A Practical Approach to choosing Formulae for Economic Index Numbers

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Research Article

Keywords: Index Numbers, Inflation, National Accounts, Revealed Preferences, Substitution Effect

Posted Date: January 13th, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-1254241/v1>

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Title Page

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Abstract

Traditional axiomatic, economic and stochastic approaches to choosing a formula for economic index numbers are based on theoretical considerations, with little regard to whether they reflect the actual circumstances affecting the markets to which they relate. This paper presents an approach to index number formulation based on how markets operate and presents a general, parameter-based formula for price and quantity indices. Two variants of this general formula cater for the "substitution effect" in different ways and one of these variants provides a practical expression for an economic-theoretic index based on purchasers' revealed preferences. Another variant applies this approach to short-term inflation indices. A final variant provides a straightforward means of estimating purchasing power parities for spatial indices. The analysis also emphasises the importance in National Accounts of using coherent price and quantity indices, whose products generate the corresponding value indices.

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MSC codes: 91B82, 62P20

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Abstract

Traditional axiomatic, economic and stochastic approaches to choosing a formula for economic index numbers are based on theoretical considerations, with little regard to whether they reflect the actual circumstances affecting the markets to which they relate. This paper presents an approach to index number formulation based on how markets operate and presents a general, parameter-based formula for price and quantity indices. Two variants of this general formula cater for the “substitution effect” in different ways and one of these variants provides a practical expression for an economic-theoretic index based on purchasers’ revealed preferences. Another variant applies this approach to short-term inflation indices. A final variant provides a straightforward means of estimating purchasing power parities for spatial indices. The analysis also emphasises the importance in National Accounts of using coherent price and quantity indices, whose products generate the corresponding value indices.

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1. Introduction

Since Dutot (1738) presented what is generally regarded as the first price index number formula, a large variety of price and quantity index number formulae have been proposed. The first important survey of all extant index number formulae was by Fisher (1922), a major classic of index number theory. Balk (2008) presents a more up-to-date analysis of the history and current state of play of index number theory, concentrating on the axiomatic or “test” approach but with brief references to the economic and stochastic approaches, for which see Frisch (1936) and Selvanathan (1994) respectively.

Throughout this time, most of the discussion regarding index number formulae has been with regard to theoretical considerations, such as whether arithmetic or geometric formulae are preferable, with little regard for what is appropriate for the actual market to which these indices are to be applied. It is the purpose of this paper to take a practical look at how economic markets actually work and develop formulae that reflect these practical considerations.

In principle, previous theoretical work should have no bearing on the findings of this paper but the index number formulae generated turn out to be based on the well-known Laspeyres and Paasche indices, so it is worthwhile defining these formulae and the notation I use at this stage.

I define P_{12} and Q_{12} to be, respectively, the price and quantity indices from period 1 to period 2. The additional suffices (L) and (P) indicate, respectively, the corresponding Laspeyres and Paasche indices. For price and quantity vectors $\mathbf{p}_1, \mathbf{q}_1$ for period 1 and $\mathbf{p}_2, \mathbf{q}_2$ for period 2, we have:

$$P_{12(L)} = \frac{\mathbf{p}_2' \mathbf{q}_1}{\mathbf{p}_1' \mathbf{q}_1}; \quad Q_{12(L)} = \frac{\mathbf{p}_1' \mathbf{q}_2}{\mathbf{p}_1' \mathbf{q}_1}; \quad P_{12(P)} = \frac{\mathbf{p}_2' \mathbf{q}_2}{\mathbf{p}_1' \mathbf{q}_2}; \quad Q_{12(P)} = \frac{\mathbf{p}_2' \mathbf{q}_2}{\mathbf{p}_2' \mathbf{q}_1}$$

Note that the price and quantity vectors must all have the same length. Occurrences of products with zero value in any period may be accommodated by assigning a zero price or quantity for the relevant product.

Section 2 derives general index number formulae to accommodate the different ways that prices and quantities are determined in different product markets. This section is concerned only with the mechanics of price and quantity determination. Section 3 modifies these formulae to correct for inadequacies in the pricing model used in section 2. Section 4 presents a further modification to accommodate aggregate trends, leading to an index formula based on relative changes in purchasers’ product choices. This formula is shown to be a practical expression for an economic-theoretic index. Section 5 uses the modification of section 4 to create a short-term inflation index that is coherent with National Accounts’ quantity measures. Section 6 applies the formulae of section 4 to spatial comparisons. Finally, section 7 presents a summary and review of the formulae presented and makes a plea for greater coherence between short-term inflation measures and National Accounts’ quantity measures.

2. Formulae based on market mechanisms for determining prices and quantities

Markets operate in diverse ways. In some, vendors set prices and purchasers choose what quantities to buy at those prices. In others, a fixed set of quantities are available for sale and purchasers bid for (that is, choose) which prices to pay for the available quantities. In some markets, neither quantities nor prices are fixed but are determined by negotiation between vendors and purchasers. I therefore assume the following three types of market:

- A. in any period, vendors present products at pre-determined fixed prices and purchasers choose what quantities to buy at these fixed prices;
- B. in any period, vendors present fixed quantities of products and purchasers choose what prices to pay for these fixed quantities;
- C. in any period, vendors present products at negotiable prices and purchasers negotiate with vendors to determine what quantities to buy at what prices.

Markets of type A are typical of retail and intermediate goods and services markets in developed economies. In such markets, prices and quantities develop as follows: first, prices in period 1 (p_1) are fixed and purchasers then decide the quantities purchased in period 1 (q_1); on the basis of these quantities and other factors, vendors then adjust their prices for period 2 (p_2) and purchasers then decide the quantities purchased for period 2 (q_2); and so on. This process may be illustrated diagrammatically using the arrow $\rightarrow$ to indicate the time direction of influence between prices and quantities:

$$p_1 \rightarrow q_1 \rightarrow p_2 \rightarrow q_2 \rightarrow p_3 \rightarrow q_3 \rightarrow \dots$$

In such a market, it is clear that prices p_2 are determined before quantities q_2 are known. It would therefore be inappropriate to use quantities q_2 to calculate an index of price change from p_1 to p_2 because the quantities q_2 are not relevant to that price change, as they are decided only after the price change has occurred. The appropriate index is the Laspeyres index $\frac{p'_2 q_1}{p_1 q_1}$, as indicated by the fact that the quantities q_1

intermediate between the prices p_1 and p_2 in the sequence $p_1 \rightarrow q_1 \rightarrow p_2$.

By a similar argument, in markets of type A the appropriate index to use for the quantity change from q_1 to q_2 is the Paasche index $\frac{p'_2 q_2}{p_2 q_1}$, with the prices p_2 intermedating between the quantities q_1 and q_2 in the sequence $q_1 \rightarrow p_2 \rightarrow q_2$.

In developed economies, markets of type B mainly comprise wholesale markets of perishable commodities, such as meats, fish, fruit, vegetables and grains, which have to be sold as they come to market before they deteriorate. For many of these commodities, the quantities available can change daily because of variations in maturation rates or the vagaries of the weather and prices change accordingly. Retail auction markets may also be of type B, although the occasional specification of reserve prices means that not all quantities are fixed but are subject to implicit price negotiation, giving them some of the character of a type C market. On the other hand, the supply to wholesale markets of non-perishable commodities such as petroleum oil and minerals may be controlled to allow negotiation over quantities and prices as in a market of type C but supply constraints, cash flow requirements and political considerations mean that these markets often function more like a market of type B.

In markets of type B, prices and quantities develop as follows: first, quantities in period 1 (q_1) are fixed and purchasers then decide the prices they are willing to pay for the quantities available in period 1 (p_1); on the basis of these quantities and other factors, vendors then adjust the supplied quantities for period 2 (q_2) and purchasers then decide what prices to pay for period 2 (p_2); and so on. The diagrammatic illustration of this sequence is:

$$q_1 \rightarrow p_1 \rightarrow q_2 \rightarrow p_2 \rightarrow q_3 \rightarrow p_3 \rightarrow \dots$$

In such a market, it is clear that quantities q_2 are determined before prices p_2 are known. It would therefore be inappropriate to use prices p_2 to calculate an index of quantity change from q_1 to q_2 because the prices p_2 are not relevant to that quantity change, as they are decided only after the quantity change has occurred. The appropriate index is the Laspeyres index $\frac{p_1'q_2}{p_1'q_1}$, as indicated by the fact that the prices p_1 intermediate

between the quantities q_1 and q_2 in the sequence $q_1 \rightarrow p_1 \rightarrow q_2$.

By a similar argument, in markets of type B the appropriate index to use for the price change from p_1 to p_2 is the Paasche index $\frac{p_2'q_2}{p_1'q_2}$, with the quantities q_2 intermediating between the prices p_1 and p_2 in the sequence

$p_1 \rightarrow q_2 \rightarrow p_2$.

In markets of type C, prices and quantities are decided simultaneously: vendors and purchasers negotiate what quantities they are willing to exchange at whatever mutually acceptable prices they can agree on. As a consequence, prices and quantities in each period are mutually influential, as well as being respectively influential on quantities and prices in the subsequent period. Flea-markets and bazaars are generally of type C but in developed economies the most economically significant market of this type is that for owner-occupied housing. The sequence of influence between prices and quantities is illustrated in the following diagram:

For market C, the appropriate index number formula for prices or quantities lies somewhere between the corresponding Laspeyres and Paasche indices. There are many ways of approaching the problem of combining Laspeyres and Paasche indices to produce the required composite index. The approach I adopt in this paper is aimed at maintaining the aggregative properties of the Laspeyres and Paasche indices. In practice, ease of aggregation is a useful property in National Accounts calculations. As is shown below, this also provides a coherent framework for a variety of indices in different contexts.

Before developing this framework, it is appropriate to indulge in a short philosophical digression by asking the question: what is a period?

As noted above, under type B markets, prices may change on a daily basis. This presents no problem for the analysis above if price indices are calculated on a daily basis but, in practice, most economic indices are calculated monthly, quarterly or annually. As a consequence, for a monthly index, some of the quantities in type B markets will be affected by changes in prices within the month (to the extent that changes in quantities are affected by human decisions rather than the providence of nature) giving the monthly index a flavour of a type C index. However, prices will **only** depend on quantities in the same month, maintaining the crucial one-way relationship $q_1 \rightarrow p_1$ that supports the conclusion made above.

On the other hand, if quantities are determined on a weekly basis but prices vary daily, a few prices will depend on quantities in the preceding month, removing the $q_1 \rightarrow p_1$ one-way relationship and requiring an index closer to a type C index, although the deviation from a type B index will depend on the relative influence of such quantities.

