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Article

Keywords:

Posted Date: January 26th, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-1281168/v1>

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Version of Record: A version of this preprint was published at Communications Physics on July 1st, 2022.
See the published version at <https://doi.org/10.1038/s42005-022-00946-8>.

Mechanism for particle fractionalization and universal edge physics in quantum Hall fluids

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Advancing a microscopic field-theoretic framework that rigorously unveils the underlying topological hallmarks of fractional quantum Hall (FQH) fluids is a prerequisite for making progress in the classification of strongly-coupled topological matter. Here we advance such a framework that helps reveal an exact fusion mechanism for particle fractionalization in FQH fluids, and uncover the fundamental structure behind the condensation of non-local operators characterizing topological order in the lowest-Landau-level (LLL). We show the first exact analytic computation of the quasielectron Berry connection and perform Monte Carlo simulations that numerically confirm the fusion mechanism for quasiparticles. Thus, two quasiholes plus one electron leads to an exact quasielectron state of fractional charge $e/3$ in a $\nu = 1/3$ Laughlin fluid. We express, in a compact manner, the sequence of (both bosonic and fermionic) Laughlin second-quantized states in a manner that highlights the lack of local condensation. Furthermore, we present a rigorous constructive subspace bosonization dictionary for the bulk fluid and establish the universal long-distance behavior of edge excitations by formulating a conjecture based on the DNA, or root state, of the FQH fluid.

I. INTRODUCTION

Fractional quantum Hall (FQH) fluids have long constituted the best known paradigm of strongly-correlated topological systems [1]. Nonetheless, several fundamental issues remain unresolved. These include the exact mechanism leading to the quasiparticle (or fractional electron) excitations and viable universal signatures in edge transport that are rooted in the topological characteristics of the bulk FQH fluid. This state of affairs is partially due to the lack of a rigorous microscopic framework capable of dealing with these highly entangled systems. In [2], the *Entangled Pauli Principle* (EPP) was advanced as an organizing principle for FQH ground states. The EPP provides information about the pattern of entanglement of the complete subspace of zero-energy modes, i.e., ground states, of quantum Hall parent Hamiltonians for both Abelian and non-Abelian fluids. Those states are generated from the so-called “DNA” [2], or root patterns [3, 4], which encode the elementary topological characteristics of the fluid.

In this work we advance second-quantized many-body techniques that allow for new fundamental insights into the nature of quasiparticle excitations of FQH liquids. In particular, we present an exact fractionalization procedure that allows for a very natural fusion mechanism of quasiparticle generation. We determine the quasihole and quasiparticle operators that explicitly flesh out Laughlin’s flux insertion/removal mechanism and provide the associated quasielectron wave function. The quasielectron that we find differs from Laughlin’s original proposal [5]. We determine the Berry connection of

this quasielectron wave function, considered as an Ehresmann connection on a principal fiber bundle, and as a result a natural fusion mechanism gets unfolded. This, in turn, leads to the exact determination of the quasielectron fractional charge. We perform Monte Carlo simulations to numerically confirm this fusion mechanism of fractionalization. In addition, we introduce an unequivocal diagnostic for characterizing and detecting the topological order of the FQH fluid in terms of a *condensation of a non-local operator* and present a constructive subspace bosonization (fermionization) dictionary for the bulk fluid that highlights the topological nature of the underlying theory. Our organizing EPP and the corresponding fluid’s DNA encode universal features of the bulk FQH state and its edge excitations. Here we formulate a conjecture that enables a demonstration of the universal long-distance behavior of edge excitations in weak confining potentials. This is based on the exact computation of the edge Green’s function over the DNA or root state of the topological fluid.

Although our main results are derived in a field-theoretical manner, we will reformulate some of our conclusions in a first quantization language, where states become wave functions. For clarity, we will occasionally use a mixed representation.

