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RESEARCH

An Energy-based Study of the Embedded Element Method for Explicit Dynamics

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Abstract

The embedded finite element technique provides a unique approach for modeling of fiber-reinforced composites. Meshing fibers as distinct bundles represented by truss elements embedded in a matrix material mesh allows for the assignment of more specific material properties for each component rather than homogenization of all of the properties. However, the implementations of the embedded element technique available in commercial software do not replace the material of the matrix elements with the material of the embedded elements. This causes a redundancy in the volume calculation of the overlapping meshes leading to artificially increased stiffness and mass. This paper investigates the consequences in the energy calculations of an explicit dynamic model due to this redundancy. A method for the correction of the redundancy within a finite element code is suggested which removes extra energy and is shown to be effective at correcting the energy calculations for large amounts of redundant volume.

Keywords: Finite element analysis (FEA); Fibers; Polymer-matrix composites; Embedded elements

1 Introduction

The embedded element method is a superposition technique in finite element analysis (FEA) where two independent element meshes are superimposed on one another. One mesh is embedded in the other by tying the degrees of freedom of the embedded nodes to the degrees of freedom of the host elements. The embedded element method has evolved both in its implementation and application since it originated and has been used under many different names. Initially, it was used as a localization method, similar to the extended finite element method, where the element

9 shape functions were modified to include embedded regions that could accommo-
10 date highly localized strain fields [1, 2]. Later, these embedded regions became a
11 separate mesh that replaced a coarse mesh in localized regions [3, 4]. This tech-
12 nique was used in various ways to model woven fiber reinforced composites, from
13 multi-scale localization models [5] to microscale models of fiber regions meshed
14 with continuum elements embedded in a matrix material [6, 7, 8]. Rather than us-
15 ing a solid element mesh, some authors used truss or spring elements to represent
16 the stiffness of fibers embedded in continuum element mesh of a matrix material
17 [9, 10, 11]. This has been a popular method for modeling rebar reinforced concrete
18 as well [12, 13, 14, 15]. In the last few years, embedded truss elements have also
19 been used to model bio-material such as white matter in the brain [16, 17, 18].

20 However, in the implementation of the embedded element method, the volume of
21 the host elements is not modified to “make room” for the embedded elements. The
22 volumes of the two meshes occupy the same space. If this is unaccounted for, it
23 causes internal energy and kinetic energy to be miscalculated by double counting
24 volume of the embedded mesh [18, 19]. The amount of redundant volume is equal to
25 the volume of the embedded mesh so in some cases, such as in reinforced concrete,
26 the redundancy is considered negligible. However, as the volume of the embedded
27 regions increases the additional mass and stiffness of the redundant volume need to
28 be addressed. There have been several different methods introduced to account for
29 the redundant volume [2, 5, 8, 6, 19, 7]. The most common way is, rather than us-
30 ing the matrix material’s true properties, to use an effective medium. This is found
31 from taking measured properties (stiffness, density) of the total composite and sub-
32 tracting the stiffness of the embedded elements [20, 6, 19, 7, 17]. Some authors have
33 used modified integration schemes, averaging or ignoring overlapping integration
34 points [2, 8, 21]. However, these methods were created for elastic problems, and
35 none for time-dependent dynamic problems or explicit finite element techniques. In
36 a paper on modeling brain matter, Garimella et al. [18] proposed a modification to
37 the explicit finite element method to internally correct the volume redundancy that
38 examined the stress response but did not analyze energy.

39 We are extending Garimella’s work to study energy to investigate the possibility
40 of using the embedded element method of FEA to create meso-scale models of fiber
41 reinforced composites, particularly for ballistic applications. If fibers could be mod-

eled as an independent mesh of truss elements, it would simplify mesh generation while maintaining a distinction between the fiber and matrix material, which is useful for meso or macro-scale modeling [3, 19]. By using a continuum material to represent the composite's matrix and embedded line elements to represent bundles of fibers, the orthotropic nature of the material would be naturally captured. It could also have the ability to model curved plates without the complexity of defining local material axes. The element mesh is easy to create, yet retains distinction between the fiber and matrix components for meso-scale analysis. An example of this type of mesh is shown in Figure 1. Since fiber reinforced composites have a large fiber volume fraction (87% is common), it is important to take into account the effects of the volume redundancy in the embedded element method in order to get accurate measures of the amount of energy absorbed and dispersed by the material.

This paper investigates the energy terms affected by volume redundancy in the embedded element method and how the volume redundancy effects problem solutions, particularly in dynamic problems with high fiber volume fraction materials.