Consider another example for a type A market. Suppose that vendors make their price decisions in the middle of each calendar month, based on observed quantities in the preceding calendar month. In this case, quantities purchased will depend not only on prices established in the current month but also on prices from the preceding month, removing the $p_1 \rightarrow q_1$ one-way relationship and requiring an index closer to a type C index, with the deviation from a type A index also depending on the relative influence of such prices.

Clearly, in practice, there are unlikely to be any purely type A and type B markets in the context of index number construction but, although all of them may be nominally of type C, most are likely to veer towards a type A or type B market to varying degrees.

To develop an appropriate index number formula, I therefore assume that a certain proportion λ_i of product i is subject to a Laspeyres price index and the complementary proportion $1-\lambda_i$ is subject to a Paasche price index. Similarly, a proportion λ_i of product i is subject to a Paasche quantity index and the complementary proportion $1-\lambda_i$ is subject to a Laspeyres quantity index. The proportion λ_i represents the degree to which the market approaches type A. Thus: in a market where both prices and quantities are freely negotiable, as in the residential housing market, $\lambda_i=1/2$; in a market where prices are mainly fixed but some informal price negotiation occurs, λ_i is close to 1; in a market where quantities are mainly fixed but there is some leeway to vary them, as in auctions with reserve prices, λ_i is close to 0. Clearly, in the two extreme cases, $\lambda_i=1$ corresponds to a type A market and $\lambda_i=0$ corresponds to a type B market. In the more general case, including those cases where the markets are nominally of type A or B but where price and quantity changes do not align exactly with period boundaries, it may be difficult to determine a precisely correct value for λ_i and some rough-and-ready estimate be have to be used. In practice, it may be sufficient to assume that $\lambda_i \in \{0, 1/2, 1\}$ and to assign each product to a market of type A, C or B according to whichever seems most appropriate.

This approach means that the aggregation of price and quantity indices from markets of different types is easily achieved through the same aggregation process as for Laspeyres and Paasche indices, namely through the summation of the numerators and denominators of the constituent indices. Defining λ as the vector of values $\{\lambda_i\}$ for all products in the target index and letting $\Lambda = \text{diag}(\lambda)$, first define the following (in some cases, notional) expenditures that are the basic components in the construction of the required indices:

$$\begin{pmatrix} A_{11} & A_{21} & A_{22} \\ B_{11} & B_{12} & B_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{p}'_1 \Lambda \mathbf{q}_1 & \mathbf{p}'_2 \Lambda \mathbf{q}_1 & \mathbf{p}'_2 \Lambda \mathbf{q}_2 \\ \mathbf{p}'_1 (\mathbf{I} - \Lambda) \mathbf{q}_1 & \mathbf{p}'_1 (\mathbf{I} - \Lambda) \mathbf{q}_2 & \mathbf{p}'_2 (\mathbf{I} - \Lambda) \mathbf{q}_2 \end{pmatrix} \quad (1)$$

Using this notation for the basic expenditure components eases reference to these components in some of the discussion below. I have chosen the notation $\{A_{st}, B_{st} : s, t \in \{1, 2\}\}$ to reflect the markets defined as type A and type B. As discussed above, type A markets have Laspeyres price indices and Paasche quantity indices (each requiring the notional expenditure A_{21}) whereas type B markets have Paasche price indices and Laspeyres quantity indices (each requiring the notional expenditure B_{12}).

Note, also, that $A_{11} + B_{11} = \mathbf{p}'_1 \mathbf{q}_1 = E_1$ and $A_{22} + B_{22} = \mathbf{p}'_2 \mathbf{q}_2 = E_2$, where E_1 and E_2 are the total expenditures covered by the index in periods 1 and 2 respectively.

The composite indices may then be written as:

$$\begin{aligned} P_{12} &= \frac{A_{21} + B_{22}}{A_{11} + B_{12}} = \frac{\mathbf{p}'_2 \Lambda \mathbf{q}_1 + \mathbf{p}'_2 (\mathbf{I} - \Lambda) \mathbf{q}_2}{\mathbf{p}'_1 \Lambda \mathbf{q}_1 + \mathbf{p}'_1 (\mathbf{I} - \Lambda) \mathbf{q}_2} \\ Q_{12} &= \frac{A_{22} + B_{12}}{A_{21} + B_{11}} = \frac{\mathbf{p}'_2 \Lambda \mathbf{q}_2 + \mathbf{p}'_1 (\mathbf{I} - \Lambda) \mathbf{q}_2}{\mathbf{p}'_2 \Lambda \mathbf{q}_1 + \mathbf{p}'_1 (\mathbf{I} - \Lambda) \mathbf{q}_1} \end{aligned} \quad (2)$$

Note that weighting of the Laspeyres and Paasche components is automatic because it is implicit in the summation of the expenditure totals in the numerator and denominator. This is a convenient facility that is related to the desired ability for aggregation of the component expenditure totals within National Accounts. Other methods of combining the two components, such as a geometric mean, do not offer this convenience because the weighting must be made explicitly (and could, in principle, be arbitrary).

It is important to note that these composite indices do not, in general, satisfy the fundamental index number identity $P_{12}Q_{12} = E_2/E_1$. However, the error involved is considerably smaller than a second order difference and is probably negligible, as demonstrated in Appendix A.

When price or quantity indices are being calculated in isolation, this doesn't matter. In National Accounts calculations, a simple remedy is to calculate price indices as implied deflators, which is already a standard practice in this context. Such implied deflators should not be very different from the directly calculated price indices from expression (2). When a more equitable conformity with the fundamental index number identity is deemed essential, I recommend raising these indices to the power $(1+\varepsilon)$, where ε is very small, such that $P_{12}^{1+\varepsilon} Q_{12}^{1+\varepsilon} = E_2/E_1$. This maintains the relative contributions from price and quantity changes to the value change. Such an adjustment would have to be made separately at each stage of aggregation, allowing the aggregation of the numerators and denominators of P_{12} and Q_{12} to proceed unhindered through the hierarchy.

It is possible to modify the formulae of expression (2) to ensure that the fundamental index number identity is met exactly by modifying the weighting of the Laspeyres and Paasche components but the modifying weights are data-sensitive and require a complicated, and possibly unstable, optimisation procedure so that the negligible benefit obtained does not justify the effort involved. Appendix B provides more detail on this point.

3. Formulae based on a model of the interaction between price and quantity changes

The composite index number formulae discussed above are based on the fundamental properties of the market, considering only the mechanism that generates price and quantity changes. However, in some circumstances, this mechanism does not conform to the underlying assumption of prices and quantities following a sequence of mutual cause and effect. For example, in the retail industry, regular discount sales are often held in order to sell off old or out-of-season stock, such as in “winter sales” and “summer sales”. In such cases, the post-sale price increases are not a reaction to a sudden, unexpected surge in demand but are actually pre-planned, in order to encourage consumers to buy at sale prices before prices return to their previous level. As these sales are conducted in a market of type A, the post-sale price index number is calculated using the quantities sold during the sale period. These relatively large quantities are applied to relatively large price increases, leading to a large post-sale average price increase, which is counter-intuitive as the number of goods actually subject to the post-sale price increases is only the relatively small post-sale quantities. To some extent, over a longer time-span, this apparent overstatement of the average price increase is offset by the price reductions going into the sale period. However, as the sales discounts are only weighted by the relatively small pre-sale quantities, the resultant average price discount will not completely offset the apparently excessive post-sale average price increase.

It may be possible to apply adjustments to the formulae of section 2 to cater for such anomalies but such treatment would necessarily be arbitrary and would be complicated to implement. The model of the market mechanism described in section 2 is not always valid and a better approach would be to replace it with a more reliable model that is not subject to distortions such as that described in the previous paragraph.

To construct such a model, we need to consider how price changes and quantity changes interact in practice. For this purpose, an appropriate simplification is to treat the set of all purchasers in the population covered by the index as a collective. In reality, the actions of this collective are the aggregation of the actions of individual purchasers acting independently. However, for an individual purchaser, the interaction between price changes and quantity changes is unclear in certain circumstances. An example is the case of occasional purchases of infrequently purchased products, such as consumer durable goods. In this case, a typical monthly time series of the quantity purchased by an individual takes the form $(...0,0,0,0,1,0,0,0,...)$. In most months, the individual does not participate in the market at all for such a product. For such products, the concept of price change between successive months for an individual purchaser is meaningless. For the collective, however, it is possible to measure the changing prices and quantities of all products for consecutive periods, even if it is different individuals who are making the purchases in each period.

The collective, then, purchases, in period 1, a quantity q_{1i} of product i at price p_{1i} and, in period 2, a quantity q_{2i} of product i at price p_{2i} . What is the effect of price change on the collective's change in expenditure

between these two periods? Clearly, $p_{2i}-p_{1i}$ is the price change arising from the purchase of one item of product i in each period. If $q_{1i}=q_{2i}$ then the entire expenditure change $q_{2i}p_{2i}-q_{1i}p_{1i}$ may reasonably be attributed to price change as there is no quantity change. But what is the aggregate price change if $q_{1i}\neq q_{2i}$?

If $q_{1i}<q_{2i}$ then the additional cost in period 2 of buying the quantities purchased in period 1, $q_{1i}p_{2i}-q_{1i}p_{1i}$, may reasonably be attributed to price change but the other component of expenditure change, $q_{2i}p_{2i}-q_{1i}p_{2i}$, is actually a quantity change, reflecting the additional cost of the extra quantity purchased in period 2, whatever the reason for that extra purchase.

If $q_{1i}>q_{2i}$ then the collective has chosen to reduce the quantity of product i purchased, for whatever reason, preferring to accept an aggregate price change $q_{2i}p_{2i}-q_{2i}p_{1i}$ on the quantity q_{2i} only, leaving the expenditure $q_{1i}p_{1i}-q_{2i}p_{1i}$, foregone from period 1, available for the purchase of other products in period 2. The negative of this expenditure, $q_{2i}p_{1i}-q_{1i}p_{1i}$, a saving, is the other component of expenditure change, a negative quantity change representing substitution away from product i in favour of other products.

