_{\infty}^2 t^2} \right) - \left(\exp(-\zeta Re) \left(\frac{r}{\sqrt{R^2 + 4\nu t}} - 1 \right)^2 + \frac{\zeta Re \sqrt{\left(1 - \frac{R^2}{r^2}\right)^2 + 4 \frac{R^2 \sin^2 \theta_W}{r^2} + c} \left(\frac{r(t)}{R} - 1\right)}{8} \right) \left(\sin^2 \theta_W + \eta \frac{\exp \left(- \frac{r^2}{4V_{\infty}^2 t^2} \right)}{2} \right) \right] \ln \left(\frac{r}{R}\right) \\
&- \frac{\partial \phi_p}{\partial \theta},
\end{aligned} \tag{4.4}$$

where

$$j = 1/5,$$

$$\zeta = \frac{130}{Re^2},$$

$$\iota = \frac{125}{Re},$$

$$r(t) = r \sqrt{\left(\sin \theta(t) - \frac{V_{\infty} t}{r} \left(\frac{R^2}{r^2} \sin \theta(t) \cos \theta(t) \right) \right)^2 + \left(\cos \theta(t) - \frac{V_{\infty} t}{r} \left(1 + \frac{R^2}{r^2} (\sin^2 \theta(t) - \cos^2 \theta(t)) \right) \right)^2},$$

$$\theta(t) = \theta_W - \frac{\chi V_{\infty} t \sin \theta_W}{r} \left(1 + \frac{R^2}{r^2} \right),$$

$$r = \sqrt{x^2 + y^2} ,$$

$$\eta = \begin{cases} 1 , & -\frac{\pi}{2} < \arctan[X, y] < \frac{\pi}{2} \\ -1 , & \frac{\pi}{2} \geq \arctan[X, y] \text{ || } \arctan[X, y] \leq -\frac{\pi}{2} \end{cases} ,$$

$$\chi = \begin{cases} 1 , & \arctan[X, y] \geq 0 \\ -1 , & \arctan[X, y] < 0 \end{cases} ,$$

and X is the body surface coordinate along the x -axis (Amoloye 2020).

For ease of coding and cost of evaluation, χ and η are parameterized as

$$\chi = (-1)^{\exp(-\exp(N \arctan[X, y]))} , \quad (4.5)$$

and

$$\eta = (-1)^{\exp(-\exp(N \arctan[Y, x]))} \quad (4.6)$$

respectively where Y is the body surface coordinate along the y -axis. The surface inclination to a local flow is factored in with $\arctan[X, y]$ and $\arctan[Y, x]$ (Amoloye 2020). N is a very large number on the order of 10^6 such that when $\arctan[X, y]$ and $\arctan[Y, x]$ are positive,

$$\chi = (-1)^{\exp\left(\underbrace{-\exp(N \arctan[X, y])}_{\rightarrow \infty}\right)} = (-1)^{\exp(-\infty)} = 1 ,$$

and

$$\eta = (-1)^{\exp\left(\underbrace{-\exp(N \arctan[Y, x])}_{\rightarrow \infty}\right)} = (-1)^{\exp(-\infty)} = 1$$

respectively (Amoloye 2020). When they are negative,

$$\chi = (-1)^{\exp\left(\underbrace{-\exp(N \arctan[X, y])}_{\rightarrow 0}\right)} = (-1)^{\exp(-0)} = -1 ,$$

and

$$\eta = (-1)^{\exp\left(\underbrace{-\exp(N \arctan[Y, x])}_{\rightarrow 0}\right)} = (-1)^{\exp(-0)} = -1$$

respectively (Amoloye 2020).

From Eqn. 4.3 and subsequently, R refers to R_{3D} . However, the freestream Reynolds number, Re_∞ , is based on the cylinder center plane radius, R_{2D} . Therefore, the local Reynolds number, Re , in Eqn. 4.4 is substituted with

$$Re = \frac{V_\infty 2R}{\nu_\infty} .$$

Then, the order of differentiation can be interchanged in the right hand sides of Eqn. 4.2

in terms of $\tilde{\kappa}$ as

$$\begin{aligned}\frac{\partial V_x}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\partial \tilde{\kappa}}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial \tilde{\kappa}}{\partial t} \right) \\ V_x \frac{\partial V_x}{\partial x} &= \frac{1}{2} \frac{\partial (V_x^2)}{\partial x} \\ V_y \frac{\partial V_x}{\partial y} &= V_y \frac{\partial}{\partial y} \left(\frac{\partial \tilde{\kappa}}{\partial x} \right) = V_y \frac{\partial}{\partial x} \left(\frac{\partial \tilde{\kappa}}{\partial y} \right) = V_y \frac{\partial V_y}{\partial x} = \frac{1}{2} \frac{\partial (V_y^2)}{\partial x} \\ \frac{\partial^2 V_x}{\partial y^2} &= \frac{\partial^2}{\partial y^2} \left(\frac{\partial \tilde{\kappa}}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial^2 \tilde{\kappa}}{\partial^2 y} \right) \\ \frac{\partial^2 V_x}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial^2 \tilde{\kappa}}{\partial^2 x} \right)\end{aligned}$$

because the time and space variables are decoupled (Amoloye 2018, 2020, 2021). Then, the x -component of Eqn. 4.2 is rewritten as