II. STATES AND OPERATOR ALGEBRA IN THE LLL

The LLL is spanned by single-particle orbitals $\phi_r(x, y)$ whose functional form depends on geometry [4]. We consider genus zero manifolds such as those of the disk and the cylinder. Lengths are measured in units of the magnetic length $\ell = \sqrt{\frac{\hbar c}{|e|B}}$, where B is the magnetic field

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strength, $\hbar$ the reduced Planck constant, c the speed of light, and $|e|$ the magnitude of the elementary charge. For ease of presentation, we will primarily focus on the disk geometry [6]. Then, $\phi_r(z = x + iy) = z^r/\mathcal{N}_r$, $\mathcal{N}_r = \sqrt{2\pi 2^r r!}$, with $r \geq 0$ a non-negative integer labeling the angular momentum and $z \in \mathbb{C}$ [7]. N -particle states (elements of the Hilbert space $\mathcal{H}_{\text{LLL}}$) belong to either the totally symmetric (bosons) or anti-symmetric (fermions) representations of the permutation group S_N . Whenever results apply to either representation, we use second-quantized creation (annihilation) $a_r^\dagger$ (a_r) operators instead of the usual $c_r^\dagger$ (c_r), $b_r^\dagger$ (b_r), for fermions and bosons, respectively. The field operator

$$\Lambda(z) = \sum_{r \geq 0} \phi_r(z) a_r, \quad (1)$$

and its adjoint $\Lambda^\dagger(z)$ satisfy canonical (anti-)commutation relations: $[\Lambda(z), \Lambda(z')]_\pm = 0$, $[\Lambda(z), \Lambda^\dagger(z')]_\pm = \{z'|z\}$, where $\{z'|z\} = \frac{1}{2\pi} e^{\frac{z\bar{z}'}{2}}$ is a bilocal kernel satisfying $\int \mathcal{D}[z] \Lambda(z) \{z|z'\} = \Lambda(z')$ [8]. Many-particle states $|\psi\rangle \in \mathcal{H}_{\text{LLL}}$ are characterized by the number of particles N and the maximal occupied orbital r_{max} , defining a filling factor $\nu = (N-1)/r_{\text{max}}$. Given an antisymmetric holomorphic function ψ , one can construct the states

$$|\psi\rangle = \int \left(\prod_{i=1}^N \mathcal{D}[z_i] \right) \psi(z_1, \dots, z_N) \Psi^\dagger(z_1) \dots \Psi^\dagger(z_N) |0\rangle,$$

in terms of the fermionic field operators $\Psi(z)$. Similarly, one can construct states for bosons in terms of permanents and field operators $\Phi(z)$.

We now introduce the operator algebra necessary for the LLL operator fractionalization and constructive bosonization. We first review the operator equivalents of the multivariate power-sum, $p_d(z)$, and elementary, $s_d(z)$, symmetric polynomials ($d \geq 0$). As shown in [9], these are, respectively, given by

$$\begin{aligned} \mathcal{O}_d &= \sum_{r \geq 0} \bar{a}_{r+d}^\dagger \bar{a}_r, \quad \text{and} \\ e_d &= \frac{1}{d!} \sum_{r_1, \dots, r_d \geq 0} \bar{a}_{r_1+1}^\dagger \dots \bar{a}_{r_d+1}^\dagger \bar{a}_{r_d} \dots \bar{a}_{r_1} \end{aligned} \quad (2)$$