As the same argument applies to all products, we may decompose the change in expenditure over all products as follows:

$$\mathbf{p}'_2\mathbf{q}_2 - \mathbf{p}'_1\mathbf{q}_1 = \left\{ \begin{array}{l} \sum_{i:q_{1i}\leq q_{2i}} (p_{2i} - p_{1i})q_{1i} + \sum_{i:q_{1i}\leq q_{2i}} p_{2i}(q_{2i} - q_{1i}) \\ + \sum_{i:q_{1i}>q_{2i}} (p_{2i} - p_{1i})q_{2i} + \sum_{i:q_{1i}>q_{2i}} p_{1i}(q_{2i} - q_{1i}) \end{array} \right\} \quad (3)$$

The top line of the right hand side of this expression shows the contributions to expenditure change from those products for which $q_{1i}\leq q_{2i}$, with price changes valued at period 1 quantities and quantity changes valued at period 2 prices, as discussed above. This corresponds to a Laspeyres price index and a Paasche quantity index for these products.

The bottom line shows the contributions from those products for which $q_{1i}>q_{2i}$, with price changes valued at period 2 quantities and quantity changes valued at period 1 prices, corresponding to a Paasche price index and a Laspeyres quantity index for these products. For completeness, I have included the case $q_{1i}=q_{2i}$ with the cases $q_{1i}<q_{2i}$ rather than with the cases $q_{1i}>q_{2i}$ but the choice is arbitrary and has no effect on this linear decomposition. However, treatment of the cases where $q_{1i}=q_{2i}$ is less clear-cut when taking ratios of the expenditures for index number construction. This question is discussed in more detail below.

In the matrix of contributions in expression (3), those in the left column relate to price changes and those in the right column to quantity changes. Summing the changes for prices and quantities separately and converting the differences shown to the ratios of expenditures used for index numbers gives the following composite indices:

$$P_{12} = \frac{\sum_{i:q_{1i}\leq q_{2i}} p_{2i}q_{1i} + \sum_{i:q_{1i}>q_{2i}} p_{2i}q_{2i}}{\sum_{i:q_{1i}\leq q_{2i}} p_{1i}q_{1i} + \sum_{i:q_{1i}>q_{2i}} p_{1i}q_{2i}}; \quad Q_{12} = \frac{\sum_{i:q_{1i}\leq q_{2i}} p_{2i}q_{2i} + \sum_{i:q_{1i}>q_{2i}} p_{1i}q_{2i}}{\sum_{i:q_{1i}\leq q_{2i}} p_{2i}q_{1i} + \sum_{i:q_{1i}>q_{2i}} p_{1i}q_{1i}}$$

Regarding the treatment of cases where $q_{1i}=q_{2i}$, we find that the following equality holds for the price index because moving these cases from one summation to the other has no effect on the numerator and denominator totals because $q_{1i}=q_{2i}$ for these cases:

$$\frac{\sum_{i:q_{1i}\leq q_{2i}} p_{2i}q_{1i} + \sum_{i:q_{1i}>q_{2i}} p_{2i}q_{2i}}{\sum_{i:q_{1i}\leq q_{2i}} p_{1i}q_{1i} + \sum_{i:q_{1i}>q_{2i}} p_{1i}q_{2i}} = \frac{\sum_{i:q_{1i}<q_{2i}} p_{2i}q_{1i} + \sum_{i:q_{1i}\geq q_{2i}} p_{2i}q_{2i}}{\sum_{i:q_{1i}<q_{2i}} p_{1i}q_{1i} + \sum_{i:q_{1i}\geq q_{2i}} p_{1i}q_{2i}}$$

but the corresponding equality does not hold for the quantity index:

$$\frac{\sum_{i:q_{1i} \leq q_{2i}} p_{2i}q_{2i} + \sum_{i:q_{1i} > q_{2i}} p_{1i}q_{2i}}{\sum_{i:q_{1i} \leq q_{2i}} p_{2i}q_{1i} + \sum_{i:q_{1i} > q_{2i}} p_{1i}q_{1i}} \neq \frac{\sum_{i:q_{1i} < q_{2i}} p_{2i}q_{2i} + \sum_{i:q_{1i} \geq q_{2i}} p_{1i}q_{2i}}{\sum_{i:q_{1i} < q_{2i}} p_{2i}q_{1i} + \sum_{i:q_{1i} \geq q_{2i}} p_{1i}q_{1i}}$$

because there is a difference between the two expressions of $(p_{1j} - p_{2j})q_{2j} = (p_{1j} - p_{2j})q_{1j}$ in both numerator and denominator for those quantities for which $q_{1j} = q_{2j}$. Although the magnitude of this difference is the same in both numerator and denominator, the relative differences are not generally identical because the other components are usually not the same in the numerator as in the denominator. This leads to a (probably small) difference in the aggregate index.

The occurrence of identical quantities in successive periods is likely to be rare and the effect of choosing one expression in favour of the other is likely to be small, so this uncertainty is not a major concern. Nonetheless, it is helpful to consider how to resolve this problem. To do so, we may use the parameters $\{\lambda_i\}$ introduced for expression (2) but define them as:

$$\lambda_i = \begin{cases} 1 & : q_{1i} < q_{2i} \\ v_i & : q_{1i} = q_{2i} \\ 0 & : q_{1i} > q_{2i} \end{cases} \quad (4)$$

where v_i is an arbitrary value in the range $[0,1]$.

A sensible approach may be to set $v_i = 1/2$ but it may also be possible to choose a set of $\{v_i\}$ that creates a value of Q_{12} that satisfies, or nearly satisfies, the fundamental index number identity $P_{12}Q_{12} = E_2/E_1$. Bearing in mind the conclusion of Appendix A, whether it is worthwhile expending the effort required to do so is moot. The next section presents an alternative set of parameters $\{\lambda_i\}$ where the value of v_i is determined by the observed data.

Note that this approach means that the resulting indices are given by expression (2) with the caveat that the parameters $\{\lambda_i\}$ are defined by definitions (4) rather than by relation to the operational properties of the market, as discussed in section 2. This difference is important. In the original definition, the $\{\lambda_i\}$ are defined according to the operational properties of the market, without regard to the observed data. In definitions (4), they are defined according to observed changes in quantities, without regard to the mechanisms that led to these changes.

This formulaic identity means that the discussion after expression (2) regarding the very close approximation to the fundamental index number identity $P_{12}Q_{12} = E_2/E_1$ and the comments on how to handle any slight divergences that arise also apply when using $\{\lambda_i\}$ defined by definitions (4) because the conclusions of that discussion depend only on the concept of splitting the component parts of the index between Laspeyres and Paasche formulations, not on how that split is determined.

Because expression (2) is an amalgam of Laspeyres and Paasche indices, it appears similar in concept to indices that are symmetric or “superlative” (in the sense defined by Diewert (1976)), such as the Marshall-Edgeworth and Fisher indices. However, indices arising from $\{\lambda_i\}$ defined as in definitions (4) are neither symmetric nor “superlative”. In boom times, when all quantities are increasing, they revert to a Laspeyres price index and a Paasche quantity index. In times of depression, when all quantities are decreasing, they revert to a Paasche price index and a Laspeyres quantity index. This tendency is intuitively appealing: in boom times, there is more “froth” in the economy, inflation is likely to take off and people may be willing to pay more than they might otherwise do, which is consistent with the faster moving Laspeyres price index; in

times of depression, people are more cautious about spending and are looking for bargains, which is consistent with the slower moving Paasche price index.

4. Formulae based on the effects of relative price and quantity changes on purchasers' welfare

The point of view implicit in definitions (4) in the preceding section is that substitution between products relates to absolute changes in quantities – that is, quantities q_{1i} and q_{2i} are compared directly to determine whether product i has been substituted in or substituted out. A possible objection to this point of view is that the concept of “substitution” actually relates to relative changes in quantities, excluding secular changes arising from changes in the size of the target population or changes in aggregate income that affect the total quantity of products purchased. Under this point of view, the relative comparator for q_{2i} is not q_{1i} but q_{1i} adjusted for the average change in quantities purchased between periods 1 and 2 – that is, q_{1i} multiplied by the quantity index Q_{12} . On this basis, the appropriate $\{\lambda_i\}$ to use in expression (2) would be:

$$\lambda_i = \begin{cases} 1 & : Q_{12}q_{1i} < q_{2i} \\ v_i & : Q_{12}q_{1i} = q_{2i}, [0 \leq v_i \leq 1] \\ 0 & : Q_{12}q_{1i} > q_{2i} \end{cases} \quad (5)$$

However, the right hand side of expression (2) is now dependent on Q_{12} , giving an implicit rather than explicit formula for Q_{12} , namely:

$$Q_{12} = \frac{\sum_{i:Q_{12}q_{1i} < q_{2i}} p_{2i}q_{2i} + \sum_{i:Q_{12}q_{1i} = q_{2i}} \{v_i p_{2i}q_{2i} + (1-v_i) p_{1i}q_{2i}\} + \sum_{i:Q_{12}q_{1i} > q_{2i}} p_{1i}q_{2i}}{\sum_{i:Q_{12}q_{1i} < q_{2i}} p_{2i}q_{1i} + \sum_{i:Q_{12}q_{1i} = q_{2i}} \{v_i p_{2i}q_{1i} + (1-v_i) p_{1i}q_{1i}\} + \sum_{i:Q_{12}q_{1i} > q_{2i}} p_{1i}q_{1i}} \quad (6)$$

To assess whether any solution to this implicit formula exists, consider figure 1 below. Figure 1 provides a schematic illustration of the relationship between Q_{12} and a variable Q ($0 < Q < \infty$) when Q is used instead of Q_{12} in definitions (5) to generate the right hand side of expression (2). Clearly, the required solution occurs where $Q_{12}=Q$.

Figure 1

Schematic Diagram of Quantity Index Q_{12} against cut-off Quantity Q

When $Q < \min(\{q_{2i}/q_{1i}\})$ the quantity index from expression (2) is a Paasche index. Similarly, when $Q > \max(\{q_{2i}/q_{1i}\})$ the quantity index is a Laspeyres index. Under standard economic theory, there is a negative correlation between between price changes and quantity changes, implying that, if the aggregate quantity is increasing, a Laspeyres index is greater than a Paasche index (see, for example, Allen 1975). This implies an increasing relationship between Q_{12} and Q , as shown in line X . This line is step-like because the index value changes only when a product moves from one summation to the other in the numerator and denominator of equation (6). Note that, for clarity, the number of products shown is much smaller than normally occurs in practice and the height of the steps is exaggerated. The index value on the vertical parts of the line is determined by the value of v_i for those products with the relevant ratio $Q=q_{2i}/q_{1i}$. Note that use of the parameter v_i allows a one-to-many mapping $Q \rightarrow Q_{12}$ at the points $\{Q: Q=q_{2i}/q_{1i}\}$ and thereby ensures that Q_{12} is a continuous function of Q . So there is always at least one solution to equation (6).