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \left[- \left(\frac{\partial \tilde{\kappa}}{\partial t} + \frac{V_x^2}{2} + \frac{V_y^2}{2} \right) + \nu \left(\frac{\partial^2 \tilde{\kappa}}{\partial^2 x} + \frac{\partial^2 \tilde{\kappa}}{\partial^2 y} \right) \right]. \quad (4.7)$$

Similar manipulations of the y -component of Eqn. 4.2 gives

$$\frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{\partial}{\partial y} \left[- \left(\frac{\partial \tilde{\kappa}}{\partial t} + \frac{V_x^2}{2} + \frac{V_y^2}{2} \right) + \nu \left(\frac{\partial^2 \tilde{\kappa}}{\partial^2 x} + \frac{\partial^2 \tilde{\kappa}}{\partial^2 y} \right) \right]. \quad (4.8)$$

So, the pressure field is obtained as

$$p = \rho \left[- \left(\frac{\partial \tilde{\kappa}}{\partial t} + \frac{V_x^2}{2} + \frac{V_y^2}{2} \right) + \nu \left(\frac{\partial^2 \tilde{\kappa}}{\partial^2 x} + \frac{\partial^2 \tilde{\kappa}}{\partial^2 y} \right) \right] + F(t) \quad (4.9)$$

where $F(t)$ is a function of time such as the total pressure, p_0 (Amoloye 2018, 2020, 2021).

The physically consistent pressure distribution is defined in the wind circumferential coordinate, $\tilde{\theta}$. $\tilde{\theta}$ is

$$\tilde{\theta} = \begin{cases} |\arctan(x, y + y_n)|, & \text{for } |\arctan(x, y)| < \theta_s & \text{(Cylinder)} \\ -|\arctan(x, y + y_n)|, & \text{for } |\arctan(x, y)| < \theta_s & \text{(Sphere)} \\ \tilde{\theta}_n, & |\arctan(x, y)| \geq \theta_s, \end{cases} \quad (4.10)$$

where θ_s is the circumferential coordinate of the flow separation point measured in radians from the front stagnation point (Amoloye 2020). $\tilde{\theta}$ is illustrated in figure 28. It is used in

$$\theta_W = \arctan \left[\cos \tilde{\theta} \sqrt{x^2 + y^2}, \sin \tilde{\theta} \sqrt{x^2 + y^2} \right]$$

to compute the pressure. In Eqn. 4.10, y_n and $\tilde{\theta}_n$ are geometrical constants that primarily control the pressure variation in the wake. From investigations of experimentally observed cylinder crossflow pressure distributions, they were determined to be $3R/4$ and $54\pi/180$ (rads) respectively (Amoloye 2020).

In figure 28, the positive directions of the geometric variables are denoted with the arrowheads. All the angular coordinates are in radians. θ_W is observed to vary from the windward direction. Therefore, the constant σ that appropriately places the counter-rotating vortices at the cylinder rear in ψ_{vis} (figure 6) is not necessary in Eqn. 4.4 ($\tilde{\kappa}$) (Amoloye 2020). This is evident in figure 29 that illustrates the real part of Eqn. 4.4

Figure 28: Pictorial representation of y_m , y_n , θ_W and $\tilde{\theta}$ in the $z = 0$ plane (Amoloye 2020).

normalized by $V_\infty R$ at $Re_\infty = 26$, $V_\infty t/R = 7$ and $z = 0$. The theoretical time is arbitrary. Ten linearly-spaced contour levels from -4 to $+4$ are plotted and compared to an experimentally observed flow. The theoretical stationary vortex is revealed, and the flow pattern is qualitatively similar to the experimentally observed flow which is visualized with aluminum powder in water (Anderson 2011; Van Dyke 1988; Amoloye 2020).

Figure 30a compares the real part of the normalized *Kwasu* function to an experimentally observed flow photographed at $Re_\infty = 100$ by Wu *et al.* (Wu *et al.* 2004). They visualized the streamlines with $50\mu\text{m}$ polyamide tracer particles in a mixture of glycerol and water (Wu *et al.* 2004; Amoloye 2020). The theoretical streamlines were evaluated in the $z = 0$ plane and at an arbitrary $V_\infty t/R = 57.86$. They were plotted with eighty-six linearly-spaced contour levels from -10 to $+10$. The shed vortices are *free* (Amoloye 2020). Figure 30b also compares an experimentally observed vortex street at $Re_\infty = 105$ to the normalized absolute value of Eqn. 4.4 at the same Re , $V_\infty t/R = 165.3$ and $z = 0$. The experimentally observed flow streaklines are shown by electrolytic precipitation in water (Van Dyke 1988; Amoloye 2020). The theoretical time is also arbitrary, and twenty-one linearly-spaced contour levels from 0 to 4 are shown to illustrate the *forced* vortices. The observable qualitative similarity between refined potential theory and experiment in these figures is agreeable (Amoloye 2020).

