

Nilpotent Bi-centers in Continuous Piecewise \mathbb{R}^2 -equivariant Cubic Polynomial Hamiltonian Systems

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Nilpotent bi-centers in continuous piecewise \mathbb{Z}_2 -equivariant cubic polynomial Hamiltonian systems

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Abstract One of the classical and difficult problems in the theory of planar differential systems is to classify their centers. Here we classify the global phase portraits in the Poincaré disc of the class continuous piecewise differential systems separated by one straight line and formed by two \mathbb{Z}_2 -equivariant cubic Hamiltonian systems with nilpotent bi-centers at $(\pm 1, 0)$. The main tools for proving our results are the Poincaré compactification, the index theory, and the theory of sign lists for determining the exact number of real roots or negative real roots of a real polynomial in one variable.

Keywords Nilpotent · Bi-centers · Hamiltonian · Phase portrait.

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1 Introduction and statement of the main results

The problem in the qualitative theory of planar differential equations of distinguishing between a focus and a center is known as the *center-focus problem*. This classical problem started with Poincaré [36] in 1881 and

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Dulac [15] in 1908, and nowadays the center-focus problem remains as one of main subjects in the qualitative theory of planar polynomial differential systems.

We say that a singular point p of a planar differential system is a *center* if it has a neighborhood U filled with periodic orbits with the unique exception of this singular point.

If a planar polynomial differential system has a *linear type center*, or a *nilpotent center*, or a *degenerate center* at the origin of coordinates, after making a time rescaling and a linear change of variables, this differential system can be written as

$$(\dot{x}, \dot{y}) = \begin{Bmatrix} (-y, x) \\ (y, 0) \\ (0, 0) \end{Bmatrix} + (f(x, y), g(x, y)), \quad (1)$$

respectively. Here the dot denotes derivative with respect to the time t , and $f(x, y)$ and $g(x, y)$ are real polynomials without constant and linear terms.

The focus-center problem for the quadratic polynomial differential systems has been solved see [6, 15, 23, 24, 37, 42]. There are partial results in the classification of the centers for the cubic polynomial differential systems, see for instance [10, 32, 40, 43, 44], but the focus-center problem for the general cubic polynomial differential systems still remains open.

Recently Colak *et al.* [12, 13] studied the phase portraits of some cubic Hamiltonian differential systems with a linear type center and a nilpotent center at the origin, respectively. Li and Liu [28] investigated the linear type bi-centers problem for \mathbb{Z}_2 -equivariant differential systems. Here we shall study the \mathbb{Z}_2 -equivariant polynomial systems having two centers at the singular points $(\pm 1, 0)$.

The study of \mathbb{Z}_q -equivariant polynomial systems, whose phase portraits are unchanged by a rotation of

$2q$ ($q \in \mathbb{Z}^+$) radians around one point, is closely related to the well-known Hilbert 16th problem, for more details see [22, 27, 28]. Chen *et al.* [11] provided all possible phase portraits of \mathbb{Z}_2 -equivariant cubic polynomial Hamiltonian vector fields with a linear type bi-centers. Chen *et al.* [8] and Li *et al.* [25, 26] studied the bi-centers and isochronous bi-centers problems in some \mathbb{Z}_2 -equivariant cubic systems.

However many natural phenomena have been modeled more accurately by dynamical systems whose differential systems are non-smooth (see for instance [1, 5, 31]), due to their applications appeared an increasing interest in studying the qualitative theory of the non-smooth differential systems, see [2, 17]. In this paper we deal with the following family of piecewise smooth differential systems

$$(\dot{x}, \dot{y}) = \begin{cases} (f^+(x, y), g^+(x, y)) & \text{if } S(x, y) \geq 0, \\ (f^-(x, y), g^-(x, y)) & \text{if } S(x, y) \leq 0, \end{cases} \quad (2)$$

where $S : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^∞ function and $(f^\pm(x, y), g^\pm(x, y))$ are smooth vector fields. In fact, systems (2) have two different regions $\Gamma^\pm = \{(x, y) \in \mathbb{R}^2 : \pm S(x, y) > 0\}$ separated by the discontinuity line $\Gamma = S^{-1}(0)$.

The focus-center problem in piecewise smooth differential systems becomes much more difficult and complicated than for the smooth differential systems. For example, a singular point of systems (2) on the discontinuous curve $S(x, y) = 0$ can be a center even it is neither a center for the first system, nor a center for the second system of (2). There are some results for studying the linear type focus-center problem of the piecewise smooth differential systems (2), see [7, 14, 20, 21, 39]

The focus-center problem for the nilpotent singular points is much more challenging compared to the study for the linear type singular points. Computationally efficient methods have been developed for studying the focus-center problem of the planar smooth differential systems with nilpotent singular points, see [18, 29, 30, 38]. However there is rarely the work for studying the nilpotent focus-center problem in piecewise smooth polynomial systems. Recently Chen *et al.* [9] studied center conditions in the quadratic piecewise smooth polynomial systems with a nilpotent singular point.

In this paper we will study the global dynamics of a class of piecewise \mathbb{Z}_2 -equivariant differential systems formed by two cubic Hamiltonian systems separated by the straight line $y = 0$, and having nilpotent bi-centers at the points $(\pm 1, 0)$. In section 3 we prove that such

class of piecewise differential systems can be written as

$$(\dot{x}, \dot{y}) = \begin{cases} \begin{pmatrix} -a_{21}y + 3b_{03}y^2 + a_{21}x^2y \\ -3b_{03}xy^2 - 2(1 + a_{21}^2)y^3 \\ -\frac{1}{2}x + \frac{1}{2}x^3 - a_{21}xy^2 + b_{03}y^3 \end{pmatrix} & \text{if } y \geq 0, \\ \begin{pmatrix} -a_{21}y - 3b_{03}y^2 + a_{21}x^2y \\ -3b_{03}xy^2 - 2(1 + a_{21}^2)y^3 \\ -\frac{1}{2}x + \frac{1}{2}x^3 - a_{21}xy^2 + b_{03}y^3 \end{pmatrix} & \text{if } y \leq 0, \end{cases} \quad (3)$$

where $b_{03} < 0$ and the singular point $(1, 0)$ of the first system of (3) is a third-order singular point, see section 3 for the definition of third-order singular point. The Hamiltonian functions for these two Hamiltonian systems are

$$H(x, y)^+ = \frac{1}{4}x^2 - \frac{1}{8}x^4 - \frac{1}{2}a_{21}y^2 + b_{03}y^3 + \frac{1}{2}a_{21}x^2y^2 - b_{03}xy^3 - \frac{1}{2}(1 + a_{21}^2)y^4,$$

for the Hamiltonian system in $y \geq 0$, and

$$H(x, y)^- = \frac{1}{4}x^2 - \frac{1}{8}x^4 - \frac{1}{2}a_{21}y^2 - b_{03}y^3 + \frac{1}{2}a_{21}x^2y^2 - b_{03}xy^3 - \frac{1}{2}(1 + a_{21}^2)y^4,$$

for the Hamiltonian system in $y \leq 0$.

Note that the piecewise differential systems (3) only are continuous on the straight line $y = 0$, so they are non-smooth piecewise differential systems. We also remark that the piecewise differential systems (3) only depends on two parameters a_{21} and b_{03} .

For the differential systems in the Poincaré disc it is known that the separatrix ℓ is an orbit such that the orbits in one side of ℓ have different α - or ω -limit than the orbits in the other side of ℓ in any small neighborhood of it. The separatrices include all the infinite orbits, all the finite singular points, the two orbits at the boundary of the hyperbolic sectors of the finite and infinite singular points, and the limit cycles, see for more details on the separatrices [33, 34]. If Σ denotes the set of all separatrices in the Poincaré disc \mathbb{D}^2 , Σ is a closed set and the components of $\mathbb{D}^2 \setminus \Sigma$ are called the canonical regions. The union of the set Σ with an orbit of each canonical region form the separatrix configuration. We denote by S and R the number of separatrices and canonical regions, respectively.

