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Murat Kirişci (✉ mkirisci@hotmail.com)

Istanbul University-Cerrahpasa

Necip Simsek

Istanbul Commerce University

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Interval-Valued Fermatean Hesitant Fuzzy Sets and Infectious Diseases Application

Murat Kirişci^{a*} and Necip Simsek^b

^aIstanbul University-Cerrahpasa, Department of Biostatistics and Medical Informatics, Istanbul, Turkey

^bIstanbul Commerce University, Department of Mathematics, Istanbul, Turkey

*Corresponding author E-mail: mkirischi@hotmail.com

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Abstract

The Hesitant Fuzzy Set, a generalization of fuzzy sets, is a crucial tool for resolving the challenges associated with defining an element's membership in a set when there is uncertainty regarding several separate values in decision-making situations. To ensure that the conditions under which experts evaluate an alternative in likely membership values and non-membership values are operational in this study, a Fermatean hesitant fuzzy set is provided. To implement multi-attributed group decision-making problems, new defined sets' aggregate operators are defined. The primary characteristics of the new sets were investigated. Two interval-valued values are compared using a new score function and accuracy function. Finally, a numerical example exemplifies the viability, applicability, and efficacy of the suggested approach.

1. Introduction

Numerous academic fields, including psychology, philosophy, cognitive science, and artificial intelligence, are interested in how humans reason and make decisions in the face of everyday situations. Usually, numerous mathematical and statistical models are used to try to characterize these processes. The issue of decision-making ($\mathcal{DM}$) surfaces in this process. $\mathcal{DM}$ is the process of choosing one or more of the various kinds of conduct that a person or institution faces in order to accomplish a certain objective. According to research, while many daily judgments may be made instinctively, complicated and important choices require more than this. The $\mathcal{DM}$ process in discrete circumstances when the options reviewed in the decision issue are finite and clearly defined is referred to as multi-attribute decision making ($MADM$). The number of choices in $MADM$ issues is preset. In decision issues involving ranking, comparing, and choosing among alternatives, $MADM$ techniques are commonly utilized. Because they enable speedy decision-making without needing complex mathematical calculations and the use of a package software, they are commonly favored approaches. The $MADM$ approach serves a single objective. The goal is to identify the best (most advantageous, least expensive) option for the choice dilemma. The decision problem's objective for the aforementioned example problem is "finding of the best suitable supplier option."

Using the collective knowledge and experience of individuals engaged in the group to make decisions that are likely to have positive outcomes is known as group decision-making (GDM). One of the main benefits of GDM is its capacity to include individuals with various thinking processes and backgrounds, allowing for a more comprehensive examination of the group's problems. To accomplish their goals, people aspire to overcome the challenges they encounter. Sometimes, this duty grows to be too big and complicated for the person to do alone. In these circumstances, employing collective power to make decisions is a more logical course of action. The group synergy that develops when people are working remotely or at a desk is a valuable tool for making better judgments and resolving issues. People use organizations to accomplish certain of their requirements and goals that they are unable to accomplish on their alone. The process of selecting the best alternative according to a single decision-maker will no longer be the problem when a group wants to solve a problem, even when each member has their ideas and reasons. The ensuing collective decision-making process would be broadened to take into consideration the conflicts of various interest groups, aims, and objectives, criteria, political conduct, etc. At this stage, all decision-makers are responsible for the ultimate solution; it is no longer up to a single decision-maker to take the initiative.

Uncertainty, in general, refers to a scenario where different outcomes of a certain occurrence are possible but the probability of those outcomes are unknown. As a result, uncertainty is a crucial concept for the $\mathcal{DM}$ process. Understanding the chances of events occurring in reality is difficult. Therefore, there is ambiguity present during the $\mathcal{DM}$ process. A robust logical inference framework is put out by fuzzy logic theory [37] in the face of ambiguous and imprecise knowledge. Computers can now analyze human linguistic input and draw on human experience thanks to fuzzy logic theory. It switches from using numerical expressions to symbolic expressions when attaining this ability. The term "fuzzy sets" refers to these symbolic expressions (FS). It is known that choice variables including probability states make up the

components of fuzzy sets. Fuzzy sets develop when each possibility is given an objective membership degree rather than the probability values associated with those possibilities.

The degree of not belonging in the FS A is $(1 - \zeta(A))$, while the MD of the set is $\zeta(A)$. Consequently, $MD + ND = 1$. This circumstance, however, falls short of adequately explaining the ambiguity in several issues. Atanassov [1] suggested the IFS theory as a result. MD and ND make up IFSs ($MD + ND \leq 1$). PFS is described by Yager [33] as being a broader and more inclusive collection than IFS. The formula for PFS is $MD^2 + ND^2 \leq 1$. There is a wide range of research on FS, IFS, and PFS, including studies [2]-[5], [9]-[12], [19], [20], [27], [32], [34], [35], [39].

The q-step orthopair fuzzy set was first described by Yager [36]. The fundamental tenet of this set theory is that the product of MD and ND must not exceed 1. Senapati and Yager [22] introduced the Fermatean fuzzy set (FFS) and looked at its fundamental characteristics in light of this concept. As new transactions for FFS, division and subtraction in Fermatean arithmetic are defined and some of their features are investigated in [23]. New weighted aggregated operators for FFSs are defined in [24]. Entropy measurements and the Fermatean fuzzy soft set (FFSS) have been defined by [13]. With regard to the FFSS, Shahzadi and Akram [25] proposed a new decision support algorithm and described the new aggregated operators. By using the t-norm and t-conorm, Garg et al. [6] constructed new FFS type aggregation operators.

By Torra [28], the FS definition was expanded to include the HFS notion. This new set of FS can handle circumstances when the difficulty in constructing the MD arises from uncertainty among a small number of values rather than from a margin of error or specific probability distribution of the probable values [38]. Because of this, the HFS, as opposed to the FS and its other generalizations, can more accurately reflect people's reluctance to express their preferences over objects. A new HFS called IHFS was created later by combining HFS with IFS [21]. The basic idea is to create a scenario in which people pause between a group of MD and ND rather than an individual MD and ND, and they need to represent such a reluctance. The idea of a dual HFS was enhanced and given various features in [40]. The dual IVHFS concept was expanded upon using the HIVIFS strategy [17]. IHFS to GDM problems were applied in [18] utilizing fuzzy cross-entropy. Khan et al. first provided the PFHS [15]. When the total of its MDs is less than 1, PHFS makes up the difference. By Kirisci [89], the Fermatean hesitant fuzzy set has been defined.

The goal of this work is to improve *MAGDM* processes in *IVFHH* environments via aggregation operators and to extend FHFSSs to *IVFHH*. Accuracy and score functions are described. The definition of *IVFHH* is reviewed along with the fundamental characteristics. By introducing the scenario outlining the concept of *MAGDM* in *IVFHH* settings, an algorithm is provided. A medical example is provided to demonstrate the viability and usefulness of the suggested technique.

2. Preliminaries

U will be used to refer to the initial universe set throughout the study.

For $\zeta_{\mathcal{F}} : U \rightarrow [0, 1]$ and $\eta_{\mathcal{F}} : U \rightarrow [0, 1]$, the FFS $\mathcal{F}$ is indicated by $\mathcal{F} = \{(u, \zeta_{\mathcal{F}}(u), \eta_{\mathcal{F}}(u)) : u \in U\}$. For the FFS, the condition $0 \leq \zeta_{\mathcal{F}}^3(u) + \eta_{\mathcal{F}}^3(u) \leq 1$ [22] is holds.

The ID of u to $\mathcal{F}$ is described as $\theta_{\mathcal{F}}(u) = \sqrt[3]{1 - (\zeta_{\mathcal{F}}^3(u) + \eta_{\mathcal{F}}^3(u))}$, for any FFS $\mathcal{F}$ and $u \in U$.

For FFSs $\mathcal{F} = \{\zeta_{\mathcal{F}}, \eta_{\mathcal{F}}\}$, $\mathcal{F}_1 = \{\zeta_{\mathcal{F}_1}, \eta_{\mathcal{F}_1}\}$ and $\mathcal{F}_2 = \{\zeta_{\mathcal{F}_2}, \eta_{\mathcal{F}_2}\}$, some operations as follows [22]:

- i. $\mathcal{F}_1 \cap \mathcal{F}_2 = (\min\{\zeta_{\mathcal{F}_1}, \zeta_{\mathcal{F}_2}\}, \max\{\eta_{\mathcal{F}_1}, \eta_{\mathcal{F}_2}\})$;
- ii. $\mathcal{F}_1 \cup \mathcal{F}_2 = (\max\{\zeta_{\mathcal{F}_1}, \zeta_{\mathcal{F}_2}\}, \min\{\eta_{\mathcal{F}_1}, \eta_{\mathcal{F}_2}\})$;
- iii. $\mathcal{F}^t = (\eta_{\mathcal{F}}, \zeta_{\mathcal{F}})$;
- iv. $\mathcal{F}_1 \boxplus \mathcal{F}_2 = (\sqrt[3]{\zeta_{\mathcal{F}_1}^3 + \zeta_{\mathcal{F}_2}^3 - \zeta_{\mathcal{F}_1}^3 \zeta_{\mathcal{F}_2}^3}, \eta_{\mathcal{F}_1} \eta_{\mathcal{F}_2})$;
- v. $\mathcal{F}_1 \boxtimes \mathcal{F}_2 = (\zeta_{\mathcal{F}_1}^3 \zeta_{\mathcal{F}_2}^3, \sqrt[3]{\eta_{\mathcal{F}_1}^3 + \eta_{\mathcal{F}_2}^3 - \eta_{\mathcal{F}_1}^3 \eta_{\mathcal{F}_2}^3})$;
- vi. $\alpha_{\mathcal{F}} = (\sqrt[3]{1 - (1 - \zeta_{\mathcal{F}}^3)^{\alpha}}, \eta_{\mathcal{F}}^{\alpha})$;
- vii. $\mathcal{F}^{\alpha} = (\zeta_{\mathcal{F}}^3, \sqrt[3]{1 - (1 - \eta_{\mathcal{F}}^3)^{\alpha}})$.

The properties of complement of FFS as follows [22]:

- i. $(\mathcal{F}_1 \cap \mathcal{F}_2)^c = \mathcal{F}_1^c \cup \mathcal{F}_2^c$;
- ii. $(\mathcal{F}_1 \cup \mathcal{F}_2)^c = \mathcal{F}_1^c \cap \mathcal{F}_2^c$;
- iii. $(\mathcal{F}_1 \boxplus \mathcal{F}_2)^c = \mathcal{F}_1^t \boxtimes \mathcal{F}_2^t$;
- iv. $(\mathcal{F}_1 \boxtimes \mathcal{F}_2)^c = \mathcal{F}_1^c \boxplus \mathcal{F}_2^c$;
- v. $\alpha(\mathcal{F})^c = (\mathcal{F}^{\alpha})^c$;
- vi. $(\mathcal{F}^c)^{\alpha} = (\alpha \mathcal{F})^c$.

Definition 2.1. [22] Choose a FFS $\mathcal{F} = \{\zeta_{\mathcal{F}}, \eta_{\mathcal{F}}\}$. For FFS $\mathcal{F}$,

$$SF = \zeta_{\mathcal{F}}^3 - \eta_{\mathcal{F}}^3. \quad (2.1)$$

is said to be a score function.

The function SF is in $[-1, 1]$.

Take the two FFSs $\mathcal{F}_1 = \{\zeta_{\mathcal{F}_1}, \eta_{\mathcal{F}_1}\}$ and $\mathcal{F}_2 = \{\zeta_{\mathcal{F}_2}, \eta_{\mathcal{F}_2}\}$. If the following condition (A) is hold, then it is called a natural quasi-ordering concerning the FFS [22]:

$$(A) \quad \mathcal{F}_1 \geq \mathcal{F}_2 \quad \Leftrightarrow \quad \zeta_{\mathcal{F}_1} \leq \eta_{\mathcal{F}_2}.$$

For the two FFSs $\mathcal{F}_1$ and $\mathcal{F}_2$:

- (a) $SF_{\mathcal{F}_1} < SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 < \mathcal{F}_2$,
- (b) $SF_{\mathcal{F}_1} > SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 > \mathcal{F}_2$,
- (c) $SF_{\mathcal{F}_1} = SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 \sim \mathcal{F}_2$.

Definition 2.2. [22] For a FFS $\mathcal{F} = \{\zeta_{\mathcal{F}}, \eta_{\mathcal{F}}\}$,

$$AF = \zeta_{\mathcal{F}}^3 + \eta_{\mathcal{F}}^3. \quad (2.2)$$

is said to be an accuracy function.

