Estimation And Fault Detection of Discrete Event Systems Modeled By Time Interval Petri Nets

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Estimation and fault detection of discrete event systems modeled by time interval Petri nets

Atef Khedher* · Kamal BenOthman

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Abstract This work is interested in studying discrete event systems modeled by time interval Petri nets. Indeed, a new state model describing the time evolution of the system is proposed. The simulation of the proposed state model gives the upper and lower bounds of each element of the **whole** system state and ~~the system~~ outputs. After that, a new state estimator is proposed allowing estimating the whole system state and inputs. This estimator gives, also, an upper and lower bounds for each element~~s~~ of the state vector and the input. The state model and the estimator are both proposed along, successively, count and dater approaches. For count approach, the system state is the number of transition firing and for dater approach, it is the dates of transitions firing. The proposed state model and estimator are applied to a **model of** baking process described by a time interval Petri net~~s~~. This application shows the robustness of the state and input estimation. The proposed state model and estimator are used also for fault detection, localization and estimation and it is shown that they give acceptable results.

Keywords Discrete event systems · Time interval Petri nets · fault detection and localization · fault estimation · state estimator · baking process.

1 Introduction

Many processes are described by discrete events systems and can be modeled by various kinds of Petri nets. Generally, time can be added to Petri nets to describe their time evolution [33]. Time interval Petri nets are used to model several process in case where transitions firing is allowed **only** in a particular time interval or the ~~the~~ sojourn time of token is governed by a special time interval [12, 14, 33]. Generally, Petri nets can be used for state estimation [1, 17, 28], discrete event systems diagnosis [7, 10, 11, 31] and fault detection [5, 30, 32, 36].

State models elaboration is important to study the system time evolution. ~~This its~~ elaboration must be made thoroughly because it is **the** base of the system simulation and it gives the initialization states values in the case of state estimation [15]. State estimation **becomes is** very important if the whole system state can not be measured [15, 16].

The aim of this work is the state estimation and the fault detection, localization and estimation for discrete events systems described by time interval Petri nets. ~~The considered system state, in this work, is the date of transition firing or their number depending on the followed approach (dater or count).~~ The system state depends on the considered approach. If it is elaborated according to a count approach, this state is the number of firing of each transition. If the dater approach is adopted, the system state is the **dates of transitions firing**. Two important parts will be presented successively. The first part proposes a new state model giving an upper and lower bounds of the system state. The second focuses on the design of a new estimator able to estimate the upper and lower bounds of the state vector and input.

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The first contribution ~~which is~~ presented in this work is ~~to elaborate~~ the elaboration of a new state model outlining the system behavior. This state model is conceived following successively a count and dater approaches. The proposed state model presents two inequalities where the first one leads to compute the upper and lower bonds of the system state and the second one allows computing the upper and lower bounds of the system's outputs.

The second contribution concerns a state estimator elaboration. This elaboration is made also following successively the count approach and the dater approach. The proposed state estimator is also composed of two inequalities where the first one estimates the upper and lower bounds of the state vector and the second one gives an estimation of the upper and lower bounds of the system inputs. So, ~~The~~ the conceived estimator is really a state and input estimator. In a next step, the residuals are ~~calculated~~ computed as the intersection between the intervals given by the simulation of the state model and the estimator in order to test the robustness of the estimation.

The proposed state model and estimator are applied to a baking process modeled by time interval Petri net in order to test ~~the proposed state model and estimator~~ and ~~to~~ apply them for fault detection, localization and estimation.

This paper is organized as ~~following~~ follows. Section 2 summarizes some related works. The third section ~~3~~ presents the used notations and variables and recalls some important definitions dealing with Petri nets. A preliminary example is also presented in this section. The section 4 presents an extension of the classical state model to time interval Petri nets following count and dater approaches. This extension is presented in two different forms. Section 5 details the design of a new state model which is proposed following successively the count and ~~the~~ dater approaches. The elaboration of the estimator is the objective of section 6. This estimator is proposed following, also, the count and dater approaches. An application of the proposed state model and estimator to a baking process modeled by a time interval petri net is presented in section 7 in order to show the robustness of the estimation. Their application to the fault estimation, detection and localization is also presented.

2 Related works

Many works discussed the problem of diagnosis and state estimation of discrete event systems modeled by different classes of Petri nets with regards to the current Petri net marking as the system state. In [7, 18], authors tried to make the state estimation under the assumption that the initial marking is known. In [38], authors proposed to bound the estimated marking number for the case where the Petri net contains unobservable transitions sharing the same label. Under the assumption that transitions have non-negative costs, authors in [29] focused on least cost firing sequences. For the class of partial observed Petri nets, reduced observer was used in [8, 23] to computes only a subset of the possible current markings, which is different from this work where a complete observer able to estimate the whole state vector is proposed. Under the assumption that Petri net does not contain cycles of unobservable transitions, Ghazel et al. [17] proposed a method of state estimation, ~~-This this~~ hypothesis is not considered in this work. The problem of on line estimation of the current markings probabilities was treated in [6, 34] by developing an appropriate recursive algorithm. The results of marking estimation are used for diagnosis in [28, 31].

Works cited above focused on the estimation and/or the boundedness of the current marking of the Petri net after each iteration in presence of unobservable transition. The considered system state in these works is the current marking which is different from this work where the considered state is the date of transitions firing in the case of dater approach or their firing number if a count approach is considered. Indeed, the state model and the estimator are proposed in this work following two different approaches : dater and count approaches.

In [15, 16], authors proposed an algebraic form allowing describing the whole solution set for a sequence of observed events. The uniqueness and computation of optimal solutions are analyzed in the case of forward/backward Conflict Free Petri nets. ~~The proposed algebraic model is applied after that in [13] using counter approach for the estimation of optimal sequences in partially observable untimed Petri nets.~~

More related works dealing with some particular classes of Petri nets can be also cited. Indeed, only some works dealing with P-time Petri nets and time interval Petri nets, which are the subject of this work, will be cited in the rest of this section. In [4], author treated the problem of marking estimation for the class of P-Time Petri net using a linear programming problem leading to check the set of firing sequences. The determination of the extremum cycle times was the subject of [24] using a particular extended graph associated with the studied P-time Petri net. In [26, 27], authors consider a particular model of P-time Event Graph named Negative Event to analyze the maximum and minimum cycle times by checking the earliest and latest possible regular firing timetables using ~~both standard and~~ max-plus algebra. The liveness is also

discussed. In [21], authors analyzed a particular process where time has only a minimum delay and not a maximum duration like the case of P-time Event Graphs. Indeed, authors proposed to consider only one common period to all tasks in order to minimize largest period of the events. [Time reachability is considered also in \[39\]](#). Many works [9,19,22,26] used the technique of linear programming to describe problems relative to discrete event system modeled by Petri nets in order to compute cycle time and model duality principle

In [37], authors proposed to add more restriction on each iteration cycle in order to compute bounds in the case of Stochastic Event Graphs. In [2], authors computed the basic quantities depending on a particular experiment interval.

These works are interesting in the analyze of cycle time contrary to this work which the aim is the state estimation and the fault detection and localization of discrete events system described by time interval Petri nets.

[Colored Petri nets are also treated recently in \[35\] and applied for technological facilities and the evolution of systems maintenance.](#)

[P-time Petri nets are considered in \[4\] and recently in \[3\] of decentralized state estimation in real time and diagnosis. The main difference with this work is that we consider P-time-interval Petri nets.](#)

3 Elementary background on Petri nets

3.1 Notations

In this section, notations used along the paper are presented. Main definition related with Petri net formalism will be also presented.

$|X|$ denotes the cardinal of the set X . A place/transition net (a P/TR net) is a structure $N = (P, TR, W^+, W^-)$, where P is a set of $|P|$ places and TR is a set of $|TR|$ transitions which are denoted by x (notation t corresponds to the current time and T to the transposition of a matrix). T_l^+ (resp. T_l^-) denotes the upper bound (resp. the lower bound) of the token sojourn time of a given place $p_l \in P$.

Post- and pre-incidence matrices W^+ and W^- are $|P| \times |TR|$ over $\mathbb{N}$ where each row $l \in \{1, \dots, |P|\}$ corresponds to the weight of the incoming and outgoing arcs of places $p_l \in P$ respectively. The incidence matrix is computed by $W = W^+ - W^-$.

In this work the weight of each arc is supposed unitary, i.e., $W_{ij} \in \{-1, 0, 1\}$.

The marking of set P is a vector $m \in \mathbb{N}^{|P|}$ that assigns to each place of a P/TR net a non-negative integer number of tokens. m_l is the marking of place p_l with $l \in \{1, \dots, |P|\}$. A net system (N, m) is a net N with an initial marking m .

In this paper, the incidence matrices and the initial marking are supposed known. The internal transitions are denoted x_i , u_i are the input transitions and y_i are the output transitions. The places set P can be decomposed on four sets:

- P_{ux} : the set of places between the input and the internal transitions, their upper bounds (resp. lower bounds) of temporizations are denoted T_{ux}^+ (resp. T_{ux}^-) and their marking is denoted m_{ux}
- P_{xx} : the set of places between the internal transitions, their upper bounds (resp. lower bounds) of temporizations are denoted T_{xx}^+ (resp. T_{xx}^-) and their marking is denoted m_{xx}
- P_{xy} : the set of places between the output and the internal transitions, their upper bounds (resp. lower bounds) of temporizations are denoted T_{xy}^+ (resp. T_{xy}^-) and their marking is denoted m_{xy}
- P_{uy} : the set of places which link directly the output and the input, their upper bounds (resp. lower bounds) of temporizations are denoted T_{uy}^+ (resp. T_{uy}^-) and their marking is denoted m_{uy}

3.2 Definitions

Definition 1 [15] Given a Petri net $N = (P, TR, W^+, W^-)$:

The N Petri net is Forward Conflict Free (FCF), i.e., any two distinct unobservable transitions have no common input place.