By far the most likely occurrence is that shown by line X , with a unique solution defined by a partition that is generated by a relatively wide range of values of Q . Line Y shows a case with more than one solution. This case is much less likely than the case shown by line X , bearing in mind that the heights of the steps are exaggerated. In this case, it would seem appropriate to choose the median solution by selecting the parameter v_i that ensures $Q_{12}=Q=q_{2i}/q_{1i}$. This minimises the maximum deviation from the true value. In practice, there is likely to be little difference in the values of Q_{12} for the different solutions, so the choice of solution may not be important.

Although Laspeyres indices are generally greater than Paasche indices, it can occur that moving a specific product from the Paasche component to the Laspeyres component creates a reduction in the index rather than an increase. This situation is shown in line Z , with the example of a solution on a vertical part of the line, like the example for line Y . In this case, a unique solution exists, determined by the appropriate value of v_i for the relevant product i .

Other situations, not shown, with larger numbers of multiple solutions are also theoretically possible. For example, in line Y or in line Z if the line $Q_{12}=Q$ cuts through more than one vertical part. However, in practice, such cases are likely to be very rare, if they occur at all, and the preferred solutions may then be decided on a bespoke basis or by choosing the median solution. Crucially, however many solutions there are, there is always at least one, allowing the creation of the required index.

Another approach to choosing a preferred solution from multiple solutions (where they exist) is to re-interpret equation (6) as an economic-theoretic solution, as defined in Frisch (1936). This is justified through analysis of purchasers' revealed preferences for q_2 over q_1 , as described in the analysis below.

Multiplying both sides of equation (6) by the right-hand side denominator leads to this re-arranged equation:

$$\sum_{i: Q_{12}q_{1i} < q_{2i}} p_{2i}(q_{2i} - Q_{12}q_{1i}) = \sum_{i: Q_{12}q_{1i} > q_{2i}} p_{1i}(Q_{12}q_{1i} - q_{2i}) \quad (7)$$

This is a practical expression of the concept of substitution, where the cost of additional, more popular products in period 2 above the expected value $\sum_{i: Q_{12}q_{1i} < q_{2i}} p_{2i}Q_{12}q_{1i}$ is exactly matched by the funds released

from period 1 by the less popular products that are reduced below the expected value $\sum_{i: Q_{12}q_{1i} > q_{2i}} p_{1i}Q_{12}q_{1i}$. In

the expressions $\sum_{i: Q_{12}q_{1i} < q_{2i}} p_{2i}(q_{2i} - Qq_{1i})$ and $\sum_{i: Qq_{1i} > q_{2i}} p_{1i}(Qq_{1i} - q_{2i})$, if Q is slightly greater than Q_{12} , but not so much that it leads to a change in the numbers of products in the summations, then $\sum_{i: Qq_{1i} < q_{2i}} p_{2i}(q_{2i} - Qq_{1i}) < \sum_{i: Qq_{1i} > q_{2i}} p_{1i}(Qq_{1i} - q_{2i})$. This is an inefficient substitution because the funds released from the discarded products are not fully used to supplement purchases of the more attractive

products. In order to make full use of the funds released, greater quantities than those actually observed would have had to be bought. If Q is, in a similar manner, slightly less than Q_{12} then $\sum_{i: Q_{1i} < q_{2i}} p_{2i}(q_{2i} - Q_{1i}) > \sum_{i: Q_{1i} > q_{2i}} p_{1i}(Q_{1i} - q_{2i})$. This is an ineffective substitution because the funds released from the discarded products are not sufficient to supplement purchases of the more attractive products. For the funds released to be adequate, smaller quantities than those actually observed would have had to be bought. So the solutions of equation (6) accord with the economic theory of index numbers, in that they produce perfect substitution of discarded products by replacement products. In the case of multiple solutions, since economic theory assumes that purchasers maximise utility for a given cost it follows that the appropriate solution to choose for an economic-theoretic index is the maximum solution, indicating the maximum increase in utility, as measured by the maximum value of Q_{12} that produces perfect substitution.

It is interesting to note that the indices defined by expression (2), using definitions (4) or (5) under appropriate conditions, satisfy the time reversal test: that is, $P_{12}P_{21} = Q_{12}Q_{21} = 1$. See Appendix C for a proof of this.

One aspect of indices based on definitions (4) and (5) that merits further comment is the treatment of zero quantities, which can arise from new and obsolete products in temporal indices and from cultural and legal differences in spatial comparisons. (See section 6 below for the application of this approach to spatial comparisons.) These indices assign a zero weight to such products in the price index, attributing the entire expenditure difference to a difference in quantities. This avoids the need to impute prices where quantities are zero. The form of the indices automatically and coherently provides a method of handling this question. The practical rationale for this approach, implicit in the presentation above, is that when there is zero quantity, there is no economic transaction relating to the affected product and the concept of an associated price is meaningless. In such a case, any difference represents a substitution away from or toward the product and is, therefore, a difference in quantity.

5. Formulae for short-term price indices

For short-term price indices, expression (2) is inconvenient because current quantities are not usually available on a timely basis. It is therefore useful to consider whether an approximation to this equation can be found that does not require knowledge of current quantities. This implies the need to make some assumption about the behaviour of quantities with respect to changes in prices and a commonly used assumption is that of constant expenditure shares. There is some empirical evidence that this is a reasonable approximation for annual data. However, it is not necessarily valid for monthly or quarterly data because of seasonal effects. Nonetheless, it is possible to use this approximation in conjunction with estimated seasonal factors to produce estimates of the quantity vector $\mathbf{q}_2$ and thus allow the application of the formulae of sections 2-4 to short-term inflation indices.

Normally, short-term price indices are expressed in the form $P_{12} = \sum_i w_{0i} \frac{p_{2i}}{p_{1i}}$ where the weights

$w_{0i} = \frac{p_{0i}q_{0i}}{\sum_j p_{0j}q_{0j}}$ are derived from earlier, known data, usually for a whole year to ensure the inclusion of all

seasonal products and a more stable set of weights. Nonetheless, it is possible to interpret these weights as

$w_{0i} = w_{1i} = \frac{p_{1i}q_{1i}}{\sum_j p_{1j}q_{1j}}$ for the known price vector $\mathbf{p}_1$ and an assumed quantity vector $\mathbf{q}_1$, defined by the

preceding expression. The quantity vector $\mathbf{q}_1$ is subject to an arbitrary scaling factor but this doesn't matter provided that the quantity vector $\mathbf{q}_2$ is also subject to the same factor, so that the two quantity vectors are on an equivalent basis. The procedure described below assumes that this is true.

Under the assumption of constant expenditure shares we have $\{p_{2i}q_{2i} = \alpha p_{1i}q_{1i}(1 + s_{2i}) : \forall i\}$ for some constant α , with the seasonal adjustment factors $\{(1 + s_{2i})\}$ allowing for deviations from the expected

average expenditure for the month or quarter. These seasonal adjustment factors would normally be estimated from seasonal patterns for previous years' data, usually averaged over several years to ensure an average representative pattern as seasonal patterns may vary markedly from year to year because of variations in weather, fashion, economic conditions, etc.

Under the fundamental index number identity $P_{12}Q_{12} = \frac{E_2}{E_1} = \frac{\sum_i p_{2i}q_{2i}}{\sum_i p_{1i}q_{1i}} = \alpha$. The value of α therefore depends

on the value of Q_{12} . As the value of Q_{12} is of no interest in this context, an obvious assumption to make is that $Q_{12}=1$, giving $\alpha=P_{12}$. This is consistent with the concept of a fixed basket of products, as used for traditional short-term price indices. However, as the underlying assumption here is that quantities change as prices change, a fixed basket is not possible but no change in aggregate quantity is an appropriate equivalent.

Note also that $\frac{\sum_i p_{2i}q_{2i}}{\sum_i p_{1i}q_{1i}} = \alpha$ implies that $\sum_i w_{1i}s_{2i} = 0$. That is, the aggregate effect of the seasonal adjustment factors is zero, their influence being felt only with regard to the relative contributions of the different products to expenditure in period 2.

The price index equation in expression (2) may be written in summation form as:

$$P_{12} = \frac{\sum_i \lambda_i p_{2i} q_{1i} + \sum_i (1 - \lambda_i) p_{2i} q_{2i}}{\sum_i \lambda_i p_{1i} q_{1i} + \sum_i (1 - \lambda_i) p_{1i} q_{2i}}$$

Applying the transformation $q_{2i} = \alpha q_{1i} p_{1i} (1 + s_{2i}) / p_{2i}$ gives:

$$P_{12} = \frac{\sum_i \lambda_i w_{1i} \frac{p_{2i}}{p_{1i}} + \sum_i \{(1 - \lambda_i) w_{1i} (1 + s_{2i})\} \alpha}{\sum_i \lambda_i w_{1i} + \sum_i \left\{ (1 - \lambda_i) w_{1i} \frac{p_{1i}}{p_{2i}} (1 + s_{2i}) \right\} \alpha}$$

For the $\{\lambda_i\}$ considered in section 2, this is straightforward to calculate, provided that α does not depend on the price index P_{12} . Unfortunately, under the assumption of a fixed aggregate quantity $Q_{12}=1$ and $\alpha=P_{12}$. This leads to the the following quadratic equation in P_{12} :

$$\sum_i \left\{ (1 - \lambda_i) w_{1i} \frac{p_{1i}}{p_{2i}} (1 + s_{2i}) \right\} P_{12}^2 + \sum_i \{(2\lambda_i - 1) w_{1i} (1 + s_{2i})\} P_{12} - \sum_i \lambda_i w_{1i} \frac{p_{2i}}{p_{1i}} = 0$$

for which the only positive root is:

$$P_{12} = \frac{\left(-\sum_i \left\{ \left(\lambda_i - \frac{1}{2} \right) w_{1i} (1 + s_{2i}) \right\} + \sqrt{\left[\sum_i \left\{ \left(\lambda_i - \frac{1}{2} \right) w_{1i} (1 + s_{2i}) \right\} \right]^2 + \sum_i \left\{ \left(1 - \lambda_i \right) w_{1i} \frac{p_{1i}}{p_{2i}} (1 + s_{2i}) \right\} \sum_i \lambda_i w_{1i} \frac{p_{2i}}{p_{1i}}} \right)}{\sum_i \left\{ \left(1 - \lambda_i \right) w_{1i} \frac{p_{1i}}{p_{2i}} (1 + s_{2i}) \right\}}$$

with the proviso that if $\{\lambda_i = 1 : \forall i\}$ P_{12} defaults to a Laspeyres index.