(with $\bar{a}_r^\dagger = \mathcal{N}_r a_r^\dagger$, $\bar{a}_r = \mathcal{N}_r^{-1} a_r$). The operator Newton-Girard relations $de_d + \sum_{k=1}^d (-1)^k \mathcal{O}_k e_{d-k} = 0$ (with $e_0 = \mathbb{1}$) link these operators with each other. The second-quantized extensions of the Newton-Girard relations are similar to dualities [10–12] in that applying them twice in a row yields back the original operators. Interestingly, the operators $\mathcal{O}_d$ can be expressed in terms of Bell polynomials in e_d 's (Appendix A). Consequently, any quantity expressible in terms of $\mathcal{O}_d$'s can be also written in terms of e_d 's and vice versa. Both the $\mathcal{O}_d$ and e_d operators generate the same commutative algebra $\mathbf{A}$. Furthermore, they satisfy the commutation relations $[\mathcal{O}_d, \bar{a}_r]_- = -\bar{a}_{r-d}$, $[\mathcal{O}_d, \bar{a}_r^\dagger]_- = \bar{a}_{r+d}^\dagger$ and $[e_d, \bar{a}_r^\dagger]_- = \bar{a}_{r+1}^\dagger e_{d-1}$, $[e_d, \bar{a}_r]_- = -e_{d-1} \bar{a}_{r-1}$.

A set of first-quantized symmetric operators, of relevance to Laughlin's quasielectron and conformal algebras, involves derivatives in z . Similar to the operators defined above, we introduce symmetric polynomials $p_d(\partial_z)$ and $s_d(\partial_z)$ whose second-quantized representations are

$$\begin{aligned} \mathcal{Q}_d &= \sum_{r > d} r(r-1) \dots (r-d) \bar{a}_{r-d}^\dagger \bar{a}_r, \quad \text{and} \\ f_d &= \frac{1}{d!} \sum_{r_1, \dots, r_d > 0} r_1 \dots r_d \bar{a}_{r_1-1}^\dagger \dots \bar{a}_{r_d-1}^\dagger \bar{a}_{r_d} \dots \bar{a}_{r_1}, \end{aligned}$$

and are Newton-Girard-related, $df_d + \sum_{k=1}^d (-1)^k \mathcal{Q}_k f_{d-k} = 0$, with $f_0 = \mathbb{1}$. One can, analogously, define operators mixing polynomials and derivatives as in the positive ($d, d' \geq 0$) Witt algebra $[\ell_d, \ell_{d'}]_- = (d-d')\ell_{d+d'}$. These Witt algebra generators are $\ell_d = -\sum_{i=1}^N z_i^{d+1} \partial_{z_i}$. Their second-quantized version is $\hat{\ell}_d = -\sum_{r \geq 0} r \bar{a}_{r+d}^\dagger \bar{a}_r$. Physically, the operators $\mathcal{O}_d, e_d, \hat{\ell}_d$ ($\mathcal{Q}_d, f_d$) increase (decrease) the total angular momentum or “add (subtract) fluxes”. Rigorous mathematical proofs appear in Appendix A.

Symmetric operators stabilizing incompressible FQH fluids as their eigenvector with lowest eigenvalue are known as “parent FQH Hamiltonians”. The EPP [2, 13] is an organizing principle for generating both Abelian and non-Abelian FQH states as zero modes (ground states) of frustration-free positive-semidefinite microscopic Hamiltonians. The Hamiltonian stabilizing Laughlin states of filling factor $\nu = 1/M$, with M a positive integer, is $H_M = \sum_m H_m$. Here H_m are the Haldane pseudopotentials and the sum is performed over all $0 \leq m < M$ sharing the (even/odd) parity of M . As demonstrated in [4], $H_m = \sum_{0 < j < r_{\text{max}}} T_{j,m}^+ T_{j,m}^-$, where $T_{j,m}^- = \sum_k \eta_k(j, m) a_{j-k} a_{j+k} = (T_{j,m}^+)^\dagger$ with $j = \frac{1}{2}, 1, \frac{3}{2}, \dots, r_{\text{max}} - \frac{1}{2}$ ($2k$ shares the parity of $2j$), and $\eta_k(j, m)$ are geometry-dependent form factors. For odd (even) m , the operator $a_r = c_r$ (b_r).

