Almost-quasifibrations and Fundamental Groups of Orbit Configuration Spaces

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ALMOST-QUASIFIBRATIONS AND FUNDAMENTAL GROUPS OF ORBIT CONFIGURATION SPACES

S.K. ROUSHON

ABSTRACT. In this article we introduce the notion of a k -almost-quasifibration and give many examples. We also show that a large class of these examples are not quasifibrations. As a consequence, supporting the Asphericity conjecture of [19], we deduce that the fundamental group of the orbit configuration space of an effective and properly discontinuous action of a discrete group, on an aspherical 2-manifold with isolated fixed points is torsion free. Furthermore, if the 2-manifold has at least one puncture then it is poly-free, and hence has an iterated semi-direct product of free groups structure, which generalizes a result of Xicoténcatl ([27], Theorem 6.3).

1. INTRODUCTION AND STATEMENTS OF RESULTS

Given a fibration $f : X \rightarrow Y$, it is a standard method to compute the homotopy groups of X in terms of the homotopy groups of Y and a fiber of f , using the long exact sequence of homotopy groups induced by f . But, for this computation we need the existence of this exact sequence, instead of the stronger fibration hypothesis. Also, a surjective map $f : X \rightarrow Y$ is called a *quasifibration* ([11], [[13], chap 4, p. 479]) if for all $y \in Y$ and for all $x \in f^{-1}(y) := F_y$, the map $f : (X, F_y, x) \rightarrow (Y, y)$ is a weak homotopy equivalence. Hence, a quasifibration induces a long exact sequence of homotopy groups as well, although it is weaker than a fibration. However, for computational purposes, it is enough if we have this exact sequence for some $y \in Y$, when Y is path connected.

Together with the main results of [18] and [19], this motivate us to define the following generalization of a quasifibration.

Definition 1.1. Let $f : X \rightarrow Y$ be a map between path connected topological spaces and $k \geq 1$ be an integer. Then f is called a *k -almost-quasifibration*, if for some $y \in Y$ and for $x \in F_y$, there is the following exact sequence.

$$\begin{aligned} 1 \longrightarrow \pi_k(F_y, x) \xrightarrow{i_*} \pi_k(X, x) \xrightarrow{f_*} \pi_k(Y, y) \xrightarrow{\partial} \pi_{k-1}(F_y, x) \longrightarrow \cdots \\ \cdots \longrightarrow \pi_2(Y, y) \xrightarrow{\partial} \pi_1(F_y, x) \xrightarrow{i_*} \pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, y) \longrightarrow 1. \end{aligned}$$

f is called an *almost-quasifibration* if the above sequence can be extended on the left up to infinity.

We now recall the definition of an orbit configuration space from [26]. We will construct many examples of 1-almost-quasifibrations and almost-quasifibrations between orbit configuration spaces of 2-manifolds (Theorem 1.3 and Proposition 1.6).

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We will also show that a large class of them are not quasifibrations (Theorem 1.8). Then we derive some consequences on the structure of the fundamental groups of orbit configuration spaces of 2-manifolds.

In a later work we will study k -almost-quasifibrations in detail, and generalize some basic results from [11] on quasifibrations, to the case of k -almost-quasifibrations.

Definition 1.2. ([26]) Let G be a discrete group acting on a connected smooth manifold M without boundary, and $k \geq 1$ be an integer. The *orbit configuration space* $\mathcal{O}_k(M, G)$ of the pair (M, G) is the space of all k -tuple of points of M with distinct orbits. That is,

$$\mathcal{O}_k(M, G) = \{(x_1, x_2, \dots, x_k) \in M^k \mid Gx_i \cap Gx_j = \emptyset, \text{ for } i \neq j\}.$$

If G is the trivial group, then $\mathcal{O}_k(M, G)$ is abbreviated as $\mathcal{O}_k(M)$, and it is the usual configuration space ([12]) of ordered k -tuple of distinct points of M . $*_n \in \mathcal{O}_n(M, G)$ will denote a point whose coordinates have trivial isotropy groups, unless otherwise mentioned. $\bar{*}_n$ will denote its image in $\mathcal{O}_n(M/G)$.