Definition 2 [15] The system of linear inequalities $Ax \leq b$ is sup-monotone if each row of matrix A has one strictly positive element at most.

Definition 3 [15] Let Γ be the solution set of a sup-monotone system $Ax \leq b$. Set Γ is a sup-semilattice since each pair of elements has an upper bound.

Moreover, the following lemma guarantees that Γ has an extremum element that is a greatest solution which belongs to Γ .

Lemma 1 [15] *Let Γ be the solution set of a sup-monotone system $Ax \leq b$. Set Γ has a greatest element if the set is non-empty and has a majorant.*

3.3 Preliminary example

Example 1 Let us consider the following elementary time interval Petri net given by figure 1.

Fig. 1 Elementary example on time interval Petri net

The time evolution of the system trajectory can be described by the equations 1 and 2:

$$x_1 + 1 \leq x_2 + x_3 \leq x_1 + 2 \quad (1)$$

$$\left. \begin{array}{l} x_2 + 3 \leq x_4 \leq x_2 + 4 \\ x_3 + 2 \leq x_4 \leq x_3 + 3 \end{array} \right\} \Rightarrow \max(x_2 + 3, x_3 + 2) \leq x_4 \leq \min(x_2 + 4, x_3 + 3) \quad (2)$$

In order to estimate the system trajectory, the equations 3, 4 and 5 can be used:

$$x_2 + x_3 - 2 \leq x_1 \leq x_2 + x_3 - 1 \quad (3)$$

$$x_4 - 4 \leq x_2 \leq x_4 - 3 \quad (4)$$

$$x_4 - 3 \leq x_3 \leq x_4 - 2 \quad (5)$$

The system trajectory presents an upper bound and a lower bound computed using the ~~time~~ interval of the token sojourn time in each place. The main objective of this work is to conceive a general state model allowing computing the upper and lower bound of each element of the system state and to conceive an observer allowing estimating the upper and lower bound of the system trajectory. The obtained result can be used after that for the fault detection.

4 Extension of the classical state model

4.1 Introduction

In this section, we present an extension of the the classical state model presented in [25] to time interval petri nets. This extension is proposed following the count approach and dater approach. This state model is presented in two different forms for both. The first proposed form is the "compact form" which uses the whole state vector denoted x , and pre- and post-incidence matrices. The second proposed form gives the equation of each element of the state vector by making explicit the state model. The used state vector $x(t)$ includes the system inputs, outputs and internal transitions. Each element of the state vector $x(t)$ is denoted $x_i(t)_{\{1, \dots, n\}}$, the state vector becomes : $x(t) = (x_1(t), x_2(t), \dots, x_i(t), \dots, x_n(t))^T$ where n is the transitions number.

4.2 Count approach

4.2.1 Compact form

~~The used state vector $x(t)$ includes the system input, output and internal transitions. Each element of the state vector $x(t)$ is noted $x_i(t)$, the state vector becomes : $x(t) = (x_1(t), x_2(t), \dots, x_i(t), \dots, x_n(t))^T$ where n is the transitions number.~~

In the case of time interval Petri nets, the system evolution in time is governed by the following inequality:

$$m + W^+ x(t - T_l^-) \leq W^- x(t) \leq m + W^+ x(t - T_l^+) \quad (6)$$

where W^+ and W^- are the incidence matrices, m is the initial marking and T_l^+ and T_l^- are respectively the upper and lower bounds of the tokens sojourn time in places. The inequality (6) describes **only** the system trajectory evolution in time. Since the behavior of the system trajectory must be non-decreasing, the inequality (8) is added. The global state model allowing the system simulation is:

$$m + W^+ x(t - T_l^-) \leq W^- x(t) \leq m + W^+ x(t - T_l^+) \quad (7)$$

$$x(t - 1) \leq x(t) \quad (8)$$

The inequality (7) has the form : $b^- \leq Ax \leq b^+$ with: $A = W^-$, $b^- = m + W^+ x(t - T_l^-)$ and $b^+ = m + W^+ x(t - T_l^+)$. As it is considered that m , W^+ , $x(t - T_l^-)$ and $x(t - T_l^+)$ are known, b^+ and b^- are known.

~~$\sum x_j(t - T_l)$ is the sum of the output arcs of the place p_i , T_l^+ and T_l^- are respectively the upper and lower bounds of the place temporization interval and m_l is the initial marking of the place p_l .~~

4.2.2 Explicit form

The explicit form let to details each term $x_i(t)$ of the state vector x . If the studied system is FCF, then it is sup-monotone and the matrix W^- presents only one 1 by row, we obtain:

$$\max_{j \in \bullet x_j} (m_i + \sum_{l \in \bullet p_i} x_l(t - T_l^-)) \leq x_j(t) \leq \min_{j \in \bullet x_j} (m_i + \sum_{l \in \bullet p_i} x_l(t - T_l^+)), \quad j \in \{1, \dots, L\} \quad (9)$$

where L is the number of column of the matrix W^- .

Let us denote $x_j^+(t)$ the greatest state and $x_j^-(t)$ the smallest state. These particular states are given by the following equations:

$$x_j^-(t) = \max_{j \in \bullet x_j} (m_i + \sum_{l \in \bullet p_i} x_l^-(t - T_l^-)), \quad j \in \{1, \dots, L\} \quad (10)$$

$$x_j^+(t) = \min_{j \in \bullet x_j} (m_i + \sum_{l \in \bullet p_i} x_l^+(t - T_l^+)), \quad j \in \{1, \dots, L\} \quad (11)$$

$\sum x_j(t - T_l)$ is the sum of the output arcs of the place p_i , T_l^+ and T_l^- are respectively the upper and lower bounds of the place temporization interval and m_l is the initial marking of the place p_l .

4.3 Dater approach

4.3.1 Compact form

~~The used state vector $x(t)$ includes the system input, output and internal transitions. Each element of the state vector $x(t)$ is noted $x_i(t)$, the state vector becomes : $x(t) = (x_1(t), x_2(t), \dots, x_i(t), \dots, x_n(t))^T$ where n is the transitions number.~~ In the case of time interval Petri nets, the system evolution in time is governed by the following inequality:

$$T^+ + W^+ x(k - m_l) \geq W^- x(k) \geq T^- + W^+ x(k - m_l) \quad (12)$$

where W^+ and W^- are the incidence matrices, T^+ is the upper bound places temporization vector, T^- is the lower bound places temporization vector and $m_l \in \{0, 1\}$. The inequality (12) describes the system evolution in time. Since the behavior of the system trajectory must be non-decreasing, the inequality (14) is added in order to obtain the following global state model:

$$T^+ + W^+ x(k - m_l) \geq W^- x(k) \geq T^- + W^+ x(k - m_l) \quad (13)$$

$$x(k - 1) \leq x(k) \quad (14)$$

4.3.2 Explicit form

The explicit form let to details each term $x_i(t)$ of the state vector x . If the studied system is FCF, then it is sup-monotone and the matrix W^- presents only one 1 by row, we obtain:

$$\max_{j \in \bullet_{x_j}} (T_i^+ + \sum_{l \in \bullet_{p_i}} x_l(k - m_l)) \geq x_j(k) \geq \min_{j \in \bullet_{x_j}} (T_i^- + \sum_{l \in \bullet_{p_i}} x_l(k - m_l)), \quad j \in \{1, \dots, L\}, \quad m_l \in \{0, 1\} \quad (15)$$

where L is the number of column of the matrix W^- .

Let us denote $x_j^+(t)$ the greatest state and $x_j^-(t)$ the smallest state. These particular states are given by the following equations:

$$x_j^-(k) = \min_{j \in \bullet_{x_j}} (T_i^- + \sum_{l \in \bullet_{p_i}} x_l^+(k - m_l)), \quad j \in \{1, \dots, L\}, \quad m_l \in \{0, 1\} \quad (16)$$

$$x_j^+(k) = \max_{j \in \bullet_{x_j}} (T_i^+ + \sum_{l \in \bullet_{p_i}} x_l^-(k - m_l)), \quad j \in \{1, \dots, L\}, \quad m_l \in \{0, 1\} \quad (17)$$

5 Design of a new state model

5.1 Objective

~~This part has the~~ The objective of this part is to develop a new form of the state model. The developed state model can be used to analyze and simulate the evolution of the system state and output. The obtained state model is component of two inequalities. The first inequality let to compute the upper and lower bounds of the system state and the second one let to compute the upper and lower bounds of the system outputs. The proposed state model is an extension to time interval Petri nets of the state model developed initially for timed Petri nets in [25]. This state model is conceived following successively a count and a dater approaches.