Figure 31a presents a schematic of a vortex street behind a finite cylinder showing streamwise vortex pairs that are characteristic of the three-dimensional nature of the flow. These streamwise vortices have been observed in actual experiments and direct numerical simulation (DNS) (Williamson 1996; Amoloye 2020). Figure 31b presents an experimental streakline visualization of the streamwise vortices in a plane parallel to the cylinder axis and perpendicular to the flow direction as illustrated with section AA in figure 31a (Williamson 1996; Amoloye 2020). The freestream Reynolds number is 300. The streamwise distance of section AA from the cylinder and the time stamp of the visualization were not reported (Williamson 1996; Amoloye 2020). So, arbitrary $y/D \approx 6.2$ and $V_\infty t/R \approx 495.9$ were used for the present theoretical evaluation with Eqn. 4.4 at the same Re_∞ shown in figure 31c. Nine linearly-spaced contour levels from 0 to 3 of the absolute value of Eqn. 4.4 normalized with $V_\infty R$ are shown. The span of the

Figure 29: Comparison of an experimentally observed flow (upper image - Source: Photograph by Sadatoshi Taneda (Van Dyke 1988)) and present theoretical streamline (Eqn. 4.4 (lower image)) at $Re_\infty = 26$ (and $V_\infty t/R = 7$ in Eqn. 4.4) (Amoloye 2020).

theoretical cylinder is $2\pi R$. This is also arbitrary. The vortices are *forced*. The figure captures the streamwise vortices (Amoloye 2020). The conditions in figure 31c are the same for the theoretical evaluation in figure 32b that is obtained in an inclined plane to the cylinder axis and flow direction as depicted in figure 32a. The planar angle of inclination is approximately 21° , and thirty-one linearly-spaced contour levels from 0 to 3 are plotted. It can also be observed that figure 32b is in agreement with an actual visualization of the plan view of the three-dimensional vortex shedding in figure 32c (Amoloye 2020).

The *Kwasu* function models the three-dimensional unsteady incompressible cylinder flow field in an Eulerian frame. It simultaneously satisfies the continuity equation and NSE. It exhibits a singularity at the absolute spatio-temporal origin where $x = y = z = t = 0$ (Amoloye 2018, 2020, 2021). However, it is continuous in time and space away from the origin except at $\theta = 0$. Therefore, ϵ is incorporated to remove singularities at $r = \theta = 0$ as (Amoloye 2020)

$$\epsilon = \begin{cases} +O(10^{-3}), & r = 0; \theta \geq 0 \\ -O(10^{-3}), & \theta < 0 \end{cases}. \quad (4.11)$$

Other parameters employed in the *Kwasu* function are summarized in table 3 (Amoloye 2020).

5. The Lagrangian Kwasu Function

5.1. Streaklines

Experimentally observed flows are usually dynamically unsteady as they become turbulent with localized fluctuations (Pope 2000; White 2006; Amoloye 2020). A Lagrangian description is used to model these dynamic and localized fluctuations in RPT (Amoloye 2020). The overlay of the real part of Eqn. 3.10 on its absolute value in figure 23a bears some resemblance to the simultaneous visualization of experimental streamlines and streaklines in figure 19c. This suggests that the normalized stream function theorizes the streaklines as follow (Amoloye 2020).

The *Kwasu* function is a stream function that has different constant values c_1, c_2 and so on for each streamline as depicted in figure 33 for a two-dimensional uniform flow. The flow is moving with a velocity, V . ΔA is a rectangular cross-sectional area of the

(a) An experimentally observed flow (upper image (Wu *et al.* 2004)) and Eqn. 4.4 (lower image) at $Re_\infty = 100$ ($V_\infty t/R = 57.86$, *free* vortices).

(b) An experimentally observed flow (upper image - Source: Photograph by Sadatoshi Taneda (Van Dyke 1988)) and absolute value of Eqn. 4.4 (lower image) at $Re_\infty = 105$ ($V_\infty t/R = 165.3$, *forced* vortices).

Figure 30: Comparison of an experimentally observed flows with present theoretical streamlines (the real and absolute values of Eqn. 4.4) (Amoloye 2020).

(a) Schematic of three-dimensional vortex shedding (Williamson 1996) in (b) and (c).

(b) An experimentally observed flow (Williamson 1996). (c) Eqn. 4.4 at $V_{\infty}t/R \approx 495.9$ $y/D \approx 6.2$, forced vortices (Amoloye 2020).