In this article, we always consider effective and properly discontinuous action of G on M ([22], Definition 3.5.1).

The class of orbit configuration spaces is a very important class of spaces, and has connection with other branches of Mathematics and Physics ([1] - [4], [6], [9], [10], [16] and [23] - [25]). Much work has been done in this area during the last couple of decades, in the particular case when the action is properly discontinuous and free. See [27] and [15] for some works and relevant literatures on orbit configuration spaces. Also see [7] for some non-free action case. However, most of these studies are related to (co)homological computations or to know the homotopy type of the orbit configuration spaces. The homotopy groups of the orbit configuration spaces are not yet studied well. We have an Asphericity conjecture in [19] in this regard (see Remark 1.5). The fundamental groups of orbit configuration spaces are interesting only in the cases of 2-manifolds. Since, $\mathcal{O}(M, G)$ is the complement of codimension ≥ 3 submanifolds of M^k if dimension of M is ≥ 3 , and hence M^k and $\mathcal{O}(M, G)$ will have isomorphic fundamental groups.

Since G is acting effectively and properly discontinuously on M , M/G has an orbifold structure and $M \rightarrow M/G$ is an orbifold covering map ([21], Proposition 5.2.6). An effective action of a finite group on M has this property. See [21] for some introductory results on orbifolds.

We recall that in [[18], Lemma 2.9] and in [19] we had constructed homomorphisms between configuration Lie groupoids ([18], Definition 2.8) of a large class of Lie groupoids. The orbit configuration space $\mathcal{O}_k(M, G)$ was the object space of the configuration Lie groupoid, defined in [[18], Definition 2.6], of the translation Lie groupoid, corresponding to the action of G on M ([18], Example 2.2). These homomorphisms induced maps on their classifying spaces are not quasifibrations (in the sense of [11]), if the action has a fixed point (see Lemma 2.3). But still they induce short exact sequences of fundamental groups ([18], §2.2] and [[19], Theorem 2.2]). Here, we will see that the same properties hold for the object level maps of these homomorphisms.

Before we state our main results, recall that the underlying topological space of a 2-dimensional orbifold has a manifold structure ([20], p. 422, last para.). Hence, the

genus of a connected 2-dimensional orbifold is defined as the genus of its underlying space.

Using the results of [18], [19] and a non-abelian version of the Snake Lemma (Lemma 2.1), we prove the following.

Theorem 1.3. *Let S be a connected 2-manifold without boundary. Let a discrete group G be acting on S effectively, properly discontinuously and with isolated fixed points. In addition, assume that $S \neq \mathbb{S}^2$ or $\mathbb{RP}^2$, and if the genus of S/G is zero, then assume that S/G has at least one puncture. Let*

$$p : \mathcal{O}_n(S, G) \rightarrow \mathcal{O}_{n-1}(S, G)$$

be the projection to the first $n - 1$ coordinates, $*_{n-1} = p(*_n)$, $F = p^{-1}(*_{n-1})$ and $i : F \rightarrow \mathcal{O}_n(S, G)$ is the inclusion map. Then, p induces the following short exact sequence.

$$1 \longrightarrow \pi_1(F, *_n) \xrightarrow{i_*} \pi_1(\mathcal{O}_n(S, G), *_n) \xrightarrow{p_*} \pi_1(\mathcal{O}_{n-1}(S, G), *_{n-1}) \longrightarrow 1.$$

That is, p is a 1-almost-quasifibration. Furthermore, p is an almost-quasifibration if, in addition, one of the following conditions is satisfied.

- G acts freely on S ([26]).
- S/G is the complex plane with at the most two cone points of order 2 each.