Theorem 1 *In the Poincaré disc the phase portraits of the continuous piecewise \mathbb{Z}_2 -equivariant cubic Hamiltonian systems (3) with nilpotent bi-centers at $(\pm 1, 0)$ are topologically equivalent to one of the 10 phase portraits showed in Figure 1.*

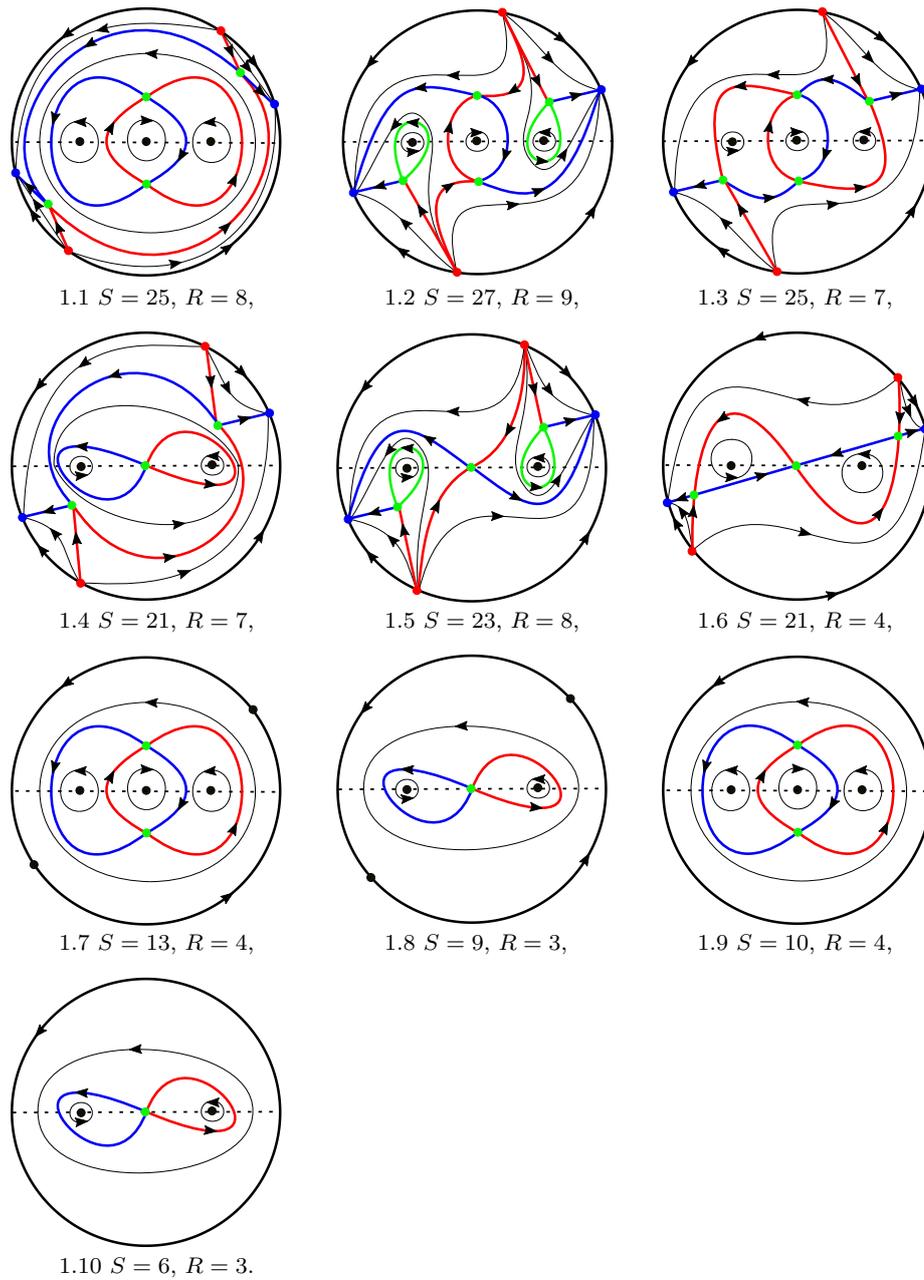


Fig. 1 The ten topological phase portraits in the Poincaré disc of Theorem 1.

In section 2 we provide a brief introduction to the Poincaré compactification, a summary on how to determine the phase portrait using the separatrix configuration, and some basic results on the topological indices that we shall need for proving Theorem 1. In section 3 we show how to obtain the continuous piecewise \mathbb{Z}_2 -equivariant cubic Hamiltonian systems (3). Finally in section 4 we characterize the global phase portraits of systems (3) in the Poincaré disc, that is we prove Theorem 1.

2 Preliminaries

2.1 Poincaré compactification

Roughly speaking this compactification identifies the plane \mathbb{R}^2 with the interior of the closed unit disc \mathbb{D}^2 centered at the origin of \mathbb{R}^2 , and allows to extend analytically a polynomial differential system to the boundary of \mathbb{D}^2 , usually called the circle of the infinity, denoted by \mathbb{S}^1 . More details are described in chapter 5 of [16]. Now we provide the equations of the Poincaré

compactification for a piecewise differential systems in \mathbb{R}^2 . The closed disc \mathbb{D}^2 is called the Poincaré disc.

We consider the piecewise polynomial differential systems in \mathbb{R}^2 of the form

$$(\dot{x}_1, \dot{x}_2) = (P^\pm(x_1, x_2), Q^\pm(x_1, x_2)), \quad \pm x_2 \geq 0, \quad (4)$$

where $(P^\pm(x_1, x_2), Q^\pm(x_1, x_2))$ are real vector fields of degree d^+ and d^- , respectively.

For studying the neighborhood of the infinity of \mathbb{R}^2 , we use the following four local charts to do the calculations, which are given by $U_i = \{(s_1, s_2) \in \mathbb{D}^2 : s_i > 0\}$ and $V_i = \{(s_1, s_2) \in \mathbb{S}^2 : s_i < 0\}$, for $i = 1, 2$. The corresponding diffeomorphisms

$$\varphi_i : U_i \rightarrow \mathbb{R}^2, \quad \psi_i : V_i \rightarrow \mathbb{R}^2, \quad (5)$$

defined by $\varphi_1(s_1, s_2) = \psi_1(s_1, s_2) = (s_2/s_1, 1/s_1) = (u, v)$ and $\varphi_2(s_1, s_2) = \psi_2(s_1, s_2) = (1/s_2, s_1/s_2) = (u, v)$. Thus the coordinates (u, v) will play different roles in the distinct local charts.

The expression of the vector field (4) in the local chart U_1 is

$$(\dot{u}, \dot{v}) = \left(v^{d^\pm} \left(P^\pm \left(\frac{u}{v}, \frac{1}{v} \right) - u Q^\pm \left(\frac{u}{v}, \frac{1}{v} \right) \right), -v^{d^\pm+1} Q^\pm \left(\frac{u}{v}, \frac{1}{v} \right) \right).$$

The corresponding expression in U_2 is

$$(\dot{u}, \dot{v}) = \left(v^{d^\pm} \left(Q^\pm \left(\frac{1}{v}, \frac{u}{v} \right) - u P^\pm \left(\frac{1}{v}, \frac{u}{v} \right) \right), -v^{d^\pm+1} P^\pm \left(\frac{1}{v}, \frac{u}{v} \right) \right).$$

The expression of the vector field (4) in the local chart V_i is equal to the expressions in U_i multiplied by $(-1)^{d^\pm-1}$.

For smooth polynomial vector fields of degree d if p is an infinite singular point, then $-p$ is another infinite singular point. Thus the number of infinite singular points is even and the local behavior of one is that of the other multiplied by $(-1)^{d+1}$. This symmetry property in general does not hold for piecewise smooth differential systems (2) because the singular points at infinity are not diametrically opposite. But in our case systems (3) are symmetry with respect to the origin, so we just need to analyze the phase portraits of the infinite singular points in the local chart $U_1|_{v=0}$ and at the origin of the local chart U_2 .

2.2 Topological equivalence and index

We say that two flows φ_1 and φ_2 on the Poincaré disc \mathbb{D}^2 are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ which sends orbits of φ_1 to orbits of φ_2 preserving or reversing the direction of all orbits.

Let Σ be the set of all separatrices of a flow on the Poincaré disc \mathbb{D}^2 . The canonical regions are the open connected components of $\mathbb{D}^2 \setminus \Sigma$. A separatrix configuration of a flow on the Poincaré disc \mathbb{D}^2 is the union of Σ with one orbit in each canonical region.

Let C_1 and C_2 be the separatrix configurations of the flows (\mathbb{D}^2, ϕ_1) and (\mathbb{D}^2, ϕ_2) respectively. We say that C_1 and C_2 are *topologically equivalent* if there exists a homeomorphism $h : C_1 \rightarrow C_2$ which sends orbits of C_1 to orbits of C_2 preserving or reversing the direction of all orbits.

From Markus [33], Neumann [34] and Peixoto [35] it shows that two continuous flows (\mathbb{D}^2, ϕ_1) and (\mathbb{D}^2, ϕ_2) with only isolated singular points are topologically equivalent if and only if their separatrix configurations are topologically equivalent.

Then we introduce the following theorems, the Index Poincaré Formula and the Poincaré–Hopf Theorem, which are useful tools for determining the local phase portrait of the differential systems.

Theorem 2 *Let q , h and e be the number of parabolic, hyperbolic and elliptic sectors of an isolated singular point p , respectively, which has the finite sectorial decomposition property. Then the topological index of p equals $1 + (e - h)/2$.*

Corollary 1 *The topological indices of a cusp, a center, a node and a saddle equal 0, 1, 1 and -1 , respectively.*

If we identify point to point the boundary of the Poincaré disc we get a 2-dimensional sphere \mathbb{S}^2 , and the flow defined by the Poincaré compactification on the disc \mathbb{D}^2 can be extended to the 2-dimensional sphere \mathbb{S}^2 , having a copy of the initial flow of the polynomial differential system in \mathbb{R}^2 in each one of the two components of $\mathbb{S}^2 \setminus \mathbb{S}^1$.