The space $\mathcal{Z}_M$ of all zero modes of H_M is generated by the states $|\psi\rangle \in \mathcal{H}_{\text{LLL}}$ satisfying $T_{j,m}^- |\psi\rangle = 0$. This space contains the Laughlin state $|\psi_M^N\rangle$ as its minimal total angular momentum, $J = MN(N-1)/2$, state. All other zero modes are obtained by the action of some linear combination of products of $\mathcal{O}_d$, equivalently e_d , operators onto $|\psi_M^N\rangle$ [4, 9]. Inward squeezing is an angular-momentum preserving operation generated by

$$A_{r,r'}^d = \bar{a}_r^\dagger \bar{a}_{r'}^\dagger \bar{a}_{r'+d} \bar{a}_{r-d}, \quad r \leq r', \quad \text{and } d > 0, \quad (3)$$

whose multiple actions on the root partition $|\tilde{\psi}_M^N\rangle = \prod_{i=1}^N \bar{a}_{M(N-i)}^\dagger |0\rangle$ generate all occupation number eigenstates $|\lambda\rangle$ in the expansion of Laughlin state $|\psi_M^N\rangle = |\tilde{\psi}_M^N\rangle + \sum_\lambda C_\lambda |\lambda\rangle$, with integers C_λ [4, 9]. By angular momentum conservation, $\langle \psi_M^N | \bar{a}_r^\dagger \bar{a}_s | \psi_M^N \rangle = \alpha(r) \delta_{r,s} \|\psi_M^N\|^2$. In the thermodynamic limit ($N, r_{\text{max}} \rightarrow \infty$ such that ν remains constant) $\alpha = N/(r_{\text{max}} + 1) \rightarrow \nu$.

III. OPERATOR FRACTIONALIZATION AND TOPOLOGICAL ORDER

Our next goal is to construct second-quantized quasihole and quasiparticle operators. Following Laughlin's insertion/removal of magnetic fluxes, fractionalization is the notion behind that construction. Repeating this procedure M times should yield an object with quantum numbers corresponding to a hole or a particle. Surprisingly, as we will show, the case of quasielectron excitations does not coincide with Laughlin's proposal (nor other proposals). As a byproduct, we will obtain a compact representation of Laughlin states (bosonic and fermionic) that emphasizes a sort of *condensation* of a non-local quantity relating to the topological nature of the FQH fluid.

As shown in [9], the second-quantized version of the quasihole operator $U_N(\eta) = \prod_{i=1}^N (z_i - \eta)$, $\eta \in \mathbb{C}$, is $\hat{U}_N(\eta) = \sum_{d=0}^N (-\eta)^{N-d} e_d$, and satisfies $[\mathcal{O}_d, \hat{U}_N(\eta)]_- = 0$. Moreover [9], $\hat{U}_N(\eta) \bar{a}_r^\dagger = -\eta \bar{a}_r^\dagger \hat{U}_N(\eta) + \bar{a}_{r+1}^\dagger \hat{U}_N(\eta)$ and $\bar{a}_r \hat{U}_N(\eta) = -\eta \hat{U}_N(\eta) \bar{a}_r + \hat{U}_N(\eta) \bar{a}_{r-1}$ (see Appendix B). The action of the quasihole operator on the field operator is given by [14]

$$\begin{aligned} \hat{U}_N(\eta) \Lambda^\dagger(z) &= -\eta \Lambda^\dagger(z) \hat{U}_N(\eta) + \sum_{r \geq 1} \mathcal{N}_{r-1}^{-1} \phi_{r-1}^*(z) \bar{a}_r^\dagger \hat{U}_N(\eta) \\ &= (D^{(z)} - \eta) \Lambda^\dagger(z) \hat{U}_N(\eta), \end{aligned} \quad (4)$$

where $D^{(z)} = 2\partial_{z^*}$. The latter operator identity can be replaced by $\hat{U}_N(\eta) \Lambda^\dagger(z) \stackrel{\int}{=} (z - \eta) \Lambda^\dagger(z) \hat{U}_N(\eta)$, where the symbol $\stackrel{\int}{=}$ stresses validity following a z integration [8].