A group G is called *poly-free* if there is a normal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots \trianglelefteq G_k = G$, such that the successive quotients G_{i+1}/G_i are free, for $i = 0, 1, \dots, k - 1$. Rehmulla and Rolfsen proved in [17] that poly-free groups have the nice properties like locally indicable and (hence) right orderable.

Corollary 1.4. *Let S and G be as in Theorem 1.3, then the fundamental group of the orbit configuration space $\mathcal{O}_n(S, G)$ is torsion free, and furthermore, it is poly-free if S has at least one puncture.*

Proof. The exact sequence of Theorem 1.3 and that an extension of groups preserves both poly-free and torsion free properties, prove the corollary. $\square$

If G is finite, S has a puncture and $\pi_1(S, *_1)$ is finitely generated, then it immediately follows that $\pi_1(\mathcal{O}_n(S, G), *_n)$ admits a normal series with free and finitely generated successive quotients.

Remark 1.5. The first part of Corollary 1.4 supports the Asphericity conjecture of [19], which says that $\pi_k^{orb}(\mathcal{O}_n(S/G), *_n) = \langle 1 \rangle$ for all $k \geq 2$, equivalently, $\pi_k(\mathcal{O}_n(S, G), *_n) = \langle 1 \rangle$ for all $k \geq 2$ ([19], Lemma 4.4). Since $\mathcal{O}_n(S, G)$ is finite dimensional, and a finite dimensional aspherical manifold has a torsion free fundamental group.

Since any poly-free group has an iterated semi-direct product of free groups structure, the second conclusion of Corollary 1.4 establishes a general version of [[27], Theorem 6.3]. Recall that, in [[27], Theorem 6.3] Xicoténcatl proved that if S/G is a closed surface other than $\mathbb{S}^2$ or $\mathbb{RP}^2$, and G is acting on S as a group of deck transformations (hence free and properly discontinuous action), so that S is the universal cover of S/G , then $\pi_1(\mathcal{O}_n(S, G), *_n)$ has an iterated semi-direct product of free groups structure. This was achieved using the fact that the fibration p has a section, in this particular case.

Next, we mention an extension of Theorem 1.3.

Let $p_l : \mathcal{O}_n(S, G) \rightarrow \mathcal{O}_l(S, G)$ be the projection to the first l coordinates, for some $1 \leq l \leq n-1$ and $p_l(*_n) = *_l = (x_1, x_2, \dots, x_l)$. Then, $p_l^{-1}(*_l)$ is homeomorphic to $\mathcal{O}_{n-l}(S_l, G)$, where $S_l = S - \cup_{j=1}^l Gx_j$.

Proposition 1.6. *Theorem 1.3 is also valid for p_l with F replaced by $p_l^{-1}(*_l)$.*

Corollary 1.7. *The Asphericity conjecture implies that p_l is an almost-quasifibration, for all S and G as in the statement of Theorem 1.3.*

Proof. If S and G are as in the statement of Theorem 1.3, and if the Asphericity conjecture is true then all the higher homotopy groups (that is, for $k \geq 2$) of the total space, base space and of the fiber of p_l vanish. See Remark 1.5. Next, since by Proposition 1.6, p_l induces the short exact sequence of fundamental groups, clearly p_l is an almost-quasifibration. $\square$

Finally, the finite homological type property (Lemma 2.5) and a calculation of the Euler characteristic (Lemma 2.6) of $\pi_1(\mathcal{O}_n(S, G), *_n)$ give the following.

Theorem 1.8. *Let S and G be as in Theorem 1.3. Assume that S has finitely generated fundamental group, G is non-trivial finite, and is acting on S with at least one fixed point. Then p_l is not a quasifibration.*

We end this section with the following question, for all pairs (M, G) . Compare it with the Quasifibration conjecture of [19].

Question. *Is $p_l : \mathcal{O}_n(M, G) \rightarrow \mathcal{O}_l(M, G)$ an almost-quasifibration?*

2. PROOFS

For the proof of Theorem 1.3 we need to prove a general version of the Snake Lemma.