Figure 31: Comparison of an experimentally observed flow with present theoretical streamlines (absolute values of Eqn. 4.4) showing three-dimensional vortex shedding at $Re_{\infty} = 300$.

depicted control volume with a unit depth into the page. ΔA is chosen such that V is constant across it. Because $\tilde{\kappa}$ is an incompressible stream function, its change, $\Delta \tilde{\kappa}$ along ΔA between the depicted streamlines ab and cd is equal to the volume flow rate through

(a) Plane of theoretical flow in (b) (Amoloye 2020).

(b) Eqn. 4.4 at $Re_\infty = 300$, $V_\infty t/R = 495.9$ (Amoloye 2020), (*forced vortices*).

(c) An experimentally observed flow (Williamson 1996).

Figure 32: Comparison of an experimentally observed flow with present theoretical streamlines (absolute value of Eqn. 4.4) showing three-dimensional vortex shedding.

the control volume between these streamlines (Amoloye 2020). That is

$$\Delta\tilde{\kappa} = c_2 - c_1 \equiv V\Delta x(1),$$

and

$$\frac{\Delta\tilde{\kappa}}{\Delta x} = V. \quad (5.1)$$

The limit as $\Delta x \rightarrow 0$ gives

$$V = \lim_{\Delta x \rightarrow 0} \frac{\Delta\tilde{\kappa}}{\Delta x} \equiv \frac{\partial\tilde{\kappa}}{\partial x}. \quad (5.2)$$

That is

$$\frac{\partial\tilde{\kappa}}{\partial x} = V.$$

Table 3: A summary of the semi-empirical parameters in the RPT (Amoloye 2020).

Parameter	Value	Purpose
c	2	removes discontinuity
q	$\frac{130}{Re^2} (0.03061 Re)^{\exp(-10/Re)}$	wake dimension
χ	$(-1)^{\exp(-\exp(N \arctan[X,y]))}$	symmetry about wind $\tilde{x}$ -axis
j	1/5	coefficient of vortex shedding function
ζ	$130/Re^2$	extends vortex shedding to very high Re
ι	$125/Re$	triggers vortex shedding at $Re \approx 50$
η	$(-1)^{\exp(-\exp(N \arctan[Y,x]))}$	vortex shedding in the wake
N	1.8×10^6	controls χ and η
σ	$\begin{cases} -\pi, & \theta \geq 0 \\ \pi, & \theta < 0 \end{cases}$	wake positioning in ψ_{ss}
ϵ	$\begin{cases} +O(10^{-3}), & r = 0; \theta \geq 0 \\ -O(10^{-3}), & \theta < 0 \end{cases}$	removes discontinuities at $r = \theta = 0$
y_m	$7R/20$	controls the flow axis
y_n	$3R/4$	controls the flow axis
$\tilde{\theta}_n$	$54\pi/180$ (rads)	controls wake pressure variation

The freestream values of V is constant (Amoloye 2020). So,

$$\frac{\partial \tilde{\kappa}}{\partial x} = V_\infty \, , \tag{5.3}$$

and the *Kwasu* function for the uniform flow is integrated as

$$\tilde{\kappa} = V_\infty x + c \, , \tag{5.4}$$

where c is an arbitrary constant that can be set to zero for $x = 0$ (Amoloye 2020). This means for any specific x location, $\tilde{\kappa}$ describes the volume flow rate tracing the particles that have passed through that location. Therefore, the x locus of these particles as they flow along the y -axis, that is the streakline, is obtained by normalizing Eqn. 5.4 as

$$x = \frac{\tilde{\kappa}}{V_\infty} \, .$$

It is observable in figure 33 that this results in lines of constant x -coordinate for the uniform flow (Amoloye 2020). The geometric x -axis is equivalent to the wind $\tilde{y}$ -axis in the present theory (Amoloye 2018, 2020, 2021). Thus, for the cylinder flow described

Figure 33: A uniform freestream flow.

with Eqn. 4.4, the wind $\tilde{y}$ -axis is

$$\tilde{y} = \frac{\tilde{\kappa}}{V_\infty} . \quad (5.5)$$

Figure 34 compares an experimentally observed flow to the present theoretical streamlines and streaklines in the $z = 0$ plane, and at $Re_\infty = 2,000$ and $V_\infty t/R = 165.35$. The flow was evaluated on a static meshgrid with one hundred linearly-spaced points for $-1.5 \leq x/D \leq 1.5$, the same number of points for $-4 \leq y/D \leq 2$, and eleven points for $-\pi \leq z/D \leq \pi$ (Amoloye 2020). The theoretical streamlines in figure 34a are contours of the normalized absolute value of Eqn. 4.4. The theoretical time is arbitrary, and forty-four linearly-spaced contour levels from 0 to 1.3 are shown. The theoretical streamline pattern is qualitatively similar to the experimentally observed flow visualized in figure 34b (Amoloye 2020). The quiescent recirculation flow in the near-wake between the shedding vortices from the cylinder crests is noticeable. However, the separation points are further downstream in the theory than in the experiment. This is likely because of other experimental conditions that are not modeled. It may also be because flow separation is unsteady (Amoloye 2020). Figure 34c is a plot of the streakline crossflow coordinate, $\tilde{y}/D$ (wind y -axis) corresponding to the flow conditions in figure 34a. The flow was evaluated on the same grid as in figure 35. Twenty-one streaklines with starting coordinates $(x_0/D, y_0/D) = (-0.9242 \geq x/D \geq -0.3182, 2)$ and the same number with starting coordinates $(x_0/D, y_0/D) = (0.2879 \leq x/D \leq 0.8939, 2)$ are plotted in red and blue lines respectively (Amoloye 2020). Figure 34d provides a close-up view of the freestream of figure 34c (Amoloye 2020).