5.2 Count approach

The system behavior can be described, for the four sets P_{ux} , P_{xx} , P_{xy} and P_{uy} , by the following inequalities where the matrices W_{ux}^+ , ~~W_{ux}^-~~ , W_{xx}^+ , ~~W_{xx}^-~~ , W_{xy}^+ , ~~W_{xy}^-~~ , W_{uy}^+ and ~~W_{uy}^-~~ are elements of the incidence ~~matrix~~ matrices W^+ and W^- .

$$(m_{ux}) + W_{ux}^+ u(t - T_{l,ux}^+) \leq W_{ux}^- x(t) \leq (m_{ux}) + W_{ux}^+ u(t - T_{l,ux}^-) \quad (18)$$

$$(m_{xx}) + W_{xx}^+ x(t - T_{l,xx}^+) \leq W_{xx}^- x(t) \leq (m_{xx}) + W_{xx}^+ x(t - T_{l,xx}^-) \quad (19)$$

$$(m_{yx}) + W_{xy}^+ x(t - T_{l,xy}^+) \leq W_{xy}^- y(t) \leq (m_{yx}) + W_{xy}^+ x(t - T_{l,xy}^-) \quad (20)$$

$$(m_{uy}) + W_{uy}^+ u(t - T_{l,uy}^+) \leq W_{uy}^- y(t) \leq (m_{uy}) + W_{uy}^+ u(t - T_{l,uy}^-) \quad (21)$$

To guarantee the non decreasing system behavior, the following inequalities are added:

$$x(t - T_{l,xx}^+) \leq x(t) \leq x(t - T_{l,xx}^-) \quad (22)$$

$$y(t - T_{l,uy}^+) \leq y(t) \leq y(t - T_{l,uy}^-) \quad (23)$$

Inequalities (18) and (19) give the upper and lower bounds of the system state $x(t)$, they can be transform on the matrix inequality 24:

$$\begin{pmatrix} m_{ux} \\ m_{xx} \end{pmatrix} + \begin{pmatrix} W_{ux}^+ & 0 \\ 0 & W_{xx}^+ \end{pmatrix} \begin{pmatrix} u(t - T_{l,ux}^+) \\ x(t - T_{l,xx}^+) \end{pmatrix} \leq \begin{pmatrix} W_{ux}^- \\ W_{xx}^- \end{pmatrix} x(t) \leq \begin{pmatrix} m_{ux} \\ m_{xx} \end{pmatrix} + \begin{pmatrix} W_{ux}^+ & 0 \\ 0 & W_{xx}^+ \end{pmatrix} \begin{pmatrix} u(t - T_{l,ux}^-) \\ x(t - T_{l,xx}^-) \end{pmatrix} \quad (24)$$

Inequality (24) let to compute an upper bound of the system state $x(t)$ denoted $x^+(t)$ and a lower bound of the system state $x(t)$ denoted $x^-(t)$.

In the same manner, Inequalities (20) and (21) giving the upper and lower bounds of the system output $y(t)$ can be put on the matrix inequality (25):

$$\begin{pmatrix} m_{xy} \\ m_{uy} \end{pmatrix} + \begin{pmatrix} W_{xy}^+ & 0 \\ 0 & W_{uy}^+ \end{pmatrix} \begin{pmatrix} x(t - T_{l,xy}^+) \\ u(t - T_{l,uy}^+) \end{pmatrix} \leq \begin{pmatrix} W_{xy}^- \\ W_{uy}^- \end{pmatrix} y(t) \leq \begin{pmatrix} m_{xy} \\ m_{uy} \end{pmatrix} + \begin{pmatrix} W_{xy}^+ & 0 \\ 0 & W_{uy}^+ \end{pmatrix} \begin{pmatrix} x(t - T_{l,xy}^-) \\ u(t - T_{l,uy}^-) \end{pmatrix} \quad (25)$$

The obtained Inequality (25) let to compute an upper bound of the system output $y(t)$ denoted $y^+(t)$ and a lower bound of the system output $y(t)$ denoted $y^-(t)$.

Inequalities (22) and (24) can be gathered in the augmented matrix inequality (26):

$$A_2^c x(t - T_{l,xx}^+) + M_x^c + B^c u(t - T_{l,xx}^+) \leq A_1^c x(t) \leq A_2^c x(t - T_{l,xx}^-) + M_x^c + B^c u(t - T_{l,xx}^-) \quad (26)$$

where :

$$A_1^c = \begin{pmatrix} I \\ W_{ux}^- \\ W_{xx}^- \end{pmatrix}, \quad A_2^c = \begin{pmatrix} I \\ 0 \\ W_{xx}^+ \end{pmatrix}, \quad M_x^c = \begin{pmatrix} 0 \\ m_{ux} \\ m_{xx} \end{pmatrix}, \quad \text{and} \quad B^c = \begin{pmatrix} 0 \\ W_{ux}^+ \\ 0 \end{pmatrix}$$

Similarly ~~Using the same approach~~, inequalities (23) and (25) can be gathered as following:

$$C_2^c x(t - T_{l,xy}^-) + M_y^c + D^c u(t - T_{l,uy}^-) + I^c y(t - T_{l,uy}^-) \leq C_1^c y(t) \leq C_2^c x(t - T_{l,xy}^-) + M_y^c + D^c u(t - T_{l,uy}^-) + I^c y(t - T_{l,uy}^-) \quad (27)$$

where:

$$C_1^c = \begin{pmatrix} I \\ W_{xy}^- \\ W_{uy}^- \end{pmatrix}, \quad C_2^c = \begin{pmatrix} 0 \\ W_{xy}^+ \\ 0 \end{pmatrix}, \quad M_y^c = \begin{pmatrix} 0 \\ m_{xy} \\ m_{uy} \end{pmatrix}, \quad I^c = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad D^c = \begin{pmatrix} 0 \\ 0 \\ W_{uy}^+ \end{pmatrix}$$

~~Using the i~~Inequalities (26) and (27), ~~form~~ the state model (28) ~~is obtained~~ given as follows :

$$\begin{cases} A_2^c x(t - T_{l,xx}^+) + M_x^c + B^c u(t - T_{l,xx}^+) \leq A_1^c x(t) \leq A_2^c x(t - T_{l,xx}^-) + M_x^c + B^c u(t - T_{l,xx}^-) \\ C_2^c x(t - T_{l,xy}^-) + M_y^c + D^c u(t - T_{l,uy}^-) + I^c y(t - T_{l,uy}^-) \leq C_1^c y(t) \leq \\ C_2^c x(t - T_{l,xy}^-) + M_y^c + D^c u(t - T_{l,uy}^-) + I^c y(t - T_{l,uy}^-) \end{cases} \quad (28)$$

The system (28) describes the obtained state model. It is possible to simulate this model in order to analyze the system's dynamic behavior. This model let to search simultaneously the upper and lower bounds of the system state $x(t)$ and the system output $y(t)$. **The upper and lower bounds of $x(t)$ and $y(t)$ are finite. It is possible to conclude that the system state and outputs are bounded and the system trajectory is non-decreasing.**

5.3 Dater approach

The system behavior can be described, for the four sets P_{ux} , P_{xx} , P_{xy} and P_{uy} , by the following inequalities where the matrices W_{ux}^+ , ~~W_{ux}^-~~ , W_{xx}^+ , ~~W_{xx}^-~~ , W_{xy}^+ , ~~W_{xy}^-~~ , W_{uy}^+ and ~~W_{uy}^-~~ are elements of the incidence ~~matrix~~ matrices W^+ and W^- .

$$T_{ux}^- + W_{ux}^+ u(k - m_{ux}) \leq W_{ux}^- x(k) \leq T_{ux}^+ + W_{ux}^+ u(k - m_{ux}) \quad (29)$$

$$T_{xx}^- + W_{xx}^+ x(k - m_{xx}) \leq W_{xx}^- x(k) \leq T_{xx}^+ + W_{xx}^+ x(k - m_{xx}) \quad (30)$$

$$T_{xy}^- + W_{xy}^+ x(k - m_{xy}) \leq W_{xy}^- y(k) \leq T_{xy}^+ + W_{xy}^+ x(k - m_{xy}) \quad (31)$$

$$T_{uy}^- + W_{uy}^+ u(k - m_{uy}) \leq W_{uy}^- y(k) \leq T_{uy}^+ + W_{uy}^+ u(k - m_{uy}) \quad (32)$$

To guarantee the non decreasing system behavior, the following inequalities are added:

$$T_{xx}^- + x(k - m_{xx}) \leq x(k) \leq T_{xx}^+ + x(k - m_{xx}) \quad (33)$$

$$T_{uy}^- + y(k - m_{uy}) \leq y(k) \leq T_{uy}^+ + y(k - m_{uy}) \quad (34)$$

Inequalities (29) and (30) give the upper and lower bounds of the system state $x(k)$, they can be transform on the matrix inequality (35):

$$\begin{pmatrix} T_{ux}^- \\ T_{xx}^- \end{pmatrix} + \begin{pmatrix} W_{ux}^+ & 0 \\ 0 & W_{xx}^+ \end{pmatrix} \begin{pmatrix} u(k - m_{ux}) \\ x(k - m_{xx}) \end{pmatrix} \leq \begin{pmatrix} W_{ux}^- \\ W_{xx}^- \end{pmatrix} x(k) \leq \begin{pmatrix} T_{ux}^+ \\ T_{xx}^+ \end{pmatrix} + \begin{pmatrix} W_{ux}^+ & 0 \\ 0 & W_{xx}^+ \end{pmatrix} \begin{pmatrix} u(k - m_{ux}) \\ x(k - m_{xx}) \end{pmatrix} \quad (35)$$

Inequality (35) let to compute an upper bound of the system state $x(k)$ denoted $x^+(k)$ and a lower bound of the system state $x(k)$ denoted $x^-(k)$.

In the same manner, Inequalities (31) and (32) giving the upper and lower bounds of the system output $y(k)$ can be put on the matrix inequality (36):

$$\begin{pmatrix} T_{xy}^- \\ T_{uy}^- \end{pmatrix} + \begin{pmatrix} W_{xy}^+ & 0 \\ 0 & W_{uy}^+ \end{pmatrix} \begin{pmatrix} x(k - m_{xy}) \\ u(k - m_{uy}) \end{pmatrix} \leq \begin{pmatrix} W_{xy}^- \\ W_{uy}^- \end{pmatrix} y(k) \leq \begin{pmatrix} T_{xy}^+ \\ T_{uy}^+ \end{pmatrix} + \begin{pmatrix} W_{xy}^+ & 0 \\ 0 & W_{uy}^+ \end{pmatrix} \begin{pmatrix} x(k - m_{xy}) \\ u(k - m_{uy}) \end{pmatrix} \quad (36)$$

The obtained Inequality (36) let to compute an upper bound of the system output $y(k)$ denoted $y^+(k)$ and a lower bound of the system output $y(k)$ denoted $y^-(k)$.