Lemma 2.1. (Snake Lemma) *Consider the following commutative diagram of discrete groups with exact rows and columns.*

$$\begin{array}{ccccccc}
 & & G_1 & \longrightarrow & G_2 & \xrightarrow{s} & G_3 & \longrightarrow & 1 \\
 & & \downarrow v & & \downarrow w & & \downarrow r & & \\
 1 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & & \\
 & & \downarrow u & & \downarrow & & \downarrow & & \\
 & & K_1 & \xrightarrow{t} & K_2 & \longrightarrow & K_3 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
 \end{array}$$

Diagram 1

Then, there is a connecting homomorphism $\Delta : \ker(r) \rightarrow K_1$ which makes the following sequence exact.

$$\ker(v) \longrightarrow \ker(w) \xrightarrow{s|_{\ker(w)}} \ker(r) \xrightarrow{\Delta} K_1 \xrightarrow{t} K_2 \longrightarrow K_3.$$

Consequently, if t is injective, then $s|_{\ker(w)} : \ker(w) \rightarrow \ker(r)$ is surjective. Also, if $G_1 \rightarrow G_2$ is injective, then so is $\ker(v) \rightarrow \ker(w)$.

Proof. The proof is a simple diagram chase. The lemma is well-known when the groups are abelian. See [[14], Ex. 1, p. 337]. The proof in the general case follows the same line of the proof of the abelian case. The only difference to note here is that in the case of abelian groups the cokernels K_i , for $i = 1, 2, 3$, always exist, but in the non-abelian case we need this as an assumption.

We give the definition of the connecting homomorphism and then leave the rest of the checking to the reader.

Let $\alpha \in \ker(r)$. Since s is surjective there is $\beta \in G_2$ such that $s(\beta) = \alpha$. Clearly, using exactness at H_1 , $w(\beta) \in H_1$. Define $\Delta(\alpha) = u(w(\beta))$. For well-definedness of Δ , let $\beta' \in G_2$, such that $s(\beta') = \alpha$. Using exactness at G_2 and H_1 , we see that $w(\beta^{-1}\beta')$ is the image of an element of G_1 , and hence lies in the kernel of u . Therefore, $u(w(\beta)) = u(w(\beta'))$. $\square$

Proof of Theorem 1.3. Since F is homeomorphic to S minus some orbits, the action of G induces an action on F . Let $\bar{F} = F/G$ and $\bar{S} = S/G$. Note that, for all k , there is an induced action of G^k on $\mathcal{O}_k(S, G)$ and $\mathcal{O}_k(\bar{S}) = \mathcal{O}_k(S, G)/G^k$. Also if $q : G^n \rightarrow G^{n-1}$ is the projection to the first $n-1$ coordinates, then clearly p is q -equivariant. Hence p induces a map $\bar{p} : \mathcal{O}_n(\bar{S}) \rightarrow \mathcal{O}_{n-1}(\bar{S})$. Similar statements are true for the inclusion map $i : F \rightarrow \mathcal{O}_n(S, G)$ and that it induces an inclusion map $\bar{i} : \bar{F} \rightarrow \mathcal{O}_n(\bar{S})$. Since G^k also acts effectively and properly discontinuously on $\mathcal{O}_k(S, G)$, $\mathcal{O}_k(\bar{S})$ is an orbifold. Hence we have orbifold covering maps $F \rightarrow \bar{F}$ and $\mathcal{O}_k(S, G) \rightarrow \mathcal{O}_k(\bar{S})$, for all k .

$$\begin{array}{ccccc} F & \xrightarrow{i} & \mathcal{O}_n(S, G) & \xrightarrow{p} & \mathcal{O}_{n-1}(S, G) \\ \downarrow & & \downarrow & & \downarrow \\ \bar{F} & \xrightarrow{\bar{i}} & \mathcal{O}_n(\bar{S}) & \xrightarrow{\bar{p}} & \mathcal{O}_{n-1}(\bar{S}) \end{array}$$