5.2. Pathlines

The fluid element pathline coordinates are obtained from the Eulerian velocity field as

$$\begin{aligned} \tilde{X} &= \int \frac{\partial \tilde{X}}{\partial t} dt = \int V_{\tilde{x}} dt = V_{\tilde{x}} t - \int \underset{\substack{\uparrow \\ u}}{t d(V_{\tilde{x}})} \\ \tilde{Y} &= \int \frac{\partial \tilde{Y}}{\partial t} dt = \int V_{\tilde{y}} dt = V_{\tilde{y}} t - \int \underset{\substack{\uparrow \\ v}}{t d(V_{\tilde{y}})} \\ \tilde{Z} &= \int \frac{\partial \tilde{Z}}{\partial t} dt = \int V_{\tilde{z}} dt = V_{\tilde{z}} t - \int \underset{\substack{\uparrow \\ w}}{t d(V_{\tilde{z}})} \end{aligned} \quad (5.6)$$

The Incompressible Kwasu Function

(a) Present theoretical streamlines (*forced vortices*) (Amoloye 2020)

(b) An experimental flow (ONERA photograph (Van Dyke 1988))

(c) Present theoretical streaklines (*forced vortices*) (Amoloye 2020)

(d) Theoretical streaklines in the freestream of (c) (Amoloye 2020).

Figure 34: The present theoretical cylinder crossflow compared with a photograph of an experimentally observed flow at $Re_\infty = 2,000$.

using integration by parts (Amoloye 2020). To obtain the underset integrals in Eqn. 5.6, the displacements of each fluid element along its path in three-dimensional space (u , v and w corresponding to x , y and z directions respectively) are assumed to be independently governed by one-dimensional wave equations in each of the coordinate directions (Amoloye 2020). Then, the wave equations (Zill & Wright 2014, p. 700) for these displacements are specified as

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= V_\infty^2 \frac{\partial^2 u}{\partial x^2}, & u(t_0, x, y, z) &= x_0, & \frac{\partial u}{\partial t} &\equiv \frac{1}{V_\infty} \frac{\partial \tilde{\kappa}}{\partial t}, & t > 0 \\ \frac{\partial^2 v}{\partial t^2} &= V_\infty^2 \frac{\partial^2 v}{\partial y^2}, & v(t_0, x, y, z) &= y_0, & \frac{\partial v}{\partial t} &\equiv \frac{1}{V_\infty} \frac{\partial \tilde{\kappa}}{\partial t}, & t > 0 \\ \frac{\partial^2 w}{\partial t^2} &= V_\infty^2 \frac{\partial^2 w}{\partial z^2}, & w(t_0, x, y, z) &= z_0, & \frac{\partial w}{\partial t} &\equiv \frac{1}{V_\infty} \frac{\partial \tilde{\kappa}}{\partial t}, & t > 0.\end{aligned}\quad (5.7)$$

The solutions to these equations are obtained with the d'Alembert's solution for one-dimensional transverse wave equations (Zill & Wright 2014, p. 705) and the Eulerian *Kwasu* function as

$$\begin{aligned}u(t, x, y, z) &= \frac{1}{2V_\infty} \int_{x-V_\infty t}^{x+V_\infty t} V_{\tilde{x}} dx = \frac{\tilde{\kappa}(t, x + V_\infty t, y, z) - \tilde{\kappa}(t, x - V_\infty t, y, z)}{2V_\infty} \\ v(t, x, y, z) &= \frac{1}{2V_\infty} \int_{y-V_\infty t}^{y+V_\infty t} V_{\tilde{y}} dy = \frac{\tilde{\kappa}(t, x, y + V_\infty t, z) - \tilde{\kappa}(t, x, y - V_\infty t, z)}{2V_\infty} \\ w(t, x, y, z) &= \frac{1}{2V_\infty} \int_{z-V_\infty t}^{z+V_\infty t} V_{\tilde{z}} dz = \frac{\tilde{\kappa}(t, x, y, z + V_\infty t) - \tilde{\kappa}(t, x, y, z - V_\infty t)}{2V_\infty}\end{aligned}\quad (5.8)$$

That is, for $u(t, x, y, z)$, the substitutions

$$r = \sqrt{(x + V_\infty t)^2 + y^2 + z^2} \quad \theta = \arctan[y, (x + V_\infty t)]$$

and

$$r = \sqrt{(x - V_\infty t)^2 + y^2 + z^2} \quad \theta = \arctan[y, (x - V_\infty t)]$$

are made in $\tilde{\kappa}(t, x + V_\infty t, y, z)$ and $\tilde{\kappa}(t, x - V_\infty t, y, z)$ in Eqn. 5.8 respectively. Similar substitutions are used in the expressions for v and w (Amoloye 2020).