Methods

Energy and the Finite Element Method

In the finite element method for structural mechanics, we wish to find the displacement field of a structural system given the system's initial position and a set of boundary conditions on the displacement and/or stress on a surface of the system. The displacement field can then be used to calculate stress and strain from the strain displacement equation and a constitutive equation. Normally, this problem results in solving the partial differential stress equilibrium equation. The principle of virtual work allows the solution of the problem via integrals. As an expression of the conservation of energy, the principle of virtual work states that the work done on the system by external forces is equal to the total amount of internal energy stored in the system plus the kinetic energy and any dissipated energy. Starting with a system with known boundary conditions, the principle of virtual work can be derived from conservation of energy as follows. For an elastic structural problem with negligible amounts of heat transfer involved and no energy dissipation, the

internal energy is equal to the strain energy of the structure, which is found by Eq (1).

$$W_{int} = \int C_{ijkl} u_{i,j} u_{k,l} dV \quad (1)$$

Here u_i is the displacement field of the system and C_{ijkl} is the material stiffness tensor. The kinetic energy and the work done by external forces can be expressed as Eq (2) and Eq (3).

$$W_{kin} = \int \rho \frac{d^2 u_i}{dt^2} u_i dV \quad (2)$$

$$W_{ext} = \int \rho b_i u_i dV + \int_{S_2} t_i u_i dA \quad (3)$$

In these equations, ρ is the material density, b_i is a vector representing body forces (such as gravity), and t_i is the vector of the surface traction (applied force per unit area) on the surface S_2 . The total system potential energy is equal to the summation of these work terms.

$$E_{total} = W_{int}(u_i) - W_{ext}(u_i) + W_{kin}(u_i) \quad (4)$$

According to the principle of minimum total potential energy, the system will move to an equilibrium position that will minimize the total potential energy. One way to tell if a system is at equilibrium is to perturb it by some amount and see if the system returns to the original state. By perturbing the potential energy equation by some virtual displacement field δu_i , we get the virtual work equation with the virtual values of internal, kinetic, and external energy. An equilibrium solution will be one that satisfies this equation for all kinematically admissible virtual displacements [22, 23, 24].

$$\int C_{ijkl} u_{i,j} \delta u_{k,l} dV - \int \rho b_i \delta u_i dV - \int_{S_2} t_i \delta u_i dA + \int \rho \frac{d^2 u_i}{dt^2} \delta u_i dV = 0 \quad (5)$$

The virtual work equation is then solved by discretizing the system into a set of nodes. The displacement field between the nodes is assumed to be a linear interpolation of the nodal displacements. This done by using interpolation functions, often called shape functions, $\mathbf{N}^a(x_j)$. The displacement field $u_i(x_j)$ is then defined by Eq (6).

$$u_i(x_j) = \sum_{a=1}^n \mathbf{N}^a(x_j) u_i^a \quad (6)$$

The summation is over all nodes in the model, where u_i^a is the displacement at node a. Substituting this definition of u_i into the virtual work equation gives a linear system that can be expressed in matrix form as in Eq (7).

$$\mathbf{M}_{ab} u_{i,tt}^b + \left(\mathbf{K}_{aibk} u_k^b - F_i^a \right) \delta u_i^a = 0 \quad (7)$$

Where

$$\mathbf{K}_{aibk} = \int \mathbf{C}_{ijkl} \frac{\delta N^b}{\delta x_j} \frac{\delta N^a}{\delta x_l} dV \quad (8)$$

$$F_i^a = \int \rho b_i \mathbf{N}^a \delta u_i^a dV + \int_{S_2} t_i \mathbf{N}^a \delta u_i^a dA \quad (9)$$

$$\mathbf{M}_{ab} = \int \rho \mathbf{N}^b \mathbf{N}^a \frac{d^2 u_i^b}{dt^2} \delta u_i^a dV \quad (10)$$

90 In this system, $\mathbf{K}$ is known as the element stiffness matrix, F is a vector of applied
 91 forces, and $\mathbf{M}$ is the element mass matrix. $\mathbf{K}$ and $\mathbf{M}$ are precomputed based on
 92 known material properties and interpolation functions [22].

93 Implementation of the Embedded Element Method

94 In the embedded element method, one finite element mesh is superimposed on
 95 another. The embedded mesh is constrained to follow the motion of the other mesh
 96 (the host mesh) by tying the degrees of freedom of the embedded nodes to the
 97 degrees of freedom of the host elements. Additionally, the stiffness of the embedded
 98 mesh resists the motion of the host mesh, thus creating a coupling system between
 99 the two meshes.

100 *Tying the Displacement Degrees of Freedom*

If slip between the host and embedded nodes is not allowed, then the displacement degrees of freedom of the embedded elements can be tied to the host element motion. Displacements anywhere in the host element are interpolated from the displacement of the host element nodes via the element shape functions, usually denoted in matrix form as $\mathbf{N}$. For the embedded element method, this same interpolation can be used to find the displacement of the embedded nodes. This is represented by Eq (11).

$$\underline{u}_i = \mathbf{N}^a_H(\underline{\xi}_i) \underline{u}_H^a \quad (11)$$

101 where $\underline{u}_i$ is the displacement of the embedded node, $\underline{\xi}_i$ is the natural coordinates
 102 of the embedded node, and $\underline{u}_H^a$ are the known displacements of the host element
 103 nodes. Figure 2 shows a two-dimensional equivalent of the scenario: the host element
 104 with nodal coordinates $\underline{x}_H^a$ and embedded node (located at $\underline{x}_i$) are shown on the
 105 right as they are in the global finite element model. The natural space of the host
 106 element, which includes the nodes of the embedded element is shown on the left. $\underline{\xi}_i$
 107 are the coordinates of the embedded node in the natural space of the host element.
 108 Eq (11) adds another constraint equation to the combined finite element model,

109 which reduces the total degrees of freedom to only the degrees of freedom associated
 110 with the host mesh.