Having established the properties of $\hat{U}_N$, we introduce the operator $\hat{K}_{\mathcal{M}}(\eta) = \Lambda^\dagger(\eta) \hat{U}_N(\eta)^{\mathcal{M}}$, for any positive integer $\mathcal{M}$. For odd $\mathcal{M} = M$ and $\Lambda(\eta) = \Psi(\eta)$, it agrees with Read's non-local operator for the LLL [15]. One can show that for odd (even) $\mathcal{M}$, the commutator (anticommutator) $[\hat{K}_{\mathcal{M}}(\eta), \hat{K}_{\mathcal{M}}(\eta')]_- = 0$ ($[\hat{K}_{\mathcal{M}}(\eta), \hat{K}_{\mathcal{M}}(\eta')]_+ = 0$), in the fermionic case, while opposite commutation relations hold for bosons. This is a consequence of the composite particle nature induced by the flux insertion mechanism [1].

One can prove (Appendix C) that Laughlin state can be expressed as

$$|\psi_M^N\rangle = \frac{1}{N!} K_{M,N-1} K_{M,N-2} \dots K_{M,0} |0\rangle, \quad (5)$$

where $K_{M,N} = \int \mathcal{D}[z] \hat{K}_M(z)$. This indicates, contrary to the proposal in [15], that the Laughlin state does not feature a local particle condensate. This impossibility is made evident by a counting argument. Each operator $K_{M,N}$ adds a maximum of MN units of angular momentum. Thus, a condensation of these objects would lead to a state with maximum total angular momentum MN^2 [15]. On the other hand, a state such as (5) has angular

momentum $\sum_{i=0}^{N-1} Mi = J$, as it should. This illustrates the above noted impossibility.

The Laughlin state, however, can be understood as a condensate of non-local objects. Consider $\mathcal{K}_M = \int \mathcal{D}[z] \hat{\mathcal{K}}_M(z)$ with $\hat{\mathcal{K}}_M(z) = \Lambda^\dagger(z) \hat{\mathcal{U}}_M(z)$, and $\hat{\mathcal{U}}_M(z) = \sum_{N \geq 0} \hat{U}_N(z)^M |\psi_M^N\rangle \langle \psi_M^N|$ the flux-number non-conserving quasihole operator. Then, for both bosons and fermions,

$$|\psi_M^N\rangle = \frac{1}{N!} \mathcal{K}_M^N |0\rangle. \quad (6)$$

Although illuminating [15], this representation depends on $|\psi_M^N\rangle$ itself through $\hat{\mathcal{U}}_M(z)$ (Appendix C). This condensation of non-local objects is behind the intrinsic topological order of Laughlin fluids. One can show this by studying the long-range order behavior of Read's operator [15]. Before doing so, we need a result (Appendix D) that justifies calling $\hat{U}_N(\eta)$ the quasihole operator. Had one created M quasiholes at position η one should generate an object with the quantum numbers of a hole [16]. That is,

$$\Lambda(\eta) |\psi_M^{N+1}\rangle = \hat{U}_N(\eta)^M |\psi_M^N\rangle. \quad (7)$$