Then, $AF \in [0, 1]$. Clearly, $0 \leq AF = \zeta_{\mathcal{F}}^3 + \eta_{\mathcal{F}}^3 \leq 1$.

For the two FFSs $\mathcal{F}_1 = (\zeta_{\mathcal{F}_1}, \eta_{\mathcal{F}_1})$ and $\mathcal{F}_2 = (\zeta_{\mathcal{F}_2}, \eta_{\mathcal{F}_2})$:

- (a) $SF_{\mathcal{F}_1} < SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 < \mathcal{F}_2$,
- (b) $SF_{\mathcal{F}_1} > SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 > \mathcal{F}_2$,
- (c) $SF_{\mathcal{F}_1} = SF_{\mathcal{F}_2} \Rightarrow \mathcal{F}$ then

$$(i) \quad AF_{\mathcal{F}_1} < AF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 < \mathcal{F}_2,$$

$$(ii) \quad AF_{\mathcal{F}_1} > AF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 > \mathcal{F}_2,$$

$$(iii) \quad AF_{\mathcal{F}_1} = AF_{\mathcal{F}_2} \Rightarrow \mathcal{F}_1 = \mathcal{F}_2.$$

For two FFS $\mathcal{F}, \mathcal{G}$, from a binary relation $\leq_{(SF, AF)}$, it may be shown as $\mathcal{F} \leq_{(SF, AF)} \mathcal{G}$ iff the condition (B) holds:

$$(B) \quad (SF_{\mathcal{F}} < SF_{\mathcal{G}}) \vee (SF_{\mathcal{F}} = SF_{\mathcal{G}} \wedge AF_{\mathcal{F}} \leq AF_{\mathcal{G}}).$$

Definition 2.3. [29] The set

$$\Gamma = \{(u, \tau_{\Gamma}(u)) : u \in U\} \quad (2.3)$$

is called HFS, where $\tau_{\Gamma}(u)$ indicates the set of some values in unit interval, that is probable MD of $u \in U$ to Γ .

From now on, HFN will be used as $\tau = \tau_{\Gamma}(u)$ throughout the paper.

Definition 2.4. The following operations are hold for three HFNs τ, τ_1, τ_2 :

1. $\tau^c = \cup_{\delta \in \tau} \{1 - \delta\}$;
2. $\tau_1 \cap \tau_2 = \cup_{\delta_1 \in \tau_1, \delta_2 \in \tau_2} \min\{\delta_1, \delta_2\}$;
3. $\tau_1 \cup \tau_2 = \cup_{\delta_1 \in \tau_1, \delta_2 \in \tau_2} \max\{\delta_1, \delta_2\}$.

Definition 2.5. The set

$$P_{\Gamma} = \{(u, \zeta_{P_{\Gamma}}(u), \eta_{P_{\Gamma}}(u)) : u \in U\} \quad (2.4)$$

is called PHFS in U , where $(\zeta_{P_{\Gamma}}(u), \eta_{P_{\Gamma}}(u))$ are functions from U to $[0, 1]$, showing a probable MD and ND of $u \in U$ in P_{Γ} respectively. Further, for each element $u \in U$

- (i.) $\forall t_{P_{\Gamma}}(u) \in \zeta_{P_{\Gamma}}(u), \exists t'_{P_{\Gamma}}(x) \in \eta_{P_{\Gamma}}(x)$, such that $0 \leq \tau_{P_{\Gamma}}^2(u) + \tau_{P_{\Gamma}}'^2(u) \leq 1$
- (ii.) $\forall \tau_{P_{\Gamma}}'(u) \in \eta_{P_{\Gamma}}(u), \exists \tau_{P_{\Gamma}}(u) \in \zeta_{P_{\Gamma}}(u)$, such that $0 \leq \tau_{P_{\Gamma}}^2(u) + \tau_{P_{\Gamma}}'^2(u) \leq 1$.

Definition 2.6. Chosse two interval-numbers $\rho = [\rho^-, \rho^+]$ and $\varsigma = [\varsigma^-, \varsigma^+]$.

$$P(\rho \succ \varsigma) = \max \left\{ 1 - \max \left\{ \frac{\varsigma^+ - \rho^-}{J(\rho) - J(\varsigma)}, 0 \right\}, 0 \right\} \quad (2.5)$$

is stated as the likelihood of $\rho \succ \varsigma$, where $J(\rho) = \rho^+ - \rho^-$ and $J(\varsigma) = \varsigma^+ - \varsigma^-$ and holds the following items:

- i. $0 \leq P(\rho \succ \varsigma) \leq 1$,
- ii. $P(\rho \succ \varsigma) = P(\varsigma \succ \rho) = 1/2$, if $P(\rho \succ \varsigma) = P(\varsigma \succ \rho)$,
- iii. $P(\rho \succ \varsigma) + P(\varsigma \succ \rho) = 1$.

3. New Hesitant Fuzzy Sets

In this section, IVFHI will be introduced and its properties will be examined in order to get better results in preventing information loss and to increase the flexibility and applicability of decision-making techniques when dealing with qualitative information.

Definition 3.1.

$$\mathcal{F} = \{(u, h_{\mathcal{F}}(u)) : u \in U\}, \quad (3.1)$$

is called an IVFHI $\mathcal{F}$ on U , where

$$h_{\mathcal{F}}(u) = \left\{ (\hat{\zeta}_{\mathcal{F}}(u), \hat{\eta}_{\mathcal{F}}(u)) : \hat{\zeta}_{\mathcal{F}}(u) = [\zeta_{\mathcal{F}}^-, \zeta_{\mathcal{F}}^+] \in D[0, 1], \hat{\eta}_{\mathcal{F}}(u) = [\eta_{\mathcal{F}}^-, \eta_{\mathcal{F}}^+] \in D[0, 1], (\zeta_{\mathcal{F}}^+)^3 + (\eta_{\mathcal{F}}^+)^3 \leq 1 \right\},$$

where $\hat{\zeta}_{\mathcal{F}}(u)$ is the possible Fermatean membership interval and $\hat{\eta}_{\mathcal{F}}(u)$ is the possible Fermatean non-membership intervals of $\mathcal{F}$. Throughout this article, Υ will show the set of all IVFHs.

Apparently,

if There is only one pair of intervals in $h_{\mathcal{F}}(u)$, the IVFHI converts into an IVFFS,

if both $(\hat{\zeta}_{\mathcal{F}}(u), \hat{\eta}_{\mathcal{F}}(u))$ converts one singleton, the IVFHI may be viewed as an FHFS,

if $\hat{\eta}_{\mathcal{F}}(u) = [0, 0]$, the IVFHI may be viewed as an IVHFS,

if $\zeta_{\mathcal{F}}^+ + \eta_{\mathcal{F}}^+ \leq 1$ the IVFHI can be seen as an IVIHFS,

for each $u \in U$.

For $\widehat{\mathcal{F}} = \{(\hat{\zeta}, \hat{\eta}) : \hat{\zeta} = [\zeta^-, \zeta^+], \hat{\eta} = [\eta^-, \eta^+]\}$, each pair $\widehat{\mathcal{F}} = h_{\mathcal{F}}(u)$ is said to be an interval-valued Fermatean hesitant fuzzy element (IVFE).

Definition 3.2. Choose be three IVFEs $\widehat{\mathcal{F}} = \{(\hat{\zeta}, \hat{\eta}) : \hat{\zeta} = [\zeta^-, \zeta^+], \hat{\eta} = [\eta^-, \eta^+]\}$, $\widehat{\mathcal{F}}_1 = \{(\hat{\zeta}_1, \hat{\eta}_1) : \hat{\zeta}_1 = [\zeta_1^-, \zeta_1^+], \hat{\eta}_1 = [\eta_1^-, \eta_1^+]\}$, $\widehat{\mathcal{F}}_2 = \{(\hat{\zeta}_2, \hat{\eta}_2) : \hat{\zeta}_2 = [\zeta_2^-, \zeta_2^+], \hat{\eta}_2 = [\eta_2^-, \eta_2^+]\}$ and $\alpha > 0$. Then, we get:

- i. $\widehat{\mathcal{F}}^C = \{([\eta^-, \eta^+], [\zeta^-, \zeta^+]) : (\hat{\zeta}, \hat{\eta}) \in \widehat{\mathcal{F}}\}$,
- ii. $\widehat{\mathcal{F}}^\alpha = \{([\zeta^-]^\alpha, [\zeta^+]^\alpha), [\sqrt[3]{1 - (1 - (\eta^-)^3)^\alpha}, \sqrt[3]{1 - (1 - (\eta^+)^3)^\alpha}]\} : (\hat{\zeta}, \hat{\eta}) \in \widehat{\mathcal{F}}\}$,
- iii. $\alpha\widehat{\mathcal{F}} = \{([\sqrt[3]{1 - (1 - (\zeta^-)^3)^\alpha}, \sqrt[3]{1 - (1 - (\zeta^+)^3)^\alpha}], [(\eta^-)^\alpha, (\eta^+)^\alpha]) : (\hat{\zeta}, \hat{\eta}) \in \widehat{\mathcal{F}}\}$
- iv. $\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2 = \left\{ \left(\left[\sqrt[3]{(\zeta_1^-)^3 + (\zeta_2^-)^3 - (\zeta_1^-)^3(\zeta_2^-)^3}, \sqrt[3]{(\zeta_1^+)^3 + (\zeta_2^+)^3 - (\zeta_1^+)^3(\zeta_2^+)^3} \right], [\eta_1^-, \eta_2^-, \eta_1^+, \eta_2^+] \right) : (\hat{\zeta}_i, \hat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\}$
- v. $\widehat{\mathcal{F}}_1 \boxtimes \widehat{\mathcal{F}}_2 = \left\{ \left([\zeta_1^- \zeta_2^-, \zeta_1^+ \zeta_2^+], \left[\sqrt[3]{(\eta_1^-)^3 + (\eta_2^-)^3 - (\eta_1^-)^3(\eta_2^-)^3}, \sqrt[3]{(\eta_1^+)^3 + (\eta_2^+)^3 - (\eta_1^+)^3(\eta_2^+)^3} \right] \right) : (\hat{\zeta}_i, \hat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\}$
- vi. $\widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2 = \left\{ ([\max\{\zeta_1^-, \zeta_2^-\}, \max\{\zeta_1^+, \zeta_2^+\}], [\min\{\eta_1^-, \eta_2^-\}, \min\{\eta_1^+, \eta_2^+\}]) : (\hat{\zeta}_i, \hat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\}$
- vii. $\widehat{\mathcal{F}}_1 \cap \widehat{\mathcal{F}}_2 = \left\{ ([\min\{\zeta_1^-, \zeta_2^-\}, \min\{\zeta_1^+, \zeta_2^+\}], [\max\{\eta_1^-, \eta_2^-\}, \max\{\eta_1^+, \eta_2^+\}]) : (\hat{\zeta}_i, \hat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\}$

Proposition 3.3. Choose be three IVFEs $\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2$ and $\alpha > 0$, then $\widehat{\mathcal{F}}^C, \widehat{\mathcal{F}}^\alpha, \alpha\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2, \widehat{\mathcal{F}}_1 \boxtimes \widehat{\mathcal{F}}_2, \widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2, \widehat{\mathcal{F}}_1 \cap \widehat{\mathcal{F}}_2$ are all IVFEs.

Proof. It is clear that $\widehat{\mathcal{F}}^C$ is an IVFE.

For any $(\hat{\zeta}, \hat{\eta}) \in \widehat{\mathcal{F}}$, since $(\zeta^+)^3 + (\eta^+)^3 \leq 1$

$$((\zeta^+)^3 + (\eta^+)^3)^\alpha = (\sqrt[3]{1 - (1 - (\eta^+)^3)^\alpha})^2 = (\zeta^+)^{3\alpha} + 1 - (1 - (\eta^+)^3)^\alpha \leq (\zeta^+)^{3\alpha} + 1 - (\zeta^+)^{3\alpha} = 1.$$

So, $\widehat{\mathcal{F}}^\alpha$ is an IVFE. Similarly, $\alpha\widehat{\mathcal{F}}$ is also an IVFE. As for $\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2$,

$$\begin{aligned} (\zeta_1^+)^3 + (\zeta_2^+)^3 - (\zeta_1^+)^3(\zeta_2^+)^3 + (\eta_1^+)^3(\eta_2^+)^3 &\leq (\zeta_1^+)^3 + (\zeta_2^+)^3 - (\zeta_1^+)^3(\zeta_2^+)^3 + (1 - (\zeta_1^+)^3)(1 - (\zeta_2^+)^3) \\ &= (\zeta_1^+)^3 + (\zeta_2^+)^3 - (\zeta_1^+)^3(\zeta_2^+)^3 + 1 - (\zeta_1^+)^3 - (\zeta_2^+)^3 + (\zeta_1^+)^3(\zeta_2^+)^3 = 1 \end{aligned}$$

So, $\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2$ is an IVFE. Similarly, $\widehat{\mathcal{F}}_1 \boxtimes \widehat{\mathcal{F}}_2$ is also an IVFE.