Inequalities (33) and (35) can be gathered in the augmented matrix inequality (37):

$$A_2^d x(k - m_{xx}) + T_x^- - + B^d u(k - m_{ux}) \leq A_1^d x(k) \leq A_2^d x(k - m_{xx}) + T_x^+ + B^d u(k - m_{ux}) \quad (37)$$

where :

$$A_1^d = \begin{pmatrix} I \\ W_{ux}^- \\ W_{xx}^- \end{pmatrix}, \quad A_2^d = \begin{pmatrix} I \\ 0 \\ W_{xx}^+ \end{pmatrix}, \quad T_x^+ = \begin{pmatrix} T_{xx}^+ \\ T_{ux}^+ \\ T_{xx}^+ \end{pmatrix}, \quad T_x^- = \begin{pmatrix} T_{xx}^- \\ T_{ux}^- \\ T_{xx}^- \end{pmatrix} \quad \text{and} \quad B^d = \begin{pmatrix} 0 \\ W_{ux}^+ \\ 0 \end{pmatrix}$$

Using the same approach, Inequalities (34) and (36) can be also gathered as following:

$$C_2^d x(k - m_{xy}) + T_y^- + D^d u(k - m_{uy}) + I^d y(k - m_{uy}) \leq C_1^d y(k) \leq C_2^d x(k - m_{xy}) + T_y^+ + D^d u(k - m_{uy}) + I^d y(k - m_{uy}) \quad (38)$$

where:

$$C_1^d = \begin{pmatrix} I \\ W_{xy}^- \\ W_{uy}^- \end{pmatrix}, \quad C_2^d = \begin{pmatrix} 0 \\ W_{xy}^+ \\ 0 \end{pmatrix}, \quad I^d = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix}, \quad T_y^- = \begin{pmatrix} T_{uy}^- \\ T_{xy}^- \\ T_{uy}^- \end{pmatrix}, \quad T_y^+ = \begin{pmatrix} T_{uy}^+ \\ T_{xy}^+ \\ T_{uy}^+ \end{pmatrix}, \quad \text{and} \quad D^d = \begin{pmatrix} I \\ 0 \\ W_{uy}^+ \end{pmatrix}$$

Using the inequalities (37) and (38), form the state model (39) is obtained:

$$\begin{cases} A_2^d x(k - m_{xx}) + T_x^- - + B^d u(k - m_{ux}) \leq A_1^d x(k) \leq A_2^d x(k - m_{xx}) + T_x^+ + B^d u(k - m_{ux}) \\ C_2^d x(k - m_{xy}) + T_y^- + D^d u(k - m_{uy}) + I^d y(k - m_{uy}) \leq C_1^d y(k) \leq \\ C_2^d x(k - m_{xy}) + T_y^+ + D^d u(k - m_{uy}) + I^d y(k - m_{uy}) \end{cases} \quad (39)$$

The system (39) describes the obtained state model. It is possible to simulate this model in order to analyze the system's dynamic behavior. This model let to search simultaneously the upper and lower bounds of the system state $x(k)$ and the system output $y(k)$. The upper and lower bounds of $x(k)$ and $y(k)$ are finite. It is possible to conclude that the system state and outputs are bounded and the system trajectory is non-decreasing.

Inequalities (28) and (39) describe the state models successively using the count and dater approaches. These models allow obtaining an upper and lower bounds of each elements of the state vector. At each moment, state vector must belong in the interval between the upper and the lower bounds.

5.4 Numerical example

Example 2 Let us consider the example of Petri net given in figure 1 of section 3.3

For this example, the system input is $u = x_1$ and the system output is $y = x_4$. Matrices W_{ux}^+ , W_{ux}^- , W_{xx}^+ , W_{xx}^- , W_{xy}^+ , W_{xy}^- , W_{uy}^+ and W_{uy}^- are the following:

$$W_{ux}^- = W_{xy}^+ = W_{uy}^+ = W_{ux}^- = W_{xx}^+ = W_{xx}^- = 0 \quad (40)$$

$$W_{ux}^+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad W_{xy}^- = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (41)$$

For the system simulation, $u = x_1$ is assumed to be known. For example we can obtain the following table presenting 5 iterations.

Table 1 Simulation of example-1

Dates	0	1	2	3	4
$u(k) = x_1(k)$	1	3	6	10	20
$x_2(k)$	1	2	3	5	10
$x_3(k)$	2	3	4	6	11
$y(k) = x_4(k)$	[4,5]	[5,6]	[6,7]	[8,9]	[13,14]

6 Design of a state and input estimator

6.1 Objective

The aim of this part is to conceive a new estimator able to estimate simultaneously the system state and input. The obtained estimator consists of two inequalities. The first inequality allows ~~estimating~~ obtaining an upper and lower bounds of the system state and the second one let to estimate the system output by computing its upper and lower bounds. The proposed estimator is conceived following successively a count and a dater approaches.

6.2 Count approach

To estimate the system state and input, it is supposed that the system output is known. Inequalities (18) to (21) can be rewritten in the following forms.

$$-(m_{ux}) + W_{ux}^- \hat{x}(t + T_{l,ux}^-) \leq W_{ux}^+ \hat{u}(t) \leq -(m_{ux}) + W_{ux}^- \hat{x}(t + T_{l,ux}^+) \quad (42)$$

$$-(m_{xx}) + W_{xx}^- \hat{x}(t + T_{l,xx}^-) \leq W_{xx}^+ \hat{x}(t) \leq -(m_{xx}) + W_{xx}^- \hat{x}(t + T_{l,xx}^+) \quad (43)$$

$$-(m_{yx}) + W_{xy}^- y(t + T_{l,xy}^-) \leq W_{xy}^+ \hat{x}(t) \leq -(m_{yx}) + W_{xy}^- y(t + T_{l,xy}^+) \quad (44)$$

$$-(m_{uy}) + W_{uy}^- y(t + T_{l,uy}^-) \leq W_{uy}^+ \hat{u}(t) \leq -(m_{uy}) + W_{uy}^- y(t + T_{l,uy}^+) \quad (45)$$

Inequalities (43) and (44) can be gathered in the matrix inequality (47) which permits to estimate the system input by computing a lower bound of $u(t)$ denoted $u^-(t)$ and an upper bound denoted $u^+(t)$.

$$\begin{pmatrix} W_{uy}^- & 0 \\ 0 & W_{ux}^- \end{pmatrix} \begin{pmatrix} y(t + T_{l,uy}^-) \\ x(t + T_{l,ux}^-) \end{pmatrix} - \begin{pmatrix} m_{uy} \\ m_{ux} \end{pmatrix} \leq \begin{pmatrix} W_{uy}^+ \\ W_{ux}^+ \end{pmatrix} \hat{u}(t) \leq \begin{pmatrix} W_{uy}^- & 0 \\ 0 & W_{ux}^- \end{pmatrix} \begin{pmatrix} y(t + T_{l,uy}^+) \\ x(t + T_{l,ux}^+) \end{pmatrix} - \begin{pmatrix} m_{uy} \\ m_{ux} \end{pmatrix} \quad (46)$$

In the same manner, inequalities (42) and (45) can be gathered in the matrix inequality (46) which is able to compute a lower bound of $x(t)$ denoted $x^-(t)$ and an upper bound denoted $x^+(t)$:

$$\begin{pmatrix} W_{xy}^- & 0 \\ 0 & W_{xx}^- \end{pmatrix} \begin{pmatrix} y(t + T_{l,xy}^-) \\ x(t + T_{l,xx}^-) \end{pmatrix} - \begin{pmatrix} m_{xy} \\ m_{xx} \end{pmatrix} \leq \begin{pmatrix} W_{xy}^+ \\ W_{xx}^+ \end{pmatrix} \hat{x}(t) \leq \begin{pmatrix} W_{xy}^- & 0 \\ 0 & W_{xx}^- \end{pmatrix} \begin{pmatrix} y(t + T_{l,xy}^+) \\ x(t + T_{l,xx}^+) \end{pmatrix} - \begin{pmatrix} m_{xy} \\ m_{xx} \end{pmatrix} \quad (47)$$

To guarantee the non decreasing behavior of the system trajectory, the following conditions are added:

$$\hat{x}(t + T_{l,xx}^-) \leq \hat{x}(t) \leq \hat{x}(t + T_{l,xx}^+) \quad (48)$$

$$\hat{u}(t + T_{l,uy}^-) \leq \hat{u}(t) \leq \hat{u}(t + T_{l,uy}^+) \quad (49)$$

Inequalities (47) and (48) can be aggregated in the augmented matrix inequality (50):

$$A_{02}^c \hat{x}(t + T_{l,xx}^-) - C_{01}^c y(t + T_{l,xy}^-) + M_{0x}^c \leq A_{01}^c \hat{x}(t) \leq A_{02}^c \hat{x}(t + T_{l,xx}^+) - C_{01}^c y(t + T_{l,xy}^+) + M_{0x}^c \quad (50)$$

where:

$$A_{01}^c = \begin{pmatrix} W_{xy}^+ \\ W_{xx}^+ \\ I \end{pmatrix}, \quad A_{02}^c = \begin{pmatrix} 0 \\ W_{xx}^- \\ I \end{pmatrix}, \quad M_{0x}^c = \begin{pmatrix} m_{xy} \\ m_{xx} \\ 0 \end{pmatrix}, \quad \text{and} \quad C_{01}^c = \begin{pmatrix} W_{xy}^- \\ 0 \\ 0 \end{pmatrix}$$

Moreover, inequalities (46) and (49) can be aggregated in the augmented matrix inequality (51):

$$\begin{aligned} I^c \hat{u}(t + T_{l,xx}^-) - A_{03}^c \hat{x}(t + T_{l,xx}^-) - C_{02}^c y(t + T_{l,xy}^-) - M_{0u}^c &\leq B_{01}^c \hat{u}(t) \leq \\ I^c \hat{u}(t + T_{l,xx}^+) - A_{03}^c \hat{x}(t + T_{l,xx}^+) - C_{02}^c y(t + T_{l,xy}^+) - M_{0u}^c &\end{aligned} \quad (51)$$

where:

$$B_{01}^c = \begin{pmatrix} W_{uy}^+ \\ W_{ux}^+ \\ I \end{pmatrix}, \quad A_{03}^c = \begin{pmatrix} 0 \\ W_{ux}^- \\ 0 \end{pmatrix}, \quad M_{0u}^c = \begin{pmatrix} m_{uy} \\ m_{ux} \\ 0 \end{pmatrix}, \quad C_{02}^c = \begin{pmatrix} W_{uy}^- \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad I^c = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}$$