These equations are not functions of the individual particle label characteristic of an original Lagrangian description. However, they are in consistence with Andrews' and McIntyre's Eulerian description of Lagrangian-mean flows (Andrews & McIntyre 1978; Amoloye 2020). Thus, the Eulerian *Kwasu* function is interpreted as a time-dependent mean flow because it was derived from CPT that is time-invariant (Amoloye 2020). The time-dependent pathline coordinates are subsequently used to define the time-dependent body axis as

$$\begin{aligned}x(t) &= x_{(t=0)} - \tilde{Y} \\ y(t) &= y_{(t=0)} + \tilde{X} \\ z(t) &= z_{(t=0)} - \tilde{Z}\end{aligned}\quad (5.9)$$

in which the negative sign of the $\tilde{Y}$ displacement is because the flow is from left to right (Amoloye 2020). The signs of the other displacements are set to be consistent with this flow direction. $x_{(t=0)}$, $y_{(t=0)}$ and $z_{(t=0)}$ are the coordinates of the static three-dimensional domain

$$x = r \cos \theta \sin \varphi \quad y = r \sin \theta \sin \varphi \quad z = r \cos \varphi. \quad (5.10)$$

They are the constants of integration in Eqn. 5.6 (Amoloye 2020). The pairing of $y_{(t=0)}$

Figure 35: The present theoretical cylinder crossflow pathlines at $Re_\infty = 2,000$ and $V_\infty t/R = 165.35$ (*forced vortices*) (Amoloye 2020).

with $\tilde{X}$ (and $x_{(t=0)}$ with $\tilde{Y}$) is because the freestream is aligned with the geometric y -axis in the *Kwasu* function. Equation 5.9 is illustrated in figure 35 for a cylinder crossflow at the same conditions as figure 34a (Amoloye 2020).

Equation 5.9 is referred to as a dynamic grid (Amoloye 2020). When it is used in place of the static grid in Eqn. 5.10, it introduces dynamic unsteadiness into the flow field (Amoloye 2020). Thus, the interaction of the local oscillatory motions with the statically unsteady mean flow is described in consistence with the generalized Lagrangian-mean (GLM) theory (Andrews & McIntyre 1978; Amoloye 2020). The freestream velocity can be unsteady depending on the prescribed motions of the body (Amoloye 2020). In a steady freestream as the present case, the three means by which unsteady fluctuations have been introduced in RPT are:

- (i) the transient variation in the exponential terms of Eulerian κ ;
- (ii) the oscillatory displacements of the fluid element paths; and
- (iii) the dynamic grid (Amoloye 2020).

The dynamic grid introduces large-amplitude local variations into the foregoing solution (Amoloye 2020). This results in a field equivalent to a Lagrangian disturbance field in GLM (Andrews & McIntyre 1978; van den Bremer & Breivik 2017; Amoloye 2020).

The instantaneous Lagrangian-mean *Kwasu* function, $\tilde{\kappa}^L$, is defined as

$$\tilde{\kappa}^L = \tilde{\kappa}^E + \tilde{\kappa}^S \quad (5.11)$$

where $\tilde{\kappa}^E$ represents the Eulerian-mean *Kwasu* function (Amoloye 2020), and $\tilde{\kappa}^S$ denotes the Stokes' correction to account for differences in the frame of reference for the averages (Andrews & McIntyre 1978; van den Bremer & Breivik 2017; Amoloye 2020). Both of

these are evaluated on a static grid. The definition in Eqn. 5.11 is used for all other field quantities including the velocity as (Amoloye 2020)

$$V^L = V^E + \underset{\substack{\uparrow \\ \text{drift velocity}}}{V^S} . \quad (5.12)$$

The drift velocity is the Stokes' correction for the velocity (Andrews & McIntyre 1978; van den Bremer & Breivik 2017; Amoloye 2020). The Stokes' correction is a wave property that can be evaluated from a mean flow and linearized wave solutions as in Eqn. 5.8 (Andrews & McIntyre 1978; van den Bremer & Breivik 2017; Amoloye 2020). The Lagrangian velocity components are time derivatives of the dynamic grid coordinates. The Lagrangian-mean velocity components are subsequent time-averaged integrals of Lagrangian velocity components. Thus, these components are obtained as (Amoloye 2020)

$$\begin{aligned} V_x^L &= \frac{1}{t} \int_0^t V_x dt \\ &= \frac{1}{t} \int_0^t \frac{\partial x(t)}{\partial t} dt \\ &= \frac{(x(t) - x(0))}{t\rho} \\ &= -\frac{\tilde{Y}}{t} \\ &= -V_{\tilde{y}} + \frac{v}{t} , \end{aligned} \quad (5.13)$$