Studying the long-range order of Read's operator [17] amounts to establishing that $\langle \hat{\mathcal{K}}_M(z)^\dagger \hat{\mathcal{K}}_M(0) \rangle$ approaches a non-zero constant at large $|z|$ [18], or alternatively, the condensation of $\hat{\mathcal{K}}_M(0)$ in the $U(1)$ coherent state $|\theta\rangle = \sum_{N \geq 0} \sigma_{M,N} e^{-iN\theta} |\psi_M^N\rangle$, where $\sigma_{M,N} = \alpha_N \|\psi_M^N\|^{-1}$ with $\alpha_N \in \mathbb{C}$ and $\theta \in \mathbb{R}$. We choose α_N such that $\gamma_{M,N} = \sigma_{M,N}^* \sigma_{M,N-1} \|\psi_M^N\|^2$ represents a probability distribution concentrated around (an assumed large) $\bar{N}$. Using the operator fractionalization relation, $\langle \theta | \hat{\mathcal{K}}_M(0) | \theta \rangle = e^{i\theta} \sum_{N \geq 1} \gamma_{M,N} \|\psi_M^N\|^2 \langle \psi_M^N | \Lambda^\dagger(0) \Lambda(0) | \psi_M^N \rangle$. Leading contributions to the sum come from terms with N close to $\bar{N}$, in which case $\langle \psi_M^N | \Lambda^\dagger(0) \Lambda(0) | \psi_M^N \rangle \cong \frac{\nu}{2\pi} \|\psi_M^N\|^2$. Therefore, $\langle \theta | \hat{\mathcal{K}}_M(0) | \theta \rangle \rightarrow \frac{\nu}{2\pi} e^{i\theta}$ for $\bar{N} \rightarrow \infty$.

Do we have a similar operator fractionalization relation for the quasiparticle operator $\hat{V}_N(\eta)$, which reduces to Laughlin's quasielectron in the case of fermions? Since within the LLL one has $\Lambda(z) \hat{U}_N^\dagger(\eta) = (2\partial_z - \eta^*) \hat{U}_N^\dagger(\eta) \Lambda(z)$ it seems natural, by analogy to the quasihole, to define quasiparticles as the second-quantized version of $W_N(\eta) = \prod_{i=1}^N (2\partial_{z_i} - \eta^*)$, Laughlin's original proposal [5]. Note, though, that the second-quantized representation of this operator is $\hat{W}_N(\eta) = \sum_{d=0}^N (-\eta^*)^{N-d} 2^d f_d$, and not $\hat{U}_N^\dagger(\eta)$. This proposal does not satisfy the operator fractionalization relation $\Lambda^\dagger(\eta) |\psi_M^{N-1}\rangle = \hat{W}_N(\eta)^M |\psi_M^N\rangle$ since total angular momenta do not match. A simple modification $\Lambda^\dagger(\eta) |\psi_M^{N-1}\rangle = \hat{W}_{N-1}(\eta)^M |\psi_M^N\rangle$, can be made to match total angular momenta as can be easily verified by localizing the quasiparticle at $\eta = 0$. A close inspection of the case $N = 5$ shows that such a modification cannot work since, albeit conserving the total angular momenta, individual component states display different angular momenta distributions (Appendix E). A proper

embodiment of the quasiparticle should satisfy

$$\begin{aligned}\Lambda^\dagger(\eta)|\psi_M^{N-1}\rangle &= \hat{V}_{N-1}(\eta)^M |\psi_M^N\rangle \quad \text{with} \\ \hat{V}_{N-1}(\eta)^M &= \Lambda^\dagger(\eta) \hat{U}_{N-1}(\eta)^{-M} \Lambda(\eta),\end{aligned}\quad (8)$$

as can be derived from the quasihole (i.e., hole fractionalization) relation. Indeed, this operator is well-defined when acting on the N -particle Laughlin state. Can $\hat{V}_{N-1}(\eta)^M$ be written as the M -th power of another operator? Suppose that one wants to localize a quasiparticle at $\eta = 0$, then $\hat{U}_{N-1}(0)^M = e_{N-1}^M$ and the problem reduces to proving that $\bar{a}_0^\dagger e_{N-1}^{-M} \bar{a}_0 = (\bar{a}_0^\dagger e_{N-1}^{-1} \bar{a}_0)^M$. Recall that any Laughlin state can be obtained by an inward squeezing process of a root partition. Even in the bosonic case, any term in such an expansion has the zeroth angular momentum orbital either empty or singly occupied. In the first (empty) case, the action of $\bar{a}_0$ annihilates such a term while in the second (singly occupied) case we are left with an $(N-1)$ -particle state. The action of e_{N-1}^{-1} on such a state reduces each remaining orbital component by a unit of flux. Since any such state has the smallest occupied orbital with $r \geq M$, the consecutive actions of $\bar{a}_0^\dagger$ and $\bar{a}_0$ are well defined. It follows from the above that we can replace $\bar{a}_0^\dagger e_{N-1}^{-1} \bar{a}_0$ by $\bar{a}_0^\dagger e_{N-1}^\dagger \bar{a}_0$. Therefore,