111 *Interaction of Forces*

112 The explicit finite element method does not calculate stiffness directly, rather, the
 113 element stresses are calculated based on the deformation, then those stresses are
 114 used to find the internal forces on the element nodes. Figure 3 shows the general
 115 algorithm for the implementation of explicit dynamics[25].

116 Embedded elements contribute to the stiffness of the host elements via the com-
 117 bined internal forces of the embedded and host elements on the host nodes. The
 118 easiest way to incorporate this is to calculate the internal forces of the embedded
 119 elements exactly as they would be in a normal finite element program, then dis-
 120 tribute the internal forces that are concentrated at the embedded nodes to the host
 121 element. The distribution is done using the host element shape functions which
 122 serve as weighting functions so a host node will get more force if it is closer to
 123 an embedded node and less force if it is further away. Using the element internal
 124 forces concentrated at the nodes also makes the method more versatile, since the
 125 embedded element mesh can be any size and still utilize the same distribution code.
 126 A version of `getForce_effective` that uses this method is included in Figure 4.

127 *Energy and the Embedded Element Method*

128 The classic embedded element technique superimposes an embedded mesh onto a
 129 host mesh without changing the volume of the host mesh to account for the space
 130 that is now occupied by the embedded mesh. Figure 5 shows an illustration of this.
 131 The finite element method is based on energy methods in which the internal energy
 132 of each element is calculated by an approximation of the integral of the strain energy
 133 over the element volume. Similarly, mesh mass is calculated as a volume integral
 134 of the material density. Having extra volume leads to increased mass and strain
 135 energy. This causes changes in the internal energy and the kinetic energy of the
 136 system and therefore influences the system solution [18].

137 The energy issues can be shown by how the model's energy terms are calculated.
 138 The total energy terms are the sums of the energies of the host element and of
 139 the embedded element. We will refer to the host elements as the matrix elements
 140 and the embedded elements as fiber elements. The internal energy, W_{int} , Eq (12),

141 is found by integrating the strain energy over the volumes of the two meshes, the
 142 kinetic energy, W_{kin} , Eq (13), is an integral of momentum times velocity over the
 143 volume, and the external work, W_{ext} , Eq (14), is the summation of the body forces
 144 on the two meshes plus any applied traction forces.

$$W_{int} = \underbrace{\int_M C_{ijkl} u_{i,j} u_{k,l} dV_M}_{\text{Matrix Strain Energy}} + \underbrace{\int_F C_{ijkl} u_{i,j} u_{k,l} dV_F}_{\text{Fiber Strain Energy}} \quad (12)$$

$$W_{kin} = \underbrace{\int_M \rho_M \frac{d^2 u_i}{dt^2} u_i dV_M}_{\text{Matrix Kinetic Energy}} + \underbrace{\int_F \rho_F \frac{d^2 u_i}{dt^2} u_i dV_F}_{\text{Fiber Kinetic Energy}} \quad (13)$$

$$W_{ext} = \underbrace{\int_M \rho_M b_i u_i dV_M}_{\text{Matrix External Load}} + \underbrace{\int_F \rho_F b_i u_i dV_F}_{\text{Fiber External Load}} + \underbrace{\int_{S_2} t_i u_i dA}_{\text{Body Force Load}} \quad (14)$$

145 In these equations, u_i is the displacement field of the system and C_{ijkl} is the material stiffness
 146 tensor, ρ is the material density, b_i is a vector representing body forces (such as gravity), and t_i is
 147 the vector of the surface traction (applied force per unit area) on the surface S_2 . The subscript M
 148 refers to material properties of the matrix and volume integrals over the matrix volume. Likewise,
 149 F refers to material properties of the fiber elements and volume integrals over the fiber volume.

150 To avoid volume redundancy, the integral over the matrix volume should not include the volume
 151 occupied by the fibers. Eq (15)-(17) include an extra term to remove the effects of the redundancy.
 152 However, in many finite element programs (e.g. Abaqus and LS-Dyna) these integrals are calculated
 153 via Gauss Quadrature, so the simplest way to account for the volume occupied by the embedded
 154 fiber is to subtract the volume integral of the matrix energy density over the fiber volume.