Diagram 2

Clearly, $\bar{i}$ and $\bar{p}$ are maps of orbifolds. Hence, we have Diagram 2 in the category of orbifolds with the two squares in the diagram commutative.

$$\begin{array}{ccccccc} 1 & \cdots \cdots \cdots & \pi_1(F, *_n) & \xrightarrow{i_*} & \pi_1(\mathcal{O}_n(S, G), *_n) & \xrightarrow{p_*} & \pi_1(\mathcal{O}_{n-1}(S, G), *_{n-1}) \cdots \cdots \cdots 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1^{orb}(\bar{F}, \bar{*}_n) & \xrightarrow{\bar{i}_*} & \pi_1^{orb}(\mathcal{O}_n(\bar{S}), \bar{*}_n) & \xrightarrow{\bar{p}_*} & \pi_1^{orb}(\mathcal{O}_{n-1}(\bar{S}), \bar{*}_{n-1}) \longrightarrow 1 \end{array}$$

Diagram 3

Now, we apply the orbifold fundamental group $\pi_1^{orb}(-)$ functor on Diagram 2 and get Diagram 3, with commutative squares. Since the spaces on the top sequence are manifolds, these groups reduce to the ordinary fundamental groups.

Since, $\bar{S}$ is a 2-dimensional orbifold, the isolated singular points are cone points ([20], §2). Hence, the exactness of the bottom sequence in Diagram 3 can be deduced from [[18], Theorem 2.14, Remark 2.15] if genus of $\bar{S}$ is zero and if genus of $\bar{S} \geq 1$, then we use [[19], Theorem 2.2].

Next, we extend Diagram 3 to Diagram 4, where all the squares are commutative. The vertical sequences are exact by orbifold covering space theory. See [[19], Lemma 3.13] and the result of Chen in [[8], (4) in Proposition 2.2.3] for a more general statement.

We have to show that the top dotted sequence in Diagram 4 is exact. But, this immediately follows from Lemma 2.1, by restricting it to the case of $K_1 = K_2 = K_3 = \langle 1 \rangle$.

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \cdots \rightarrow & \pi_1(F, *_n) & \xrightarrow{i_*} & \pi_1(\mathcal{O}_n(S, G), *_n) & \xrightarrow{p_*} & \pi_1(\mathcal{O}_{n-1}(S, G), *_{n-1}) \cdots \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & \pi_1^{orb}(\bar{F}, \bar{*_n}) & \xrightarrow{\bar{i}_*} & \pi_1^{orb}(\mathcal{O}_n(\bar{S}), \bar{*_n}) & \xrightarrow{\bar{p}_*} & \pi_1^{orb}(\mathcal{O}_{n-1}(\bar{S}), \bar{*_n}) \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & G & \xrightarrow{\quad} & G^n & \xrightarrow{q} & G^{n-1} \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

Diagram 4

For the last assertion in the theorem, by Corollary 1.7, we only have to show that $\mathcal{O}_n(S, G)$ is aspherical.

- When the action is also free, then note that $S \rightarrow \bar{S}$ is a covering map, and hence $\bar{S}$ has no singular point. Therefore, the following commutative diagram is in the category of manifolds.

$$\begin{array}{ccc}
 \mathcal{O}_n(S, G) & \xrightarrow{p} & \mathcal{O}_{n-1}(S, G) \\
 \downarrow & & \downarrow \\
 \mathcal{O}_n(\bar{S}) & \xrightarrow{\bar{p}} & \mathcal{O}_{n-1}(\bar{S})
 \end{array}$$

Diagram 5

In Diagram 5 the two vertical maps are covering maps, since G^k also acts freely on $\mathcal{O}_k(S, G)$ for all k . And the bottom horizontal map $\bar{p}$ is a fiber bundle projection by the Fadell-Neuwirth fibration theorem ([12], Theorem 3). By [[18], Lemma 5.1] we get that p is a fibration with aspherical fiber. Hence, by an induction argument on n and using the long exact sequence of homotopy groups induced by p , it follows that $\mathcal{O}_n(S, G)$ is aspherical.