$$\Lambda^\dagger(0)|\psi_M^{N-1}\rangle = (\Lambda^\dagger(0)\hat{U}_{N-1}^\dagger(0)\Lambda(0))^M |\psi_M^N\rangle. \quad (9)$$

Analogous considerations apply to $\eta \neq 0$, as long as one can argue that the action of $\hat{V}_{N-1}(\eta)^k$ is well-defined on the Laughlin state $|\psi_M^N\rangle$, for $k = 1, \dots, M$. Indeed, if $T(\eta)$ is the magnetic translation operator by η , the translated state $T(-\eta)|\psi_M^N\rangle$ is still a zero mode of the Laughlin state parent Hamiltonian. Thus by the same squeezing argument, $\hat{V}_{N-1}(0)^k T(-\eta)|\psi_M^N\rangle$ is well-defined. Since (up to phases) $T(\eta)\hat{U}_{N-1}(0)T(-\eta)$ equals $\hat{U}_{N-1}(\eta)$, this behavior under translation carries over to $\hat{U}_{N-1}(\eta)^{-k}$, $(\hat{U}_{N-1}(\eta)^\dagger)^k$, and $\hat{V}_{N-1}(\eta)^k$. Thus, the stated relations for the actions of these operators on the Laughlin state extend to finite η .

IV. QUASIPARTICLE WAVE FUNCTION

The field-theoretic approach provides an elegant formalism to prove the exact mechanism behind particle fractionalization. We next illustrate how this mechanism is translated in a first-quantized language. To this end, we start using a *mixed* representation of the quasiparticle wave function. In this representation the corresponding quasiparticle (quasielectron) wave function, localized at $\eta \in \mathbb{C}$, is given by

$$\Psi_\eta^{\text{qp}}(Z_N) = \Lambda^\dagger(\eta) \Psi_\eta^{(M-1)\text{qh}}(Z_{N-1}), \quad (10)$$

where $Z_N = \{z_1, z_2, \dots, z_N\}$, $\Lambda^\dagger(\eta)$ creates a particle in the state $\psi_\eta^0(z) = \mathcal{N}_0 e^{-\frac{1}{4}|z-\eta|^2}$ [19] and

$$\Psi_\eta^{(M-1)\text{qh}}(Z_{N-1}) = \mathcal{N}_{\eta, N-1}^{(M-1)\text{qh}} \prod_{k=1}^{N-1} (z_k - \eta)^{M-1} \Psi_M(Z_{N-1})$$

is the $M-1$ -quasiholes, located at η , wave function for $N-1$ particles, and Laughlin's (un-normalized) state

$$\Psi_M(Z_{N-1}) = \prod_{1 \leq i < j \leq N-1} (z_i - z_j)^M e^{-\frac{1}{4} \sum_{k=1}^{N-1} |z_k|^2}.$$

By the definition of the operator $\Lambda^\dagger(\eta)$, then,

$$\Psi_\eta^{\text{qp}}(Z_N) = \sqrt{N} \hat{\mathcal{A}} \left[\psi_\eta^0(z_N) \Psi_\eta^{(M-1)\text{qh}}(Z_{N-1}) \right], \quad (11)$$

where, for fermions for instance,

$$\hat{\mathcal{A}} \Phi(Z_N) = \frac{1}{N!} \sum_{\sigma \in S_N} \text{sgn}(\sigma) \Phi(z_{\sigma(1)}, \dots, z_{\sigma(N)}). \quad (12)$$