The $\{\lambda_i\}$ for definitions (4) become:

$$\lambda_i = \begin{cases} 1 & : p_{2i} < \alpha (1 + s_{2i}) p_{1i} \\ v_i & : p_{2i} = \alpha (1 + s_{2i}) p_{1i}, [0 \leq v_i \leq 1] \\ 0 & : p_{2i} > \alpha (1 + s_{2i}) p_{1i} \end{cases}$$

and for definitions (5) they become:

$$\lambda_i = \begin{cases} 1 & : p_{2i} < P_{12} (1 + s_{2i}) p_{1i} \\ v_i & : p_{2i} = P_{12} (1 + s_{2i}) p_{1i}, [0 \leq v_i \leq 1] \\ 0 & : p_{2i} > P_{12} (1 + s_{2i}) p_{1i} \end{cases}$$

because $\alpha = P_{12} Q_{12}$. Note that the $\{\lambda_i\}$ for definitions (5) do not depend on α .

As noted above, under the assumption of a fixed aggregate quantity $Q_{12}=1$, $\alpha=P_{12}$ and the $\{\lambda_i\}$ for definitions (4) are equal to those for definitions (5). So, for definitions (4) under the assumption that $Q_{12}=1$ and for definitions (5) under any assumption for the value of the constant α , the resultant, implicit price index formula is:

$$P_{12} = \frac{\sum_{i: p_{2i} < (1+s_{2i})P_{12}p_{1i}} w_{1i} \frac{p_{2i}}{p_{1i}} + \sum_{i: p_{2i} = (1+s_{2i})P_{12}p_{1i}} w_{1i} \left\{ v_i \frac{p_{2i}}{p_{1i}} + (1-v_i)(1+s_{2i}) \right\} + \sum_{i: p_{2i} > (1+s_{2i})P_{12}p_{1i}} w_{1i} (1+s_{2i})P_{12}}{\sum_{i: p_{2i} < (1+s_{2i})P_{12}p_{1i}} w_{1i} + \sum_{i: p_{2i} = (1+s_{2i})P_{12}p_{1i}} w_{1i} \left\{ v_i + (1-v_i) \frac{p_{1i}}{p_{2i}} (1+s_{2i}) \right\} + \sum_{i: p_{2i} > (1+s_{2i})P_{12}p_{1i}} w_{1i} \frac{p_{1i}}{p_{2i}} (1+s_{2i})P_{12}}$$

Proof of the existence of at least one solution for this equation follows a similar argument to that presented in section 4 for the existence of solutions to the implicit equation for the quantity index under definitions (5).

For definitions (2) and definitions (4), it is possible to produce simpler expressions for P_{12} by setting $\alpha=1$, that is assuming constant total expenditure rather than constant aggregate quantity. Such an assumption may better reflect consumers' actual experience of inflation, which involves adjusting their expenditure patterns according to their budgets. The general formula is then:

$$P_{12} = \frac{\sum_i \lambda_i w_{1i} \frac{p_{2i}}{p_{1i}} + \sum_i \left\{ (1 - \lambda_i) w_{1i} (1 + s_{2i}) \right\}}{\sum_i \lambda_i w_{1i} + \sum_i \left\{ (1 - \lambda_i) w_{1i} (1 + s_{2i}) \frac{p_{1i}}{p_{2i}} \right\}}$$

and the explicit formula for definitions (4) is:

$$P_{12} = \frac{\sum_{i: p_{2i} \leq (1+s_{2i})p_{1i}} w_{1i} \frac{p_{2i}}{p_{1i}} + \sum_{i: p_{2i} > (1+s_{2i})p_{1i}} w_{1i} (1+s_{2i})}{\sum_{i: p_{2i} \leq (1+s_{2i})p_{1i}} w_{1i} + \sum_{i: p_{2i} > (1+s_{2i})p_{1i}} w_{1i} \frac{p_{1i}}{p_{2i}} (1+s_{2i})}$$

6. Formulae for spatial comparisons

So far, I have only discussed index comparisons over time, implicitly over relatively short intervals and for the same intrinsic population. However, price comparisons are often made over space, such as between regions within a country or between countries. It is therefore worth discussing whether the index number formulae presented above are suitable for spatial comparisons.

The justification for using indices under the original definitions of the $\{\lambda_i\}$ in expression (2) is based on the operational properties of the market. These properties may not be the same between regions within a country or, especially, between countries. More importantly, the justification for the use of these original definitions is based on the time sequence of price and quantity changes. As this sequence does not apply spatially, indices based on these definitions of the $\{\lambda_i\}$ are not appropriate for spatial comparisons.

The objection in the previous paragraph also applies to the indices under definitions (4), presented in section 3. Although definitions (4) do not assume a time sequence of price and quantity changes they do assume a temporal relationship between similar prices and similar quantities. Such a relationship does not exist spatially. For example, if one territory is much larger than another, only the smaller territory's quantities would be used for the price comparison (with, perhaps, only a few exceptions). Even if adjustments are made for different population sizes, differences in average income could still lead to an imbalance in the quantities used. Similarly, prices for international comparisons may be defined in different currencies and this needs to be allowed for when estimating quantity changes. Indeed, the major focus of international comparisons is to estimate purchasing power parities, currency exchange rates that equalise the cost of purchasing a fixed basket of goods in different territories.

Using definitions (5), however, avoids this problem because of the comparison of relative quantities by adjusting for the quantity index Q_{12} , where the suffices 1 and 2 now refer, for the purposes of this section, to different territories. A similar adjustment using the price index P_{12} allows the comparison of relative prices. This leads to the following two simultaneous equations for P_{12} and Q_{12} .

$$\begin{aligned} P_{12} &= \frac{\sum_{i: Q_{12}q_{1i} < q_{2i}} p_{2i}Q_{12}q_{1i} + \sum_{i: Q_{12}q_{1i} = q_{2i}} \{v_i p_{2i}Q_{12}q_{1i} + (1-v_i)p_{2i}q_{2i}\} + \sum_{i: Q_{12}q_{1i} > q_{2i}} p_{2i}q_{2i}}{\sum_{i: Q_{12}q_{1i} < q_{2i}} p_{1i}Q_{12}q_{1i} + \sum_{i: Q_{12}q_{1i} = q_{2i}} \{v_i p_{1i}Q_{12}q_{1i} + (1-v_i)p_{1i}q_{2i}\} + \sum_{i: Q_{12}q_{1i} > q_{2i}} p_{1i}q_{2i}} \\ Q_{12} &= \frac{\sum_{i: Q_{12}q_{1i} < q_{2i}} p_{2i}q_{2i} + \sum_{i: Q_{12}q_{1i} = q_{2i}} \{v_i p_{2i}q_{2i} + (1-v_i)P_{12}p_{1i}q_{2i}\} + \sum_{i: Q_{12}q_{1i} > q_{2i}} P_{12}p_{1i}q_{2i}}{\sum_{i: Q_{12}q_{1i} < q_{2i}} p_{2i}q_{1i} + \sum_{i: Q_{12}q_{1i} = q_{2i}} \{v_i p_{2i}q_{1i} + (1-v_i)P_{12}p_{1i}q_{1i}\} + \sum_{i: Q_{12}q_{1i} > q_{2i}} P_{12}p_{1i}q_{1i}} \end{aligned} \quad (8)$$

Note the indexation adjustments applied to territory 1 quantities in the price index formula and to territory 1 prices in the quantity index formula. These adjustments ensure that the absolute levels of prices and quantities do not affect the calculation, which is dependent only on the relative levels of individual prices and quantities. The indexation of quantities ensures that the price index calculation is not dominated by the quantities in the larger territory. The indexation of prices ensures that the quantity index calculation is based on prices denominated in the same currency unit.

In practice, spatial indices are usually required for $n > 2$ territories. It is possible to calculate all the two-way indices but they are not guaranteed to be mutually coherent, in the sense that they are transitive and cyclical.

To ensure these desirable properties, we may define notional price and quantity levels P_r and Q_r (single subscript) for each territory r . Then, for territories r and s : $P_{rs}=P_s/P_r$ and $Q_{rs}=Q_s/Q_r$. Indices defined in this way are automatically transitive and cyclical. As the parameters $\{P_r, Q_r\}$ are used only to form the ratios $\{P_{rs}=P_s/P_r, Q_{rs}=Q_s/Q_r\}$, only their relative values are identifiable because they may all be multiplied by a constant with no effect on the result. To ensure an identifiable solution, an appropriate constraint must be applied. An obvious, simple constraint is to set $P_t=Q_t=1$ for some territory t but any suitable, alternative constraint would produce the same effective result.

As it is not generally possible to ensure that all pairwise indices of this type satisfy equations (8), it is necessary to apply some kind of optimisation procedure. To do so, first rewrite the price index equation into the general form for more than two territories, as proposed in the previous paragraph, thus:

$$P_{rs} = \frac{P_s}{P_r} = \frac{\sum_{i: \frac{Q_s}{Q_r} q_{ri} < q_{si}} p_{si} \frac{Q_s}{Q_r} q_{ri} + \sum_{i: \frac{Q_s}{Q_r} q_{ri} = q_{si}} \left\{ v_{rsi} p_{si} \frac{Q_s}{Q_r} q_{ri} + (1-v_{rsi}) p_{si} q_{si} \right\} + \sum_{i: \frac{Q_s}{Q_r} q_{ri} > q_{si}} p_{si} q_{si}}{\sum_{i: \frac{Q_s}{Q_r} q_{ri} < q_{si}} p_{ri} \frac{Q_s}{Q_r} q_{ri} + \sum_{i: \frac{Q_s}{Q_r} q_{ri} = q_{si}} \left\{ v_{rsi} p_{ri} \frac{Q_s}{Q_r} q_{ri} + (1-v_{rsi}) p_{ri} q_{si} \right\} + \sum_{i: \frac{Q_s}{Q_r} q_{ri} > q_{si}} p_{ri} q_{si}}$$

That is:

$$1 = \frac{P_r \left[\sum_{i: Q_s q_{ri} < Q_r q_{si}} p_{si} Q_s q_{ri} + \sum_{i: Q_s q_{ri} = Q_r q_{si}} \left\{ v_{rsi} p_{si} Q_s q_{ri} + (1-v_{rsi}) p_{si} Q_r q_{si} \right\} + \sum_{i: Q_s q_{ri} > Q_r q_{si}} p_{si} Q_r q_{si} \right]}{P_s \left[\sum_{i: Q_s q_{ri} < Q_r q_{si}} p_{ri} Q_s q_{ri} + \sum_{i: Q_s q_{ri} = Q_r q_{si}} \left\{ v_{rsi} p_{ri} Q_s q_{ri} + (1-v_{rsi}) p_{ri} Q_r q_{si} \right\} + \sum_{i: Q_s q_{ri} > Q_r q_{si}} p_{ri} Q_r q_{si} \right]} \quad (9)$$

Note the triple subscript for the nuisance parameters $\{v_{rsi}\}$, as the correct values for these may differ over all pair-wise comparisons.