- For the last claim, that is, when $\bar{S}$ is the complex plane with at the most two singular points of degree 2 each, by [[18] Proposition 2.4], $\mathcal{O}_n(\bar{S})$ is aspherical as an orbifold, that is $\pi_k^{orb}(\mathcal{O}_n(\bar{S}), \bar{*_n}) = \langle 1 \rangle$ for all $k \geq 2$. Hence $\mathcal{O}_n(S, G)$ is aspherical ([19], Lemma 4.4).

This completes the proof of Theorem 1.3. $\square$

We now prove Proposition 1.6, showing that, under the hypothesis of Theorem 1.3, $p_l : \mathcal{O}_n(S, G) \rightarrow \mathcal{O}_l(S, G)$ induces a short exact sequence as well.

Proof of Proposition 1.6. Note that, Diagram 6 is also valid since the middle horizontal sequence in the diagram is exact, which follows from [[18], Remark 5.2] in the case when genus of $\bar{S}$ is zero and from [[19], Remark 5.2] otherwise.

Here $\bar{S}_l = S_l/G$, $q_l : G^n \rightarrow G^l$ is the projection to the first l -coordinates, $*^{n-l} = (x_{l+1}, x_{l+2}, \dots, x_n)$ and $j : \mathcal{O}_{n-l}(S_l, G) \rightarrow \mathcal{O}_n(S, G)$ is defined by

$$j(y_1, y_2, \dots, y_{n-l}) = (x_1, x_2, \dots, x_l, y_1, y_2, \dots, y_{n-l}).$$

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \cdots \rightarrow & \pi_1(\mathcal{O}_{n-l}(S_l, G), *^{n-l}) & \xrightarrow{j_*} & \pi_1(\mathcal{O}_n(S, G), *) & \xrightarrow{p_l^*} & \pi_1(\mathcal{O}_l(S, G), *) \cdots \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \rightarrow & \pi_1^{orb}(\mathcal{O}_{n-l}(\bar{S}_l), \bar{*}^{n-l}) & \xrightarrow{\bar{j}_*} & \pi_1^{orb}(\mathcal{O}_n(\bar{S}), \bar{*}_n) & \xrightarrow{\bar{p}_l^*} & \pi_1^{orb}(\mathcal{O}_l(\bar{S}), \bar{*}_l) \rightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & G^{n-l} & \longrightarrow & G^n & \xrightarrow{q_l} & G^l \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

Diagram 6

Hence, we can apply Lemma 2.1 again to conclude that the dotted sequence in Diagram 6 is exact.

The last two parts of Theorem 1.3 for p_l , also follow from similar arguments. $\square$

Remark 2.2. In the statements of Theorem 1.3 and Proposition 1.6, when the action of G is also free, then it is evident from the proof that p_l , for $1 \leq l \leq n-1$, is a fibration without any restriction on the dimension of S . This result was also proved in [[27], Theorem 2.2] adapting the proof of the Fadell-Neuwirth fibration theorem.

For the proof of Theorem 1.8, we start with the following lemma, which exhibits the main idea behind the proof. This lemma is essentially [[18], Proposition 2.11] with a correction in its hypothesis.

Lemma 2.3. *Let G be a finite group, acting effectively on a connected manifold M of dimension ≥ 2 , with at least one fixed point. Assume that M is either compact or has finitely generated integral homology groups. Then the map $p : \mathcal{O}_n(M, G) \rightarrow \mathcal{O}_{n-1}(M, G)$ is not a quasifibration.*

Proof. If we remove different numbers of points from M , then we get manifolds with different homology groups, and hence they are not weak homotopy equivalent. On the other hand, for a quasifibration over a path connected space any two fibers are weak homotopy equivalent ([13], chap. 4, p. 479). $\square$

Next we recall a finiteness result for groups from [5].