Theorem 3 *Let φ be the extended flow of a Poincaré compactification on the sphere \mathbb{S}^2 having finitely many singular points. Then the sum of the topological indices of all its singular points is 2.*

For more details about the above theorems see Chapter 6 of [16].

3 Obtaining systems (3)

Here a vector field $\mathbf{X}(x, y)$ is \mathbb{Z}_2 -equivariant if $-\mathbf{X}(x, y) = \mathbf{X}(-x, -y)$. Then the \mathbb{Z}_2 -equivariant piecewise cubic polynomial differential systems (3) separated by the straight line $y = 0$ are differential systems of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{00} + a_{10}x + a_{01}y + a_{20}x^2 \\ + a_{11}xy + a_{02}y^2 + a_{30}x^3 \\ + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ b_{00} + b_{10}x + b_{01}y + b_{20}x^2 \\ + b_{11}xy + b_{02}y^2 + b_{30}x^3 \\ + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{pmatrix} & \text{if } y \geq 0, \\ \begin{pmatrix} -a_{00} + a_{10}x + a_{01}y - a_{20}x^2 \\ -a_{11}xy - a_{02}y^2 + a_{30}x^3 \\ + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ -b_{00} + b_{10}x + b_{01}y - b_{20}x^2 \\ -b_{11}xy - b_{02}y^2 + b_{30}x^3 \\ + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{pmatrix} & \text{if } y \leq 0. \end{cases} \quad (6)$$

Assuming that $(\pm 1, 0)$ are two singular points of systems (6), we have

$$a_{00} = -a_{20}, \quad a_{10} = -a_{30}, \quad b_{00} = -b_{20}, \quad b_{10} = -b_{30}. \quad (7)$$

Moreover, in order that the points $(\pm 1, 0)$ be nilpotent-type singular points whose Jacobian matrices are

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (8)$$

as in (1), we must $a_{20} = a_{11} = b_{20} = b_{11} = 0$. For the general Jacobian matrices with nilpotent singular point were not analyze in this paper, because when doing a change of variable that transforms these matrices into the canonical form (8) it will modify the separation straight line $y = 0$ and breaks the symmetry.

If we assume that $b_{30} = 0$, then the Jacobian matrix of systems (6) evaluated at $(\pm 1, 0)$ is

$$J = \begin{pmatrix} 2a_{30} & a_{01} + a_{21} \\ 0 & b_{01} + b_{21} \end{pmatrix}, \quad (9)$$

yields a triangular matrix having the two characteristic roots

$$\lambda_1 = 2a_{30}, \quad \lambda_2 = b_{01} + b_{21}.$$

Furthermore we take $\lambda_1 = \lambda_2 = 0$, because we want that the singular points $(\pm 1, 0)$ of the piecewise differential systems (6) be nilpotent, so we obtain

$$a_{30} = 0, \quad b_{01} = -b_{21}. \quad (10)$$

The piecewise differential systems (6) become

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y(a_{01} + a_{21}x^2 + a_{02}y + a_{12}xy) \\ + a_{03}y^2, \\ y(-b_{21} + b_{21}x^2 + b_{02}y + b_{12}xy) \\ + b_{03}y^2, \end{pmatrix} & \text{if } y \geq 0, \\ \begin{pmatrix} y(a_{01} + a_{21}x^2 - a_{02}y + a_{12}xy) \\ + a_{03}y^2, \\ y(-b_{21} + b_{21}x^2 - b_{02}y + b_{12}xy) \\ + b_{03}y^2, \end{pmatrix} & \text{if } y \leq 0. \end{cases} \quad (11)$$

Since the polynomials in (11) have a common factor y , the singular points $(\pm 1, 0)$ are not isolated singular points.

Hence in order to make the singular points $(\pm 1, 0)$ be isolated nilpotent singular points with the canonical form given in (1) we force that

$$J = \begin{pmatrix} 0 & 0 \\ 2b_{30} & 0 \end{pmatrix},$$

consequently $a_{30} = 0$, $b_{01} = -b_{21}$ and $a_{01} = -a_{21}$. Then we apply scaling on the state variables and parameters in systems (6) so that $b_{30} = 1/2$.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -a_{21}y + a_{21}x^2y + a_{02}y^2 \\ + a_{12}xy^2 + a_{03}y^3 = X^+(x, y), \\ -\frac{x}{2} + \frac{x^3}{2} - b_{21}y + b_{21}x^2y \\ + b_{02}y^2 + b_{12}xy^2 + b_{03}y^3 \\ = Y^+(x, y), \end{pmatrix} & \text{if } y \geq 0, \\ \begin{pmatrix} -a_{21}y + a_{21}x^2y - a_{02}y^2 \\ + a_{12}xy^2 + a_{03}y^3 = X^-(x, y), \\ -\frac{x}{2} + \frac{x^3}{2} - b_{21}y + b_{21}x^2y \\ - b_{02}y^2 + b_{12}xy^2 + b_{03}y^3 \\ = Y^-(x, y), \end{pmatrix} & \text{if } y \leq 0. \end{cases} \quad (12)$$

Next, let $H^+(x, y)$ be the Hamiltonian of the first system of (12). To find this Hamiltonian we integrate $X^+(x, y)$ of (12) with respect to y and obtain

$$\begin{aligned} H_1^+(x, y) &= f(x) + \int X^+(x, y) dy \\ &= f(x) - \frac{1}{2}a_{21}y^2 + \frac{1}{2}a_{21}x^2y^2 + \frac{1}{3}a_{02}y^3 \\ &\quad + \frac{1}{3}a_{12}xy^3 + \frac{1}{4}a_{03}y^4, \end{aligned} \quad (13)$$

for some real polynomials $f(x)$. And we integrate $Y^+(x, y)$ of (12) with respect to x and obtain

$$\begin{aligned} H_2^+(x, y) &= g(y) - \int Y^+(x, y) dx \\ &= g(y) + \frac{1}{4}x^2 - \frac{1}{8}x^4 + b_{21}xy - \frac{1}{3}b_{21}x^2y \\ &\quad - b_{02}xy^2 - \frac{1}{2}b_{12}x^2y^2 - b_{03}xy^3, \end{aligned} \quad (14)$$

for some real polynomials $g(y)$. Equating $H_1^+(x, y)$ to $H_2^+(x, y)$ we obtain

$$b_{12} = -a_{21}, \quad a_{12} = -3b_{03}, \quad b_{02} = b_{21} = 0, \quad (15)$$

$$f(x) = x^2/4 - x^4/8 \text{ and } g(y) = -a_{21}y^2/2 + a_{02}y^3/3 + a_{03}y^4/4.$$

Then systems (12) become the piecewise Hamiltonian systems

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{cases} \begin{pmatrix} -a_{21}y + a_{21}x^2y + a_{02}y^2 \\ -3b_{03}xy^2 + a_{03}y^3 \\ -\frac{x}{2} + \frac{x^3}{2} - a_{21}xy^2 + b_{03}y^3 \end{pmatrix} & \text{if } y \geq 0, \\ \begin{pmatrix} -a_{21}y + a_{21}x^2y - a_{02}y^2 \\ -3b_{03}xy^2 + a_{03}y^3 \\ -\frac{x}{2} + \frac{x^3}{2} - a_{21}xy^2 + b_{03}y^3 \end{pmatrix} & \text{if } y \leq 0, \end{cases} \end{aligned} \quad (16)$$

where systems (16) have the Hamiltonian

$$\begin{aligned} H(x, y)^+ &= \frac{1}{4}x^2 - \frac{1}{8}x^4 - \frac{1}{2}a_{21}y^2 + \frac{1}{2}a_{21}x^2y^2 \\ &\quad + \frac{1}{3}a_{02}y^3 - b_{03}xy^3 + \frac{1}{4}a_{03}y^4, \end{aligned} \quad (17)$$

for the Hamiltonian system in $y > 0$, and the Hamiltonian

$$\begin{aligned} H(x, y)^- &= \frac{1}{4}x^2 - \frac{1}{8}x^4 - \frac{1}{2}a_{21}y^2 + \frac{1}{2}a_{21}x^2y^2 \\ &\quad - \frac{1}{3}a_{02}y^3 - b_{03}xy^3 + \frac{1}{4}a_{03}y^4, \end{aligned} \quad (18)$$

for the Hamiltonian system in $y < 0$, respectively.

