Sufficient condition for infinite solutions of Diophantine Equation: $n! = P(s)$

DEVANSH SINGH (✉ ds0324549@gmail.com)

I.E.T. Lucknow

Research Article

Keywords: Laplace Transform, s , $f(t)$

Posted Date: January 11th, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-1246725/v1>

License: This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Sufficient condition for infinite solutions of Diophantine Equation: $n! = P(s)$

Devansh Singh

B.Tech., M.E.-4th Year

Department of Mechanical Engineering

Institute of Engineering and Technology, Lucknow, Uttar Pradesh, India

ds0324549@gmail.com , 8787011853, ORCID- 0000-0002-4947-8297

Keywords-Laplace Transform, s , $f(t)$.

Abbreviations- $P(s)$ is Polynomial of degree k in variable s . $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$, where $s > 0$. $L[f(t)]$ is called Laplace transform of $f(t)$. $f'(t)$ is first derivative of $f(t)$.

Abstract- In this article I have used method which tells that number of solutions of Diophantine equation: $n! = P(s)$ is infinite if some condition is satisfied. I have applied Inverse Laplace Transform to $n! = P(s)$ and got function $f(t)$ which is easier to deal with. . The condition is given in section below contains zero of $f(t)$ or zero of some modified function of $f(t)$: $g(t) = f(t) - h(t)$.

Introduction- I will find sufficient condition for infinite number of solutions of Wilson Primes as an example and then generalize to Diophantine equation: $n! = P(s)$. If p is Wilson Prime^[1] then $(p-1)! \equiv -1 \pmod{p^2}$ or $(p-1)! = p^2 \cdot q - 1$, where $q \geq 1$. Known Wilson primes are 5, 13, 563.

Let $s=p^2$ then $s \cdot q - 1 = (p-1)! - (i)$.

Applying Inverse Laplace Transform on equation (i):

We can apply inverse Laplace transform on equation (i) to get a function which is simpler to deal with than its predecessor.

Comparing 1st eqn. equation with $s.L[f(t)] - f(0) = L[f'(t)]$, we get $L[f(t)] = q, f(0) = 1$ and $L[f'(t)] = (p-1)!$. From, $L[f'(t)] = (p-1)!$, we get: $L[f'(t)] = L[s^p \cdot t^{p-1}]$ (as $L[t^n] = n!/s^{n+1}$) or $f'(t) = s^p \cdot t^{p-1}$ -eqn. (ii).

After integrating $f(t)$ with respect to t in eqn. (ii), we get: $f(t) = s^p \cdot \frac{t^p}{p} + c$, where c is constant.

Applying $f(0) = 1$ to $f(t)$, we get: $f(t) = s^p \cdot \frac{t^p}{p} + 1$. Let $g(t) = s^p \cdot \frac{t^p}{p} - 1$.

Existence of $L[f(t)]$:

^[2]Sufficient conditions for existence of $L[f(t)]$:

- 1) $f(t)$ is piecewise-continuous on $[0, \infty)$.
- 2) $f(t)$ is of exponential order c if there exist positive real numbers M, c : $f(t) \leq M \cdot e^{ct}$ for all $t > t'$.

We have:

- a) $f(t)$ is piecewise-continuous on $[0, \infty)$.
- b) $\because f(t) = s^p \cdot \frac{t^p}{p} + 1 < \frac{2s^p e^{pt}}{p}$ for $\forall t > 1$. There exist positive real numbers M, c such that $f(t) \leq M \cdot e^{ct}$ for all $t > 1$, where $M = \frac{2s^p}{p}$ and $c = p$. $\therefore f(t)$ is of exponential order ' c '.

So, $L[f(t)]$ exists for $s = p^2 > c$.

Now we have a function $g(t)$ which is zero at $t = t_0(p) = \frac{p^{\frac{1}{p}}}{s} = \frac{p^{\frac{1}{p}}}{p^2}$. Let set of Wilson Primes be S , then $n \cdot \min(t_0) < \sum_{p \in S} \frac{p^{\frac{1}{p}}}{p^2} = \sum t_0(p) < n \cdot \max(t_0)$, where $\max(t_0)$ is maximum $t_0(p)$ and n is number of solutions to equation (i).

$\because \sum_{p \in S} \frac{p^{\frac{1}{p}}}{p^2} < \sum_{p \in S} \frac{5^{\frac{1}{5}}}{p^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Sufficient condition for convergence).

$\therefore \sum_{p \in S} \frac{p^{\frac{1}{p}}}{p^2} = \sum_{p \in S} t_0(p)$ is convergent.

$\sum_{p \in S} t_0(p)$ is convergent $\Rightarrow n$ is finite or infinite. But if $\exists$ Sufficient condition for divergence of $\sum_{p \in S} t_0(p)$ or $\sum_{p \in S} t_0(p)$ is divergent then n is infinite. For other example read this[\[3\]](#).

Above can be generalized to $n! = P(s)$ assuming that we have $f(t)$ or $g(t)$ for it . With proper selection of s such that Sufficient conditions for existence of $L[f(t)]$ are satisfied we can make $f(t)$ or can make $g(t) = f(t) - h(t)$.

Let zero of $f(t)$ or $g(t)$ be $t_0(i)$, where ‘ i ’ is a function of s and some other variable p like in Wilson’s Prime example. Then-

Condition- If $\exists$ Sufficient condition for divergence of $\sum_i t_0(i)$ then n is infinite or number of solutions of $n! = P(s)$ is infinite.

References-

[1.] (Wilson Prime - Wikipedia)

[2.] Devansh Singh. Prime n solutions of the Brocard’s Problem, 30 December 2021, PREPRINT (Version 1) available at Research Square [<https://doi.org/10.21203/rs.3.rs-1217305/v1>]