The estimator (52) is obtained using the inequalities (50) and (51):

$$\begin{cases} A_{02}^c \hat{x}(t + T_{l,xx}^-) - C_{01}^c y(t + T_{l,xy}^-) + M_{0x}^c \leq A_{01}^c \hat{x}(t) \leq A_{02}^c \hat{x}(t + T_{l,xx}^+) - C_{01}^c y(t + T_{l,xy}^+) + M_{0x}^c \\ I^c \hat{u}(t + T_{l,xx}^-) - A_{03}^c \hat{x}(t + T_{l,xx}^-) - C_{02}^c y(t + T_{l,xy}^-) - M_{0u}^c \leq B_{01}^c \hat{u}(t) \leq \\ I^c \hat{u}(t + T_{l,xx}^+) - A_{03}^c \hat{x}(t + T_{l,xx}^+) - C_{02}^c y(t + T_{l,xy}^+) - M_{0u}^c \end{cases} \quad (52)$$

The obtained estimator (52) is an algebraic model giving simultaneously an estimation of upper and lower bounds of the system state and input. State and input estimation obtained using this estimator ~~are~~ is generally useful for fault estimation, detection and localization. The upper and lower bounds of the estimation of $x(t)$ and $y(t)$ are finite. The estimation of the system state and inputs are bounded and the system trajectory estimation is non-decreasing.

6.3 Dater approach

To estimate the system state and input, inequalities (29) to (32) can be rewritten in the following form.

$$-T_{ux}^+ + W_{ux}^- \hat{x}(k + m_{ux}) \leq W_{ux}^+ \hat{u}(k) \leq -T_{ux}^- + W_{ux}^- \hat{x}(k + m_{ux}) \quad (53)$$

$$-T_{xx}^+ + W_{xx}^- \hat{x}(k + m_{xx}) \leq W_{xx}^+ \hat{x}(k) \leq -T_{xx}^- + W_{xx}^- \hat{x}(k + m_{xx}) \quad (54)$$

$$-T_{xy}^+ + W_{xy}^- y(k + m_{xy}) \leq W_{xy}^+ \hat{x}(k) \leq -T_{xy}^- + W_{xy}^- y(k + m_{xy}) \quad (55)$$

$$-T_{uy}^+ + W_{uy}^- y(k + m_{uy}) \leq W_{uy}^+ \hat{u}(k) \leq -T_{uy}^- + W_{uy}^- y(k + m_{uy}) \quad (56)$$

Inequalities (54) and (55) can be gathered in the matrix inequality (57) which permits to estimate the system input by computing a lower bound of $u(k)$ denoted $u^-(k)$ and an upper bound denoted $u^+(k)$.

$$\begin{pmatrix} W_{ux}^- & 0 \\ 0 & W_{uy}^- \end{pmatrix} \begin{pmatrix} \hat{x}(k + m_{ux}) \\ y(k + m_{uy}) \end{pmatrix} - \begin{pmatrix} T_{ux}^+ \\ T_{uy}^+ \end{pmatrix} \leq \begin{pmatrix} W_{ux}^+ \\ W_{uy}^+ \end{pmatrix} \hat{u}(k) \leq \begin{pmatrix} W_{ux}^- & 0 \\ 0 & W_{uy}^- \end{pmatrix} \begin{pmatrix} \hat{x}(k + m_{ux}) \\ y(k + m_{uy}) \end{pmatrix} - \begin{pmatrix} T_{ux}^- \\ T_{uy}^- \end{pmatrix} \quad (57)$$

In the same manner, inequalities (53) and (56) can be gathered in the matrix inequality (58) which is able to compute a lower bound of $x(k)$ denoted $x^-(k)$ and an upper bound denoted $x^+(k)$:

$$\begin{pmatrix} W_{xx}^- & 0 \\ 0 & W_{xy}^- \end{pmatrix} \begin{pmatrix} \hat{x}(k + m_{xx}) \\ y(k + m_{xy}) \end{pmatrix} - \begin{pmatrix} T_{xx}^+ \\ T_{xy}^+ \end{pmatrix} \leq \begin{pmatrix} W_{xx}^+ \\ W_{xy}^+ \end{pmatrix} \hat{x}(k) \leq \begin{pmatrix} W_{xx}^- & 0 \\ 0 & W_{xy}^- \end{pmatrix} \begin{pmatrix} \hat{x}(k + m_{xx}) \\ y(k + m_{xy}) \end{pmatrix} - \begin{pmatrix} T_{xx}^- \\ T_{xy}^- \end{pmatrix} \quad (58)$$

To guarantee the non decreasing behavior of the system trajectory, the following conditions are added:

$$T_{xx}^- + \hat{x}(k + m_{xx}) \leq \hat{x}(k) \leq T_{xx}^+ + \hat{x}(k + m_{xx}) \quad (59)$$

$$T_{uy}^- + \hat{u}(k + m_{uy}) \leq \hat{u}(k) \leq T_{uy}^+ + \hat{u}(k + m_{uy}) \quad (60)$$

Inequalities (58) and (59) can be aggregated in the augmented matrix inequality (61):

$$A_{02}^d \hat{x}(k + m_{xx}) + C_{01}^d y(k + m_{xy}) - T_{0x}^+ \leq A_{01}^d \hat{x}(k) \leq A_{02}^d \hat{x}(k + m_{xx}) + C_{01}^d y(k + m_{xy}) - T_{0x}^- \quad (61)$$

where:

$$A_{01}^d = \begin{pmatrix} W_{xx}^+ \\ W_{xy}^+ \\ I \end{pmatrix}, \quad A_{02}^d = \begin{pmatrix} W_{xx}^- \\ 0 \\ I \end{pmatrix}, \quad T_x^- = \begin{pmatrix} T_{xx}^- \\ T_{xy}^- \\ -T_{xx}^+ \end{pmatrix}, \quad T_x^+ = \begin{pmatrix} T_{xx}^+ \\ T_{xy}^+ \\ -T_{xx}^- \end{pmatrix}, \quad \text{and} \quad C_{01}^d = \begin{pmatrix} 0 \\ W_{xy}^- \\ 0 \end{pmatrix}$$

Moreover, inequalities (57) and (60) can be aggregated in the augmented matrix inequality (62):

$$A_{03}^d \hat{x}(k + m_{ux}) + I^d \hat{u}(k + m_{uy}) + C_{02}^d y(k + m_{xy}) - T_{0u}^+ \leq B_{01}^d \hat{u}(k) \leq A_{03}^d \hat{x}(k + m_{ux}) + I^d \hat{u}(k + m_{uy}) + C_{02}^d y(k + m_{xy}) - T_{0u}^- \quad (62)$$

where:

$$B_{01}^d = \begin{pmatrix} W_{ux}^+ \\ W_{uy}^+ \\ I \end{pmatrix}, \quad A_{03}^d = \begin{pmatrix} W_{ux}^- \\ 0 \\ 0 \end{pmatrix}, \quad T_{0u}^- = \begin{pmatrix} T_{ux}^- \\ T_{uy}^- \\ -T_{ux}^+ \end{pmatrix}, \quad T_{0u}^+ = \begin{pmatrix} T_{ux}^+ \\ T_{uy}^+ \\ -T_{ux}^- \end{pmatrix}, \quad C_{02}^d = \begin{pmatrix} 0 \\ W_{uy}^- \\ 0 \end{pmatrix}, \quad \text{and} \quad I^d = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}$$

The estimator (63) is obtained using the inequalities (61) and (62):

$$\begin{cases} A_{02}^d \hat{x}(k + m_{xx}) + C_{01}^d y(k + m_{xy}) - T_{0x}^+ \leq A_{01}^d \hat{x}(k) \leq A_{02}^d \hat{x}(k + m_{xx}) + C_{01}^d y(k + m_{xy}) - T_{0x}^- \\ A_{03}^d \hat{x}(k + m_{ux}) + I^d \hat{u}(k + m_{uy}) + C_{02}^d y(k + m_{xy}) - T_{0u}^+ \leq B_{01}^d \hat{u}(k) \leq A_{03}^d \hat{x}(k + m_{ux}) + I^d \hat{u}(k + m_{uy}) + C_{02}^d y(k + m_{xy}) - T_{0u}^- \end{cases} \quad (63)$$

The obtained estimator (63) is an algebraic model giving simultaneously an upper and lower bounds of the system state and input. State and input estimation obtained using this estimator are generally useful for fault estimation, detection and localization.

The upper and lower bounds of the estimation of $x(k)$ and $y(k)$ are finite. The estimation of the system state and inputs are bounded and the system trajectory estimation is non-decreasing.

Inequalities (52) and (63) describe the elaborated estimator using successively the count and dater approaches. These models allow obtaining an estimation of the upper and lower bounds of each elements of the state vector. At each moment, state vector estimation must belong in the interval between the upper and the lower bounds.

6.4 Numerical example

Example 3 Let us consider the example of Petri net given in figure 1 of section 3.3

For this example, the system input is $u = x_1$ and the system output is $y = x_4$. Matrices W_{ux}^+ , W_{ux}^- , W_{xx}^+ , W_{xx}^- , W_{xy}^+ , W_{xy}^- , W_{uy}^+ and W_{uy}^- are given in equations (40) and (41). For the system simulation, $y = x_4$ is assumed to be known. For example we can obtain the following table presenting 5 iterations.