Introducing the transformation $x \rightarrow x + 1$ into systems (16) we get

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{cases} \begin{pmatrix} 2a_{21}xy + a_{21}x^2y + (a_{02} \\ -3b_{03})y^2 + a_{12}xy^2 \\ + a_{03}y^3 = \Psi^+(x, y), \\ x + \frac{3x^2}{2} + \frac{x^3}{2} + a_{21}y^2 \\ - a_{21}xy^2 + b_{03}y^3 \\ = x + \Phi^+(x, y), \end{pmatrix} & \text{if } y \geq 0, \\ \begin{pmatrix} 2a_{21}xy + a_{21}x^2y - (a_{02} \\ +3b_{03})y^2 + a_{12}xy^2 \\ + a_{03}y^3 = \Psi^-(x, y), \\ x + \frac{3x^2}{2} + \frac{x^3}{2} + a_{21}y^2 \\ - a_{21}xy^2 + b_{03}y^3 \\ = x + \Phi^-(x, y), \end{pmatrix} & \text{if } y \leq 0, \end{cases} \end{aligned} \quad (19)$$

and so the singular point $(1, 0)$ of systems (16) is moved to the origin of systems (19). Then we assume that

$$f^\pm(y) = \sum_{k=2}^{\infty} c_k^\pm y^k$$

are the unique solutions of the implicit function equations $x + \Phi^\pm(x, y) = 0$ in a neighborhood of the origin, respectively. In order to determine the local phase portraits of the nilpotent points $(\pm 1, 0)$ we write

$$\begin{aligned} \Psi^\pm(f^\pm(y), y) &= \sum_{k=2}^{\infty} \alpha_k^\pm y^k, \\ \left[\frac{\partial \Psi^\pm}{\partial x} + \frac{\partial \Phi^\pm}{\partial y} \right]_{(f^\pm(y), y)} &= \sum_{k=1}^{\infty} \beta_k^\pm y^k, \end{aligned} \quad (20)$$

where

$$\beta_n^\pm \equiv 0, \quad \alpha_2^\pm = \pm a_{02} - 3b_{03}, \quad \alpha_3^\pm = a_{03} + 2a_{21}^2. \quad (21)$$

For the polynomial differential systems if $\alpha_2 = \alpha_3 = \dots = \alpha_{k-1} = 0$ and $\alpha_k \neq 0$, then the multiplicity of the nilpotent singular point is exactly k , for more details see [30]. It follows from Theorem 3.5 in [16] that if $\beta_n = 0$ and $\alpha_m \neq 0$ this nilpotent singular point is a

$$\begin{cases} \text{a cusp if } m = 2k, \\ \text{a saddle if } m = 2k + 1 \text{ and } \alpha_m > 0, \\ \text{a center or a focus if } m = 2k + 1 \text{ and } \alpha_m < 0. \end{cases} \quad (22)$$

Since the multiplicity of a nilpotent center or focus (i.e. of a monodromic singular point) of a differential system is an odd positive integer greater than one, it

follows that the smallest multiplicity of $(1, 0)$ must be 3 if the singular point $(1, 0)$ is a nilpotent focus or a center in the first system of (16). For convenience we will call this singular point a *third-order singular point*. More precisely we have the following statement: The singular point $(1, 0)$ of the first system of (16) is a monodromic critical point with multiplicity 3 if and only if

$$\alpha_2^+ = 0, \quad \alpha_3^+ < 0,$$

namely,

$$a_{02} = 3b_{03}, \quad a_{03} + 2a_{21}^2 < 0.$$

Setting $\alpha_3^+ = -2$ yields $a_{03} = -2a_{21}^2 - 2$. Then we have that the singular point $(1, 0)$ of the first system of (16) is monodromic. Therefore we obtain systems (3) and we have

$$\alpha_2^- = -6b_{03}, \quad \alpha_3^- = -2.$$

If $b_{03} = 0$, i.e. $\alpha_2^- = 0$, then the piecewise differential systems (3) are smooth. If $b_{03} \neq 0$, i.e., $\alpha_2^- \neq 0$, then the singular point $(1, 0)$ of the second differential system of (3) is a cusp. But the singular points $(\pm 1, 0)$ of the piecewise differential systems (3) cannot be monodromic when $b_{03} > 0$, so we only consider $b_{03} < 0$. Here we present the following phase portrait of systems (3) as an illustration.

Example 1 The phase portrait of systems (16), as depicted in Fig. 2, shows the singular points $(\pm 1, 0)$ to be two cusps.

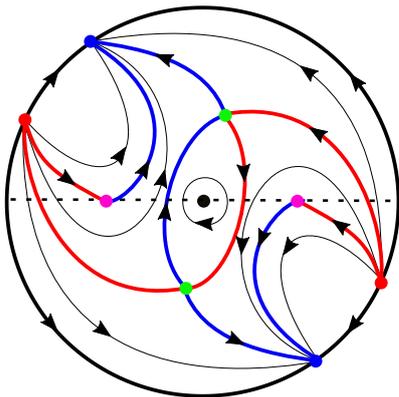


Fig. 2 The phase portrait of systems (16) for $a_{21} = -1$ and $b_{03} = 2$, showing that the singular points $(\pm 1, 0)$ are two cusps.

In summary we have obtained the continuous piecewise differential system (3) that we study in this paper.

Furthermore from Proposition 2.1 of [7] we have that the Hamiltonians of the first and second systems

of (3) satisfy $H^+(x, 0) \equiv H^-(x, 0)$. Hence systems (3) have nilpotent bi-centers at $(\pm 1, 0)$. Remark that when $\alpha_2^+ \neq 0$, i.e. $a_{03} \neq 3b_{03}$ systems (3) can also have nilpotent bi-centers at $(\pm 1, 0)$, but this case becomes more complicated we do not provide its analysis in this paper.

4 Global phase portraits of systems (3)

Now we consider the finite singular points of systems (3). The singular points $p_{1,2} = (\pm 1, 0)$ are two centers, the origin p_3 of systems (3) is also a singular point, whose Jacobian matrix is

$$\begin{pmatrix} 0 & -a_{21} \\ -\frac{1}{2} & 0 \end{pmatrix}. \quad (23)$$

From (23) we have that the origin is a saddle when $a_{21} \geq 0$ (a nilpotent saddle when $a_{21} = 0$), or a center when $a_{21} < 0$. Now we need to study if there are additional singular points.

Since systems (3) are symmetric with respect to the origin of coordinates p_3 , we just need to study the phase portrait of the first system in (3).

The Jacobian matrix of the first system of (3) at a finite singular point (x, y) is

$$\begin{pmatrix} y(2a_{21}x - 3b_{03}y) & N_1 \\ \frac{1}{2}(-1 + 3x^2 - 2a_{21}y^2) - y(2a_{21}x - 3b_{03}y) & \end{pmatrix}, \quad (24)$$

where

$$N_1 = -a_{21} + a_{21}x^2 + 6b_{03}y - 6b_{03}xy - 6y^2 - 6a_{21}y^2.$$

We claim that there are no finite singular points for the first system of (3) whose linear part be identically zero. Indeed, we obtain that $-1 + 3x^2 - 2a_{21}y^2$ and $y(2a_{21}x - 3b_{03}y)$ have no common solutions, because the Gröbner basis for the polynomials \dot{x} , \dot{y} , $-1 + 3x^2 - 2a_{21}y^2$ and $y(2a_{21}x - 3b_{03}y)$ is 1. We again calculate the Gröbner basis for four polynomials \dot{x} , \dot{y} , $y(2a_{21}x - 3b_{03}y)$ and N_1 , then we obtain seven polynomials $a_{21}y$, $b_{03}y^2$, y^3 , $-a_{21} + a_{21}x^2 + 6b_{03}y - 6b_{03}xy - 6y^2$, $xy - b_{03} + b_{03}x + y$, $x(1+x)y^2$ and $(-1+x)x(1+x)$. It means that there are no other nilpotent singular points different from p_k for $k = 1, 2, 3$, because these are the unique solutions of the previous polynomial system. Hence all the remaining finite singular points are hyperbolic, or semi-hyperbolic, or centers and by Theorems 2.15 and 2.19 of [16] the remaining finite singular points must be saddles or centers because the systems are Hamiltonian.

The explicit expressions of the finite singular points different from p_k for $k = 1, 2, 3$, and their eigenvalues in terms of parameters a_{21} and b_{03} are complicated, it is hard to study their existence and their local phase

portraits. Thus we need to present more algebraic tools for solving this problem.

From the first system in (3) we compute the Gröbner basis for \dot{x} and \dot{y} and we obtain eight polynomials, in particular the following two polynomials

$$y^2 \left[3a_{21}b_{03} - 3a_{21}b_{03}x - 9b_{03}^2y + (2a_{21} + 9b_{03}^2)xy + 2b_{03}(3 + 4a_{21}^2)y^2 \right]$$

and

$$y^3 \left[6a_{21}b_{03} + (-2a_{21} - 18b_{03}^2 + 15a_{21}^2b_{03}^2)y + (18b_{03} + 12a_{21}^2b_{03} - 54a_{21}b_{03}^3)y^2 + (-4 - 4a_{21}^2 + 36a_{21}b_{03}^2 + 32a_{21}^3b_{03}^2 + 27b_{03}^4)y^3 \right] \quad (25)$$

$$= y^3 f(y).$$

are enough for our analysis. We note that polynomial (25) is not identically zero, because in order that it be identically zero we need that $a_{21} = b_{03} = 0$, but then the resultant reduces to $-4y^6 \neq 0$. Now in order to study the number of the real roots of the polynomial $f(y)$ we shall use the method of the *discriminant sequence* associated to $f(y)$ developed in [41].