155 If the fibers and matrix have the same material properties, both integrals over the fiber volume
 156 cancel and what remains are the equations for only the matrix energy, which would be the same
 157 situation as having the matrix elements without any embedded elements.

$$W_{int} = \underbrace{\int_M C_{ijkl} u_{i,j} u_{k,l} dV_M}_{\text{Matrix Strain Energy}} - \underbrace{\int_F C_{ijkl} u_{i,j} u_{k,l} dV_F}_{\text{"Extra" Matrix Strain Energy}} + \underbrace{\int_F C_{ijkl} u_{i,j} u_{k,l} dV_F}_{\text{Fiber Strain Energy}} \quad (15)$$

$$W_{kin} = \underbrace{\int_M \rho_M \frac{d^2 u_i}{dt^2} u_i dV_M}_{\text{Matrix Kinetic Energy}} - \underbrace{\int_F \rho_M \frac{d^2 u_i}{dt^2} u_i dV_F}_{\text{"Extra" Matrix Kinetic Energy}} + \underbrace{\int_F \rho_F \frac{d^2 u_i}{dt^2} u_i dV_F}_{\text{Fiber Kinetic Energy}} \quad (16)$$

$$W_{ext} = \underbrace{\int_M \rho_M b_i u_i dV_M}_{\text{Matrix External Load}} - \underbrace{\int_F \rho_M b_i u_i dV_F}_{\text{"Extra" Matrix External Load}} + \underbrace{\int_F \rho_F b_i u_i dV_F}_{\text{Fiber External Load}} + \underbrace{\int_{S_2} t_i u_i dA}_{\text{Body Force Load}} \quad (17)$$

158 Volume redundancy is not corrected in most commercial codes. This is acknowledged [26], leaving
 159 it up to the analyst to account for the redundant mass. When the embedded element method is
 160 used in quasi-static models, the redundancy can be addressed by reducing the density and stiffness
 161 of the host or embedded material [11, 27, 6, 7, 15]. For dynamic applications, this is unacceptable,
 162 since it will alter the wave speed of the material and any internal energy calculations that may be
 163 used in material damage models. The negative effects of volume redundancy in embedded elements
 164 can be shown by a simple test. A model of a unit cube with embedded truss elements ought to
 165 behave as if it was a single homogeneous cube if the cube and the embedded truss elements have
 166 the same material properties (i.e., the embedded element should have no effect on the model).

167 Comparing this model with an identical cube with no embedded truss, any differences will be
 168 due to the volume redundancy. Four versions of a unit cube model were created with 0, 2, 10
 169 or 25 embedded truss elements to represent bundles of fibers. All of the truss elements had the
 170 same cross-sectional area so increasing the number of trusses would increase their volume fraction.
 171 Twenty-five truss elements result in a fiber volume ratio of 0.5. Examples of the models are shown
 172 in Figure 6a.

173 The simulations were run using the dynamic explicit solver in Abaqus with the boundary con-
 174 ditions shown in Figure 6b. Abaqus was chosen as a representative commercial code because it
 175 is widely used in industry. A displacement boundary condition of 0.05 m was applied to the four
 176 nodes of the positive y face, while the nodes of the negative x, y, and z faces were pinned in those
 177 directions.

178 Proof of Volume Redundancy Impact

179 In the initial Abaqus tests, the embedded elements and the host elements are the same material,
 180 according to the rule of mixtures there should be no difference between the stiffness and behavior
 181 of the models even as more embedded elements are added. Figure 7 shows a plot of the strain
 182 energy of each Abaqus model as a function of displacement. For a given displacement, adding
 183 more embedded elements increases the amount of stored internal energy.

184 Figure 7 also shows a few other types of energy plots. The energy balance shows that in all
 185 cases energy is conserved. The difference between the models is that the total amount of energy
 186 increases with the number of embedded elements. The increase in kinetic energy with increase in
 187 fiber elements shows that the redundancy in the mass matrix calculation does have an effect. The
 188 applied displacement ensures that all three models will have the same velocity as a function of
 189 time. Therefore, the only difference in their kinetic energies is due to their differences in mass. In a
 190 low strain rate analysis such as this, incorrect kinetic energies have little effect on the final result,
 191 but might be a significant problem in problems with large velocities or high strain rates.

192 Method of Volume Redundancy Correction

193 Garimella [18] suggested a way to correct the volume redundancy in the algorithm for dynamic
 194 explicit finite element analysis. The implementation of Garimella's method, shown in Figure 8, was
 195 modified to make the incorporation into the embedded element method easier. While Garimella
 196 used a calculation of the embedded element internal force in the host element's natural coordinate
 197 system to find the amount of force needed to correct for the internal energy of the redundant
 198 volume, we chose to use a method similar to the force interaction method described earlier. Here,
 199 the internal force of the redundant volume is calculated as if it were a truss element made of the
 200 host element material as the embedded element is oriented in the global coordinate system. This
 201 results in the correction force which is distributed to the host nodes using the host element shape
 202 functions, the same way as the embedded element force. By using the standard calculation for the
 203 internal force of the embedded element in the global coordinate system, we avoid any confusion in
 204 attempting to convert the one-dimensional axial stress of a truss element to a stress measure that
 205 would need to exist in three dimensions. Additionally, the accuracy of the force transfer is now
 206 directly correlated with the embedded element mesh density. Increasing the mesh density of the
 207 embedded nodes will correlate with approaching the true, non-discretized, constraint condition
 208 between the fibers and the mesh where they are bonded along the entire length of the fiber. The
 209 differences between Garimella's [18] method and the method used here are shown in the flowchart
 210 of Figure 4 in step 4.a.e.v.A-D and Figure 8 steps 4.e.i-iv.