The dimensionless nature of equation (9) makes it suitable as the basis for an objective function for optimising the parameters $\{P_r, Q_r\}$. A least squares approach would require minimisation of the following double summation:

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^n \left(\frac{P_r \left[\sum_{i: Q_s q_{ri} < Q_r q_{si}} p_{si} Q_s q_{ri} + \sum_{i: Q_s q_{ri} = Q_r q_{si}} \left\{ v_{rsi} p_{si} Q_s q_{ri} + (1-v_{rsi}) p_{si} Q_r q_{si} \right\} + \sum_{i: Q_s q_{ri} > Q_r q_{si}} p_{si} Q_r q_{si} \right]}{P_s \left[\sum_{i: Q_s q_{ri} < Q_r q_{si}} p_{ri} Q_s q_{ri} + \sum_{i: Q_s q_{ri} = Q_r q_{si}} \left\{ v_{rsi} p_{ri} Q_s q_{ri} + (1-v_{rsi}) p_{ri} Q_r q_{si} \right\} + \sum_{i: Q_s q_{ri} > Q_r q_{si}} p_{ri} Q_r q_{si} \right]} - 1 \right)^2$$

Note that this objective function only optimises on the price level parameters, although the quantity level parameters are a necessary input to the calculation. This is because it is the price level comparisons that are of most interest. Discrepancies between the quantity indices estimated from the parameters $\{P_r, Q_r\}$ and the correct quantity indices determined by equations (9) are usually not important. If smaller such discrepancies are required, a corresponding double summation for the quantity indices may be added to the objective function.

Alternatively, it is possible to make the objective function dependent only on the $\{P_r\}$ by defining the $\{Q_r\}$ as $\{Q_r : Q_r = E_r/P_r \forall r\}$, where E_r is the total expenditure in territory r , at territory r prices, for the data used in the analysis. As the $\{Q_r\}$ are no longer parameters, it is not necessary and not appropriate to specify any

constraint on them for an identifiable solution. If the simple constraint suggested above is used ($P_t=1$ for some territory t), the $\{Q_r\}$ become the expenditures $\{E_r\}$ expressed in the currency of territory t .

The objective function then becomes:

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^n \left[\frac{\sum_{i: \frac{P_r q_{ri}}{E_r} < \frac{P_s q_{si}}{E_s}} \frac{P_r}{P_s} E_s p_{si} q_{ri} + \sum_{i: \frac{P_r q_{ri}}{E_r} = \frac{P_s q_{si}}{E_s}} \left\{ v_{rsi} \frac{P_r}{P_s} E_s p_{si} q_{ri} + (1-v_{rsi}) E_r p_{si} q_{si} \right\} + \sum_{i: \frac{P_r q_{ri}}{E_r} > \frac{P_s q_{si}}{E_s}} E_r p_{si} q_{si}}{\sum_{i: \frac{P_r q_{ri}}{E_r} < \frac{P_s q_{si}}{E_s}} E_s p_{ri} q_{ri} + \sum_{i: \frac{P_r q_{ri}}{E_r} = \frac{P_s q_{si}}{E_s}} \left\{ v_{rsi} E_s p_{ri} q_{ri} + (1-v_{rsi}) \frac{P_s}{P_r} E_r p_{ri} q_{si} \right\} + \sum_{i: \frac{P_r q_{ri}}{E_r} > \frac{P_s q_{si}}{E_s}} \frac{P_s}{P_r} E_r p_{ri} q_{si}} - 1 \right]^2$$

As the parameters $\{v_{rsi}\}$ have no intrinsic interest and are only rarely required, some simplification is appropriate, to avoid unnecessary complications to the optimisation process. An obvious simplification is to set $\{v_{rsi} = \frac{1}{2} \forall r, s, i\}$ but it would be prudent to apply some check on the potential impact of variations in these parameters.

7. Summary and Discussion

The purpose of this paper is to present a set of index number formulae derived from consideration of how markets operate and how their participants behave, in practice, as opposed to the purely theoretical considerations used in traditional index number formulae. An important result of the analysis in the paper is the discovery that it is neither necessary nor appropriate to treat all products in the same way, as has been the case for all previously published index number formulae.

Section 2 showed that markets may be categorised into two basic types: those where purchasers make decisions on quantities for a given set of prices (type A) and those where purchasers make decisions on prices for a given set of quantities (type B), although many markets have both these characteristics to varying extents (described as type C). For type A markets, Laspeyres price indices are appropriate (with corresponding Paasche quantity indices). For type B markets, Laspeyres quantity indices are appropriate (with corresponding Paasche price indices). For type C, some kind of amalgamation of the two types A and B is necessary and I have chosen a formula based on weighted averages of the numerators and denominators of the Laspeyres/Paasche components, as given in expression (2). The rationale for this is the practical convenience for aggregating the components within National Accounts, notwithstanding any theoretical preferences there might be for other amalgamation methods.

Although expression (2) takes account of quantity and price changes, its formulation is based only on the properties of the market and does not depend on the nature of these changes. However, under commonly-occurring circumstances, the assumed model of quantity and price changes is invalid. Section 3 addresses this problem by considering what we mean by a “price change” in practice and presents an argument that a price change is only valid as a price change if it applies to the same quantity in each period. This quantity is necessarily the minimum of the quantities in the two periods. The difference between the two quantities then represents the quantity change, measured at prices in the period with the larger quantity. This leads to an alternative set of weights, as given in definitions (4), for use in expression (2).

Section 3 implicitly assumes that substitution from less popular to more popular products relates to the observed quantity changes *per se*. However, changes in population size and average income can lead to secular changes in the total quantities purchased independently of any substitution. To cater for this, the appropriate quantity change to use for allocating quantity changes to the Laspeyres or Paasche component is not the absolute quantity change (up or down) but the change relative to the observed aggregate change, measured by the target index Q_{12} .

Section 4 presents an examination of this alternative approach, leading to the different allocation basis given in definitions (5) and a consequent implicit equation for Q_{12} , equation (6). The discussion in section 4 demonstrates that there is always at least one solution for Q_{12} , that occurrences of multiple solutions are rare and that there is a simple practical process for obtaining a unique solution from multiple solutions. I also demonstrate that equation (7), a variant of equation (6), is a direct expression of the substitution effect and must be satisfied by a true economic-theoretic index.

Before considering the results of sections 5 and 6, it is worth examining at this stage which of the definitions of the $\{\lambda_{ij}\}$ from sections 2-4 would be most appropriate for use in National Accounts. If the aim is simply to reflect average quantity and price movements as such, then clearly the $\{\lambda_{ij}\}$ from section 2 are most appropriate. This raises practical problems of assessing the $\{\lambda_{ij}\}$ to an acceptable degree of accuracy, although once assessed they are not likely to change much over time for individual products. A simpler approach is to adopt the suggestion made in section 2 that $\lambda_{ij} \in (0, \frac{1}{2}, 1) \forall i$. This would make assessment much easier and implementation more practical.

If the aim is produce an economic-theoretic index, the choice is between the $\{\lambda_{ij}\}$ from sections 3 and 4. Clearly, the $\{\lambda_{ij}\}$ from section 4 offer the best approach to such an index but this creates the additional complication of a non-linear estimation process for every sub-index required. There are also complications regarding how to aggregate up the product classification hierarchy. As aggregation methods are not central to the main topic of this paper, I confine the discussion on this matter to Appendix D.

The $\{\lambda_{ij}\}$ from section 3 offer a possible, much simpler, alternative but the resultant index is an inferior choice, being half way between a pure quantity and price movements index and the desired economic-theoretic index. Nonetheless, it is much simpler to implement than the index from section 4 and goes some way to recognising the effect of substitution between products and may be a suitable choice where National Accounts resources are limited. Where sufficient resources are available, the section 4 index would be my preferred choice.

Eurostat, the statistical office of the European Union, specifies in paragraph 10.20 of the European System of Accounts 2010 that National Accounts are to be based on Laspeyres quantity indices and Paasche price indices. The reasons for this, given in paragraph 10.21, are “the interpretation and calculation simplicity and the additivity property in the supply and use balances”. No mention is made of accuracy or validity.

The implication from the arguments presented in sections 2-4 of this paper is that the Laspeyres index number formula is not suitable for quantity indices in National Accounts: under section 2, as most markets covered by National Accounts are fundamentally of type A, requiring Laspeyres price indices and Paasche quantity indices, an index number formula closer to a Paasche index should be preferred; under sections 3 and 4, a formula somewhere between Laspeyres and Paasche indices should be preferred, for both price and quantity indices. The only circumstance justifying a Laspeyres quantity index is under section 3 if all quantities are declining, which is clearly not appropriate as the basis for a quantity index measurement policy.

The conclusion from this discussion is that the use of Laspeyres index numbers in National Accounts overstates GDP growth, thereby misleading economic decision-makers and the general public. A suitable alternative would be a Paasche quantity index, which is appropriate for markets of type A and has the same “interpretation and calculation simplicity and the additivity property” as a Laspeyres index. However, as not all markets are perfectly of type A, the compound indices described in this paper would be more appropriate, although their interpretation and calculation are less simple and their additivity property is subject to the constraints described in Appendix D.

Having considered National Accounts requirements in sections 2-4, section 5 addresses the problem of producing a short-term inflation index that is coherent with the indices proposed for use in National Accounts. Using the plausible assumption that value shares are approximately constant allows the development of short-term inflation index number formulae that have similar structures to those of sections 2-4, according to preferred specifications for the behaviour of the unknown short-term quantity index, making due allowance for seasonal effects.

For the European Union's Harmonised Index of Consumer Prices (HICP), the HICP Methodological Manual specifies a "Laspeyres-type" index formula and, in section 3.5, specifies price-updating to the reference period, partly for harmonisation between member states but also "for any price change over the interval between [the weights'] reference period and the price reference period.". There is no specification of provision for any quantity change over the same period, despite the fact that there is a well-known and well-established negative correlation between price movements and quantity movements. Such quantity changes would, in large part, negate the effect of price-updating on the weights. The assumption in section 5 is that this negation would be complete. This assumption is, of course, only an approximation but I contend that it is a closer approximation to the truth than the price-updated weights.