A group Γ is said to be of *finite homological type* ([5], chap. IX, §6), if it has finite virtual cohomological dimension and for any Γ -module Λ , which is finitely generated as an abelian group, $H_i(\Gamma, \Lambda)$ is finitely generated for all i . The advantage of this definition is that one can define Euler characteristic of a torsion free group Γ of finite homological type, as follows. See [5], p. 247 for notations and details.

$$\chi(\Gamma) = \sum_i (-1)^i \text{rk}_{\mathbb{Z}}(H_i \Gamma).$$

This fact is crucial in our proof of Theorem 1.8.

An application of the Hochschild-Serre spectral sequence is the following important multiplicative property of the Euler characteristic.

Lemma 2.4. ([5], (d) of Proposition IX.7.3) *Consider the following short exact sequence of discrete groups.*

$$1 \rightarrow \Gamma' \rightarrow \Gamma \rightarrow \Gamma'' \rightarrow 1.$$

If both Γ' and Γ'' are of finite homological type and Γ is virtually torsion free, then Γ is of finite homological type and the following holds.

$$\chi(\Gamma) = \chi(\Gamma') \cdot \chi(\Gamma'').$$

Now note that, although we do not yet know if $\mathcal{O}_n(S, G)$ is aspherical, we can still prove the following.

Lemma 2.5. *Let S and G be as in Theorem 1.3, the fundamental group of S is finitely generated and G is finite. Then $\pi_1(\mathcal{O}_n(S, G), *_n)$ is of finite homological type.*

Proof. The proof is by induction on n . First note that, $\pi_1(\mathcal{O}_1(S, G), *_1) = \pi_1(S, *_1)$ is of finite homological type, since S has a Eilenberg-MacLane space which is a finite complex ([5], chap. IX, §6). Next, assume that for all S and G as in the hypothesis, $\pi_1(\mathcal{O}_{n-1}(S, G), *_{n-1})$ is of finite homological type.

Now consider the map p_1 and the associated short exact sequence given by Proposition 1.6.

$$1 \rightarrow \pi_1(\mathcal{O}_{n-1}(S_1, G), *_{n-1}) \xrightarrow{j_*} \pi_1(\mathcal{O}_n(S, G), *_n) \xrightarrow{p_1^*} \pi_1(\mathcal{O}_1(S, G), *_1) \rightarrow 1.$$

Clearly, S_1 also satisfies the hypothesis and hence by the induction hypothesis $\pi_1(\mathcal{O}_{n-1}(S_1, G), *_{n-1})$ is of finite homological type. Since, by Corollary 1.4, $\pi_1(\mathcal{O}_n(S, G), *_n)$ is torsion free, we can apply Lemma 2.4, to complete the proof of the Lemma. $\square$

We now explicitly calculate $\chi(\pi_1(\mathcal{O}_n(S, G), *_n))$ under some conditions.

Lemma 2.6. *Let S be a connected 2-manifold with finitely generated free fundamental group of rank k . Let G be a finite group of order m , acting effectively on S with isolated fixed points. Then,*

$$\chi(\pi_1(\mathcal{O}_n(S, G), *_n)) = \prod_{i=1}^n (1 - k - (i-1)m).$$

Proof. Recall the notation $S_{n-1} = S - \cup_{i=1}^{n-1} Gx_i$. Since, for all i , the isotropy group of x_i is trivial, S_{n-1} has free fundamental group of rank $k + (n-1)m$.

The proof of the lemma is by induction on n . As S is an Eilenberg-MacLane space, we have $\chi(\pi_1(\mathcal{O}_1(S, G), *_1)) = \chi(S) = 1 - k$ ([5], p. 247).