$$\begin{aligned} V_y^L &= \frac{1}{t} \int_0^t V_y dt \\ &= \frac{1}{t} \int_0^t \frac{\partial y(t)}{\partial t} dt \\ &= \frac{(y(t) - y(0))}{t} \\ &= \frac{\tilde{X}}{t} \\ &= V_{\tilde{x}} - \frac{u}{t} , \end{aligned} \quad (5.14)$$

and

$$\begin{aligned} V_z^L &= \frac{1}{t} \int_0^t V_z dt \\ &= \frac{1}{t} \int_0^t \frac{\partial z(t)}{\partial t} dt \\ &= \frac{(z(t) - z(0))}{t} \\ &= -\frac{\tilde{Z}}{t} \\ &= V_{\tilde{z}} - \frac{w}{t} . \end{aligned} \quad (5.15)$$

The radial and tangential Lagrangian-mean velocity components are (Amoloye 2020)

$$\begin{aligned}
 V_r^L &= V_x^L \cos \theta + V_y^L \sin \theta \\
 &= -\frac{\tilde{Y}}{t} \cos \theta + \frac{\tilde{X}}{t} \sin \theta \\
 &= -(-V_{\tilde{y}} + \frac{v}{t}) \cos \theta + (V_{\tilde{x}} - \frac{u}{t}) \sin \theta \\
 &= \underbrace{(V_{\tilde{y}} \cos \theta + V_{\tilde{x}} \sin \theta)}_{V_r^E} - \underbrace{(\frac{v}{t} \cos \theta + \frac{u}{t} \sin \theta)}_{V_r^S} \\
 V_\theta^L &= -V_x^L \sin \theta + V_y^L \cos \theta \\
 &= \frac{\tilde{Y}}{t} \sin \theta + \frac{\tilde{X}}{t} \cos \theta \\
 &= (-V_{\tilde{y}} + \frac{v}{t}) \sin \theta + (V_{\tilde{x}} - \frac{u}{t}) \cos \theta \\
 &= \underbrace{(-V_{\tilde{y}} \sin \theta + V_{\tilde{x}} \cos \theta)}_{V_\theta^E} + \underbrace{(\frac{v}{t} \sin \theta - \frac{u}{t} \cos \theta)}_{V_\theta^S} .
 \end{aligned} \tag{5.16}$$

Other Lagrangian-mean quantities like the vorticity are expressed based on these Lagrangian-mean velocity components accordingly. For instance, the magnitude of the Lagrangian-mean specific angular momentum of a fluid particle is (Amoloye 2020)

$$h^L = rV_\theta^L . \tag{5.17}$$

In the perifocal frame, h is equivalent to the negative of the stream function (see page 24). Thus, the Lagrangian-mean *Kwasu* function is (Amoloye 2020)

$$\tilde{\kappa}^L = -h^L = -rV_\theta^L . \tag{5.18}$$

6. Discussion and Conclusion

The motion of fluids presents interesting phenomena including flow separation, wakes, turbulence etc. The physics of these are enshrined in the continuity equation and the NSE. Therefore, their studies are important in mathematics and physics. They also have engineering applications. These studies can either be carried out experimentally, numerically or theoretically. Theoretical studies using CPT have some gaps when compared to experiments table 1. These justified the proposition of RPT (Amoloye 2020, 2021). Table 4 provides a summary of the comparison between CPT, RPT and experimentally observed flows (Amoloye 2020). (Amoloye 2020, 2021) both discuss the items in the upper block of table 4.

The present publication is part of a series introducing RPT (Amoloye 2020, 2021). It covers the items in the middle block of table 4. It leverages experimental observations, physical deductions and the match between CPT and experimentally observed flows in table 1 to aid the theoretical development. In an analytical imitation of the numerical source/vortex panel method (Anderson 2011, secs. 3.17 and 4.10), it describes how wall bounded eddies, the likes advanced by Alan Townsend (Birnie *et al.* 2021; McKeon 2019), in a three-dimensional cylinder crossflow are linked to the detached wake eddies. However, unlike discrete and arbitrary vortices/sources on the cylinder surface whose strengths are numerically determined in the panel method, the vortices/sources/sinks in

RPT are mutually concentric and continuously distributed on the cylinder surface. Their strengths are analytically determined from CPT using physical deductions starting from Re -dependence (Amoloye 2018, 2020, 2021). This results in ψ_{vis} which has a non-zero vorticity field, resolves the boundary layer, and produces a stationary pair of counter-rotating wake vortices that is dependent on time and/or Re . As Re increases to about 50, the stationary counter-rotating wake vortices begin to alternately shed into the von Kármán vortex street.

Theoretical simulations of the shedding process are difficult when the street is from a bluff body. In the present study, an exponential function with a complex-valued argument is exploited to introduce spatial and temporal periodicity in the wake (Amoloye 2020).