The use of a Laspeyres short-term prices index is not consistent with the use of a Laspeyres quantity index in National Accounts. This inconsistency is exacerbated by the probable over-estimation of the short-term prices index because of the use of price-updated weights, which give greater weight to those products with faster rising prices. The result is overstatements of both GDP growth and consumer price inflation, misleading economic decision-makers and the general public on both indices. As these are the most important and well-known economic statistics their simultaneous overstatement gives a false general impression of how fast the economy is growing in money terms.

I have used Eurostat's policies on these economic indicators merely as an illustrative example of this problem because these policies are well documented and readily available but the same problem is likely to be replicated within many, perhaps most, national statistical institutes. The solution, of course, is to modify one or both of these measures so that they are both consistent. One way of doing so is to use Paasche indices for GDP estimates but this may underestimate GDP because Paasche indices are not necessarily appropriate for all products within the coverage of GDP, whose coverage is much greater than that of consumer price indices. This paper presents three sets of index number formulae, based on the models presented in sections 2, 3 and 4, that can provide such a solution. Other approaches may also be possible.

Finally, section 6 presents a coherent method of estimating international price and quantity comparisons, based on the approach of section 4. Note that this method, as discussed at the end of section 4, neatly and automatically incorporates cases with zero quantities. This avoids the need to use reduced product sets for international comparisons because of the absence of certain products in some countries.

Appendix A

Proof that expression (2) almost satisfies the fundamental index number identity

To demonstrate that the indices defined by expression (2) approximately satisfy the fundamental index number identity, consider this decomposition of the product $P_{12}Q_{12}$:

$$\begin{aligned}
P_{12}Q_{12} &= \frac{\mathbf{p}'_2 \Lambda \mathbf{q}_1 + \mathbf{p}'_2 (I - \Lambda) \mathbf{q}_2}{\mathbf{p}'_1 \Lambda \mathbf{q}_1 + \mathbf{p}'_1 (I - \Lambda) \mathbf{q}_2} \times \frac{\mathbf{p}'_2 \Lambda \mathbf{q}_2 + \mathbf{p}'_1 (I - \Lambda) \mathbf{q}_1}{\mathbf{p}'_2 \Lambda \mathbf{q}_1 + \mathbf{p}'_1 (I - \Lambda) \mathbf{q}_1} \\
&= \frac{\mathbf{p}'_2 \mathbf{q}_2 + \mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_1 \mathbf{q}_2 + \mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)} \times \frac{\mathbf{p}'_1 \mathbf{q}_2 + (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_1 + (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_1} \\
&= \frac{\mathbf{p}'_2 \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_2} \times \frac{1 + \frac{\mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_2 \mathbf{q}_2}}{1 + \frac{\mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_1 \mathbf{q}_2}} \times \frac{\mathbf{p}'_1 \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_1} \times \frac{1 + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_1}}{1 + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_1}{\mathbf{p}'_1 \mathbf{q}_1}} \\
&= \frac{\mathbf{p}'_2 \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_1} \times \frac{1 + \frac{\mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_2 \mathbf{q}_2}}{1 + \frac{\mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_1 \mathbf{q}_2}} \times \frac{1 + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_1}}{1 + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_1}{\mathbf{p}'_1 \mathbf{q}_1}}
\end{aligned}$$

Note that the terms $\frac{\mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_2 \mathbf{q}_2}$ and $\frac{\mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_1 \mathbf{q}_2}$ are relative expenditure differences at different sets of prices ($\mathbf{p}_2$ and $\mathbf{p}_1$, respectively). Under normal conditions, these terms are small relative to 1 and of similar magnitudes, so the ratio $\left[1 + \frac{\mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_2 \mathbf{q}_2} \right] / \left[1 + \frac{\mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_1 \mathbf{q}_2} \right]$ should be very close to 1. A similar argument applies to the ratio $\left[1 + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_2} \right] / \left[1 + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_1}{\mathbf{p}'_1 \mathbf{q}_1} \right]$.

To assess the effect of these ratios, consider the difference:

$$\left\{ \left[1 + \frac{\mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_2 \mathbf{q}_2} \right] \times \left[1 + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_2} \right] \right\} - \left\{ \left[1 + \frac{\mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_1 \mathbf{q}_2} \right] \left[1 + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_1}{\mathbf{p}'_1 \mathbf{q}_1} \right] \right\}$$

Expanding this gives:

$$\begin{aligned}
& \left\{ \begin{aligned} & \left[1 + \frac{\mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_2 \mathbf{q}_2} + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_2} + \frac{\mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2) (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_2}{\mathbf{p}'_2 \mathbf{q}_2 \mathbf{p}'_1 \mathbf{q}_2} \right] \\ & - \left[1 + \frac{\mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2)}{\mathbf{p}'_1 \mathbf{q}_2} + \frac{(\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_1}{\mathbf{p}'_1 \mathbf{q}_1} + \frac{\mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2) (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_1}{\mathbf{p}'_1 \mathbf{q}_2 \mathbf{p}'_1 \mathbf{q}_1} \right] \end{aligned} \right\} \\
& = \left\{ \begin{aligned} & \left(\frac{\mathbf{p}'_2}{\mathbf{p}'_2 \mathbf{q}_2} - \frac{\mathbf{p}'_1}{\mathbf{p}'_1 \mathbf{q}_2} \right) \Lambda (\mathbf{q}_1 - \mathbf{q}_2) + (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \left(\frac{\mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_2} - \frac{\mathbf{q}_1}{\mathbf{p}'_1 \mathbf{q}_1} \right) \\ & + \frac{\mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2) (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_2}{\mathbf{p}'_2 \mathbf{q}_2 \mathbf{p}'_1 \mathbf{q}_2} - \frac{\mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2) (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_1}{\mathbf{p}'_1 \mathbf{q}_2 \mathbf{p}'_1 \mathbf{q}_1} \end{aligned} \right\} \\
& = \frac{1}{\mathbf{p}'_1 \mathbf{q}_2} \left\{ \begin{aligned} & \left(\frac{\mathbf{p}'_1 \mathbf{q}_2}{\mathbf{p}'_2 \mathbf{q}_2} \mathbf{p}'_2 - \mathbf{p}'_1 \right) \Lambda (\mathbf{q}_1 - \mathbf{q}_2) - (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \left(\frac{\mathbf{p}'_1 \mathbf{q}_2}{\mathbf{p}'_1 \mathbf{q}_1} \mathbf{q}_1 - \mathbf{q}_2 \right) \\ & + \frac{\mathbf{p}'_2 \Lambda (\mathbf{q}_1 - \mathbf{q}_2) (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_2}{\mathbf{p}'_2 \mathbf{q}_2} - \frac{\mathbf{p}'_1 \Lambda (\mathbf{q}_1 - \mathbf{q}_2) (\mathbf{p}'_2 - \mathbf{p}'_1) \Lambda \mathbf{q}_1}{\mathbf{p}'_1 \mathbf{q}_1} \end{aligned} \right\}
\end{aligned}$$

Note that the four terms within the final pair of braces are, in essence, scalar products of the vector differences $\mathbf{p}_2 - \mathbf{p}_1$ and $\mathbf{q}_1 - \mathbf{q}_2$, all relative to the base expenditure $\mathbf{p}'_1 \mathbf{q}_2$ in the denominator. There is some slight rescaling of some of the component vectors and the matrix Λ acts as a weighting matrix, with product i being assigned the weight λ_i . As a consequence, each term is a second order difference.

Furthermore, the first two terms are almost identical, apart from slight scaling differences for $\mathbf{p}'_2$ and $\mathbf{q}_1$, and have opposite signs. So their combined effect should be negligible, perhaps fourth order or smaller. A similar argument also applies to the last two terms.

In aggregate, therefore, the price and quantity indices of expression (2) may reasonably be deemed to conform with the fundamental index number identity for all practical purposes, as the small errors involved are within the domain of rounding errors.

Appendix B

Ways of satisfying the fundamental index number identity exactly

Notwithstanding the comments at the end of section 2, this appendix briefly considers how a scientifically more justifiable and, in some sense, optimal approach to maintaining the fundamental index number identity might be attained. As an example, the following adjusted formulae for P_{12} and Q_{12} display one possible way of maintaining this identity:

$$P_{12} Q_{12} = \frac{\gamma A_{21} + B_{22}}{\gamma A_{11} + B_{12}} \frac{A_{22} + \kappa B_{12}}{A_{21} + \kappa B_{11}} = \frac{E_2}{E_1} \quad (\text{B1})$$

The factors γ and κ adjust the weighting of the component indices but, to some degree, maintain the aggregative nature of the numerators and denominators. It is not necessary to adjust both price and quantity indices. Indeed, adjusting only one index in this way would allow a straightforward linear solution for the single parameter but would be arbitrary, inequitable and scientifically unjustifiable. Given that it is desirable to adjust both P_{12} and Q_{12} for the sake of balance between prices and quantities, it doesn't matter which component indices are adjusted in the way shown because each index P_{12} , Q_{12} is invariant to multiplication of both numerator and denominator by the same constant. I have chosen to adjust the components as shown

because this choice allows the following two simple solutions to equation (B1): $(\gamma, \kappa) = \left(\frac{A_{22}}{A_{21}}, \frac{A_{21}}{A_{11}} \right)$ and $(\gamma, \kappa) = \left(\frac{B_{12}}{B_{11}}, \frac{B_{22}}{B_{12}} \right)$. This is easily verified.

It is interesting that both these solutions comprise a quantity index and a price index, from market type A and market type B respectively, and that they have the effect of revaluing the affected components to period 2 prices and quantities. Either solution could be used instead of the suggestions made in the main body of the paper but, again, this would require an arbitrary choice between the two and there is no obvious rationale for making such a choice.

There is clearly an infinite number of other solutions but it is possible to obtain a practical, optimal compromise. From equation (B1), it is straightforward to verify that the parameters γ and κ are related through the hyperbolic equation:

$$\gamma\kappa \left(1 - \frac{A_{11}B_{11}E_2}{A_{21}B_{12}E_1} \right) + \gamma \left(\frac{A_{22}}{B_{12}} - \frac{A_{11}E_2}{B_{12}E_1} \right) + \kappa \left(\frac{B_{22}}{A_{21}} - \frac{B_{11}E_2}{A_{21}E_1} \right) + \left(\frac{A_{22}B_{22}}{A_{21}B_{12}} - \frac{E_2}{E_1} \right) = 0$$

We can obtain an optimal solution by minimising the distance of the point (γ, κ) from the preferred point (1,1) subject to the constraint imposed by the equation above. Note, however, that all the coefficients in the equation above are very small, being differences between numbers of similar magnitude. As any solution of this equation must be based on ratios of these very small numbers, the results are likely to be unstable. As Appendix A demonstrates that the need for a more refined solution is negligible, pursuing this approach any further has little value.