We associate to the polynomial

$$f(y) = a_0y^k + a_1y^{k-1} + \cdots + a_k \quad (26)$$

the $(2k+1) \times (2k+1)$ matrix

$$M = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_k \\ 0 & ka_0 & (k-1)a_1 & \cdots & a_{k-1} \\ & a_0 & a_1 & \cdots & a_{k-1} & a_k \\ & & na_0 & \cdots & 2a_{k-2} & a_{k-1} \\ & & & \cdots & \cdots & \\ & & & \cdots & \cdots & \\ & & & & a_0 & a_1 & \cdots & a_k \\ & & & & 0 & ka_0 & \cdots & a_{k-1} \\ & & & & & a_0 & a_1 & a_{k-1} & a_k \end{pmatrix}.$$

We define d_j as the determinant of the submatrix of M constructed with the first j rows and columns of the matrix M for $j = 1, \dots, 2k+1$. Thus we have the sequence

$$\{d_1, d_2, \dots, d_{2k+1}\}. \quad (27)$$

Consider the discriminant sequence $\{d_2, d_4, \dots, d_{2k}\}$ and the sequence of its signs

$$[\text{sign}(d_2), \text{sign}(d_4), \dots, \text{sign}(d_{2k})],$$

called *sign list*, where as usual the sign function is

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases} \quad (28)$$

For a sign list $[s_1, s_2, \dots, s_n]$ of $f(y)$ we define its *revised sign list* $[l_1, l_2, \dots, l_n]$ as follows:

1. If $s_k \neq 0$ we write $l_k = s_k$.
2. If subsection $[s_i, s_{i+1}, \dots, s_{i+j}]$ of this sign list, which satisfies with $s_{i+1} = \cdots = s_{i+j-1} = 0$ and $s_i s_{i+j} \neq 0$, we replace the subsection $[s_{i+1}, s_{i+2}, \dots, s_{i+j-1}]$ with $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, -s_i, \dots]$ keeping the number of terms.

For convenient we denote by RSL and SL the revised sign list and the sign list of the discriminant sequence, respectively. Then the RSL $[l_1, l_2, \dots, l_n]$ has no zeros between two nonzero members.

From Theorems 2.1 and 3.3 of [41] we obtain the following two theorems.

Theorem 4 *Let $f(y)$ be the polynomial (26) with real coefficient. If the number of the sign changes of the RSL $[d_2, d_4, \dots, d_{2k}]$ is equal to m , and the number of nonzero elements of this RSL is equal to ℓ , then the number of the distinct real roots of the polynomial $f(y)$ is $\ell - 2m$.*

Theorem 5 *Let $f(y)$ be the polynomial (26) with real coefficient such that $f(0) \neq 0$. If the number of the sign changes of the RSL $[d_1d_2, d_2d_3, \dots, d_{2k}d_{2k+1}]$ is equal to m , and the number of nonzero elements of this RSL is equal to l , the number of the negative roots of the polynomial $f(y)$ is $l/2 - m$.*

We separate the study of the roots of the polynomial $f(y)$ of (25) in two cases.

Case 1: The coefficient of the cubic term of $f(y)$ in (25) is zero, i.e.

$$N_2 = -4 - 4a_{21}^2 + 36a_{21}b_{03}^2 + 32a_{21}^3b_{03}^2 + 27b_{03}^4 = 0.$$

Then we have

$$b_{03}^2 = \frac{2}{27} \left(-9a_{21} - 8a_{21}^3 + \sqrt{(3 + 4a_{21}^4)^3} \right). \quad (29)$$

Now we calculate the resultant of the coefficient of y^2 of $f(y)$ in (25) with N_2 with respect to the variable a_{21} and obtain $6912b_{03}^2(1 + 27b_{03}^4)^3 \neq 0$. Thus the coefficient of y^2 in $f(y)$ is nonzero when the coefficient of

y^3 is zero. Multiply this quadratic coefficient and the constant term of $f(y)$ we obtain

$$\begin{aligned} & \frac{8}{3}a_{21} \left(-9a_{21} - 8a_{21}^3 + \sqrt{27 + 108a_{21}^2 + 144a_{21}^4 + 64a_{21}^6} \right) \\ & \times \left[3 + 2a_{21}^2 - \frac{2}{3}a_{21} \left(-9a_{21} - 8a_{21}^3 \right. \right. \\ & \left. \left. + \sqrt{27 + 108a_{21}^2 + 144a_{21}^4 + 64a_{21}^6} \right) \right] \leq 0. \end{aligned} \quad (30)$$

Hence $f(y)$ has at most one positive root. Actually if $a_{21} = 0$ we have $b_{03} = -\sqrt{2}/(3^{3/4})$, and $f(y)$ has no positive roots.

Case 2: The coefficient of the cubic term of $f(y)$ in (25) is nonzero, i.e. $N_2 \neq 0$. Then finding the number of the positive roots of $f(y)$ in (25) is equivalent to find the number of the negative roots of $-f(-y)$. Now we shall compute the negative roots of the polynomial $-f(-y)$ using Theorem 5. So we consider the sequence

$$\{d_1d_2, d_2d_3, d_3d_4, d_4d_5, d_5d_6, d_6d_7\} \quad (31)$$

associated to $-f(-y)$, and we have

$$\begin{aligned} d_1 &= N_2, & d_2 &= 3N_2^2, \\ d_3 &= -6b_{03}(3 + 2a_{21}^2 - 9a_{21}b_{03}^2)N_2^2, \\ d_4 &= -6N_2^2N_3, & d_5 &= -4N_2^2N_4, \\ d_6 &= -4(a_{21} + 9b_{03}^2 + 6a_{21}^2b_{03}^2)N_2^2N_5, \\ d_7 &= 24a_{21}b_{03}(a_{21} + 9b_{03}^2 + 6a_{21}^2b_{03}^2)N_2^2N_5, \end{aligned} \quad (32)$$

where

$$\begin{aligned} N_3 &= 8a_{21} + 8a_{21}^3 - 36b_{03}^2 - 204a_{21}^2b_{03}^2 - 172a_{21}^4b_{03}^2 \\ & - 54a_{21}b_{03}^4 + 396a_{21}^3b_{03}^4 - 480a_{21}^5b_{03}^4 - 486b_{03}^6 - 4 \\ & - 567a_{21}^2b_{03}^6, \\ N_4 &= -16a_{21}^2 - 16a_{21}^4 + 198a_{21}b_{03}^2 + 852a_{21}^3b_{03}^2 \\ & + 656a_{21}^5b_{03}^2 + 162b_{03}^4 - 567a_{21}^2b_{03}^4 - 5724a_{21}^4b_{03}^4 \\ & - 5088a_{21}^6b_{03}^4 + 2673a_{21}b_{03}^6 - 486a_{21}^3b_{03}^6 \\ & + 3456a_{21}^5b_{03}^6 + 7200a_{21}^7b_{03}^6 + 8748b_{03}^8 \\ & + 5103a_{21}^2b_{03}^8 - 4860a_{21}^4b_{03}^8, \\ N_5 &= 32a_{21} + 32a_{21}^3 - 36b_{03}^2 - 240a_{21}^2b_{03}^2 - 208a_{21}^4b_{03}^2 \\ & + 2052a_{21}b_{03}^4 + 5040a_{21}^3b_{03}^4 + 3000a_{21}^5b_{03}^4 \\ & - 1944b_{03}^6 - 2025a_{21}^2b_{03}^6. \end{aligned} \quad (33)$$

From Theorem 5 we obtain that the polynomial $-f(-y)$ has three distinct negative roots if and only if the revised sign list of (31) is $[1, 1, 1, 1, 1]$ or $[-1, -1, -1, -1, -1]$ (i.e., $d_i < 0$, $i = 1, 2, \dots, 7$), which cannot be obtained varying the parameters a_{21} and b_{03} . Therefore the polynomial $-f(-y)$ has at most two negative roots.