Therefore, we assume the lemma for $n-1$ and prove its validity for n . The first step of the proof is the following short exact sequence, from Proposition 1.6.

$$1 \rightarrow \pi_1(\mathcal{O}_1(S_{n-1}, G), *_1) \xrightarrow{j_*} \pi_1(\mathcal{O}_n(S, G), *_n) \xrightarrow{p_*} \pi_1(\mathcal{O}_{n-1}(S, G), *_{n-1}) \rightarrow 1.$$

Now, Lemma 2.4 implies the following.

$$\chi(\pi_1(\mathcal{O}_n(S, G), *_n)) = \chi(\pi_1(\mathcal{O}_1(S_{n-1}, G), *_1)) \chi(\pi_1(\mathcal{O}_{n-1}(S, G), *_{n-1})).$$

Again, since S_{n-1} is an Eilenberg-MacLane space,

$$\chi(\pi_1(\mathcal{O}_1(S_{n-1}, G), *^1)) = \chi(S_{n-1}) = (1 - k - (n - 1)m).$$

By the induction hypothesis the proof is now complete as follows.

$$\begin{aligned} \chi(\pi_1(\mathcal{O}_n(S, G), *_n)) &= \chi(\pi_1(\mathcal{O}_1(S_{n-1}, G), *^1)) \prod_{i=1}^{n-1} (1 - k - (i - 1)m) = \\ (1 - k - (n - 1)m) \prod_{i=1}^{n-1} (1 - k - (i - 1)m) &= \prod_{i=1}^n (1 - k - (i - 1)m). \quad \square \end{aligned}$$

We are now ready to prove Theorem 1.8. Under the hypothesis of Lemma 2.6, we denote the number $\chi(\pi_1(\mathcal{O}_n(S, G), *_n))$ by $\chi(k, n, m)$.

Proof of Theorem 1.8. Let $\dot{*}_n = (\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)$ and $\hat{*}_n = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ be two points in $\mathcal{O}_n(S, G)$, such that $\dot{x}_i$ and $\hat{x}_j$ for $i = 1, 2, \dots, n$ and $j = 2, 3, \dots, n$ have trivial isotropy groups. And $\hat{x}_1$ is a fixed point of the action of G on S . Let $\dot{S} = S - \cup_{i=1}^l G\dot{x}_i$ and $\hat{S} = S - \cup_{i=1}^l G\hat{x}_i$. Then, $p_l^{-1}(\dot{*}_l)$ is homeomorphic to $\mathcal{O}_{n-l}(\dot{S}, G)$ and $p_l^{-1}(\hat{*}_l)$ is homeomorphic to $\mathcal{O}_{n-l}(\hat{S}, G)$. Also, the coordinates in $\dot{*}^{n-l}$ and in $\hat{*}^{n-l}$ have trivial isotropy groups. By Lemma 2.5, $\pi_1(\mathcal{O}_{n-l}(\dot{S}, G), \dot{*}^{n-l})$ and $\pi_1(\mathcal{O}_{n-l}(\hat{S}, G), \hat{*}^{n-l})$ are of finite homological types. Therefore, we can calculate their Euler characteristics using the formula $\chi(k, n, m)$, since $\dot{S}$ and $\hat{S}$ satisfy the hypothesis of Lemma 2.6. Clearly, $|\cup_{i=1}^l G\dot{x}_i| \neq |\cup_{i=1}^l G\hat{x}_i|$, which implies $rk(\pi_1(\dot{S}, \dot{*}^1)) \neq rk(\pi_1(\hat{S}, \hat{*}^1))$. Hence,

$$\chi(rk(\pi_1(\dot{S}, \dot{*}^1)), n, m) \neq \chi(rk(\pi_1(\hat{S}, \hat{*}^1)), n, m).$$

Thus, the fundamental groups of $p_l^{-1}(\dot{*}_l)$ and $p_l^{-1}(\hat{*}_l)$ are not isomorphic. Hence, the fibers of p_l are not of the same weak homotopy types. Consequently, p_l is not a quasifibration (see Lemma 2.3).

This completes the proof of Theorem 1.8. $\square$

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