Table 2 Estimation of example-1

Dates	0	1	2	3	4
$u(k) = x_1(k)$	[0,4]	[1,6]	[3,8]	[7,12]	[17,22]
$x_2(k)$	[0,2]	[1,3]	[2,4]	[4,6]	[9,11]
$x_3(k)$	[1,3]	[2,4]	[3,5]	[5,7]	[10,12]
$y(k) = x_4(k)$	[4,5]	[5,6]	[6,7]	[8,9]	[13,14]

7 Application to a baking process

7.1 Introduction

In this section, the proposed state model (39) and estimator (63) will be applied to a model of baking process. The objective of this application is to validate their efficiency. A dater approach is followed in order to validate the proposed system state and estimator. Indeed, the application is divided into four parts. The first part is an application of the proposed state model (39). The objective of the second part is to apply the proposed estimator. Residuals are computed in the third part in order to validate the proposed state model and estimator (63). In the ~~last~~ **fourth** part of this application, we propose to use the proposed state model and estimator to the fault detection and localization.

The model of the baking process is proposed in [14, 20]. It describe two ranges of bread production with two different qualities. The baking process is modeled by a time interval Petri net and its operating is detailed in [14, 20]. The model of the baking process is given in figure 2 [14].

Fig. 2 Plant bakery modeled by time interval Petri net

7.2 System simulation

The matrices W_{ux}^+ , W_{ux}^- , W_{xx}^+ , W_{xx}^- , W_{xy}^+ , W_{xy}^- , W_{uy}^+ and W_{uy}^- are the following:

$$W_{ux}^- = W_{xy}^+ = W_{uy}^+ = W_{uy}^- = 0 \quad (64)$$

$$\begin{aligned}
W_{ux}^+ = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad W_{xy}^- = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad W_{xx}^+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad W_{xx}^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (65)
\end{aligned}$$

Simulation results are presented in table 3. Indeed, it is supposed that the system input $u(k)$ is known, system state and output are computed after. Table 3 gives the upper and lower bounds for each element of the system state and the output. These upper and lower bounds are obtained using the system state proposed in the equation (39) where the matrices W_{ux}^+ , W_{ux}^- , W_{xx}^+ , W_{xx}^- , W_{xy}^+ , W_{xy}^- , W_{uy}^+ and W_{uy}^- are given in (64) and (65).

Table 3 System simulation

Dates	0	1	2	3	4	5	6	7	8
$u(k)$	0	1	2	4	7	10	15	20	100
$x_1(k)$	[25,27]	[28,35]	[29,45]	[31,48]	[34,49]	[37,51]	[42,54]	[47,57]	[125,127]
$x_2(k)$	[35,42]	[38,50]	[39,60]	[41,63]	[44,64]	[47,66]	[52,69]	[57,72]	[135,142]
$x_3(k)$	[215,235]	[218,243]	[219,253]	[221,256]	[224,257]	[227,259]	[232,262]	[237,265]	[315,335]
$x_4(k)$	[230,255]	[233,263]	[234,273]	[236,276]	[239,277]	[242,279]	[247,282]	[252,285]	[330,355]
$x_5(k)$	[410,495]	[413,503]	[414,513]	[416,516]	[419,517]	[422,519]	[427,522]	[432,525]	[510,595]
$x_6(k)$	[430,520]	[433,528]	[434,538]	[436,541]	[439,542]	[442,544]	[447,547]	[452,550]	[530,620]
$x_7(k)$	[15,20]	[16,21]	[17,22]	[19,24]	[22,27]	[25,30]	[30,35]	[35,40]	[115,120]
$x_8(k)$	[25,50]	[35,51]	[38,52]	[39,54]	[41,57]	[44,60]	[47,65]	[52,70]	[125,150]
$x_9(k)$	[35,65]	[45,66]	[48,67]	[49,69]	[51,72]	[54,75]	[57,80]	[62,85]	[135,165]
$x_{10}(k)$	[50,85]	[86,230]	[87,233]	[89,234]	[92,236]	[95,239]	[100,242]	[105,247]	[185,252]
$x_{11}(k)$	[65,105]	[101,250]	[102,253]	[104,254]	[107,256]	[110,259]	[115,262]	[120,267]	[200,272]
$x_{12}(k)$	[185,225]	[221,400]	[222,403]	[224,404]	[227,406]	[230,409]	[235,412]	[240,417]	[320,422]
$x_{13}(k)$	[205,280]	[241,425]	[242,428]	[244,429]	[247,431]	[250,434]	[255,437]	[260,442]	[345,447]
$y(k)$	–	430,+∞[	433,+∞[	434,+∞[	436,+∞[	439,+∞[	442,+∞[	447,+∞[	452,+∞[

For each date, each element x_i of the state vector x must belong in the correspondent interval. For example $x_3(k=4) \in [224, 257]$ and $x_7(k=2) \in [17, 22]$ etc.

7.3 State estimation

The table 4 gives the estimation result of the system state and input. The symbol $()$ means that it is an estimated variable. For the estimation, it is supposed that only the system output $y(k)$ is known. The results given in table 4 are obtained using the estimator proposed in the equation (63) where the matrices W_{ux}^+ , W_{ux}^- , W_{xx}^+ , W_{xx}^- , W_{xy}^+ , W_{xy}^- , W_{uy}^+ and W_{uy}^- are given in (64) and (65).

Table 4 State estimation

Dates	0	1	2	3	4	5	6	7	8
$u(k)$	0	[0,3]	[0,4]	[0,6]	[0,7]	[0,12]	[17,172]	[22,172]	[250,300]
$\hat{x}_1(k)$	[0,25]	[0,28]	[0,29]	[0,31]	[0,34]	[0,37]	[0,42]	[0,47]	[160,325]
$\hat{x}_2(k)$	[0,35]	[0,38]	[0,39]	[0,41]	[0,44]	[0,47]	[0,52]	[0,57]	[175,335]
$\hat{x}_3(k)$	[145,215]	[148,218]	[149,219]	[151,221]	[154,224]	[157,227]	[162,232]	[167,237]	[310,355]
$\hat{x}_4(k)$	[165,230]	[168,233]	[169,234]	[171,236]	[174,239]	[177,242]	[182,247]	[187,252]	[330,370]
$\hat{x}_5(k)$	[0, 410]	[0,413]	[0,414]	[0,416]	[0,419]	[0,422]	[0,427]	[0,432]	[0,510]
$\hat{x}_6(k)$	[0, 430]	[0, 433]	[0, 434]	[0, 436]	[0, 439]	[0, 442]	[0, 447]	[0, 452]	[0, 530]
$\hat{x}_7(k)$	[0,143]	[0,144]	[0,156]	[0,159]	[0,167]	[0,172]	[2,177]	[192,262]	[270,340]
$\hat{x}_8(k)$	[13,168]	[14,169]	[16,171]	[19,174]	[22,177]	[27,182]	[32,187]	[222,272]	[300,350]
$\hat{x}_9(k)$	[28,178]	[29,179]	[31,181]	[34,184]	[37,187]	[42,192]	[47,197]	[237,282]	[315,360]
$\hat{x}_{10}(k)$	[198,240]	[199,243]	[201,244]	[204,246]	[207,249]	[212,252]	[217,257]	[257,297]	[335,375]
$\hat{x}_{11}(k)$	[218,255]	[219,258]	[221,259]	[224,261]	[227,264]	[232,267]	[237,272]	[277,312]	[355,390]
$\hat{x}_{12}(k)$	[0, 410]	[0,413]	[0,414]	[0,416]	[0,419]	[0,422]	[0,427]	[0,432]	[0,510]
$\hat{x}_{13}(k)$	[0, 430]	[0, 433]	[0, 434]	[0, 436]	[0, 439]	[0, 442]	[0, 447]	[0, 452]	[0, 530]
$y(k)$	–	[430,+∞[	[433,+∞[	[434,+∞[	[436,+∞[	[439,+∞[	[442,+∞[	[447,+∞[	[452,+∞[

Table 4 gives a possible estimation of the state vector at each date k . The estimation is based on the interval of $y(k)$.

7.4 Residuals computation

Simulation and estimation results give for each variable a lower and upper bound. Residuals are generally computed to know the precision of the state estimation. In this work, residuals are computed as the intersection between the two obtained intervals. If the intersection is different of $\emptyset$, it is supposed that the estimation is accepted. ~~Residuals presented in table 5 show that estimation is accepted in more than 89% of cases.~~

Table 5 Residuals computation

Dates	0	1	2	3	4	5	6	7	8
$r_u(k)$	0	1	2	4	7	10	15	$\emptyset$	$\emptyset$
$r_1(k)$	25	28	29	31	34	37	42	47	$\emptyset$
$r_2(k)$	35	38	39	41	44	47	52	57	$\emptyset$
$r_3(k)$	215	218	219	221	224	227	232	237	[315,335]
$r_4(k)$	230	233	234	236	239	242	247	252	[330,355]
$r_5(k)$	410	413	414	416	419	422	427	432	510
$r_6(k)$	430	433	434	436	439	442	447	452	530
$r_7(k)$	[15,20]	[16,21]	[17,22]	[19,24]	[22,27]	[25,30]	[30,35]	$\emptyset$	$\emptyset$
$r_8(k)$	[25,50]	[35,51]	[38,52]	[39,54]	[41,57]	[44,60]	[47,65]	$\emptyset$	$\emptyset$
$x_9(k)$	[35,65]	[45,66]	[48,67]	[49,69]	[51,72]	[54,75]	[57,80]	$\emptyset$	$\emptyset$
$x_{10}(k)$	[50,85]	[86,230]	[87,233]	[89,234]	[92,236]	[95,239]	[100,242]	[105,247]	[335,252]
$x_{11}(k)$	$\emptyset$	[219,250]	[221,253]	[224,254]	[227,256]	[232,259]	[237,262]	$\emptyset$	$\emptyset$
$r_{12}(k)$	[185,225]	[221,400]	[222,403]	[224,404]	[227,406]	[230,409]	[235,412]	[240,417]	[320,422]
$x_{13}(k)$	[205,280]	[241,425]	[242,428]	[244,429]	[247,431]	[250,434]	[255,437]	[260,442]	[345,447]
$y(k)$	–	[430,+∞[	[433,+∞[	[434,+∞[	[436,+∞[	[439,+∞[	[442,+∞[	[447,+∞[	[452,+∞[

Residuals presented in table 5 show that estimation is accepted in more than 89% of cases. Indeed, the estimation is not precise only for few cases. In more than 89% of cases, the estimation is considered precise since the intersection between the simulation result and the estimated values is not empty. Residuals computation show that the proposed system state (39) and estimator (63) are sufficiently precise. This result can be used for fault detection and localization and/or fault estimation.