Now we study the case when the polynomial $-f(-y)$ has distinct negative roots. We denote by $R[f(\alpha), i]$ the i -th real root of the polynomial $f(\alpha)$ with respect to α , and these roots are ordered as follows $R[f(\alpha), i] < R[f(\alpha), j]$ if and only if $i < j$. We describe the possible revised sign lists of the associated discriminant sequences as we show in Tables 1,2,3, when the polynomial $-f(-y)$ has two negative roots, one negative root and no negative roots, respectively, where

$$\begin{aligned} N_6 &= -36 - 21a_{21}^2 + 20a_{21}^4, \\ N_7 &= -108 - 27a_{21}^2 + 99a_{21}^4 + 5a_{21}^6, \\ N_8 &= -81 + 1026a_{21}^2 + 3429a_{21}^4 + 3498a_{21}^6 + 1175a_{21}^8, \\ N_9 &= -648 + 837a_{21}^2 + 3942a_{21}^4 + 3468a_{21}^6 + 1000a_{21}^8, \\ N_{10} &= -128490624 - 62227804500a_{21}^2 - 515496116628a_{21}^4 \\ & - 916466231925a_{21}^6 + 210402679464a_{21}^8 \\ & + 1653908444856a_{21}^{10} + 863216641008a_{21}^{12} \\ & - 432308074320a_{21}^{14} - 139867591104a_{21}^{16} \\ & + 313035878400a_{21}^{18} + 137815040000a_{21}^{20}, \\ N_{11} &= -243 - 4536a_{21}^2 - 9180a_{21}^4 - 3168a_{21}^6 \\ & + 4260a_{21}^8 + 2540a_{21}^{10}. \end{aligned} \quad (34)$$

In Table 1 there are only two subcases, i.e the revised sign lists of (31) are $[1, 1, 1, -1, -1, -1]$ and $[1, 1, 1, 1, 1, -1]$, when the polynomial $-f(-y)$ has two negative roots. In fact, with a computer algebra system such as Mathematica, we can obtain the corresponding conditions of the revised sign lists by solving the inequalities with either $d_{1,2,3} > 0$, $d_{4,5} \leq 0$ and $d_6 < 0$, or $d_{1,2,3,4,5} > 0$ and $d_6 < 0$, respectively.

In summary, the first system of the piecewise differential systems (3) has at most two singular points different from p_j for $j = 1, 2, 3$. Next we shall determine the local phase portraits of these additional finite singular points using the information provided by the infinite singular points.

In the local chart U_2 the first system of (3) becomes

$$\begin{aligned} u' &= \frac{1}{2}(-4 - 4a_{21}^2 - 8b_{03}u + 4a_{21}u^2 - u^4 + 6b_{03}v \\ & - 2a_{21}v^2 + u^2v^2), \\ v' &= -\frac{1}{2}v(2b_{03} - 2a_{21}u + u^3 - uv^2). \end{aligned} \quad (35)$$

Clearly the origin of U_2 is not a singular point. In the local chart U_1 the first system of (3) has the form

$$\begin{aligned} u' &= \frac{1}{2}(1 - 4a_{21}u^2 + 8b_{03}u^3 + 4u^4 + 4a_{21}^2u^4 - 6b_{03}u^3v \\ & - v^2 + 2a_{21}u^2v^2), \\ v' &= uv(-a_{21} + 3b_{03}u + 2u^2 + 2a_{21}^2u^2 - 3b_{03}uv + a_{21}v^2). \end{aligned}$$

Table 1 The conditions in order that the revised sign list (RSL) of (31) has two distinct negative roots.

RSL	Conditions
[1, 1, 1, -1, -1, -1]	$R[N_6, 1] < a_{21} < R[N_7, 1], b_{03} \leq R[N_4, 1],$ or $a_{21} = R[N_7, 1], b_{03} < R[N_4, 1],$ or $R[N_7, 1] < a_{21} < 0, b_{03} < R[N_2, 1];$
[1, 1, 1, 1, 1, -1]	$a_{21} \leq R[N_6, 1], b_{03} < R[N_2, 1],$ or $R[N_6, 1] < a_{21} < R[N_7, 1], R[N_4, 1] < b_{03} < R[N_2, 1].$

Table 2 The conditions in order that the revised sign list (RSL) of (31) has one negative root.

RSL	Conditions
[1, -1, 1, 1, 1, 1]	$R[N_{11}, 2] < a_{21} \leq \sqrt{\frac{3}{2}}, R[N_3, 1] < b_{03} \leq R[N_3, 2],$ or $a_{21} > \sqrt{\frac{3}{2}}, R[N_3, 1] < b_{03} < -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}};$
[1, 1, -1, 1, 1, 1]	$\sqrt{\frac{3}{2}} < a_{21} \leq R[N_6, 2], -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}} < b_{03} \leq R[N_4, 1],$ or $a_{21} > R[N_6, 2], -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}} < b_{03} \leq R[N_5, 2];$
[1, 1, 1, -1, 1, 1]	$R[N_8, 2] < a_{21} \leq R[N_9, 2], R[N_5, 1] < b_{03} < R[N_2, 1],$ or $R[N_9, 2] < a_{21} < R[N_{10}, 2], -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}} < b_{03} < R[N_2, 1],$ or $R[N_{10}, 2] \leq a_{21} < \sqrt{\frac{3}{2}}, -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}} < b_{03} < R[N_2, 1], b_{03} \neq R[N_4, 2],$ or $\sqrt{\frac{3}{2}} \leq a_{21} \leq R[N_6, 2], R[N_4, 2] < b_{03} < R[N_2, 1],$ or $a_{21} > R[N_6, 2], R[N_4, 3] < b_{03} < R[N_2, 1];$
[1, 1, 1, 1, -1, 1]	$R[N_{10}, 2] < a_{21} \leq \sqrt{\frac{3}{2}}, R[N_4, 1] < b_{03} < R[N_4, 2],$ or $\sqrt{\frac{3}{2}} < a_{21} \leq R[N_6, 2], R[N_3, 2] < b_{03} < R[N_4, 2],$ or $a_{21} > R[N_6, 2], R[N_3, 2] < b_{03} < R[N_4, 3];$
[1, -1, -1, 1, 1, 1]	$a_{21} > R[N_6, 2], b_{03} \leq R[N_4, 1],$ or $a_{21} > \sqrt{\frac{3}{2}}, b_{03} = -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}};$
[1, 1, 1, -1, -1, 1]	$0 < a_{21} \leq R[N_8, 2], -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}} < b_{03} < R[N_2, 1],$ or $R[N_8, 2] < a_{21} < R[N_9, 2], -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}} < b_{03} < R[N_5, 1],$ or $R[N_{10}, 2] \leq a_{21} < \sqrt{\frac{3}{2}}, b_{03} = R[N_6, 1],$ or $R[N_{10}, 2] < a_{21} \leq R[N_6, 2], b_{03} = R[N_6, 2],$ or $a_{21} > R[N_6, 2], b_{03} = R[N_6, 3];$
[1, -1, -1, -1, 1, 1]	$R[N_9, 2] < a_{21} < R[N_{11}, 2] = 1.20891, R[N_5, 1] < b_{03} \leq -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}},$ or $a_{21} = R[N_{11}, 2], R[N_5, 1] < b_{03} \leq -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}, b_{03} \neq R[N_3, 1],$ or $R[N_{11}, 2] < a_{21}, R[N_5, 1] < b_{03} < R[N_3, 1],$ or $R[N_{11}, 2] < a_{21} < \sqrt{\frac{3}{2}}, R[N_3, 2] < b_{03} \leq -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}};$
[1, 1, -1, -1, -1, 1]	$\sqrt{\frac{3}{2}} < a_{21} \leq R[N_6, 2], R[N_4, 1] < b_{03} \leq R[N_3, 2],$ or $a_{21} > R[N_6, 2], R[N_4, 2] < b_{03} \leq R[N_3, 2];$
[1, -1, -1, -1, -1, 1]	$0 < a_{21} < R[N_9, 2], b_{03} \leq -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}},$ or $a_{21} = R[N_9, 2], b_{03} < -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}},$ or $R[N_9, 2] < a_{21} < R[N_6, 2], b_{03} < R[N_5, 1],$ or $a_{21} > R[N_6, 2], R[N_4, 1] < b_{03} < R[N_5, 1];$
[1, -1, -1, -1, 0, 0]	$a_{21} = R[N_9, 2], b_{03} = -\sqrt{\frac{3+2a_{21}^2}{9a_{21}}},$ or $R[N_9, 2] < a_{21}, b_{03} = R[N_5, 1];$
[-1, 1, 1, -1, -1, -1]	$R[N_7, 1] < a_{21} < 0, R[N_2, 1] < b_{03} < R[N_4, 1];$
[-1, 1, 1, 1, 1, -1]	$a_{21} \leq R[N_7, 1], R[N_2, 1] < b_{03} < 0$ and $b_{03} \neq \sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}},$ or $R[N_7, 1] < a_{21} < 0, R[N_4, 1] < b_{03} < 0$ and $b_{03} \neq \sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}};$
[-1, 1, 1, 1, 0, 0]	$a_{21} < 0, b_{03} = \sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}}.$

Table 3 The conditions in order that the revised sign list (RSL) of (31) has no negative roots.