211 The mass correction was carried out as shown in the flowchart of Figure 9, this is the same
 212 algorithm as used in Garimella [18]. This only requires calculating the mass of the redundant
 213 volume by multiplying the volume of the embedded mesh by the density of the host mesh. This
 214 mass is then subtracted from the total mass of the host element before it is scattered to the lumped
 215 mass matrix.

216 To assess the accuracy of this correction, it was incorporated into a dynamic explicit code
 217 implemented in Matlab. This code is a modified version of Bonet *et al.*'s [28] FLaGSHyP code
 218 that was written to go along with their textbook. When this work started only the static implicit
 219 version of FLaGSHyP was available, so a dynamic explicit solver modified to include the volume
 220 redundancy correction was added. The algorithm suggested by Garimella was implemented so that
 221 the user can choose to use the correction if desired. The resulting FE code will be referred to as
 222 FLaGSHyP Modified (FM). When the volume correction feature is in use, the code will be referred
 223 to as FM Corrected. This code can be found on GitHub at
 224 https://github.com/Valerie96/flagshyp/tree/Flagshyp2_element_types.

225 Results

226 Verification of FLaGSHyP Modified

227 To verify FLaGSHyP Modified, a single continuum element in tension was compared with an
 228 identical element but with an embedded truss. The truss had the same material properties as the
 229 host element, and so should have no effect on the total energy or stiffness. This test case was also
 230 run in Abaqus. Since the truss element and the host element have the same material properties,
 231 they should produce exactly the same response. As shown previously, this is not the case for
 232 the Abaqus models. However, the results in Figure 10 show that with the volume redundancy
 233 correction algorithm turned on, FM was able to reproduce the results for a single solid element
 234 when the truss and the host materials were identical.

235 Specific Effects of Redundancy

236 To explore the amount of difference the volume redundancy makes in different cases, several more
 237 applied displacement models were run to create plots of energy vs. fiber volume fraction and energy
 238 vs. strain rate. The redundant volume should affect the internal and kinetic energies. With extra
 239 volume the model can store more internal energy for a given displacement. The extra volume also
 240 comes with extra mass, which will show up as increased kinetic energy. As the fiber volume fraction
 241 increases, the amount of redundant volume increases and should make these energy differences
 242 larger. Increasing the strain rate will increase the nodal velocities. In this case, the strain energy
 243 should not be changing but the kinetic energy, which depends on velocity, will increase.

244 Fiber Volume Fraction Effects

245 The tests on fiber volume fraction involved running the displacement simulation in both FM
 246 and Abaqus with zero, two, ten, and twenty-five embedded truss elements, where the host and
 247 embedded elements had the same material type. The truss elements all had the same cross-sectional
 248 area so adding more trusses increased their volume fraction. Additionally, the embedded element
 249 simulations were rerun in FM, but this time with the volume redundancy correction turned on.
 250 Figure 11 shows a schematic of the set up and Table 1 shows all of the test cases. To compare
 251 the effect of the embedded elements, the internal and kinetic energy of each model was compared
 252 with the solid element model to get a percent difference in energy. Abaqus embedded element
 253 models were compared to the Abaqus solid element model and FM embedded element models
 254 were compared to the FM solid element model. With the embedded elements having the same
 255 material as the host, they should all be equivalent to a solid element, which is why the zero truss
 256 case is used as the reference. These comparisons resulted in a relative difference, or error introduced
 257 by added the embedded elements which is shown in the plots in Figure 12. As expected, as the
 258 fiber volume fraction increased, both the total internal kinetic and internal energies increased as
 259 well as their relative error. The exception to this was the models using FM Corrected. As shown
 260 in Figure 12, the correction nearly eliminated the error in internal energy and drastically reduced
 261 the kinetic energy error. Considering that the uncorrected FM models almost exactly match the
 262 data from Abaqus, this shows that the volume redundancy is causing error, we know where it is,
 263 and we know how to correct it.

264 *Strain Rate (Velocity Effects)*

265 A second set of tests was run to check the dependency of energy on strain rate. These consisted of
 266 an applied displacement boundary condition. The total displacement in each case was the same,
 267 but the speed at which it was applied was changed to approximate strain rates of 5, 10, 50, and
 268 200 1/s. A reference set of models at each of these strain rates was run in both Abaqus and FM
 269 using a solid hex element. Each test was then repeated with 25 embedded truss elements of the
 270 same material type as the host element. Finally, the 25 truss element models were repeated again
 271 in FM Corrected. Figure 13 shows a schematic of the set up and Table 2 lists all of the test cases.