7.5 Fault detection and localization

The main objective of this part is to apply the results of estimation and simulation to ~~make~~ fault detection and localization. It is important also to deduce if the fault detection is made at the same date or with a delay. To validate this ~~approach work~~, it is supposed that the studied system is affected by a fault. In this work, the fault is modeled by an unknown input transition for the place P_{11} . This transition is denoted

$\alpha(k)$. It is supposed that this transition can take, at many dates, very high values. For simulation, it is supposed that the fault $\alpha(k)$ takes the values given in the table 6. The following inequality is added.

$$x_{10}(k) + \alpha(k) + T_{11}^- \leq x_{11}(k) \leq x_{10}(k) + \alpha(k) + T_{11}^+ \quad (66)$$

Table 6 Fault detection and localization

Dates	0	1	2	3	4	5	6	7	8
$u(k)$	0	1	2	4	7	10	15	20	100
$\alpha(k)$	0	0	5000	0	0	5000	0	5000	100
$x_1(k)$	[25,27]	[28,35]	[29,45]	[31,48]	[34,49]	[37,51]	[42,54]	[47,57]	[125,127]
$x_2(k)$	[35,42]	[38,50]	[39,60]	[41,63]	[44,64]	[47,66]	[52,69]	[57,72]	[135,142]
$x_3(k)$	[215,235]	[218,243]	[219,253]	[256,5102]	[224,257]	[227,259]	[262,5110]	[237,265]	[335,5120]
$x_4(k)$	[230,255]	[233,263]	[234,273]	[271,5122]	[239,277]	[242,279]	[277,5130]	[252,285]	[350,5150]
$x_5(k)$	[410,495]	[413,503]	[414,513]	[551,5362]	[419,517]	[422,519]	[457,5370]	[432,525]	[530,5390]
$x_6(k)$	[430,520]	[433,528]	[434,538]	[571,5387]	[439,542]	[442,544]	[477,5395]	[452,550]	[550,5415]
$x_7(k)$	[15,20]	[16,21]	[17,22]	[19,24]	[22,27]	[25,30]	[30,35]	[35,40]	[115,120]
$x_8(k)$	[25,50]	[35,51]	[38,52]	[39,54]	[41,57]	[44,60]	[47,65]	[52,70]	[125,150]
$x_9(k)$	[35,65]	[45,66]	[48,67]	[49,69]	[51,72]	[54,75]	[57,80]	[62,85]	[135,165]
$x_{10}(k)$	[50,85]	[86,230]	[87,233]	[89,234]	[92,236]	[95,239]	[100,242]	[105,247]	[185,252]
$x_{11}(k)$	[65,105]	[101,250]	[5102,5253]	[104,254]	[107,256]	[5110,5259]	[115,262]	[5120,5267]	[200,272]
$x_{12}(k)$	[185,225]	[221,400]	[5222,5403]	[224,404]	[227,406]	[5230,5409]	[235,412]	[5240,5417]	[320,422]
$x_{13}(k)$	[205,280]	[241,425]	[5242,5428]	[244,429]	[247,431]	[5250,5434]	[255,437]	[5260,5442]	[345,447]
$y(k)$	—	430,+∞	5242,+∞	434,+∞	436,+∞	5250,+∞	442,+∞	5260,+∞	452,+∞

Table 6 shows that for the dates $k = 2, 5$ and 7 , $\alpha(k)$ takes the value 5000 and the transitions x_{11}, x_{12}, x_{13} and y are immediately (at the same date) affected by the fault $\alpha(k)$. However, the transitions x_3, x_4, x_5 and x_6 are affected by the fault with a delay of one iteration. The transitions $x_1, x_2, x_7, x_8, x_9, x_{10}$ and u are not affected by the fault $\alpha(k)$.

The analysis of the result presented in table 6 let to conclude that the fault detection and localization is made immediately at the same time of the fault appearance or at least after one iteration. Transitions which are not affected by faults are also known.

The state vector presents some elements which are sensibles immediately to faults, other elements are sensibles to this fault with a delay. Some elements are not affected by this fault.

The simulation of the proposed state model let to detect the fault in real time or with a delay and let to know the transitions not affected by the fault

7.6 Fault estimation

The objective of this part is to use the proposed system state and estimator to fault estimation. The obtained results are presented in the table 7. The fault estimated $\hat{\alpha}(k)$ is considered sufficiently precise. The following inequality is added and can be used for fault estimation.

$$\hat{x}_{11}(k) - \hat{x}_{10}(k) - T_{11}^+ \leq \hat{\alpha}(k) \leq \hat{x}_{11}(k) - \hat{x}_{10}(k) - T_{11}^- \quad (67)$$

Table 7 Fault estimation

Dates	0	1	2	3	4	5	6	7	8
$u(k)$	0	1	2	4	7	10	15	20	100
$\alpha(k)$	[0,5]	[0,5]	[4995,5005]	[0,5]	[0,5]	[4995,5005]	[0,5]	[4995,5005]	[0,5]
$x_1(k)$	[25,27]	[28,35]	[29,45]	[31,48]	[34,49]	[37,51]	[42,54]	[47,57]	[125,127]
$x_2(k)$	[35,42]	[38,50]	[39,60]	[41,63]	[44,64]	[47,66]	[52,69]	[57,72]	[135,142]
$x_3(k)$	[215,235]	[218,243]	[219,253]	[256,5102]	[224,257]	[227,259]	[262,5110]	[237,265]	[335,5120]
$x_4(k)$	[230,255]	[233,263]	[234,273]	[271,5122]	[239,277]	[242,279]	[277,5130]	[252,285]	[350,5150]
$x_5(k)$	[410,495]	[413,503]	[414,513]	[551,5362]	[419,517]	[422,519]	[457,5370]	[432,525]	[530,5390]
$x_6(k)$	[430,520]	[433,528]	[434,538]	[571,5387]	[439,542]	[442,544]	[477,5395]	[452,550]	[550,5415]
$x_7(k)$	[15,20]	[16,21]	[17,22]	[19,24]	[22,27]	[25,30]	[30,35]	[35,40]	[115,120]
$x_8(k)$	[25,50]	[35,51]	[38,52]	[39,54]	[41,57]	[44,60]	[47,65]	[52,70]	[125,150]
$x_9(k)$	[35,65]	[45,66]	[48,67]	[49,69]	[51,72]	[54,75]	[57,80]	[62,85]	[135,165]
$x_{10}(k)$	[50,85]	[86,230]	[87,233]	[89,234]	[92,236]	[95,239]	[100,242]	[105,247]	[185,252]
$x_{11}(k)$	[65,105]	[101,250]	[5102,5253]	[104,254]	[107,256]	[5110,5259]	[115,262]	[5120,5267]	[200,272]
$x_{12}(k)$	[185,225]	[221,400]	[5222,5403]	[224,404]	[227,406]	[5230,5409]	[235,412]	[5240,5417]	[320,422]
$x_{13}(k)$	[205,280]	[241,425]	[5242,5428]	[244,429]	[247,431]	[5250,5434]	[255,437]	[5260,5442]	[345,447]
$y(k)$	—	430,+∞	5242,+∞	434,+∞	436,+∞	5250,+∞	442,+∞	5260,+∞	452,+∞

According to the obtained fault estimation, it is possible to conclude that this estimation is sufficiently precise since the theoretical values of the fault are belonging in the obtained interval.

The proposed estimator let to estimate the system state and inputs and also the fault affecting the system.

7.7 Discussion

The proposed state model (39) and estimator (63) are used successively to simulate and estimate the system state using the values of matrices W_{ux}^+ , W_{ux}^- , W_{xx}^+ , ~~W_{xx}^-~~ , W_{xy}^+ , W_{xy}^- , W_{uy}^+ and W_{uy}^- given by equations (64) and (65). The system input is also estimated and the obtained results are presented in form of tables. For each element of the state vector, a lower and upper bounds are computed for both cases of simulation and estimation. To analyze the robustness of the estimation, residuals are computed. In this work residuals are computed as the intersection, for each element of the state vector, of both intervals given by simulation and estimation. It is considered that the estimation is acceptable if the intersection between the intervals are different of $\emptyset$. The next step of the application is devoted to the fault detection and localization. For this case, it is supposed that an unknown transition is added to the system as an input transition of the place P_{11} . The application of the proposed state model (39) and estimator (63) show that the fault localization is possible and easy by the analysis of bounds of the state vector. The fault detection is possible ~~can be~~ at the same date of its apparition or with a delay of one iteration at max. The last step of this application is dedicated to the fault estimation. The analysis of the obtained estimation result show that the real value of the unknown input $\alpha(k)$ is included in the obtained interval, so it is supposed that the fault estimation is acceptable.

8 Conclusion and future works

This work ~~allowed proposing~~ proposes a new state model describing the time evolution ~~in time~~ of discrete events systems modeled by time interval Petri nets. A new estimator is also proposed allowing estimating the whole system state and input. The proposed state model and estimator allow computing an upper and lower bounds for each element of the ~~system~~ state vector. To test the robustness of the state estimation, residuals are computed as the intersection between the two intervals obtained by the estimation and the simulation. The proposed estimator can be used also for fault detection, localization and estimation. The state model and the estimator are proposed following successively a count and a dater approaches. In order to show the efficiency of the proposed state model and estimator, they have been applied, following a dater approach, to a baking process modeled by a time interval Petri net. ~~This The application of the proposed state model and estimator to the baking process~~ shows that the state estimation is acceptable in more than 89% of cases. It shown also that the fault localization is garanted and its detection is possible at the same ~~date~~ ~~daye~~ of its apparition or, at max, by a delay of one iteration. The fault estimation is also possible and acceptable.