Appendix C

Proof that $P_{12}P_{21} = Q_{12}Q_{21} = 1$ under definitions (4) or (5)

For quantities under definitions (4):

$$Q_{12} = \frac{\sum_{i:q_{1i} < q_{2i}} p_{2i}q_{2i} + \sum_{i:q_{1i} = q_{2i}} \{v_i p_{2i}q_{2i} + (1-v_i) p_{1i}q_{2i}\} + \sum_{i:q_{1i} > q_{2i}} p_{1i}q_{2i}}{\sum_{i:q_{1i} < q_{2i}} p_{2i}q_{1i} + \sum_{i:q_{1i} = q_{2i}} \{v_i p_{2i}q_{1i} + (1-v_i) p_{1i}q_{1i}\} + \sum_{i:q_{1i} > q_{2i}} p_{1i}q_{1i}}$$

$$Q_{21} = \frac{\sum_{i:q_{2i} < q_{1i}} p_{1i}q_{1i} + \sum_{i:q_{2i} = q_{1i}} \{v'_i p_{1i}q_{1i} + (1-v'_i) p_{2i}q_{1i}\} + \sum_{i:q_{2i} > q_{1i}} p_{2i}q_{1i}}{\sum_{i:q_{2i} < q_{1i}} p_{1i}q_{2i} + \sum_{i:q_{2i} = q_{1i}} \{v'_i p_{1i}q_{2i} + (1-v'_i) p_{2i}q_{2i}\} + \sum_{i:q_{2i} > q_{1i}} p_{2i}q_{2i}}$$

Note the use of the parameter v'_i rather than v_i in the formula for Q_{21} . As this parameter is arbitrary, it need not have the same value as the parameter v_i for Q_{12} .

Now:

$$\frac{1}{Q_{21}} = \frac{\sum_{i:q_{1i} < q_{2i}} p_{2i}q_{2i} + \sum_{i:q_{1i} = q_{2i}} \{(1-v'_i) p_{2i}q_{2i} + v'_i p_{1i}q_{2i}\} + \sum_{i:q_{1i} > q_{2i}} p_{1i}q_{2i}}{\sum_{i:q_{1i} < q_{2i}} p_{2i}q_{1i} + \sum_{i:q_{1i} = q_{2i}} \{(1-v'_i) p_{2i}q_{1i} + v'_i p_{1i}q_{1i}\} + \sum_{i:q_{1i} > q_{2i}} p_{1i}q_{1i}}$$

and $Q_{12}Q_{21} = 1$ if $\{v_i + v'_i = 1, \forall i : q_{1i} = q_{2i}\}$.

Similarly, the equivalent result for prices is $P_{12}P_{21} = 1$ if $\{v_i + v'_i = 1, \forall i : q_{1i} = q_{2i}\}$.

As the sets $\{v_i\}$ and $\{v'_i\}$ are arbitrary, it is therefore easy to ensure that both price and quantity indices under definitions (4) satisfy the time reversal test. As stated above, the most appropriate choice would be $\{v_i = v'_i = \frac{1}{2}, \forall i : q_{1i} = q_{2i}\}$, for symmetry and convenience.

For quantities under definitions (5):

$$Q_{12} = \frac{\sum_{i:Q_{12}q_{1i} < q_{2i}} p_{2i}q_{2i} + \sum_{i:Q_{12}q_{1i} = q_{2i}} \{v_i p_{2i}q_{2i} + (1-v_i) p_{1i}q_{2i}\} + \sum_{i:Q_{12}q_{1i} > q_{2i}} p_{1i}q_{2i}}{\sum_{i:Q_{12}q_{1i} < q_{2i}} p_{2i}q_{1i} + \sum_{i:Q_{12}q_{1i} = q_{2i}} \{v_i p_{2i}q_{1i} + (1-v_i) p_{1i}q_{1i}\} + \sum_{i:Q_{12}q_{1i} > q_{2i}} p_{1i}q_{1i}} \quad (C1)$$

$$Q_{21} = \frac{\sum_{i:Q_{21}q_{2i} < q_{1i}} p_{1i}q_{1i} + \sum_{i:Q_{21}q_{2i} = q_{1i}} \{v'_i p_{1i}q_{1i} + (1-v'_i) p_{2i}q_{1i}\} + \sum_{i:Q_{21}q_{2i} > q_{1i}} p_{2i}q_{1i}}{\sum_{i:Q_{21}q_{2i} < q_{1i}} p_{1i}q_{2i} + \sum_{i:Q_{21}q_{2i} = q_{1i}} \{v'_i p_{1i}q_{2i} + (1-v'_i) p_{2i}q_{2i}\} + \sum_{i:Q_{21}q_{2i} > q_{1i}} p_{2i}q_{2i}}$$

Now:

$$\frac{1}{Q_{21}} = \frac{\sum_{i:\frac{1}{Q_{21}}q_{1i} < q_{2i}} p_{2i}q_{2i} + \sum_{i:\frac{1}{Q_{21}}q_{1i} = q_{2i}} \{(1-v'_i) p_{2i}q_{2i} + v'_i p_{1i}q_{2i}\} + \sum_{i:\frac{1}{Q_{21}}q_{1i} > q_{2i}} p_{1i}q_{2i}}{\sum_{i:\frac{1}{Q_{21}}q_{1i} < q_{2i}} p_{2i}q_{1i} + \sum_{i:\frac{1}{Q_{21}}q_{1i} = q_{2i}} \{(1-v'_i) p_{2i}q_{1i} + v'_i p_{1i}q_{1i}\} + \sum_{i:\frac{1}{Q_{21}}q_{1i} > q_{2i}} p_{1i}q_{1i}} \quad (C2)$$

Clearly, equation (C2) is merely a restatement of equation (C1) with parameters $\{Q_{12}, \{v_i, \forall i : Q_{12}q_{1i} = q_{2i}\}\}$ replaced by parameters $\{\frac{1}{Q_{21}}, \{1-v'_i, \forall i : \frac{1}{Q_{21}}q_{1i} = q_{2i}\}\}$. So the solution set for parameters

$\{\frac{1}{Q_{21}}, \{1-v'_i, \forall i : \frac{1}{Q_{21}}q_{1i} = q_{2i}\}\}$ is the same as the solution set for parameters $\{Q_{12}, \{v_i, \forall i : Q_{12}q_{1i} = q_{2i}\}\}$.

When there is only one solution, as with lines *X* and *Z* in Figure 1, the result $Q_{12}Q_{21} = 1$ and $v_i + v'_i = 1, \forall i : Q_{12}q_{1i} = q_{2i}$ follows automatically. When there are multiple solutions, as with line *Y* in Figure 1, the result $Q_{12}Q_{21} = 1$ and $v_i + v'_i = 1, \forall i : Q_{12}q_{1i} = q_{2i}$ still holds, provided that complementary solutions of the two equations are chosen.

Given complementary solutions for the quantity parameters, the result $P_{12}P_{21} = 1$ also holds, using the same process as described above for quantity indices using definitions (5).

Appendix D

Aggregation within National Accounts

As noted in section 2, the main reason for the structure of the index numbers presented in this paper is to ease aggregation of quantity measures in National Accounts. The $\{\lambda_i\}$ defined in section 2 are fixed and pre-determined by the operational properties of the various markets. As a consequence, the expenditure components ($A_{11}, A_{21}, A_{22}, B_{11}, B_{12}, B_{22},$) defined in definitions (1) may be aggregated over lower levels of the hierarchy to generate the corresponding components for a higher level. The same is also true for the $\{\lambda_i\}$

defined in section 3, even if the $\{v_i\}$ within definitions (4) are chosen so as to optimise that level's value of Q_{12} with regard to the fundamental index number identity $P_{12}Q_{12} = E_2/E_1$. Any such choice would have repercussions with regard to the accumulation for higher-level indices but such repercussions are easily managed.

This ease of aggregation does not apply to indices produced using definitions (5) of section 4. All the $\{\lambda_i\}$ depend on the data such that the very partition between the Laspeyres and Paasche components is not known in advance but is dependent on the very quantity index that is to be estimated.

There are two main approaches to this problem. The first is to apply the technically correct approach, namely to calculate every required quantity index directly from the lowest-level inputs. Clearly, this is computationally intensive as it requires repeated aggregation of the same data but with different partitions. However, there are ways of reducing the computational burden. If the quantity index lies within the Laspeyres-Paasche bounds, as it should, then all products whose quantity ratio (q_{2i}/q_{1i}) is less than the Paasche index for any target index would automatically be included in the Laspeyres component and all products whose quantity ratio is greater than the Laspeyres index for any target index would automatically be included in the Paasche component. These partial sums should account for a major part of the overall index and could be accumulated up the hierarchy. Some adjustments would have to be made at each higher level to correct for the differences between the higher-level and lower-level Paasche and Laspeyres component indices.

The second approach is technically less valid but is more practical, provided its results do not stray too far from the technically correct approach. In this approach, the input quantities and prices for the higher-level indices are not the lowest-level quantities and prices but the quantity and price index Laspeyres and Paasche components from the next lower level. It is then possible to accumulate the expenditure totals for the lower-level Laspeyres and Paasche components separately. Normally, Laspeyres and Paasche totals from the lower level should accumulate into, respectively, the Laspeyres and Paasche totals for the higher level but it may happen that the lower-level totals for both the Laspeyres and Paasche components are accumulated into either the Laspeyres or the Paasche component for the higher-level index. This would arise if a lower-level index were particularly low or high relative to the other lower-level indices included in the higher-level index. This phenomenon also arises with the quantity ratios for individual products. For example, IT products famously have rapidly falling prices and rapidly rising quantities. When combined, in higher-level indices, with other products whose prices and quantities are more stable, all or almost all IT products would be included in the Paasche component. Only within the IT index would some IT products be included in the Laspeyres component. This phenomenon also indicates a plausible justification for the approach described in this paragraph: within higher-level indices we are only concerned with substitution between the component sub-indices, substitution within these sub-indices having been allowed for already.

Statements and Declarations

Funding

No funding was received for conducting this study.

Competing Interests

The author has no relevant financial or non-financial interests to disclose.

Data availability

No data were used in this paper.

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