Assume that $\zeta_1^+ < \zeta_2^+$. Then,

$$\max\{\zeta_1^+, \zeta_2^+\}^3 = (\zeta_2^+)^3 \leq 1 - (\eta_1^+)^3 \leq 1 - (\min\{\eta_1^+, \eta_2^+\})^3.$$

So, $\widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2$ is an IVFE. Similarly, $\widehat{\mathcal{F}}_1 \cap \widehat{\mathcal{F}}_2$ is also an IVFE. □

Proposition 3.4. Let $\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \widehat{\mathcal{F}}_3$ be four IVFEs and $\alpha, \alpha_1, \alpha_2 > 0$, then

- i. $\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2 = \widehat{\mathcal{F}}_2 \boxplus \widehat{\mathcal{F}}_1$,
- ii. $\widehat{\mathcal{F}}_1 \boxtimes \widehat{\mathcal{F}}_2 = \widehat{\mathcal{F}}_2 \boxtimes \widehat{\mathcal{F}}_1$,
- iii. $\alpha\widehat{\mathcal{F}}_1 \boxplus \alpha\widehat{\mathcal{F}}_2 = \alpha(\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2)$,
- iv. $\widehat{\mathcal{F}}_1^\alpha \boxtimes \widehat{\mathcal{F}}_2^\alpha = (\widehat{\mathcal{F}}_1 \boxtimes \widehat{\mathcal{F}}_2)^\alpha$,

$$\begin{aligned} \text{v. } \alpha_1 \widehat{\mathcal{F}} \boxplus \alpha_2 \widehat{\mathcal{F}} &= (\alpha_1 + \alpha_2) \widehat{\mathcal{F}}, \\ \text{vi. } \widehat{\mathcal{F}}^{\alpha_1} \boxtimes \widehat{\mathcal{F}}^{\alpha_2} &= (\widehat{\mathcal{F}}^{\alpha_1 + \alpha_2}). \end{aligned}$$

Proof. (i.) Based on Definition 3.2, item (i) and (ii) are obvious. Now let's prove item (iii):

$$\begin{aligned} \alpha \widehat{\mathcal{F}}_1 \boxplus \alpha \widehat{\mathcal{F}}_2 &= \left\{ \left(\left[\sqrt[3]{1 - (1 - (\zeta_1^-)^3)^\alpha}, \sqrt[3]{1 - (1 - (\zeta_1^+)^3)^\alpha} \right], [(\eta_1^-)^\alpha, (\eta_1^+)^\alpha] \right) : (\widehat{\zeta}_1, \widehat{\eta}_1) \in \widehat{\mathcal{F}}_1 \right\} \\ &\boxplus \left\{ \left(\left[\sqrt[3]{1 - (1 - (\zeta_2^-)^3)^\alpha}, \sqrt[3]{1 - (1 - (\zeta_2^+)^3)^\alpha} \right], [(\eta_2^-)^\alpha, (\eta_2^+)^\alpha] \right) : (\widehat{\zeta}_2, \widehat{\eta}_2) \in \widehat{\mathcal{F}}_2 \right\} \\ &= \left\{ \left(\left[\sqrt[3]{1 - (1 - (\zeta_1^-)^3)^\alpha ((\zeta_2^-)^3)^\alpha}, \sqrt[3]{1 - (1 - (\zeta_1^+)^3)^\alpha ((\zeta_2^+)^3)^\alpha} \right], [(\eta_1^- \eta_2^-)^\alpha, (\eta_1^+ \eta_2^+)^\alpha] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\}, \end{aligned}$$

and

$$\begin{aligned} \alpha(\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2) &= \alpha \left\{ \left(\left[\sqrt[3]{(\zeta_1^-)^3 + (\zeta_2^-)^3 - (\zeta_1^-)^3 (\zeta_2^-)^3}, \sqrt[3]{(\zeta_1^+)^3 + (\zeta_2^+)^3 - (\zeta_1^+)^3 (\zeta_2^+)^3} \right], [\eta_1^- \eta_2^-, \eta_1^+ \eta_2^+] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\} \\ &= \alpha \left\{ \left(\left[\sqrt[3]{1 - (1 - \zeta_1^-)^3 (1 - \zeta_2^-)^3}, \sqrt[3]{1 - (1 - \zeta_1^+)^3 (1 - \zeta_2^+)^3} \right], [\eta_1^- \eta_2^-, \eta_1^+ \eta_2^+] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\} \\ &= \left\{ \left(\left[\sqrt[3]{1 - ((1 - \zeta_1^-)^3)^\alpha (1 - (\zeta_2^-)^3)^\alpha}, \sqrt[3]{1 - (1 - (\zeta_1^+)^3)^\alpha (1 - (\zeta_2^+)^3)^\alpha} \right], [\eta_1^- \eta_2^-, \eta_1^+ \eta_2^+] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\} \end{aligned}$$

So $\alpha \widehat{\mathcal{F}}_1 \boxplus \alpha \widehat{\mathcal{F}}_2 = \alpha(\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2)$ holds. Similarly, we have $\widehat{\mathcal{F}}_1^\alpha \boxtimes \widehat{\mathcal{F}}_2^\alpha = (\widehat{\mathcal{F}}_1 \boxtimes \widehat{\mathcal{F}}_2)^\alpha$. $\square$

Proposition 3.5. Let $\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \widehat{\mathcal{F}}_3$ be four IVFEs and $\alpha, \alpha_1, \alpha_2 > 0$, then

- i. $(\widehat{\mathcal{F}}^C)^\alpha = (\alpha \widehat{\mathcal{F}})^C$,
- ii. $\alpha(\widehat{\mathcal{F}}^C) = \widehat{\mathcal{F}}^{\alpha C}$,
- iii. $(\widehat{\mathcal{F}}_1)^C \boxplus (\widehat{\mathcal{F}}_2)^C = (\widehat{\mathcal{F}}_1 \boxtimes \widehat{\mathcal{F}}_2)^C$,
- iv. $(\widehat{\mathcal{F}}_1)^C \boxtimes (\widehat{\mathcal{F}}_2)^C = (\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2)^C$,
- v. $(\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2) \boxplus \widehat{\mathcal{F}}_3 = \widehat{\mathcal{F}}_1 \boxplus (\widehat{\mathcal{F}}_2 \boxplus \widehat{\mathcal{F}}_3)$
- vi. $(\widehat{\mathcal{F}}_1 \boxtimes \widehat{\mathcal{F}}_2) \boxtimes \widehat{\mathcal{F}}_3 = \widehat{\mathcal{F}}_1 \boxtimes (\widehat{\mathcal{F}}_2 \boxtimes \widehat{\mathcal{F}}_3)$

Proof.

$$\begin{aligned} (\widehat{\mathcal{F}}^C)^\alpha &= \left\{ \left([(\eta^-)^\alpha, (\eta^+)^\alpha], \left[\sqrt[3]{1 - (1 - (\zeta^-)^3)^\alpha}, \sqrt[3]{1 - (1 - (\zeta^+)^3)^\alpha} \right] \right) : (\widehat{\zeta}, \widehat{\eta}) \in \widehat{\mathcal{F}} \right\} \\ &= \left\{ \left(\left[\sqrt[3]{1 - (1 - (\zeta^-)^3)^\alpha}, \sqrt[3]{1 - (1 - (\zeta^+)^3)^\alpha} \right], [(\eta^-)^\alpha, (\eta^+)^\alpha] \right) : (\widehat{\zeta}, \widehat{\eta}) \in \widehat{\mathcal{F}} \right\} \\ &= \alpha(\widehat{\mathcal{F}})^C. \end{aligned}$$

$$\begin{aligned} \widehat{\mathcal{F}}_1^C \boxplus \widehat{\mathcal{F}}_2^C &= \left\{ ([\eta_1^-, \eta_1^+], [\zeta_1^-, \zeta_1^+]) : (\widehat{\zeta}_1, \widehat{\eta}_1) \in \widehat{\mathcal{F}}_1 \right\} \boxplus \left\{ ([\eta_2^-, \eta_2^+], [\zeta_2^-, \zeta_2^+]) : (\widehat{\zeta}_2, \widehat{\eta}_2) \in \widehat{\mathcal{F}}_2 \right\} \\ &= \left\{ \left(\left[\sqrt[3]{(\eta_1^-)^3 + (\eta_2^-)^3 - (\eta_1^-)^3 (\eta_2^-)^3}, \sqrt[3]{(\eta_1^+)^3 + (\eta_2^+)^3 - (\eta_1^+)^3 (\eta_2^+)^3} \right], [\zeta_1^- \zeta_2^-, \zeta_1^+ \zeta_2^+] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\} \\ &= \left\{ \left(\left[\zeta_1^- \zeta_2^-, \zeta_1^+ \zeta_2^+ \right], \left[\sqrt[3]{(\eta_1^-)^3 + (\eta_2^-)^3 - (\eta_1^-)^3 (\eta_2^-)^3}, \sqrt[3]{(\eta_1^+)^3 + (\eta_2^+)^3 - (\eta_1^+)^3 (\eta_2^+)^3} \right] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2 \right\} \\ &= (\widehat{\mathcal{F}}_1 \boxtimes \widehat{\mathcal{F}}_2)^C. \end{aligned}$$

$$\begin{aligned} (\widehat{\mathcal{F}}_1 \boxplus \widehat{\mathcal{F}}_2) \boxplus \widehat{\mathcal{F}}_3 &= \left\{ \left(\left[\sqrt[3]{(\zeta_1^-)^3 + (\zeta_2^-)^3 - (\zeta_1^-)^3 (\zeta_2^-)^3}, \sqrt[3]{(\zeta_1^+)^3 + (\zeta_2^+)^3 - (\zeta_1^+)^3 (\zeta_2^+)^3} \right], [\eta_1^- \eta_2^-, \eta_1^+ \eta_2^+] \right) : (\widehat{\zeta}_1, \widehat{\eta}_1) \in \widehat{\mathcal{F}}_1, (\widehat{\zeta}_2, \widehat{\eta}_2) \in \widehat{\mathcal{F}}_2 \right\} \\ &\quad \boxplus \left\{ ([\widehat{\zeta}_3, \widehat{\zeta}_3], [\widehat{\eta}_3, \widehat{\eta}_3]) : (\widehat{\zeta}_3, \widehat{\eta}_3) \in \widehat{\mathcal{F}}_3 \right\} \\ &= \left\{ \left(\left[\sqrt[3]{(\zeta_1^-)^3 + (\zeta_2^-)^3 + (\zeta_3^-)^3 - (\zeta_1^-)^3 (\zeta_2^-)^3 - (\zeta_1^-)^3 (\zeta_3^-)^3 - (\zeta_2^-)^3 (\zeta_3^-)^3 + (\zeta_1^-)^3 (\zeta_2^-)^3 (\zeta_3^-)^3}, \right. \right. \\ &\quad \left. \left[\sqrt[3]{(\zeta_1^+)^3 + (\zeta_2^+)^3 + (\zeta_3^+)^3 - (\zeta_1^+)^3 (\zeta_2^+)^3 - (\zeta_1^+)^3 (\zeta_3^+)^3 - (\zeta_2^+)^3 (\zeta_3^+)^3 + (\zeta_1^+)^3 (\zeta_2^+)^3 (\zeta_3^+)^3}, \right. \right. \\ &\quad \left. \left. [\eta_1^- \eta_2^- \eta_3^-, \eta_1^+ \eta_2^+ \eta_3^+] \right] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, 3 \right\} \\ &= \left\{ ([\zeta_1^-, \zeta_1^+], [\zeta_2^-, \zeta_2^+]) : (\widehat{\zeta}_1, \widehat{\eta}_1) \in \widehat{\mathcal{F}}_1 \right\} \boxplus \left\{ \left(\left[\sqrt[3]{(\zeta_2^-)^3 + (\zeta_3^-)^3 - (\zeta_2^-)^3 (\zeta_3^-)^3}, \sqrt[3]{(\zeta_2^+)^3 + (\zeta_3^+)^3 - (\zeta_2^+)^3 (\zeta_3^+)^3} \right], \right. \right. \\ &\quad \left. \left. [\eta_2^- \eta_3^-, \eta_2^+ \eta_3^+] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, 3 \right\} = \widehat{\mathcal{F}}_1 \boxplus (\widehat{\mathcal{F}}_2 \boxplus \widehat{\mathcal{F}}_3) \end{aligned}$$

□

Proposition 3.6. Let $\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \widehat{\mathcal{F}}_3$ be four IVFEs and $\alpha, \alpha_1, \alpha_2 > 0$, then

- i. $\widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2 = \widehat{\mathcal{F}}_2 \cup \widehat{\mathcal{F}}_1$,
- ii. $\widehat{\mathcal{F}}_1 \cap \widehat{\mathcal{F}}_2 = \widehat{\mathcal{F}}_2 \cap \widehat{\mathcal{F}}_1$,
- iii. $\alpha \widehat{\mathcal{F}}_1 \cup \alpha \widehat{\mathcal{F}}_2 = \alpha(\widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2)$,
- vi. $(\widehat{\mathcal{F}}_1)^\alpha \cap (\widehat{\mathcal{F}}_2)^\alpha = (\widehat{\mathcal{F}}_1 \cap \widehat{\mathcal{F}}_2)^\alpha$,
- v. $(\widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2) \cup \widehat{\mathcal{F}}_3 = \widehat{\mathcal{F}}_1 \cup (\widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_3)$,
- vi. $(\widehat{\mathcal{F}}_1 \cap \widehat{\mathcal{F}}_2) \cap \widehat{\mathcal{F}}_3 = \widehat{\mathcal{F}}_1 \cap (\widehat{\mathcal{F}}_1 \cap \widehat{\mathcal{F}}_3)$,
- vii. $(\widehat{\mathcal{F}}_1 \cup \widehat{\mathcal{F}}_2) \cap \widehat{\mathcal{F}}_2 = (\widehat{\mathcal{F}}_1)$,
- viii. $(\widehat{\mathcal{F}}_1 \cap \widehat{\mathcal{F}}_2) \cup \widehat{\mathcal{F}}_2 = (\widehat{\mathcal{F}}_1)$.