272 Internal energy was expected to remain constant since the total displacement was unchanged.
 273 Kinetic energy was expected to increase due to the increase in applied velocity. Similar to the
 274 volume fraction cases, each embedded element model was compared with its' equivalent solid
 275 element at the same strain rate. These comparisons resulted in a relative difference, or error
 276 introduced by added the embedded elements which is shown in the plots in Figure 14. Again, FM
 277 Corrected was able to completely correct for the error in the internal energy. The relative error
 278 in kinetic energy was expected to increase as the strain rate and nodal velocity increased, yet the
 279 error remained constant.

280 **Discussion**

281 *Energy Implications*

282 The volume redundancy is a well-known short coming and is addressed by many authors by
 283 modifying the material properties of the host material. Using this volume correction algorithm is
 284 a much more robust method because it does not require the estimation of an "effective medium."
 285 When the stiffness of the host material is modified it complicates the stress calculation. This
 286 becomes critical where stress wave propagation is important. With our method the true material
 287 properties are used and stress calculations are not affected. Correcting the energy terms in the
 288 code itself provides a model that more accurately represents the physics of the problem instead
 289 smearing the different materials together into a representative material. The modification also
 290 ensures the correct calculation of both internal and kinetic energies which become more important
 291 as strain rate and fiber volume fraction increase.

292 *Volume Fraction Effects*

293 The volume fraction of truss element to host element volume has a large impact on the energy
 294 terms, since it adds both extra mass and the potential to store more internal energy. Our code
 295 is able to eliminate almost all of the error in the internal energy due to the volume redundancy
 296 without modification to the material properties themselves. In these tests, internal energy accounts
 297 for most of the energy in the system and is important to have accurate in applications involving
 298 stress wave propagation. The redundancy correction also reduced the error in kinetic energy to
 299 under 10 percent for fiber volume fractions up to 0.5. Based on the trends shown here, the error in
 300 the corrected version will continue to increase with volume fraction. This might only be because,
 301 as shown in Figures 7 and 14, the kinetic energy in these small models oscillates a lot which is a
 302 result of small over corrections by the explicit time stepping method, which damps out over time.
 303 The kinetic energies used in comparisons here are just the kinetic energy reported at the end of
 304 the last time increment so the oscillation in each model adds to the error. This needs to be tested
 305 on models with larger numbers of host elements, which will be done in the future.

306 *Strain Rate Effects*

307 For the strain rate tests, since the amount of volume redundancy was kept constant, the strain
 308 rate was expected to impact the kinetic energy by increasing the nodal velocities while having no
 309 effect on total internal energy stored since the total displacement remained constant. The effect of
 310 strain rate on the error in kinetic energy was small although it did increase slightly with increasing
 311 strain rate as expected. Only four different strain rates were tested with the maximum as only

200 1/s. Much larger strain rates were not possible for the small tests (due to errors caused by the coarse model discretization) but could result in increasing error. In the tested cases the corrected code was able to reduce the error in the kinetic energy to below 10 percent. In these cases the kinetic energy is several orders of magnitude smaller than the internal energy, so having it exactly corrected is less important. Future work will include tests on larger models with more nodes and higher velocities to determine if, or when, the kinetic energy becomes significant.

Limitations of the Embedded Element Method

The volume redundancy code is still in early stages and it has not yet been compared to experimental results with real fiber reinforced material to prove it has better experimental predictions than the standard embedded element method. Another obstacle is that researchers cannot implement this code as a user defined element type or subroutine in commercial codes since it is a modification to the solution process, not the material behavior. The idea is to modify mass and internal energy calculations directly to create a model that more closely matches the physics. We plan to add the volume redundancy calculation to a C based finite element code that will be more efficient than the current Matlab code.

Conclusion

It has been shown how incorrect volume and mass effect the energy calculations in the embedded element method, which in turn effects the analysis solution. Leaving this uncorrected in commercial codes can cause erroneous results especially in cases with wave propagation. A simple test of a single host element with embedded elements using the same material properties was created and compared to the solution for the same continuum element with no embedded elements. Since the embedded elements were the same material as the continuum element the effective system is equivalent to having no embedded elements at all. Due to the volume redundancies in the calculation, the model with embedded elements could store more internal energy. The volume redundancy also added mass and increases the kinetic energy of the system. The differences in internal energy between the embedded element model and the control continuum element were shown to be accounted for by the additional matrix volume in the calculation. A modified explicit finite element code was shown to be able to correct the effects of the volume redundancy for small models. This technique could be extended to model larger objects with embedded elements, particularly fiber reinforced composites. Next steps for this research should include larger models with different loading types and comparisons against experimental data to determine if this corrected version of the embedded element method will perform better than the standard version in predicting the response of dynamic events in fiber reinforced composites.

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Abbreviations

FEA - Finite Element Analysis FEM - Finite Element Method FM - FLaGSHyP Modified

356 Availability of data and materials

357 The source code for FLagSHyP Modified can be found at:
 358 https://github.com/Valerie96/flagshyp/tree/Flagshyp2_element_types

359 Competing interests

360 Reuben Kraft has a financial interest in BrainSim Technologies Inc., a company which could potentially benefit
 361 from the results of this research. This interest has been reviewed by Penn State University in accordance with
 362 its Individual Conflict of Interest policy for the purpose of maintaining the objectivity and integrity in research
 363 and is being managed.