RSL	Conditions
$[-1, 1, -1, 1, 1]$	$0 < a_{21} < R[N_{10}, 2], R[N_3, 1] < b_{03} \leq R[N_4, 1],$ or $R[N_{10}, 2] \leq a_{21} < R[N_{11}, 2], R[N_3, 1] < b_{03} < R[N_4, 3],$ or $R[N_{11}, 2] \leq a_{21} < R[N_6, 2], R[N_3, 3] < b_{03} \leq R[N_4, 3],$ or $a_{21} > R[N_6, 2], R[N_3, 1] < b_{03} < R[N_4, 4];$
$[-1, 1, 1, -1, 1]$	$0 < a_{21} \leq R[N_8, 2], R[N_5, 1] < b_{03} < R[N_3, 1],$ or $R[N_{10}, 2] \leq a_{21} < R[N_{11}, 2], R[N_2, 1] < b_{03} < R[N_3, 1],$ or $a_{21} \geq R[N_{11}, 2], R[N_2, 1] < b_{03} < R[N_3, 3];$
$[-1, 1, -1, -1, 1]$	$0 < a_{21} < R[N_{11}, 2], b_{03} = R[N_3, 1],$ or $a_{21} \geq R[N_{11}, 2], b_{03} = R[N_3, 3];$
$[-1, 1, 1, -1, -1]$	$0 < a_{21} < R[N_8, 2], R[N_2, 1] < b_{03} < R[N_5, 1];$
$[-1, 1, -1, -1, -1]$	$0 < a_{21} < R[N_{10}, 2], R[N_4, 1] < b_{03} < 0,$ or $R[N_{10}, 2] \leq a_{21} \leq R[N_6, 2], R[N_4, 3] < b_{03} < 0,$ or $a_{21} > R[N_6, 2], R[N_4, 4] < b_{03} < 0,$
$[-1, 1, 1, -1, 0, 0]$	$0 < a_{21} < R[N_8, 2], b_{03} = R[N_5, 1].$

(36)

(39)

The linear part of system (36) on $v = 0$ is

$$\begin{pmatrix} 4uN_{12} & -3b_{03}u^3 \\ 0 & uN_{12} \end{pmatrix}, \quad (37)$$

where $N_{12} = -a_{21} + 3b_{03}u + 2u^2 + 2a_{21}^2u^2$. By computing the resultant of

$$g(u) = u'|_{v=0} = 1 - 4a_{21}u^2 + 8b_{03}u^3 + (4 + 4a_{21}^2)u^4 \quad (38)$$

and uN_{12} with respect to the variable u we obtain the polynomial $-(1 + a_{21}^2)N_2$. And the possible singular points in U_1 are nilpotent when $N_2 = 0$, or nodes when $N_2 \neq 0$.

Now we shall determine the local phase portraits of the infinite singular points in the chart U_1 . We need to find the real solutions in $g(u) = 0$. But we will be able to determine the number and the type of the remaining infinite singular points using Theorems 5 and 3. Then we do not need to calculate explicitly the coordinates of these singular points.

Remark 1 When $u < 0$ the infinite singular points of the first system of (3) in U_1 are virtual points, but there are corresponding infinite singular points in V_1 with $u > 0$ by the symmetry. And the origins of U_2 and V_2 are not infinite singular points for the first system of (3). Hence we must study all real solutions of $g(u) = 0$ for studying the infinite singular points of the first system of (3).

We compute the sequence $\{\tilde{d}_2, \tilde{d}_4, \tilde{d}_6, \tilde{d}_8\}$ of $g(u)$ from (38), and have

$$\begin{aligned} \tilde{d}_2 &= 64(1 + a_{21}^2)^2, \\ \tilde{d}_4 &= 1024(1 + a_{21}^2)^2(2a_{21} + 2a_{21}^3 - 3b_{03}^2), \\ \tilde{d}_6 &= -16384(1 + a_{21}^2)^2(2a_{21} + 2a_{21}^3 + 3b_{03}^2 + a_{21}^2b_{03}^2), \\ \tilde{d}_8 &= -65536(1 + a_{21}^2)^2N_2. \end{aligned}$$

We cannot find the parameter values such that the corresponding RSL be $[1, 1, 1, 1]$, $[-1, -1, -1, -1]$, $[1, 1, 1, -1]$, $[1, -1, -1, -1]$, $[-1, -1, -1, 1]$, $[1, -1, -1, -1]$, $[1, 1, 1, 0]$ or $[-1, -1, -1, 0]$, but we know that there are at most two distinct positive roots of (38), i.e., there are at most two infinite singular points in U_1 .

(a) When the polynomial $g(u)$ has two distinct roots, we obtain that the possible RSL of $g(u)$ is $[1, 1, -1, -1]$, whose condition is $b_{03} < R[N_2, 1]$, i.e.,

$$b_{03} < -\frac{\sqrt{2(-9a_{21} - 8a_{21}^3 + \sqrt{(3 + 4a_{21}^2)^3})}}{3\sqrt{3}}.$$

Since $N_2 \neq 0$, from (37) we have that the two remaining infinite singular points are two nodes in U_1 . Since the piecewise differential systems (3) are symmetric with respect to the origin, systems (3) have two corresponding infinite singular points in V_1 . On the other hand, systems (3) are continuous because $(f^+(x, 0), g^+(x, 0)) = (f^-(x, 0), g^-(x, 0))$ in these systems.

(a.1): If $a_{21} < 0$, the origin p_3 is a center. Hence on the Poincaré sphere the sum of the indices of the known singular points is 10. By Theorem 3, the sum of the indices of the remaining finite singular points must be -8 . From the previous analysis, systems (3) have at most two finite singular points in $y > 0$, which are different from p_j for $j = 1, 2, 3$. Due to the symmetry with respect to the origin of coordinates the remaining finite singular points are four saddles p_j for $j = 4, 5, 6, 7$, where the two saddles p_4 and p_6 are in $y > 0$, and two saddles p_5 and p_7 are in $y < 0$. From (35) and (36) we have $u'|_{u=0} = -2(1 + a_{21}^2) < 0$ in U_2 and $u'|_{u=0} = 2 > 0$ in U_1 . Then we obtain that the local phase portraits at these singular points in the Poincaré disc are shown in Figure 3(a).

Since the finite singular points of the piecewise differential systems (3) are saddles or centers there must be one saddle on the boundary of period annulus of the center. Recall that the period annulus of a center is the maximal connected set formed by the periodic orbits surrounding the center and having in its boundary the center point. Assume that the saddle p_6 of systems (3) is on the boundary of period annulus of the center p_3 , by the symmetry of the \mathbb{Z}_2 -equivariant differential systems (3) with respect to the origin, the saddle p_7 of systems (3) must be on the boundary of period annulus of the center p_3 .

(a.1.1): One stable and one unstable separatrices of the saddle p_4 of systems (3) connect with saddle p_5 and with the infinite singular points A_1 and A_2 of U_1 (see Figure 3(a)), respectively, then the saddles p_6 and p_7 are also on the boundary of the period annulus of the center p_1 . We have that this phase portrait of the piecewise \mathbb{Z}_2 -equivariant cubic systems (3) in the Poincaré disc is topologically equivalent to the phase portrait 1.1 of Figure 1, which for instance can be realized when $a_{21} = -1$ and $b_{03} = -2$.

(a.1.2): The saddle p_4 of systems (3) is on the boundary of period annulus of the center p_1 , then it creates a center-loop. The phase portrait of the piecewise \mathbb{Z}_2 -equivariant cubic systems (3) is topologically equivalent to the phase portrait 1.2 of Figure 1. For instance this phase portrait can be realized when $a_{21} = -1$ and $b_{03} = -5$.

(a.1.3): By the continuity of the global phase portraits with respect to the parameter b_{03} , from the phase portraits 1.1 realized with the parameters $a_{21} = -1$ and $b_{03} = -2$ to the phase portraits 1.2 realized with the parameters $a_{21} = -1$ and $b_{03} = -5$, we have that there must exist one phase portrait that the saddles p_4 and p_6 are on the boundary of period annulus of the center p_1 . Therefore we have the phase portrait 1.3 of Figure 1.

(a.2): If $a_{21} \geq 0$ then the origin p_3 is a saddle. Hence on the Poincaré sphere the sum of the indices of the known singular points is 6. By Theorem 3 the sum of the indices of the remaining finite singular points must be -2 . Hence the finite singular points other than p_j for $j = 1, 2, 3$, can be either two saddles, or four saddles and two centers. From the previous analysis we know that when $a_{12} \geq 0$ the piecewise differential systems (3) have at most one finite singular point in $y > 0$, see Tables 2 and 3. Hence the remaining finite singular points are two saddles p_4 and p_5 , where p_4 is in $y > 0$ and p_5 is in $y < 0$. Then we obtain that the local phase portraits at these singular points in the Poincaré disc are shown in Figure 3(b).

(a.2.1): If the saddle p_3 is the unique saddle of system (3) in the boundary of the period annulus of the center p_1 , taking into account the symmetry of the \mathbb{Z}_2 -equivariant differential systems (3) with respect to the origin, p_3 is also on the boundary of the period annulus of p_2 , creating an eight-figure loop. Then one stable and one unstable separatrices of the saddle p_4 of systems (3) connect with the saddle p_5 and with the infinite singular points A_1 and A_2 of the local chart U_1 (see Figure 3(b)), respectively. Hence we have that this phase portrait of the piecewise \mathbb{Z}_2 -equivariant differential systems (3) in the Poincaré disc is topologically equivalent to the phase portrait 1.4 of Figure 1, which can be realized when $a_{21} = 1$ and $b_{03} = -0.4$.