Proposition 3.7. Let $\widehat{\mathcal{F}}, \widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \widehat{\mathcal{F}}_3$ be four IVFEs, then

- i. $(\widehat{\mathcal{F}}_1)^C \cup (\widehat{\mathcal{F}}_2)^C = (\widehat{\mathcal{F}}_2 \cap \widehat{\mathcal{F}}_1)^C$,
- ii. $(\widehat{\mathcal{F}}_1)^C \cap (\widehat{\mathcal{F}}_2)^C = (\widehat{\mathcal{F}}_2 \cup \widehat{\mathcal{F}}_1)^C$.

Definition 3.8. Take an IVFE $\widehat{\mathcal{F}} = \{(\widehat{\zeta}, \widehat{\eta}) : \widehat{\zeta} = [\zeta^-, \zeta^+], \widehat{\eta} = [\eta^-, \eta^+]\}$.

$$SC(\widehat{\mathcal{F}}) = \frac{1}{2|\widehat{\mathcal{F}}|} \sum_{(\widehat{\zeta}, \widehat{\eta}) \in \widehat{\mathcal{F}}} (\widehat{\zeta}^3 - \widehat{\eta}^3) = \left[\frac{1}{2|\widehat{\mathcal{F}}|} \sum_{(\widehat{\zeta}, \widehat{\eta}) \in \widehat{\mathcal{F}}} ((\zeta^-)^3 - (\eta^+)^3), \frac{1}{2|\widehat{\mathcal{F}}|} \sum_{(\widehat{\zeta}, \widehat{\eta}) \in \widehat{\mathcal{F}}} ((\zeta^+)^3 - (\eta^-)^3) \right].$$

is called the score function $SC(\widehat{\mathcal{F}})$. Further,

$$AF(\widehat{\mathcal{F}}) = \frac{1}{2|\widehat{\mathcal{F}}|} \sum_{(\widehat{\zeta}, \widehat{\eta}) \in \widehat{\mathcal{F}}} (\widehat{\zeta}^3 + \widehat{\eta}^3) = \left[\frac{1}{2|\widehat{\mathcal{F}}|} \sum_{(\widehat{\zeta}, \widehat{\eta}) \in \widehat{\mathcal{F}}} ((\zeta^-)^3 + (\eta^+)^3), \frac{1}{2|\widehat{\mathcal{F}}|} \sum_{(\widehat{\zeta}, \widehat{\eta}) \in \widehat{\mathcal{F}}} ((\zeta^+)^3 + (\eta^-)^3) \right].$$

is called the accuracy function $AF(\widehat{\mathcal{F}})$.

Definition 3.9. Let $\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2$ be two IVFEs.

- i. If $P(SC(\widehat{\mathcal{F}}_1) > SC(\widehat{\mathcal{F}}_2)) < 1/2$, then we say $\widehat{\mathcal{F}}_1 \prec \widehat{\mathcal{F}}_2$
- ii. If $P(SC(\widehat{\mathcal{F}}_1) > SC(\widehat{\mathcal{F}}_2)) = 1/2$, then
 - a) If $P(AF(\widehat{\mathcal{F}}_1) > AF(\widehat{\mathcal{F}}_2)) < 1/2$, we say $\widehat{\mathcal{F}}_1 \prec \widehat{\mathcal{F}}_2$,
 - b) If $P(AF(\widehat{\mathcal{F}}_1) > AF(\widehat{\mathcal{F}}_2)) = 1/2$, we say $\widehat{\mathcal{F}}_1 = \widehat{\mathcal{F}}_2$.

Proposition 3.10. Take two IVFEs $\widehat{\mathcal{F}}_1 = \{(\widehat{\zeta}_{1j}, \widehat{\eta}_{1j}) : \widehat{\zeta}_{1j} = [\zeta_{1j}^-, \zeta_{1j}^+], \widehat{\eta}_{1j} = [\eta_{1j}^-, \eta_{1j}^+], j = 1, 2, \dots, m\}$,

$\widehat{\mathcal{F}}_2 = \{(\widehat{\zeta}_{2j}, \widehat{\eta}_{2j}) : \widehat{\zeta}_{2j} = [\zeta_{2j}^-, \zeta_{2j}^+], \widehat{\eta}_{2j} = [\eta_{2j}^-, \eta_{2j}^+], j = 1, 2, \dots, m\}$. If $(\zeta_{2j}^+)^3 - (\zeta_{1j}^-)^3 + (\eta_{1j}^+)^3 - (\eta_{2j}^-)^3 \geq (\zeta_{1j}^+)^3 - (\zeta_{2j}^-)^3 + (\eta_{2j}^+)^3 - (\eta_{1j}^-)^3 \geq 0$

, for any $\zeta_{1j}^- \leq \zeta_{2j}^-$, $\zeta_{1j}^+ \leq \zeta_{2j}^+$, $\eta_{1j}^- \geq \eta_{2j}^-$, $\eta_{1j}^+ \geq \eta_{2j}^+$, then $SC(\widehat{\mathcal{F}}_1) \preceq SC(\widehat{\mathcal{F}}_2)$.

Proof. From Definition 3.8,

$$SC(\widehat{\mathcal{F}}_1) = \frac{1}{2m} \sum_{j=1}^m [(\zeta_{1j}^-)^3 - (\eta_{1j}^-)^3, (\zeta_{1j}^+)^3 - (\eta_{1j}^+)^3], \quad SC(\widehat{\mathcal{F}}_2) = \frac{1}{2m} \sum_{j=1}^m [(\zeta_{2j}^-)^3 - (\eta_{2j}^-)^3, (\zeta_{2j}^+)^3 - (\eta_{2j}^+)^3].$$

Suppose

$$\mathcal{L} = \frac{\sum_{j=1}^m [(\zeta_{2j}^+)^3 - (\eta_{2j}^-)^3 + (\eta_{1j}^+)^3 - (\zeta_{1j}^-)^3]}{\sum_{j=1}^m [(\zeta_{1j}^+)^3 - (\eta_{1j}^-)^3 + (\eta_{1j}^+)^3 - (\zeta_{1j}^-)^3 + (\zeta_{2j}^+)^3 - (\eta_{2j}^-)^3 + (\eta_{1j}^+)^3 - (\zeta_{1j}^-)^3]} \quad (3.2)$$

Hence, we obtain $P(SC(\widehat{\mathcal{F}}_1) > SC(\widehat{\mathcal{F}}_2)) = \max\{1 - \max\{\mathcal{L}, 0\}, 0\}$. $\zeta_{1j}^- \leq \zeta_{2j}^-$, $\zeta_{1j}^+ \leq \zeta_{2j}^+$, $\eta_{1j}^- \geq \eta_{2j}^-$, $\eta_{1j}^+ \geq \eta_{2j}^+$, $(\zeta_{2j}^+)^3 - (\zeta_{1j}^-)^3 + (\eta_{1j}^+)^3 - (\eta_{2j}^-)^3 \geq (\zeta_{1j}^+)^3 - (\zeta_{2j}^-)^3 + (\eta_{2j}^+)^3 - (\eta_{1j}^-)^3 \geq 0$

then $1/2 \leq \mathcal{L} \leq 1$. So, $0 \leq 1 - \max\{\mathcal{L}, 0\} = 1 - \mathcal{L} \leq 1/2$, for any $j = 1, 2, \dots, m$. Therefore, $P(SC(\widehat{\mathcal{F}}_1) > SC(\widehat{\mathcal{F}}_2)) = 1 - \mathcal{L} \leq 1/2$. That means $\widehat{\mathcal{F}}_1 \preceq SC(\widehat{\mathcal{F}}_2)$ holds. □

4. Aggregation Operators

Firstly IFWA, IFWG, GIFWA, and GIFWG operators will be given:

Choose $\widehat{\mathcal{F}}_i$ ($i = 1, 2, \dots, n$), $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ as a composition of some IVFEs and the weight vector of $\widehat{\mathcal{F}}_1$ ($\omega_i \in [0, 1]$, $\sum_{i=1}^n \omega_i = 1$, and $\alpha > 0$), respectively.

$$\begin{aligned} \text{IFWA}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) &= \omega_1 \widehat{\mathcal{F}}_1 \boxplus \dots \boxplus \omega_n \widehat{\mathcal{F}}_n \\ &= \left\{ \left(\left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_i^-)^3)^{\omega_i}}, \sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_i^+)^3)^{\omega_i}} \right], \left[\prod_{i=1}^n (\eta_i^-)^{\omega_i}, \prod_{i=1}^n (\eta_i^+)^{\omega_i} \right] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, \dots, n \right\} \end{aligned} \quad (4.1)$$

is called an IFWA operator, where $\text{IFWA} : \Upsilon^n \rightarrow \Upsilon$.

$$\begin{aligned} \text{IFWG}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) &= \omega_1 \widehat{\mathcal{F}}_1 \boxtimes \dots \boxtimes \omega_n \widehat{\mathcal{F}}_n \\ &= \left\{ \left(\left[\prod_{i=1}^n (\zeta_i^-)^{\omega_i}, \prod_{i=1}^n (\zeta_i^+)^{\omega_i} \right], \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_i^-)^3)^{\omega_i}}, \sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_i^+)^3)^{\omega_i}} \right] \right) : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, \dots, n \right\} \end{aligned} \quad (4.2)$$

is called IFWG operator, where $\text{IFWG} : \Upsilon^n \rightarrow \Upsilon$.

Example 4.1. Choose the three IVFHS $\widehat{\mathcal{F}}_1 = \{([0.6, 0.8], [0.5, 0.7])\}$, $\widehat{\mathcal{F}}_2 = \{([0.3, 0.9], [0.5, 0.8]), ([0.4, 0.7], [0.5, 0.8])\}$, $\widehat{\mathcal{F}}_3 = \{([0.6, 0.7], [0.5, 0.8]), ([0.3, 0.7], [0.4, 0.8]), ([0.5, 0.7], [0.4, 0.6])\}$. Take the weight vector for these IVFHSs as $\omega = (0.23, 0.37, 0.40)^T$.

$$\begin{aligned} \text{IFWA} &= \left\{ ([0.532, 0.87], [0.5, 0.776]), ([0.444, 0.73], [0.457, 0.776]), ([0.438, 0.8], [0.46, 0.7]), \right. \\ &\quad \left. ([0.546, 0.73], [0.5, 0.776]), ([0.444, 0.73], [0.46, 0.776]), ([0.5, 0.73], [0.457, 0.7]) \right\} \\ \text{IFWG} &= \left\{ ([0.464, 0.8], [0.5, 0.782]), ([0.352, 0.8], [0.466, 0.782]), ([0.438, 0.8], [0.466, 0.717]), \right. \\ &\quad \left. ([0.516, 0.68], [0.546, 0.782]), ([0.4, 0.722], [0.466, 0.782]), ([0.48, 0.722], [0.5, 0.744]) \right\} \end{aligned}$$

Now, we calculate the score values: $SC(\text{IFWA}) = [-0.155, 0.1761]$, $SC(\text{IFWG}) = [-0.18, 0.167]$ and so $P(SC(\text{IFWG}) \leq SC(\text{IFWA})) = 0.6023$.