It is possible to extend this work to analyze the optimality of the estimation using the principle of linear programming where we can add a criteria which must be respected in the simulation of the state model and the estimation.

9 Compliance with Ethical Standards

There is no funding in this study

Authors declare that there is no conflict of interest.

This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. F. Basile, M.P. Cabasino, and C. Seatzu, "State estimation and fault diagnosis of labeled time Petri net systems with unobservable transitions," IEEE Transactions on Automatic Control, vol. 60, no 4, pp. 997-1009, 2015.
2. S. Bernardi and J. Campos "A min-max problem for the computation of the cycle time lower bound in interval-based Time Petri Nets", IEEE Transactions on Systems, Man, and Cybernetics: systems, 43(5), 2013.
3. P. Bonhomme "Decentralized state estimation and diagnosis of p-time labeled Petri nets systems" Discrete Event Dynamic Systems, 31(1), pages137Ü162 (2021) <https://doi.org/10.1007/s10626-020-00326-w>

4. P. Bonhomme, "State Observer Synthesis of Real-Time Systems Modeled by P-Time Petri Nets," in 18th IEEE Int. Conf. on Emerging Technologies & Factory Automation, 2013.
5. A. Boussif M. Ghazel and K. Klai Fault diagnosis of discrete-event systems based on the symbolic observation graph International Journal Critical Computer-Based Systems, Vol. 8, No. 2, 2018
6. M.P. Cabasino, M. Paola, C.N. Hadjicostis, Christoforos N., and Seatzu, Carla. Probabilistic marking estimation in labeled Petri nets. Automatic Control, IEEE Transactions on, 2015, vol. 60, no 2, p. 528-533.
7. M. Cabasino, A. Giua, M. Poggi, and C. Seatzu, "Discrete event diagnosis using labeled Petri nets. An application to manufacturing systems," Control Engineering Practice, vol. 19, no. 9, pp. 989-1001, 2011.
8. M.P. Cabasino, A. Giua, and C. Seatzu. "Fault detection for discrete event systems using Petri nets with unobservable transitions," Automatica, Vol. 46, No. 9, pp. 1531-1539, September 2010.
9. S. Calvez, P. Aygalinc, and P. Bonhomme, "Transient functioning mode of manufacturing systems with operation time constraints," International Journal of Advanced Manufacturing Technology, Vol. 14, Issue 10, pp. 694-703, Oct. 1998.
10. A. Chouchane, A. Khedher, O. Nasri and A. Kamoun Diagnosis of partially observed Petri net based on analytical redundancy relationships Asian Journal of Control, Vol. 21, No. 5, pp. 1-14, 2019 DOI: 10.1002/asjc.1832
11. A. Chouchane, Ph. Declerck, A. Khedher and Anas Kamoun (2018): Diagnostic based on estimation using linear programming for partially observable petri nets with indistinguishable events, International Journal of Systems Science: Operations & Logistics, DOI: 10.1080/23302674.2018.1554169
12. S. Collart-Dutilleul. Réseaux de Petri P-temporels: Modélisation et validation d'exigences temporelles. Automatique / Robotique. Université des Sciences et Technologie de Lille - Lille I, 2008. .tel-00364666.
13. Ph. Declerck Counter approach for the estimation of optimal sequences in Partially Observable Untimed Petri Nets Discrete Event Dynamic Systems, 31(4), pages489-512 (2021) <https://doi.org/10.1007/s10626-021-00341-5>
14. Ph. Declerck, (2020): "Critical subsystems in time interval models. Application to a baking process". Discrete Event Dynamic Systems, doi.org/10.1007/s10626-020-00322-0
15. Ph. Declerck and P. Bonhomme, "State Estimation of Timed Labeled Petri Nets with Unobservable Transitions," IEEE Transactions on Automation Science and Engineering (IEEE-TASE), Special Issue on Discrete Event Systems for Automation, volume 11, No. 1, ITASC9, January 2014.
16. Ph. Declerck and A. Guezzi, "State Estimation and Detection of Changes in Time Interval Models," Journal of Discrete Event Dynamic Systems, volume 24, N°1, 53-68, Mars 2014.
17. M. Ghazel, A. Toguyeni, and P. Yim, "State observer for DES under partial observation with Time Petri Nets," Discrete Event Dynamic Systems, vol. 19, no. 2, pp. 137-165, 2009.
18. A. Giua and C. Seatzu, "Observability of place/transition nets," IEEE Trans. on Automatic Control, vol. 47, no. 9, pp. 1424-1437, 2002.
19. A. Giua, A. Piccaluga, and C. Seatzu, "Optimal Token Allocation in Timed Cyclic Event-Graphs," Discrete Event Systems, Vol. 569, pp. 209- 218, Aug. 2000.
20. C. Gros DEA Report in TBiotechnologies et Industries Alimentaires, Ecole Nationale supérieure d'Agronomie et des Industries Alimentaires, ENSAIA, INPL (2000)
21. C. Hanen and A. Munier Kordon, "Periodic schedules for linear precedence constraints," Discrete Applied Mathematics, Vol. 157, Issue 2, pp 280-291, Jan. 2009.
22. M. Jeng, "Comments on "Timed Petri Nets in Modeling and Analysis of Cluster tools," IEEE Trans. Autom. Sci. Eng., Vol. 2, Issue 1, pp. 92-93, Jan. 2005.
23. G. Jiroveanu , R. K. Boel and B. Bordbar, "On-line monitoring of large Petri net models under partial observation," Discrete Event Dynamic Systems, vol. 18, no 3, pp. 323-354, 2008.
24. W. Khansa, "Réseaux de Petri P-temporels. Contribution à l'étude des systèmes à Événements discrets," Ph.D. dissertation, Univ. Savoie, Annecy, France, Mar.1997.
25. A. Khedher, K. Ben Othman, "Modeling, simulation, estimation and boundedness analysis of discrete event systems", Soft Computing 24, 4775-4789 (2020) (DOI: 10.1007/s00500-019-04231-9)
26. T.-E. Lee and S.-H. Park, "An Extended Event Graph With Negative Places and Tokens for Time Window Constraints," IEEE Trans. Autom. Sci. Eng., Vol. 2, No. 4, pp. 319-332, Oct.2005.
27. T.-E. Lee, S.-H. Park, and C. Jung, "Steady State Analysis of Timed Event Graph with Time Window Constraints," Discrete Applied Mathematics, Vol. 167, pp. 202-216, 2014.
28. L. Li and C. N. Hadjicostis, "Minimum initial marking estimation in labeled Petri nets," IEEE Trans. on Automatic Control, vol. 58, no. 1, pp. 198-203, 2013.
29. L. Li and C. N. Hadjicostis, "Least-Cost Transition Firing Sequence Estimation in Labeled Petri Nets with Unobservable Transitions," IEEE Transactions on Automation Science and Engineering, vol. 8, no. 2, pp. 394-403, April 2011.
30. B. Liu, M. Ghazel, and A. Toguyeni, "On-the-fly and incremental technique for fault diagnosis of discrete event systems modeled by labeled Petri nets, Asian Journal of Control, Vol. 20, No. 1, pp. 1-13, January 2018 DOI: 10.1002/asjc.1508
31. B. Liu, M. Ghazel, and A. Toguyeni, "Diagnosis of Labeled Time Petri Nets Using Time Interval Splitting," in Proc. 19th IFAC World Congress, 2014.
32. J. Liu, Z. Zhou, and Z. Wang, "Online Fault Diagnosis in Discrete Event Systems with Partially Observed Petri Nets, International Journal of Control, Automation and Systems 16(1) (2018) 217-224 <http://dx.doi.org/10.1007/s12555-016-0168-6>
33. T. Murata, Petri Nets: Properties, Analysis and Applications, in PROCEEDINGS OF THE IEEE, Vol. 77, No. 4, pp. 541-580, Apr. 1989. OK Vol.77, No.4 pp.541-580, April, 1989
34. M. Notomi and T. Murata "Hierarchical reachability graph of bounded Petri nets for concurrent-software analysis". Software Engineering, IEEE Transactions on, 1994, vol. 20, no 5, p. 325-336.
35. S. P. Orlov S. V. Susarev and R. A. Uchaikin, "Application of Hierarchical Colored Petri Nets for Technological Facilities' Maintenance Process Evaluation Applied Sciences 2021, 11(11); <https://doi.org/10.3390/app1115100>
36. N. Ran, Sh. Wang, H. Su, and Ch. Wang "Fault diagnosis for discrete event systems modeled by bounded Petri nets" Asian Journal of Control, Vol. 19, No. 6, pp. 1-10, November 2017 DOI: 10.1002/asjc.1500
37. R.J. Rodriguez and J. Julvez "Accurate Performance Estimation for Stochastic Marked Graphs by Bottleneck Regrowing". From book Computer Performance Engineering - 7th European Performance Engineering Workshop, EPEW, 2010, Bertinoro, Italy, September 23-24, 2010, Proceedings, pp 175-190
38. Y. Ru and C. N. Hadjicostis, "Bounds on the Number of Markings Consistent with Label Observations in Petri Nets," IEEE Transactions on Automation Science and Engineering, vol. 6, no. 2, pp. 334-344, April 2009.
39. J. Zhou, D. Lefebvre, Z. Li "A clustering approach to approximate the timed reachability graph for a class of time Petri nets IEEE Transactions on Automatic Control, Sept. 2021, DOI: 10.1109/TAC.2021.3110010