The resultant ψ_{vis} is continuously differentiable. Physical quantities such as the velocity, the vorticity and the shear stress that are products of differentiation of the stream function are directly obtainable (Amoloye 2020). However, it is nearly impossible to analytically integrate. This is primarily due to the complexity of the cylindrical polar coordinate system and non-linear combinations of natural logarithm terms with exponential terms in its expression. The relatively simplified form of the Cartesian coordinate system is exploited in ψ_{vis} to define it on a three-dimensional (3D) principal Cartesian coordinate axis about which the vorticity is identically zero. With this, ψ_{vis} is transformed into a viscous incompressible Eulerian velocity potential/stream function, $\tilde{\kappa}$, that is used to obtain the pressure field from the NSE (Amoloye 2018, 2020, 2021).

These are novel advancements on the state of the art in the theoretical prediction of unsteady fluid dynamics phenomena. Birnir *et al.* (2021) developed a spectral theory variant that incorporates the Townsend attached eddy hypothesis to predict the mean flow velocity and fluctuations in a turbulent boundary layer. However, this prediction has not been advanced into the wake (Birnir *et al.* 2021, p. 043054-8). The predicted mean flow velocity obey the Prandtl and von Kármán log-law (Birnir *et al.* 2021; Pope 2000) that encapsulates the logarithmic dependence of the velocity with distance above the wall in a similar way that Eqn. 1.6 of the present formulation does. The log-law typifies wall-bounded turbulence (Birnir *et al.* 2021; Pope 2000). It is interesting that RPT velocity field conforms with the log-law as exemplified by Eqn. 1.6 and the quantitative verification of the RPT boundary layer profile in a cylinder crossflow at $Re_\infty = 98\,590$ and $319\,970$ (Amoloye 2020, pp. 151-153). The presence of wall-bounded turbulence in RPT is explored with the analyses that illuminates a cylinder rear shear stress distribution in a crossflow at $Re_\infty = 3\,900$ (Amoloye 2020, pp. 128-131). This is going to be further investigated with energy spectra analyses of the unsteady boundary layer velocity. Energy spectra analyses of a RPT cylinder crossflow wake at $Re_\infty = 3\,900$ display Kolmogorov's Five-Thirds law of homogeneous isotropic turbulence (Amoloye 2020, pp. 164-170).

A Lagrangian description is used to model dynamic and localized fluctuations in RPT (Amoloye 2020). The normalized stream function is used to theorize streaklines (Amoloye 2020). To obtain pathlines, the displacements of each fluid element along its path in three-dimensional space are assumed to be independently governed by one-dimensional wave equations in each of the coordinate directions (Amoloye 2020). The solutions to these equations are obtained with the d'Alembert's solution for one-dimensional transverse wave equations (Zill & Wright 2014, p. 705) and $\tilde{\kappa}$.

The interaction of the local oscillatory motions with the statically unsteady mean flow is modeled with a dynamic grid that is obtained from the time-dependent pathline coordinates. A dynamic grid is a time-dependent body axis.

A gravity analogy is used to describe the affinity between a fluid element and the cylinder wall in RPT. This is to predict the points of separation, transition, and reattachment, and the thickness of the boundary layer. The analogy introduces the perifocal frame of

Table 4: Summary of the comparison between the classical potential theory (CPT), refined potential theory (RPT) and experimentally observed flows (Amoloye 2020).

	CPT	Experimentally observed flows	RPT
Satisfies Continuity Eqn., $\nabla \cdot \mathbf{V} = 0$	✓		✓
Bounded at infinity	✓		✓
Smooth and Continuous	✓		✓
Satisfies NSE	✓		✓
Physical streamlines	✓		✓
No-slip condition	×	×, ✓	✓
Re_∞ dependence	×		✓
Vorticity, $\omega \neq 0$	×		✓
Boundary layer	×		✓
Wake	×		✓
Vortex shedding	×		✓
Three-dimensions mostly	×		✓
Unsteadiness, Turbulence	×		✓
Compressible	✓		✓
Drag	×		✓

fluid motion and stream function (Amoloye 2018, 2020). It is also exploited to predict drag occurrence that is in the last block of table 4. When verified against experimental and numerical data on a cylinder in an incompressible crossflow at $Re_\infty = 3,900$, the drag prediction is within the error bound of measured data and CFD prediction (Amoloye 2020, p. 158). The predictions of the pressure distribution, separation point and Strouhal number are also within acceptable ranges (Amoloye 2020, pp. 157-158). The details of these and compressibility effects are discussed in (Amoloye 2020). These details are not included in the present article but are subjects of other publications.

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Declarations

- Funding
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- Conflict of interest
The author is currently an employee of Kwara State University, Malete, Nigeria.
- Ethics approval
Not applicable

- Consent to participate

Not applicable

- Consent for publication

Not applicable

- Availability of data and materials

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

- Code availability

Not applicable

- Authors' contributions

Not applicable

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