(a.2.2): The saddle p_4 of systems (3) is on the boundary of the period annulus of the center p_1 , creating a center-loop. Then one stable and one unstable separatrices of the saddle p_3 connect with the infinite singular points A_2 and A_1 of the local chart U_1 , respectively. By the symmetry the phase portrait of the \mathbb{Z}_2 -equivariant piecewise cubic systems (3) is topologically equivalent to the phase portrait 1.5 of Figure 1. This phase portrait can be realized with the parameters $a_{21} = 1$ and $b_{03} = -1$.

(a.2.3): By the continuity of the global phase portraits with respect to the parameter b_{03} , from the phase portrait 1.4 realized when $a_{21} = 1$ and $b_{03} = -0.4$ to the phase portrait 1.5 realized when $a_{21} = 1$ and $b_{03} = -1$ we have that there must exist one phase portrait that the saddles p_3 and p_4 are on the boundary of the period annulus of the center p_1 . Then this phase portrait of the piecewise \mathbb{Z}_2 -equivariant cubic systems (3) is topologically equivalent to the phase portrait 1.6 of Figure 1.

(b) When the polynomial $g(u)$ has one real root, it shows that the possible RSL of $g(u)$ is $[1, 1, -1, 0]$, whose condition is $b_{03} = R[N_2, 1]$. From the previous analysis we know that the infinite singular points are nilpotent. The linear part of the two systems of (3) at an infinite singular point $(u, 0)$ of the local chart U_1 is

$$\begin{pmatrix} 4uN_{12} & \mp 3b_{03}u^3 \\ 0 & uN_{12} \end{pmatrix},$$

where in $\mp 3b_{03}u^3$ we have the minus sign for the first system of (3) and the positive sign for the second system.

In order that the point $(u, 0) \in U_1$ be a nilpotent infinite singular point the two following equations must be satisfied:

$$4uN_{12} = 0, \quad g(u) = 0. \quad (40)$$

The second equation comes from imposing that $(u, 0)$ be an infinite singular point in the local chart U_1 for

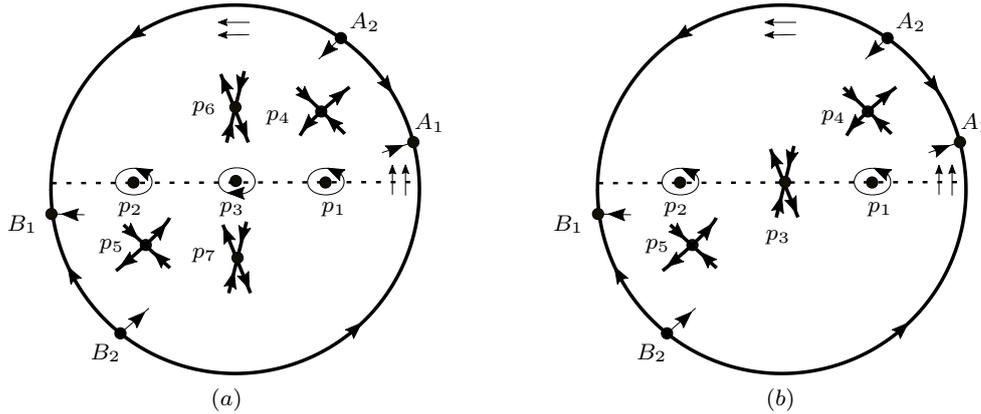


Fig. 3 The local phase portraits at all finite and infinite singular points of the piecewise differential system (3) when $g(u)$ has two distinct roots.

Table 4 The conditions of the revised sign lists of $g(u)$ in (38) without real roots.

RSL	Conditions
$[1, -1, 1, 1]$	$a_{21} < 0, -\sqrt{\frac{-2a_{21}(1+a_{21}^2)}{3+a_{21}^2}} < b_{03} < 0;$
$[1, 1, -1, 1]$	$a_{21} \leq 0, R[N_2, 1] < b_{03} < -\sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}},$ or $a_{21} > 0, R[N_2, 1] < b_{03} < 0;$
$[1, -1, -1, 1]$	$a_{21} < 0, -\sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}} < b_{03} < -\sqrt{\frac{-2a_{21}(1+a_{21}^2)}{3+a_{21}^2}}.$

both systems forming the piecewise differential system (3), see (38). This system has only one real solution

$$(u, b_{03}) = \left(\sqrt{\frac{3}{2\sqrt{4a_{21}^2 + 3} + 2a_{21}}}, \frac{-\sqrt{2(-9a_{21} - 8a_{21}^3 + \sqrt{(3 + 4a_{21}^2)^3})}}{3\sqrt{3}} \right).$$

Then the nilpotent infinite singular point is

$$(u, 0) = \left(\sqrt{\frac{3}{2\sqrt{4a_{21}^2 + 3} + 2a_{21}}}, 0 \right),$$

since its u -coordinate is positive it belongs to the first system of (3) defined in $y > 0$. Applying Theorem 3.5 of [16] such a nilpotent infinite singular point of the first system of (3) is formed by an elliptic sector and a hyperbolic sector, the hyperbolic sector has its two separatrices contained in the straight line of the infinity and its elliptic sector is outside the Poincaré disc, so it does not appear in the phase portrait of our piecewise differential system. But, in fact, the infinite singular point $(u, 0)$ of system (3) in U_1 with $v = u/y < 0$ is equal to the infinite singular point of the second system of (3) in V_1 with $v > 0$. Hence, by the symmetry, the infinite singular point $(u, 0)$ of system (3) in U_1 consists of two hyperbolic sectors, which has index 0.

(b.1): $a_{21} < 0$. Then the origin p_3 is a center. From the previous analysis the piecewise differential systems (3) have at most two finite singular points, which are different from the three centers p_j for $j = 1, 2, 3$.

(b.1.1): We can assume that the remaining finite singular points are two saddles p_4 and p_5 , where the saddle p_4 is in $y > 0$ and the saddle p_5 is in $y < 0$. These piecewise differential systems have the phase portrait 1.7 of Figure 1. For instance the piecewise differential system (3) with $a_{21} = -1$ and $b_{03} = -\frac{1}{9}\sqrt{1 + \sqrt{7}}(5 + \sqrt{7})$ realizes such a phase portrait.

(b.1.2): Assume the remaining finite singular points are two centers p_4 and p_5 , where p_4 is in $y > 0$ and p_5 is in $y < 0$. Then on the Poincaré sphere the sum of the indices of the known singular points is 10. By Theorem 3 the sum of the indices of the remaining two nilpotent infinite singular points must be -8 . But the nilpotent singular point formed by two hyperbolic sectors has index 0. In summary, it follows that the remaining finite singular points p_4 and p_5 cannot be two centers.

(b.2): $a_{21} \geq 0$. Then the origin p_3 is a saddle. Recall that p_1 and p_2 are centers, and the two nilpotent infinite singular points inside the Poincaré disc have a hyperbolic sector with their two separatrices contained in the straight line of the infinity. Then these piecewise differential systems have the phase portrait 1.8 of Figure 1. For instance the piecewise differential system (3)

with $a_{21} = 1$ and

$$b_{03} = (\sqrt{7} - 5) \sqrt{2/(3(1 + \sqrt{7}))}/3$$

realizes such a phase portrait.

(c) When the polynomial $g(u)$ has no real roots, its possible RSL of $g(u)$ are described in Table 4, i.e. $b_{03} > R[N_2, 1]$. In this case we have that there are no infinite singular points in U_1 with $u > 0$.

(c.1): If $a_{21} < 0$ then the origin p_3 is a center. Hence on the Poincaré sphere the sum of the indices of the known singular points is 6. By Theorem 3 the sum of the indices of the remaining finite singular points must be -4 . From the previous analysis, when $a_{21} < 0$ the piecewise differential systems (3) have at most two finite singular points in $y > 0$, which are different from the center p_1 , p_2 and the origin p_3 . By the symmetry the remaining finite singular points are two saddles, where one saddle is in $y > 0$ and the other is in $y < 0$. Similarly to case (b.1.1) we obtain the phase portrait 1.9 of Figure 1, which is achieved when $a_{21} = -1$ and $b_{03} = -1$.

(c.2): If $a_{21} \geq 0$ then the origin p_3 is a saddle. And we know that when $a_{21} \geq 0$ the piecewise differential systems (3) have at most one finite singular point in $y > 0$, which is a center or a saddle. In fact, by the symmetry and the sum of the indices we obtain that the piecewise differential systems (3) have no other finite singular points different from the p_j , $j = 1, 2, 3$. Similarly to case (b.1.2) we obtain the phase portrait 1.10 of Figure 1, which can be realized when $a_{21} = 1$ and $b_{03} = -0.2$.

Thus we have obtained all the phase portraits of the \mathbb{Z}_2 -equivariant cubic Hamiltonian systems (3) with nilpotent bi-centers, which are provided in Theorem 1. On the other hand we note that we can use these series of symbolic way to obtain the phase portraits of piecewise smooth differential systems having more complicated singular points.

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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