Proposition 4.2. Let $\widehat{\mathcal{F}}_i$ be a composition of IVFEs ($i = 1, 2, \dots, n$), $\omega_i > 0$, $\sum_{i=1}^n \omega_i = 1$. Therefore,

$$\text{IFWG}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) \preceq \text{IFWA}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) \quad (4.3)$$

Since $\prod_{i=1}^n (\widehat{\mathcal{F}}_i)^{\omega_i} \leq \sum_{i=1}^n \omega_i \widehat{\mathcal{F}}_i$, for $\widehat{\mathcal{F}}_i > 0$, $\omega_i > 0$, and $\sum_{i=1}^n \omega_i = 1$ (the equality holds iff $\widehat{\mathcal{F}}_1 = \widehat{\mathcal{F}}_2 = \dots = \widehat{\mathcal{F}}_n$) [30], this proposition can be easily proved.

$$\begin{aligned} \text{GIFWA}_\alpha(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) &= (\omega_1 \widehat{\mathcal{F}}_1^\alpha \boxplus \dots \boxplus \omega_n \widehat{\mathcal{F}}_n^\alpha)^{1/\alpha} = \left\{ \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_i^-)^{3\alpha})^{\omega_i}} \right]^{1/\alpha}, \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_i^+)^{3\alpha})^{\omega_i}} \right]^{1/\alpha} \right. \\ &\quad \left. \left[\sqrt[3]{1 - \left(1 - \prod_{i=1}^n (1 - (\eta_i^-)^{3\alpha})^{\omega_i}\right)^{1/\alpha}}, \sqrt[3]{1 - \left(1 - \prod_{i=1}^n (1 - (\eta_i^+)^{3\alpha})^{\omega_i}\right)^{1/\alpha}} \right] : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, \dots, n \right\} \end{aligned} \quad (4.4)$$

is called a GIFWA operator, where $\text{GIFWA} : \Upsilon^n \rightarrow \Upsilon$.

$$\begin{aligned} \text{GIFWG}_\alpha(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) &= \frac{1}{\alpha} (\alpha \widehat{\mathcal{F}}_1^{\omega_1} \boxtimes \dots \boxtimes \alpha \widehat{\mathcal{F}}_n^{\omega_n}) = \left\{ \left[\sqrt[3]{1 - \left(1 - \prod_{i=1}^n (1 - (\zeta_i^-)^{3\alpha})^{\omega_i}\right)^{1/\alpha}} \right]^{1/\alpha}, \right. \\ &\quad \left. \left[\sqrt[3]{1 - \left(1 - \prod_{i=1}^n (1 - (\zeta_i^+)^{3\alpha})^{\omega_i}\right)^{1/\alpha}} \right]^{1/\alpha}, \left[\sqrt[3]{1 - \left(1 - \prod_{i=1}^n (1 - (\eta_i^-)^{3\alpha})^{\omega_i}\right)^{1/\alpha}}, \sqrt[3]{1 - \left(1 - \prod_{i=1}^n (1 - (\eta_i^+)^{3\alpha})^{\omega_i}\right)^{1/\alpha}} \right] : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, \dots, n \right\} \end{aligned} \quad (4.5)$$

is called a GIFWG operator, where $\text{GIFWG} : \Upsilon^n \rightarrow \Upsilon$.

Example 4.3. Take the values of Example 4.1 and choose $\alpha = 4$. Then

$$\begin{aligned} \text{GIFWA}_4 = & \left\{ ([0.531, 0.844], [0.844, 0.85]), ([0.66, 0.85], [0.844, 0.531]), ([0.844, 0.531], [0.66, 0.85]), \right. \\ & \left. ([0.531, 0.726], [0.66, 0.844]), ([0.531, 0.726], [0.66, 0.85]), ([0.531, 0.726], [0.66, 0.85]) \right\} \\ \text{GIFWG}_4 = & \left\{ ([0.57, 0.914], [0.48, 0.745]), ([0.57, 0.914], [0.48, 0.745]), ([0.57, 0.914], [0.48, 0.745]), \right. \\ & \left. ([0.633, 0.833], [0.48, 0.745]), ([0.633, 0.833], [0.48, 0.745]), ([0.633, 0.833], [0.48, 0.745]) \right\} \end{aligned}$$

Now, we calculate the score values: $SC(\text{GIFWA}_4) = [-0.10571, 0.0511]$, $SC(\text{GIFWG}_4) = [-0.097, 0.205]$ and so $P(SC(\text{GIFWG}_\alpha) \leq SC(\text{GIFWA}_4)) = 0.6381$, $P(SC(\text{GIFWG}_4) \leq SC(\text{IFWA})) = 0.642$.

Proposition 4.4. Let $\widehat{\mathcal{F}}_i$ ($i = 1, 2, \dots, n$) be a composition of IVFEs, $\alpha > 0$ $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$, then

$$\begin{aligned} \text{IVWG}(\widehat{\mathcal{F}}_i) & \preceq \text{GIVWA}_\alpha(\widehat{\mathcal{F}}_i) \\ \text{GIVWG}_\alpha(\widehat{\mathcal{F}}_i) & \preceq \text{IVWA}(\widehat{\mathcal{F}}_i) \end{aligned} \quad (4.6)$$

Proposition 4.5. Let $\widehat{\mathcal{F}}_i$ ($i = 1, 2, \dots, n$) be a composition of IVFEs, $\alpha_2 \geq \alpha_1 > 0$ $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$, then

$$\begin{aligned} \text{GIVWA}_{\alpha_1}(\widehat{\mathcal{F}}_i) & \preceq \text{GIVWA}_{\alpha_2}(\widehat{\mathcal{F}}_i) \\ \text{GIVWG}_{\alpha_1}(\widehat{\mathcal{F}}_i) & \succeq \text{GIVWG}_{\alpha_2}(\widehat{\mathcal{F}}_i) \end{aligned} \quad (4.7)$$

IFOA, IFOG, GIFOA, and GIFOG operators will be defined:

Let $\widehat{\mathcal{F}}_{\sigma(i)}$ be the i th largest of $\widehat{\mathcal{F}}_i$ ($i = 1, 2, \dots, n$), and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)^T$ be the associated vector ($\kappa_i \in [0, 1]$, $\sum_{i=1}^n \kappa_i = 1$), and $\alpha > 0$.

$$\begin{aligned} \text{IFOA}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) & = (\kappa_1 \widehat{\mathcal{F}}_{\sigma(1)} \boxplus \dots \boxplus \kappa_n \widehat{\mathcal{F}}_{\sigma(n)}) \\ & = \left\{ \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^-)^3)^{\kappa_i}}, \sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^+)^3)^{\kappa_i}} \right], \left[\prod_{i=1}^n (\eta_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^n (\eta_{\sigma(i)}^+)^{\kappa_i} \right] : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, \dots, n \right\} \end{aligned} \quad (4.8)$$

is called an IFOA operator, where $\text{IFOA} : \Upsilon^n \rightarrow \Upsilon$.

$$\begin{aligned} \text{IFOG}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) & = (\widehat{\mathcal{F}}_{\sigma(1)}^{\kappa_1} \boxtimes \dots \boxtimes \widehat{\mathcal{F}}_{\sigma(n)}^{\kappa_n}) \\ & = \left[\prod_{i=1}^n (\zeta_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^n (\zeta_{\sigma(i)}^+)^{\kappa_i} \right], \left\{ \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^-)^3)^{\kappa_i}}, \sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^+)^3)^{\kappa_i}} \right] : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, \dots, n \right\} \end{aligned} \quad (4.9)$$

is called an IFOG operator, where $\text{IFOG} : \Upsilon^n \rightarrow \Upsilon$.

$$\begin{aligned} \text{GIFOA}_\alpha(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) & = (\kappa_1 \widehat{\mathcal{F}}_{\sigma(1)} \boxplus \dots \boxplus \kappa_n \widehat{\mathcal{F}}_{\sigma(n)})^{1/\alpha} = \left\{ \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^-)^{3\alpha})^{\kappa_i}} \right]^{1/\alpha}, \right. \\ & \left. \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^+)^{3\alpha})^{\kappa_i}} \right]^{1/\alpha}, \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^-)^{3\alpha})^{\kappa_i}} \right]^{1/\alpha}, \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^+)^{3\alpha})^{\kappa_i}} \right]^{1/\alpha} : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, \dots, n \right\}, \end{aligned} \quad (4.10)$$

is called a GIFOA operator, where $\text{GIFOA} : \Upsilon^n \rightarrow \Upsilon$ and $\widehat{\mathcal{F}}_{\sigma(i)}$ is the largest i th of $\widehat{\mathcal{F}}_k = n\omega_k \widehat{\mathcal{F}}_k$ ($k = 1, 2, \dots, n$).

$$\begin{aligned} \text{GIFOG}_\alpha(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) & = \frac{1}{\alpha} \left(\widehat{\mathcal{F}}_{\sigma(1)}^{\kappa_1} \boxtimes \dots \boxtimes \widehat{\mathcal{F}}_{\sigma(n)}^{\kappa_n} \right) = \left\{ \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^-)^{3\alpha})^{\kappa_i}} \right]^{1/\alpha}, \right. \\ & \left. \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^+)^{3\alpha})^{\kappa_i}} \right]^{1/\alpha}, \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^-)^{3\alpha})^{\kappa_i}} \right]^{1/\alpha}, \left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^+)^{3\alpha})^{\kappa_i}} \right]^{1/\alpha} : (\widehat{\zeta}_i, \widehat{\eta}_i) \in \widehat{\mathcal{F}}_i, i = 1, 2, \dots, n \right\}, \end{aligned} \quad (4.11)$$

is called a GIFOG operator, where $\text{GIFOG} : \Upsilon^n \rightarrow \Upsilon$ and $\widehat{\mathcal{F}}_{\sigma(i)}$ is the largest i th of $\widehat{\mathcal{F}}_k = (\widehat{\mathcal{F}}_k)^{n\omega_k}$ ($k = 1, 2, \dots, n$).

Example 4.6. Take the values of Example 4.1 and choose $\alpha = 4$. Let the aggregation-associated vector be chosen as $\kappa = (0.3, 0.38, 0.32)^T$. $SC(\widehat{\mathcal{F}}_1) = [-0.0635, 0.1935]$, $SC(\widehat{\mathcal{F}}_2) = [-0.23325, 0.2055]$, $SC(\widehat{\mathcal{F}}_3) = [-0.2, 0.13]$. Then we obtain $P(SC(\widehat{\mathcal{F}}_1) > SC(\widehat{\mathcal{F}}_2)) = 0.534$ and $P(SC(\widehat{\mathcal{F}}_3) > SC(\widehat{\mathcal{F}}_2)) = 0.6927$. Then $\widehat{\mathcal{F}}_{\sigma(1)} = \widehat{\mathcal{F}}_1$, $\widehat{\mathcal{F}}_{\sigma(2)} = \widehat{\mathcal{F}}_3$, $\widehat{\mathcal{F}}_{\sigma(3)} = \widehat{\mathcal{F}}_2$.

$$\begin{aligned} \text{IFOA} &= \left\{ ([0.53, 0.83], [0.5, 0.77]), ([0.445, 0.83], [0.465, 0.77]), ([0.5, 0.83], [0.465, 0.77]), \right. \\ &\quad \left. ([0.544, 0.74], [0.5, 0.77]), ([0.466, 0.74], [0.466, 0.77]), ([0.51, 0.74], [0.466, 0.77]) \right\} \\ \text{IFOG} &= \left\{ ([0.5, 0.776], [0.461, 0.8]), ([0.473, 0.745], [0.37, 0.8]), ([0.473, 0.724], [0.435, 0.8]), \right. \\ &\quad \left. ([0.5, 0.776], [0.514, 0.74]), ([0.473, 0.776], [0.485, 0.73]), ([0.473, 0.724], [0.485, 0.73]) \right\} \\ \text{GIFOA}_4 &= \left\{ ([0.543, 0.85], [0.624, 0.825]), ([0.543, 0.85], [0.624, 0.825]), ([0.543, 0.854], [0.624, 0.825]), \right. \\ &\quad \left. ([0.543, 0.74], [0.624, 0.825]), ([0.543, 0.74], [0.624, 0.825]), ([0.5473, 0.74], [0.624, 0.825]) \right\} \\ \text{GIFOG}_4 &= \left\{ ([0.543, 0.9], [0.485, 0.75]), ([0.543, 0.9], [0.485, 0.75]), ([0.543, 0.9], [0.485, 0.75]), \right. \\ &\quad \left. ([0.605, 0.87], [0.485, 0.75]), ([0.605, 0.87], [0.485, 0.75]), ([0.605, 0.87], [0.485, 0.75]) \right\} \end{aligned}$$

Hence, we can obtain that $SC(\text{IFOA}) = [-0.137, 0.197]$, $SC(\text{IFOG}) = [-0.147, 0.165]$, $SC(\text{GIFOA}_4) = [-0.101, 0.179]$, $SC(\text{GIFOG}_4) = [-0.116, 0.307]$ and so $P(SC(\text{IFOG}) \leq SC(\text{IFOA})) = 0.4538$, $P(SC(\text{IFOG}) \leq SC(\text{GIFOA}_4)) = 0.6127$, $P(SC(\text{GIFOG}_4) \leq SC(\text{IFOA})) = 0.5405$.