364 Authors' contributions

365 VM contributed to the design of the work, the acquisition, analysis, interpretation of data, the creation of new
 366 software used in the work, and drafted and revised the paper
 367 RK contributed to the conception and design of the work analysis, and draft revision.
 368 TH contributed to the draft revision
 369 SE contributed to the conception and design of the work

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457 Figures

458 Tables

Figure 3 Explicit time integration flowchart from Nonlinear Finite Elements for Continua and Structures by Belytschko et al.[25]

Table 1 Fiber Volume Fraction Dependency Tests

	Truss Elements	Volume Fraction	Total Mass (kg)	Applied Displacement (m)	Time Increment (s)	Strain Rate (1/s)
Abaqus	0	0	7800	0.05	0.01	5
	2	0.04	8112	0.05	0.01	5
	10	0.2	9360	0.05	0.01	5
	25	0.5	11700	0.05	0.01	5
FM	0	0	7800	0.05	0.01	5
	2	0.04	8112	0.05	0.01	5
	10	0.2	9360	0.05	0.01	5
	25	0.5	11700	0.05	0.01	5
FM	2	0.04	7800	0.05	0.01	5
Corrected	10	0.2	7800	0.05	0.01	5
	25	0.5	7800	0.05	0.01	5

Table 2 Strain Rate Dependency Tests

	Truss Elements	Volume Fraction	Total Mass (kg)	Applied Displacement (m)	Time Increment (s)	Strain Rate (1/s)
Abaqus	0	0	7800	0.05	0.01	5
		0	7800	0.05	0.002	25
		0	7800	0.05	0.001	50
		0	7800	0.05	0.00025	200
	25	0.5	11700	0.05	0.01	5
		0.5	11700	0.05	0.002	25
		0.5	11700	0.05	0.001	50
		0.5	11700	0.05	0.00025	200
FM	0	0	7800	0.05	0.01	5
		0	7800	0.05	0.002	25
		0	7800	0.05	0.001	50
		0	7800	0.05	0.00025	200
	25	0.5	11700	0.05	0.01	5
		0.5	11700	0.05	0.002	25
		0.5	11700	0.05	0.001	50
		0.5	11700	0.05	0.00025	200
FM Corrected	25	0.5	11700	0.05	0.01	5
		0.5	11700	0.05	0.002	25
		0.5	11700	0.05	0.001	50
		0.5	11700	0.05	0.00025	200

FLAGSHYP - getForceEffective

1. if computing f_{net}^0 , replace "n+1" with "0" throughout subroutine
2. Initialize: f_{net}^{n+1}
3. Update global external nodal forces f_{ext}^{n+1}
4. Loop over host elements n_h
 - a. GATHER element nodal displacements and velocities
 - b. Initialize: $f_{int,h}^{n+1}$
 - c. Loop over quadrature points n_{ζ_h}
 - i. if computing f_{net}^0 , go to (c)iv.
 - ii. Compute deformation measures of the fiber: $F^n(\zeta_h)$, $E^n(\zeta_h)$
 - iii. Compute stress: $\sigma^n(\zeta_h) = f(F^n(\zeta_h))$
 - iv. $f_{int,h}^{n+1} \leftarrow f_{int,h}^{n+1} + (B^T \sigma^n \bar{w}_\zeta^T) \zeta$
 - d. END quadrature point loop.
 - e. Loop over all the embedded fibers n_f
 - i. if computing f_{net}^0 , go to (e)v.D.
 - ii. Compute deformation measures of the fiber: $F^n(\eta_f)$
 - iii. Calculate the stress and internal force of the fiber using the fiber material properties: $\sigma_f = f(F^n(\eta_f), \rho_f)$, $f_{int,f}^{n+1} = \sigma_f a_f$
 - iv. Calculate the correction stress and force of the fiber using the host material properties: $\sigma_{fc} = f(F^n(\eta_f), \rho_h)$, $f_{int,c}^{n+1} = \sigma_{fc} a_f$
 - v. Loop over fiber nodes $x_{f,i}$
 - A. Get fiber node coordinate $(\zeta_{x_{f,i}})$ in the iso-parametric coordinate system of the host element using Newton-Raphson algorithm.
 - B. Distribute the fiber internal force on node $x_{f,i}$ to the host nodes x_h using the values of the host element shape functions at $x_{f,i}$ as weight values $f_{int,f}^{n+1} = N(\zeta_{x_{f,i}}) f_{int,f}^{n+1}$
 - C. Distribute the correction force "on" node $x_{f,i}$ to the host nodes x_h using the values of the host element shape functions at $x_{f,i}$ as weight values $f_{int,c}^{n+1} = N(\zeta_{x_{f,i}}) f_{int,c}^{n+1}$
 - D. $f_{int,h}^{n+1} \leftarrow f_{int,h}^{n+1} + f_{int,f}^{n+1} - f_{int,c}^{n+1}$
 - iv. END loop over the quadrature points of the embedded fibers.
 - f. END loop over the embedded fibers.
 - g. GATHER external nodal forces on the element, $f_{ext,h}^{n+1}$
 - h. Compute $f_{net,h}^{n+1} = f_{ext,h}^{n+1} - f_{int,h}^{n+1}$
 - i. SCATTER $f_{net,h}^{n+1}$ to global f_{net}^{n+1}