Finally IFHA, IFHG, GIFHA, and GIFHG operators will be defined as follows:

Let $\widehat{\mathcal{F}}_i$ be a composition of some IVFEs ($i = 1, 2, \dots, n$), ω, κ are the weight vector and the associated vector of $\widehat{\mathcal{F}}_i$, and $\alpha > 0$.

$$\begin{aligned} \text{IFHA}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) &= \kappa_1 \widehat{\mathcal{F}}_{\sigma(1)} \boxplus \dots \boxplus \kappa_n \widehat{\mathcal{F}}_{\sigma(n)} \quad (4.12) \\ &= \left\{ \left(\left[\sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^-)^3)^{\kappa_i}}, \sqrt[3]{1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^+)^3)^{\kappa_i}} \right], \left[\prod_{i=1}^n (\eta_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^n (\eta_{\sigma(i)}^+)^{\kappa_i} \right] \right) : (\zeta_{\sigma(i)}, \eta_{\sigma(i)}) \in \widehat{\mathcal{F}}_{\sigma(i)}, i = 1, 2, \dots, n \right\}, \end{aligned}$$

is called an IFHA operator, where $\text{IFHA} : \Upsilon^n \rightarrow \Upsilon$ and $\widehat{\mathcal{F}}_{\sigma(i)}$ is the largest i th of $\widehat{\mathcal{F}}_{\sigma(i)} = n\omega_k \widehat{\mathcal{F}}$, ($k = 1, 2, \dots, n$).

$$\begin{aligned} \text{IFHG}(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) &= \left(\widehat{\mathcal{F}}_{\sigma(1)} \right)^{\kappa_1} \boxtimes \dots \boxtimes \left(\widehat{\mathcal{F}}_{\sigma(n)} \right)^{\kappa_n} \quad (4.13) \\ &= \left\{ \left(\left[\prod_{i=1}^n (\zeta_{\sigma(i)}^-)^{\kappa_i}, \prod_{i=1}^n (\zeta_{\sigma(i)}^+)^{\kappa_i} \right], \sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^-)^3)^{\kappa_i}}, \sqrt[3]{1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^+)^3)^{\kappa_i}} \right) : (\zeta_{\sigma(i)}, \eta_{\sigma(i)}) \in \widehat{\mathcal{F}}_{\sigma(i)}, i = 1, 2, \dots, n \right\}, \end{aligned}$$

is called an IFHG operator, where $\Upsilon^n \rightarrow \Upsilon$, and $\widehat{\mathcal{F}}_{\sigma(i)}$ is the largest i th of $\widehat{\mathcal{F}}_{\sigma(i)} = \left(\widehat{\mathcal{F}} \right)^{n\omega_k}$, ($k = 1, 2, \dots, n$).

$$\begin{aligned} \text{GIFHA}_\alpha(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) &= \left(\left(\kappa_1 \widehat{\mathcal{F}}_{\sigma(1)} \right)^\alpha \boxtimes \dots \boxtimes \left(\kappa_n \widehat{\mathcal{F}}_{\sigma(n)} \right)^\alpha \right)^{1/\alpha} = \left\{ \left[\sqrt[3]{(1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^-)^{3\alpha})^{\kappa_i})^{1/\alpha}}, \right. \right. \\ &\quad \left. \sqrt[3]{(1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^+)^{3\alpha})^{\kappa_i})^{1/\alpha}} \right], \left[\sqrt[3]{1 - (1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^-)^{3\alpha})^{\kappa_i})^{1/\alpha}}, \sqrt[3]{1 - (1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^+)^{3\alpha})^{\kappa_i})^{1/\alpha}} \right] : (\zeta_{\sigma(i)}, \eta_{\sigma(i)}) \in \widehat{\mathcal{F}}_{\sigma(i)}, i = 1, 2, \dots, n \right\}, \end{aligned} \quad (4.14)$$

is called a GIFHA operator, where $\text{GIFHA} : \Upsilon^n \rightarrow \Upsilon$ and $\widehat{\mathcal{F}}_{\sigma(i)}$ is the largest i th of $\widehat{\mathcal{F}}_k = n\omega_k \widehat{\mathcal{F}}$ ($k = 1, 2, \dots, n$).

$$\begin{aligned} \text{GIFHG}_\alpha(\widehat{\mathcal{F}}_1, \widehat{\mathcal{F}}_2, \dots, \widehat{\mathcal{F}}_n) &= \frac{1}{\alpha} \left(\left(\alpha \widehat{\mathcal{F}}_{\sigma(1)} \right)^{\kappa_1} \boxtimes \dots \boxtimes \left(\alpha \widehat{\mathcal{F}}_{\sigma(n)} \right)^{\kappa_n} \right) = \left\{ \left[\sqrt[3]{1 - (1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^-)^{3\alpha})^{\kappa_i})^{1/\alpha}}, \right. \right. \\ &\quad \left. \sqrt[3]{1 - (1 - \prod_{i=1}^n (1 - (\zeta_{\sigma(i)}^+)^{3\alpha})^{\kappa_i})^{1/\alpha}} \right], \left[\sqrt[3]{1 - (1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^-)^{3\alpha})^{\kappa_i})^{1/\alpha}}, \sqrt[3]{1 - (1 - \prod_{i=1}^n (1 - (\eta_{\sigma(i)}^+)^{3\alpha})^{\kappa_i})^{1/\alpha}} \right] : (\zeta_{\sigma(i)}, \eta_{\sigma(i)}) \in \widehat{\mathcal{F}}_{\sigma(i)}, i = 1, 2, \dots, n \right\}, \end{aligned} \quad (4.15)$$

is called a GIFHG operator, where $\text{GIFHG} : \Upsilon^n \rightarrow \Upsilon$ and $\widehat{\mathcal{F}}_{\sigma(i)}$ is the largest i th of $\widehat{\mathcal{F}}_k = (\widehat{\mathcal{F}})^{n\omega_k}$ ($k = 1, 2, \dots, n$).

Example 4.7. Take the values of Example 4.1 and choose $\alpha = 4$. Let $\kappa = (0.3, 0.38, 0.32)^T$ be the aggregation-associated vector.

$$\begin{aligned}\widehat{\mathcal{F}}_1 &= (4 \times 0.23) \cdot \widehat{\mathcal{F}}_1 = \{([0.585, 0.785], [0.53, 0.72])\}, \\ \widehat{\mathcal{F}}_2 &= (4 \times 0.37) \cdot \widehat{\mathcal{F}}_2 = \{([0.341, 0.95], [0.36, 0.72]), ([0.453, 0.774], [0.36, 0.72])\} \\ \widehat{\mathcal{F}}_3 &= (4 \times 0.40) \cdot \widehat{\mathcal{F}}_3 = \{([0.68, 0.79], [0.33, 0.7]), ([0.35, 0.79], [0.231, 0.7]), ([0.58, 0.79], [0.231, 0.442])\}, \\ \widehat{\mathcal{F}}_1 &= (\widehat{\mathcal{F}}_1)^{(4 \times 0.23)} = \{([0.625, 0.814], [0.49, 0.68])\} \\ \widehat{\mathcal{F}}_2 &= (\widehat{\mathcal{F}}_2)^{(4 \times 0.37)} = \{([0.17, 0.86], [0.564, 0.87]), ([0.26, 0.59], [0.564, 0.87])\} \\ \widehat{\mathcal{F}}_3 &= (\widehat{\mathcal{F}}_3)^{(4 \times 0.40)} = \{([0.441, 0.565], [0.577, 0.88]), ([0.146, 0.565], [0.464, 0.88]), ([0.33, 0.565], [0.464, 0.686])\}\end{aligned}$$

$$\begin{aligned}SC(\widehat{\mathcal{F}}_1) &= [-0.0864, 0.167], \quad SC(\widehat{\mathcal{F}}_2) = [-0.08325, -0.06825], \quad SC(\widehat{\mathcal{F}}_3) = [-0.0365, 0.24], \\ SC(\widehat{\mathcal{F}}_1) &= [0.26, 0.0233], \quad SC(\widehat{\mathcal{F}}_2) = [-0.023, 0.21], \quad SC(\widehat{\mathcal{F}}_3) = [-0.324, 0.11].\end{aligned}$$

Using the Definition 3.9, $P(SC(\widehat{\mathcal{F}}_1) \leq SC(\widehat{\mathcal{F}}_2)) = 1$; $P(SC(\widehat{\mathcal{F}}_2) \leq SC(\widehat{\mathcal{F}}_3)) = 0.822$, $P(SC(\widehat{\mathcal{F}}_3) \leq SC(\widehat{\mathcal{F}}_2)) = 0.809$ and $P(SC(\widehat{\mathcal{F}}_2) \leq SC(\widehat{\mathcal{F}}_1)) = 1$. So, we get $\widehat{\mathcal{F}}_{\sigma(1)} = \widehat{\mathcal{F}}_3$, $\widehat{\mathcal{F}}_{\sigma(2)} = \widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}_{\sigma(3)} = \widehat{\mathcal{F}}_1$, $\widehat{\mathcal{F}}_{\sigma(1)} = \widehat{\mathcal{F}}_1$, $\widehat{\mathcal{F}}_{\sigma(2)} = \widehat{\mathcal{F}}_2$, $\widehat{\mathcal{F}}_{\sigma(3)} = \widehat{\mathcal{F}}_3$.

Then,

$$\begin{aligned}\text{IFHA} &= \left\{ ([0.57, 0.882], [0.4, 0.714]), ([0.4513, 0.882], [0.35, 0.714]), ([0.52, 0.882], [0.35, 0.616]), ([0.586, 0.783], [0.4, 0.714]), \right. \\ &\quad \left. ([0.481, 0.783], [0.351, 0.714]), ([0.542, 0.783], [0.351, 0.616]) \right\} \\ \text{IFHG} &= \left\{ ([0.5, 0.846], [0.426, 0.714]), ([0.41, 0.846], [0.414, 0.714]), ([0.4753, 0.846], [0.414, 0.67]), ([0.555, 0.712], [0.564, 0.74]), \right. \\ &\quad \left. ([0.455, 0.712], [0.558, 0.74]), ([0.53, 0.712], [0.558, 0.7]) \right\} \\ \text{GIFHA}_4 &= \left\{ ([0.626, 0.9], [0.9616, 0.86]), ([0.53, 0.9], [0.9617, 0.86]), ([0.56, 0.9], [0.9617, 0.871]), ([0.63, 0.77], [0.9616, 0.86]), \right. \\ &\quad \left. ([0.532, 0.77], [0.9617, 0.86]), ([0.561, 0.77], [0.9617, 0.871]) \right\} \\ \text{GIFHG}_4 &= \left\{ ([0.91, 0.65], [0.48, 0.714]), ([0.95, 0.65], [0.479, 0.714]), ([0.94, 0.65], [0.479, 0.7]), ([0.91, 0.8], [0.48, 0.714]), \right. \\ &\quad \left. ([0.95, 0.8], [0.479, 0.714]), ([0.94, 0.8], [0.479, 0.7]) \right\}\end{aligned}$$

5. Medical Decision-Making Application

IVFHN scenarios description:

$\mathcal{MAGDM}$ is a technique to evaluate the most convenient alternative according to professional person decisions sustaining offered attributes and to construct and solve the planning and judgemental issues. In these types of problems, professionals usually decide according to different research techniques and different structures in their background knowledge. For this reason, $\mathcal{MAGDM}$ is very important for the optimal model that can be acquired with the highest level of consensus of professionals. Here, we suggest a new $\mathcal{MAGDM}$ technique in accordance with the IVAOs. Furthermore, the ranking of integrated IVFHNs is carried out by taking up seriously the SC. The certain circumstances of the $\mathcal{MAGDM}$ technique are demonstrated in the following.

The sets of alternatives, attributes and l professional persons denoted by $A = \{A_i : i = 1, 2, \dots, m\}$, $K = \{K_j : j = 1, 2, \dots, n\}$, $P = \{P_k : k = 1, 2, \dots, l\}$.

If $\widehat{\mathcal{F}}_{ij}^{(k)} = \{(\widehat{\zeta}_{ij}^{(k)}, \widehat{\eta}_{ij}^{(k)})\}$, ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) is an IVFE served by the professional person P_k , in which $(\widehat{\zeta}_{ij}^{(k)})$ points out the probable membership intervals that the alternative A_i satisfies the attribute K_j and $(\widehat{\eta}_{ij}^{(k)})$ points out the probable non-membership intervals that the alternative A_i satisfies the attribute K_j , then let $D^{(k)} = (\widehat{\mathcal{F}}_{ij}^{(k)})_{m \times n}$ be shown as the IVFHM of the k th professional person.

For MAGDM, we can say that the larger the value of the attribute, the benefit attribute, and the smaller the value of the attribute, the cost attribute. Therefore, convert the cost attribute values into the benefit attribute values and normalize the IVFHM $D^{(k)} = (\widehat{\mathcal{F}}_{ij}^{(k)})_{m \times n}$ into the corresponding IVFHM $E^{(k)} = (\widehat{F}_{ij}^{(k)})_{m \times n}$,

$$\widehat{F}_{ij}^{(k)} = \begin{cases} \widehat{\mathcal{F}}_{ij}^{(k)} & , \quad \text{for benefit attribute } C_j, \\ (\widehat{\mathcal{F}}_{ij}^{(k)})^C & , \quad \text{for cost attribute } C_j. \end{cases}$$

where $(\widehat{\mathcal{F}}_{ij}^{(k)})^C$ is the complement of $\widehat{\mathcal{F}}_{ij}^{(k)}$.

Based on these considerations, a new technique was constructed for $\mathcal{MAGDM}$ in IVFHF environments. The algorithm for this technique is as follows:

Algorithm:

Step 1: Form the IVFHM $D^{(k)} = (\widehat{\mathcal{F}}_{ij}^{(k)})_{m \times n}$ and convert $D^{(k)}$ into the $E^{(k)} = (\widehat{F}_{ij}^{(k)})_{m \times n}$.

Step 2: For $\rho = (\rho_1, \rho_2, \dots, \rho_l)^T$ is the weight vector of professionals P_k ($k = 1, 2, \dots, l$), employ the GIFHA (or GIFHG) operator to collected all IVFHMs $E^{(k)} = (\widehat{F}_{ij}^{(k)})_{m \times n}$ into the composition IVFHM $E = (\widehat{F}_{ij})_{m \times n}$, and the certain operation is as follows:

$$\widehat{F}_{ij} = \text{GIFHA}_\alpha(\widehat{\mathcal{F}}_{ij}^{(1)}, \widehat{\mathcal{F}}_{ij}^{(2)}, \dots, \widehat{\mathcal{F}}_{ij}^{(n)}) = \left\{ \left(\left[\sqrt[3]{1 - \prod_{k=1}^l \left(1 - (\zeta_{\widehat{\mathcal{F}}_{ij}^{(k)}}^-)^{3\alpha} \right)^{\rho_k}} \right]^{1/\alpha}, \sqrt[3]{1 - \prod_{k=1}^l \left(1 - (\zeta_{\widehat{\mathcal{F}}_{ij}^{(k)}}^+)^{3\alpha} \right)^{\rho_k}} \right]^{1/\alpha} \right\}, \quad (5.1)$$

$$\left[\sqrt[3]{1 - \prod_{k=1}^l \left(1 - (1 - \eta_{\widehat{\mathcal{F}}_{ij}^{(k)}}^-)^{3\alpha} \right)^{\rho_k}} \right]^{1/\alpha}, \sqrt[3]{1 - \prod_{k=1}^l \left(1 - (1 - \eta_{\widehat{\mathcal{F}}_{ij}^{(k)}}^+)^{3\alpha} \right)^{\rho_k}} \right]^{1/\alpha} : (\zeta_{\widehat{\mathcal{F}}_{ij}^{(k)}}, \eta_{\widehat{\mathcal{F}}_{ij}^{(k)}}) \in \widehat{\mathcal{F}}_{\widehat{\mathcal{F}}_{ij}^{(k)}}, i = 1, 2, \dots, n \Big\},$$

$$\widehat{F}_{ij} = \text{GIFHG}_\alpha(\widehat{\mathcal{F}}_{ij}^{(1)}, \widehat{\mathcal{F}}_{ij}^{(2)}, \dots, \widehat{\mathcal{F}}_{ij}^{(n)}) = \left\{ \left(\left[\sqrt[3]{1 - \prod_{k=1}^l \left(1 - (\zeta_{\widehat{\mathcal{F}}_{ij}^{(k)}}^-)^{3\alpha} \right)^{\rho_k}} \right]^{1/\alpha}, \sqrt[3]{1 - \prod_{k=1}^l \left(1 - (\zeta_{\widehat{\mathcal{F}}_{ij}^{(k)}}^+)^{3\alpha} \right)^{\rho_k}} \right]^{1/\alpha} \right\}, \quad (5.2)$$

$$\left[\sqrt[3]{1 - \prod_{k=1}^l \left(1 - (\eta_{\widehat{\mathcal{F}}_{ij}^{(k)}}^-)^{3\alpha} \right)^{\rho_k}} \right]^{1/\alpha}, \sqrt[3]{1 - \prod_{k=1}^l \left(1 - (\eta_{\widehat{\mathcal{F}}_{ij}^{(k)}}^+)^{3\alpha} \right)^{\rho_k}} \right]^{1/\alpha} : (\zeta_{\widehat{\mathcal{F}}_{ij}^{(k)}}, \eta_{\widehat{\mathcal{F}}_{ij}^{(k)}}) \in \widehat{\mathcal{F}}_{\widehat{\mathcal{F}}_{ij}^{(k)}}, i = 1, 2, \dots, n \Big\},$$

Step 3: For the associated weight vector $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_n)^T$, utilize the GIFHA (or GIFHG) operator to collect all the preference values $\widehat{\mathcal{F}}_i$ all over. The certain operation is as follows:

$$\widehat{F}_i = \text{GIFHA}_\alpha(\widehat{\mathcal{F}}_{i1}, \widehat{\mathcal{F}}_{i2}, \dots, \widehat{\mathcal{F}}_{in}) = \left\{ \left(\left[\sqrt[3]{1 - \prod_{j=1}^n \left(1 - (\zeta_{\widehat{\mathcal{F}}_{i\sigma(j)}}^-)^{3\alpha} \right)^{\kappa_j}} \right]^{1/\alpha}, \sqrt[3]{1 - \prod_{j=1}^n \left(1 - (\zeta_{\widehat{\mathcal{F}}_{i\sigma(j)}}^+)^{3\alpha} \right)^{\kappa_j}} \right]^{1/\alpha} \right\}, \quad (5.3)$$

$$\left[\sqrt[3]{1 - \prod_{j=1}^n \left(1 - (1 - \eta_{\widehat{\mathcal{F}}_{i\sigma(j)}}^-)^{3\alpha} \right)^{\kappa_j}} \right]^{1/\alpha}, \sqrt[3]{1 - \prod_{j=1}^n \left(1 - (1 - \eta_{\widehat{\mathcal{F}}_{i\sigma(j)}}^+)^{3\alpha} \right)^{\kappa_j}} \right]^{1/\alpha} : (\zeta_{\widehat{\mathcal{F}}_{i\sigma(j)}}, \eta_{\widehat{\mathcal{F}}_{i\sigma(j)}}) \in \widehat{\mathcal{F}}_{\widehat{\mathcal{F}}_{i\sigma(j)}}, i = 1, 2, \dots, n \Big\},$$

$$\widehat{F}_i = \text{GIFHG}_\alpha(\widehat{\mathcal{F}}_{i1}, \widehat{\mathcal{F}}_{i2}, \dots, \widehat{\mathcal{F}}_{in}) = \left\{ \left(\left[\sqrt[3]{1 - \prod_{j=1}^n \left(1 - (\zeta_{\widehat{\mathcal{F}}_{i\sigma(j)}}^-)^{3\alpha} \right)^{\kappa_j}} \right]^{1/\alpha}, \sqrt[3]{1 - \prod_{j=1}^n \left(1 - (\zeta_{\widehat{\mathcal{F}}_{i\sigma(j)}}^+)^{3\alpha} \right)^{\kappa_j}} \right]^{1/\alpha} \right\}, \quad (5.4)$$

$$\left[\sqrt[3]{1 - \prod_{j=1}^n \left(1 - (\eta_{\widehat{\mathcal{F}}_{i\sigma(j)}}^-)^{3\alpha} \right)^{\kappa_j}} \right]^{1/\alpha}, \sqrt[3]{1 - \prod_{j=1}^n \left(1 - (\eta_{\widehat{\mathcal{F}}_{i\sigma(j)}}^+)^{3\alpha} \right)^{\kappa_j}} \right]^{1/\alpha} : (\zeta_{\widehat{\mathcal{F}}_{i\sigma(j)}}, \eta_{\widehat{\mathcal{F}}_{i\sigma(j)}}) \in \widehat{\mathcal{F}}_{\widehat{\mathcal{F}}_{i\sigma(j)}}, i = 1, 2, \dots, n \Big\},$$

Here, ω is the weight vector of the attributes K_j . Hence, $\widehat{\mathcal{F}}_{\widehat{\mathcal{F}}_{i\sigma(j)}}$ (or $\widehat{\mathcal{F}}_{\widehat{\mathcal{F}}_{i\sigma(j)}}$) can be defined as

$$\widehat{\mathcal{F}}_{ij} = (n \times \omega_j) \boxtimes \widehat{\mathcal{F}}_{ij} = \left\{ \left(\left[\sqrt[3]{1 - \left(1 - (\zeta_{\widehat{\mathcal{F}}_{ij}}^-)^3 \right)^{n \times \omega_j}} \right], \left[\sqrt[3]{1 - \left(1 - (\zeta_{\widehat{\mathcal{F}}_{ij}}^+)^3 \right)^{n \times \omega_j}} \right] \right), \left[\left(\eta_{\widehat{\mathcal{F}}_{ij}}^- \right)^{n \times \omega_j}, \left(\eta_{\widehat{\mathcal{F}}_{ij}}^+ \right)^{n \times \omega_j} \right] \right\} \quad (5.5)$$

$$: (\zeta_{\widehat{\mathcal{F}}_{ij}}, \eta_{\widehat{\mathcal{F}}_{ij}}) \in \widehat{\mathcal{F}}_{\widehat{\mathcal{F}}_{ij}}, i = 1, 2, \dots, n \Big\}.$$

$$\widehat{\mathcal{F}}_{ij} = (n \times \omega_j) \boxtimes \widehat{\mathcal{F}}_{ij} = \left\{ \left(\left[\left(\zeta_{\widehat{\mathcal{F}}_{ij}}^- \right)^{n \times \omega_j}, \left(\zeta_{\widehat{\mathcal{F}}_{ij}}^+ \right)^{n \times \omega_j} \right], \left[\sqrt[3]{1 - \left(1 - (\eta_{\widehat{\mathcal{F}}_{ij}}^-)^3 \right)^{n \times \omega_j}}, \sqrt[3]{1 - \left(1 - (\eta_{\widehat{\mathcal{F}}_{ij}}^+)^3 \right)^{n \times \omega_j}} \right] \right), \right\} \quad (5.6)$$

$$: (\zeta_{\widehat{\mathcal{F}}_{ij}}, \eta_{\widehat{\mathcal{F}}_{ij}}) \in \widehat{\mathcal{F}}_{\widehat{\mathcal{F}}_{ij}}, i = 1, 2, \dots, n \Big\}.$$

Step 4: Calculate the score values $SC(\hat{F}_i)$ and the accuracy values $AF(\hat{F}_i)$.

Step 5: Obtain the priority of the alternatives A_i by ranking $SC(\hat{F}_i)$.

Example 5.1. Let's choose the A_i ($i = 1, 2, 3$) as the set of alternatives made up of hospital management system software. Denote the set P_k ($k = 1, 2, 3$) three physicians and the set K three criteria. The first criterion is "price", which is the cost type. The second and third criterion are "speed" and "efficiency" respectively, which are the benefit type. The weight vector of the physicians is $\rho = (0.18, 0.52, 0.3)^T$. The weight vector of the criterion is $\omega = (0.5, 0.2, 0.3)^T$. Suppose that physicians provide their own IVFHM $D^{(k)} = (\hat{\mathcal{F}}_{ij}^{(k)})_{m \times n}$ (Tables 1-3), where $\hat{\mathcal{F}}_{ij}^{(k)}$ is an IVFE offered by the professionals P_k (Tables 4-7).