Figure 4 Modified algorithm for the getforce subroutine that includes the interaction between the embedded element internal forces and the host element

Figure 5 a) shows how the volume redundancy is created in the embedded element method. b) shows how to correctly model the system

Figure 6 a) Unit cube models used containing 0, 2, and 25 embedded elements. b) Applied displacement boundary conditions for the embedded element models

Figure 7 Energy plots from the 0 fiber, 2 fiber, 10 fiber, and 25 fiber analyses with the applied displacement boundary condition. Although all of the models should be equivalent systems, adding embedded elements effects all types of energies.

Box 3. Subroutine - *getForceEffective*

1. if computing f_{net}^0 , replace “ $n+1$ ” with “0” throughout subroutine
2. Initialize: f_{net}^{n+1}
3. Update global external nodal forces f_{ext}^{n+1}
4. Loop over elements n_e
 - a. GATHER element nodal displacements and velocities
 - b. Initialize: $f_{int,e}^{n+1}$
 - c. Loop over quadrature points n_{ζ}
 - i. if computing f_{net}^0 , go to (c)iv.
 - ii. Compute deformation measures $F^n(\zeta)$, $E^n(\zeta)$
 - iii. Compute stress: $\sigma^n(\zeta) = f(F^n(\zeta))$
 - iv. $f_{int,e}^{n+1} \leftarrow f_{int,e}^{n+1} + (B^T \sigma^n \bar{w}_q J)_{\zeta}$
 - d. END quadrature point loop.
 - e. Loop over all the embedded fibers n_f
 - i. if computing f_{net}^0 , go to (e)iii.E.
 - ii. Calculate the displacements of the embedded nodes from the host nodal displacements:
 $u_{embed} = [N]u_{host}$. Shape function matrix is calculated at the embedded nodes.
 - iii. Loop over the quadrature points of the fiber n_{ζ_f}
 - A. Calculate the fiber quadrature point coordinates (ζ_f) in the iso-parametric coordinate system of the host element using Newton-Raphson algorithm.
 - B. Compute deformation measures at these fiber quadrature points: $F^n(\zeta_f)$.
 - C. Calculate the fiber stress measure in the host coordinate system:
using the fiber material model.
 - D. Calculate the correction stress measure in the host coordinate system: $\sigma^n(\zeta_f) = f(F^n(\zeta_f))$ using the host element material model.
 - E. $f_{int,e}^{n+1} \leftarrow f_{int,e}^{n+1} + (B^T \sigma_f^n \bar{w}_{f,q} J_f - B^T \sigma_c^n \bar{w}_{f,q} J_f)_{\zeta_f}$ where $J_f = \frac{l_f}{2} a_f$ is the fiber volume Jacobian.
 - iv. END loop over the quadrature points of the embedded fibers.
 - f. END loop over the embedded fibers.
 - g. GATHER external nodal forces on the element, $f_{ext,e}^{n+1}$
 - h. Compute $f_{net,e}^{n+1} = f_{ext,e}^{n+1} - f_{int,e}^{n+1}$
 - i. SCATTER $f_{net,e}^{n+1}$ to global f_{net}^{n+1}

Figure 8 Algorithm suggested by Garimella[18] to address internal energy redundancy.

Subroutine - getMassEffective

1. Loop over elements n_e
 - a. For each element, calculate the volume: v_e
 - b. Calculate the mass of each element: $m_e = \rho_e v_e$
 - c. Loop over all the embedded fibers: n_f
 - i. For each fiber, calculate the volume using length l_f and cross-sectional area a_f
 - ii. Calculate the fiber mass: $m_e = \rho_f l_f a_f$
 - iii. Calculate the correction mass: $m_c = \rho_e l_f a_f$
 - d. Calculate the new effective element mass: $m_{eff} = m_e + m_f - m_c$
 - e. Divide the effective mass equally among all element nodes – form nodal mass vector
 - f. Use the nodal mass vector to form the element mass matrix $\mathbf{M}_e$ (diagonal mass matrix)
 - g. Scatter mass matrix to global mass matrix $\mathbf{M}$
2. END loop over the elements n_e

Figure 9 Algorithm suggested by Garimella[18] to address mass redundancy. **Figure 10** Raw energy data comparison from FLagSHyP Modified verification tests. A solid element is compared with an identical model with the addition of an embedded element. The embedded element was assigned the same material properties as the host element so that the system equivalent behavior should be the same as the solid element model. The volume redundancy in the embedded element method causes increases in internal energy, stiffness, and kinetic energy. The correction method shown is able to remove this extra energy and returns the same results as a solid cube.