Step 1. Convert the matrix $D^{(k)}$ into the matrix $E^{(k)} = (\hat{F}_{ij}^{(k)})_{m \times n}$ (Tables 4-6).

Step 2: Assume that $\alpha = 5$ and employ the GIFHA operator to collect the 3 IVFHM $E^{(k)} = (\hat{F}_{ij}^{(k)})_{3 \times 3}$ ($k = 1, 2, 3$) into the composition IVFHM $E = (\hat{F}_{ij})_{3 \times 3}$ (Table 7).

Step 3: For associated weighting vector $\kappa = (0.23, 0.58, 0.19)^T$, collect all the preference values $\hat{F}_{ij}$ ($j = 1, 2, 3$) in the i th line of E based on the GIFHA operators.

Step 4: Calculate the three score values: $SC(\hat{F}_1) = [0.3017, 0.3280]$, $SC(\hat{F}_2) = [0.2453, 0.3376]$, $SC(\hat{F}_3) = [0.2746, 0.3112]$.

Step 5: Obtain the priority of the alternatives by ranking the score functions. Then, the ranking order $A_1 > A_3 > A_2$. So the optimal scheme is A_1 .

Table 1: IVFHM for First Physician

	K_1	K_2	K_3
A_1	$\{([0.5, 0.6], [0.4, 0.5])\}$	$\{([0.3, 0.4], [0.6, 0.9])\}$	$\{([0.5, 0.6], [0.4, 0.5]),$ $([0.7, 0.8], [0.3, 0.4]),$ $([0.7, 0.8], [0.2, 0.3])\}$
A_2	$\{([0.1, 0.3], [0.7, 0.7]),$ $([0.2, 0.4], [0.5, 0.6])\}$	$\{([0.5, 0.6], [0.5, 0.5]),$ $([0.7, 0.8], [0.1, 0.2]),$ $([0.6, 0.8], [0.2, 0.4])\}$	$\{([0.7, 0.8], [0.2, 0.2])\}$
A_3	$\{([0.3, 0.4], [0.7, 0.7]),$ $([0.1, 0.3], [0.5, 0.8])\}$	$\{([0.5, 0.9], [0.2, 0.1])\}$	$\{([0.4, 0.6], [0.6, 0.6])\}$

Table 2: IVFHM for Second Physician

	K_1	K_2	K_3
A_1	$\{([0.6, 0.8], [0.1, 0.1]),$ $([0.7, 0.8], [0.1, 0.2]),$ $([0.6, 0.9], [0.1, 0.2])\}$	$\{([0.5, 0.7], [0.4, 0.4]),$ $([0.6, 0.8], [0.3, 0.5])\}$	$\{([0.7, 0.7], [0.3, 0.4])\}$
A_2	$\{([0.7, 0.8], [0.2, 0.3])\}$	$\{([0.7, 0.7], [0.2, 0.3])\}$	$\{([0.6, 0.8], [0.2, 0.4]),$ $([0.6, 0.8], [0.2, 0.4])\}$
A_3	$\{([0.3, 0.5], [0.6, 0.7])\}$	$\{([0.6, 0.8], [0.2, 0.3]),$ $([0.5, 0.9], [0.1, 0.1])\}$	$\{([0.2, 0.3], [0.5, 0.8]),$ $([0.3, 0.4], [0.6, 0.7])\}$

Table 3: IVFHM for Third Physician

	K_1	K_2	K_3
A_1	$\{([0.4, 0.4], [0.6, 0.6])\}$	$\{([0.3, 0.4], [0.5, 0.6]),$ $([0.2, 0.3], [0.7, 0.8])\}$	$\{([0.7, 0.8], [0.2, 0.3])\}$
A_2	$\{([0.5, 0.6], [0.3, 0.4]),$ $([0.5, 0.5], [0.5, 0.6])\}$	$\{([0.1, 0.2], [0.7, 0.8])\}$	$\{([0.3, 0.4], [0.6, 0.6]),$ $([0.2, 0.3], [0.7, 0.8]),$ $([0.3, 0.4], [0.6, 0.7])\}$
A_3	$\{([0.7, 0.8], [0.2, 0.2]),$ $([0.6, 0.7], [0.1, 0.2])\}$	$\{([0.3, 0.6], [0.4, 0.4]),$ $([0.4, 0.5], [0.4, 0.5]),$ $([0.3, 0.5], [0.4, 0.5])\}$	$\{([0.7, 0.8], [0.1, 0.1])\}$

This example can also be solved with the GIFHA operator instead of GIFHG. In addition, new solution values can be reached by changing the α values.

Table 4: Normalized IVFHM for First Physician

	K_1	K_2	K_3
A_1	$\{([0.4, 0.5], [0.5, 0.6])\}$	$\{([0.3, 0.4], [0.6, 0.9])\}$	$\{([0.5, 0.6], [0.4, 0.5]),$ $([0.7, 0.8], [0.3, 0.4]),$ $([0.7, 0.8], [0.2, 0.3])\}$
A_2	$\{([0.7, 0.7], [0.1, 0.3]),$ $([0.5, 0.6], [0.2, 0.4])\}$	$\{([0.5, 0.6], [0.5, 0.5]),$ $([0.7, 0.8], [0.1, 0.2]),$ $([0.6, 0.8], [0.2, 0.4])\}$	$\{([0.7, 0.8], [0.2, 0.2])\}$
A_3	$\{([0.7, 0.7], [0.3, 0.4]),$ $([0.5, 0.8], [0.1, 0.3])\}$	$\{([0.5, 0.9], [0.2, 0.1])\}$	$\{([0.4, 0.6], [0.6, 0.6])\}$

Table 5: Normalized IVFHM for Second Physician

	K_1	K_2	K_3
A_1	$\{([0.1, 0.1], [0.6, 0.8]),$ $([0.1, 0.2], [0.7, 0.8]),$ $([0.1, 0.2], [0.6, 0.9])\}$	$\{([0.5, 0.7], [0.4, 0.4]),$ $([0.6, 0.8], [0.3, 0.5])\}$	$\{([0.7, 0.7], [0.3, 0.4])\}$
A_2	$\{([0.2, 0.3], [0.7, 0.8])\}$	$\{([0.7, 0.7], [0.2, 0.3])\}$	$\{([0.6, 0.8], [0.2, 0.4]),$ $([0.6, 0.8], [0.2, 0.4])\}$
A_3	$\{([0.6, 0.7], [0.3, 0.5])\}$	$\{([0.6, 0.8], [0.2, 0.3]),$ $([0.5, 0.9], [0.1, 0.1])\}$	$\{([0.2, 0.3], [0.5, 0.8]),$ $([0.3, 0.4], [0.6, 0.7])\}$

Table 6: Normalized IVFHM for Third Physician

	K_1	K_2	K_3
A_1	$\{([0.6, 0.6], [0.4, 0.4])\}$	$\{([0.3, 0.4], [0.5, 0.6]),$ $([0.2, 0.3], [0.7, 0.8])\}$	$\{([0.7, 0.8], [0.2, 0.3])\}$
A_2	$\{([0.3, 0.4], [0.5, 0.6]),$ $([0.5, 0.6], [0.5, 0.5])\}$	$\{([0.1, 0.2], [0.7, 0.8])\}$	$\{([0.3, 0.4], [0.6, 0.6]),$ $([0.2, 0.3], [0.7, 0.8]),$ $([0.3, 0.4], [0.6, 0.7])\}$
A_3	$\{([0.2, 0.2], [0.7, 0.8]),$ $([0.1, 0.2], [0.6, 0.7])\}$	$\{([0.3, 0.6], [0.4, 0.4]),$ $([0.4, 0.5], [0.4, 0.5]),$ $([0.3, 0.5], [0.4, 0.5])\}$	$\{([0.7, 0.8], [0.1, 0.1])\}$

Table 7: Normalized IVFHM

	K_1	K_2	K_3
A_1	$\{([0.575, 0.766], [0.98, 0.94]),$ $([0.67, 0.766], [0.98, 0.94]),$ $([0.575, 0.865], [0.98, 0.94])\}$	$\{([0.48, 0.67], [0.945, 0.779]),$ $([0.48, 0.67], [0.9, 0.766]),$ $([0.574, 0.766], [0.943, 0.78]),$ $([0.574, 0.766], [0.9, 0.766])\}$	$\{([0.7, 0.75], [0.99, 0.97]),$ $([0.7, 0.77], [0.99, 0.98]),$ $([0.7, 0.77], [0.99, 0.98])\}$
A_2	$\{([0.67, 0.767], [0.91, 0.91]),$ $([0.67, 0.766], [0.91, 0.91]),$ $([0.67, 0.767], [0.91, 0.95]),$ $([0.67, 0.766], [0.91, 0.93])\}$	$\{([0.67, 0.672], [0.9, 0.842]),$ $([0.684, 0.73], [0.9, 0.842]),$ $([0.672, 0.78], [0.9, 0.842])\}$	$\{([0.64, 0.78], [0.94, 0.94]),$ $([0.64, 0.78], [0.9, 0.84]),$ $([0.64, 0.78], [0.94, 0.9]),$ $([0.64, 0.78], [0.94, 0.94]),$ $([0.64, 0.78], [0.9, 0.84]),$ $([0.64, 0.78], [0.94, 0.9])\}$
A_3	$\{([0.646, 0.739], [0.906, 0.88]),$ $([0.554, 0.646], [0.906, 0.88]),$ $([0.646, 0.739], [0.932, 0.848]),$ $([0.554, 0.647], [0.932, 0.848])\}$	$\{([0.58, 0.83], [0.98, 0.98]),$ $([0.58, 0.83], [0.98, 0.966]),$ $([0.58, 0.83], [0.98, 0.97]),$ $([0.56, 0.81], [0.98, 0.98]),$ $([0.49, 0.81], [0.97, 0.97]),$ $([0.49, 0.81], [0.97, 0.97])\}$	$\{([0.65, 0.74], [0.94, 0.82]),$ $([0.65, 0.74], [0.93, 0.89])\}$

6. Conclusion

This work expands IVHFSs and IVFFSs to IVFHSs. The score and accuracy functions belonging to IVFHSs are given. These functions are used for comparing the size of IVFEs according to the comparison of interval numbers. Various aggregation operators such as IFWA, IFWG, GIFWA, GIFWG, IFOA, IFOG, GIFOA, GIFOG, IFHA, IFHG, GIFHA, and GIFHG connected to IVFH information were built to be used in solving *MAGDM* problems. In order to account for the distinctions of the thought of all decision-makers in MAGDM problems, a new MAGDM approach based on IVHF has been presented. It has been demonstrated both theoretically and practically that our approach can handle MAGDM problems in a flexible and objective mode under IVHF environment.

Ethics declarations

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Authorship contributions

All authors equally contributed to the design and implementation of the research, the analysis of the results, and the writing of the manuscript.

Abbreviations

MD: membership degree
 ND: non-membership degree
 ID: degree of indeterminacy
 FS: Fuzzy set
 IFS: Intuitionistic fuzzy set
 PFS: Pythagorean fuzzy set
 HFS: hesitant fuzzy set
 HPN: Hesitant fuzzy number
 IHFS: intuitionistic hesitant fuzzy set
 PFHS: Pythagorean hesitant fuzzy set
 FFS: Fermatean fuzzy set
 FFSS: Fermatean fuzzy soft set
 IVHFS: interval-valued hesitant fuzzy set
 IVFFS: interval-valued Fermatean fuzzy set
 FFHS: Fermatean hesitant fuzzy set
 IVIHFS: interval-valued intuitionistic hesitant fuzzy set
 IVFHS: Interval-valued Fermatean hesitant fuzzy set
 IVFHN: Interval-valued Fermatean Hesitant Fuzzy Number
 IVAO: Interval-valued Fermatean Hesitant Fuzzy aggregation operators
 IVFHM: Interval-valued Fermatean hesitant fuzzy decision matrix
 IVFE : Interval-valued Fermatean hesitant fuzzy element
 IFWA: IVFHF weighted averaging operator
 IFWG: IVFHF weighted geometric operator
 GIFWA: GIVFHF weighted averaging operator
 GIFWG: GIVFHF weighted geometric operator
 IFOA: IVFHF ordered weighted averaging operator
 IFOG: IVFHF ordered weighted geometric operator
 GIFOA: GIVFHF ordered weighted averaging operator
 GIFOG: GIVFHF ordered weighted geometric operator
 IFHA: IVFHF hybrid weighted averaging operator
 IFHG: IVFHF hybrid weighted geometric operator

GIFHA: GIVFHF hybrid weighted averaging operator
 GIFHG: GIVFHF hybrid weighted geometric operator.

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