This straightforwardly gives the first quantized quasiparticle wave function

$$\begin{aligned}\Psi_\eta^{\text{qp}}(Z_N) &= \sqrt{N} \mathcal{N}_{\eta, N-1}^{(M-1)\text{qh}} \mathcal{N}_0 e^{-\frac{|\eta|^2}{4}} e^{-\frac{1}{4} \sum_{k=1}^N |z_k|^2} \\ &\times \hat{\mathcal{A}} \left[e^{\frac{z_N \eta^*}{2}} \prod_{k=1}^{N-1} (z_k - \eta)^{M-1} \prod_{1 \leq i < j \leq N-1} (z_i - z_j)^M \right],\end{aligned}\quad (13)$$

with all normalization factors included. We claim that this wave function is properly normalized. Indeed, we have

$$\langle \Psi_\eta^{\text{qp}} | \Psi_\eta^{\text{qp}} \rangle = \langle \Psi_\eta^{(M-1)\text{qh}} | \Lambda(\eta) \Lambda^\dagger(\eta) | \Psi_\eta^{(M-1)\text{qh}} \rangle. \quad (14)$$

Since the orbital ψ_η^0 is unoccupied in $\Psi_\eta^{(M-1)\text{qh}}$, $|\Psi_\eta^{(M-1)\text{qh}}\rangle$ is an eigenstate of $\Lambda(\eta) \Lambda^\dagger(\eta)$ with eigenvalue 1. Therefore,

$$\langle \Psi_\eta^{\text{qp}} | \Psi_\eta^{\text{qp}} \rangle = \langle \Psi_\eta^{(M-1)\text{qh}} | \Psi_\eta^{(M-1)\text{qh}} \rangle \quad (15)$$

and Ψ_η^{qp} is normalized if $\Psi_\eta^{(M-1)\text{qh}}$ is normalized.

One can re-write the (un-normalized) quasiparticle (quasielectron) wave function $\bar{\Psi}_\eta^{\text{qp}}$ in an enlightening manner

$$\bar{\Psi}_\eta^{\text{qp}}(Z_N) = \Gamma_\eta^\dagger(Z_N) \Psi_M(Z_N), \quad (16)$$

with the *quasiparticle (quasielectron) operator*

$$\Gamma_\eta^\dagger(Z_N) = \sum_{i=1}^N e^{\frac{z_i \eta^*}{2}} \prod_{j \neq i} \frac{(z_j - \eta)^{M-1}}{(z_j - z_i)^M}, \quad (17)$$

which clearly shows how it differs significantly from prior proposals [5, 20–23] (see Appendix E). But this is not all the story. It is even more illuminating to understand the precise mechanism leading to this remarkable quasiparticle. Before doing so, we will first compute the charge of this excitation using the Berry connection idea proposed in [24] and further elaborated in Section 2.4 of [8] for the quasihole, that is the Aharonov-Bohm effective charge coupled to magnetic flux. We will then show a remarkable exact property of the charge density that will shed light on the underlying fractionalization mechanism.

A. Berry connection

For pedagogical reasons, we next focus on the fermionic (electron) case. Consider an adiabatic process (in time t) where the position of the quasiparticle, $\eta = \eta(t)$, is encircling an area enclosing a magnetic flux ϕ . We will next show that the Berry connection decomposes into

$$\left\langle \Psi_\eta^{\text{qp}} \left| \frac{d}{dt} \Psi_\eta^{\text{qp}} \right. \right\rangle = i\mathcal{A}_1 + i\tilde{\mathcal{A}}_{M-1}. \quad (18)$$

As we will explain, $\mathcal{A}_1$ describes the Berry phase contribution from a single particle (electron) and $\tilde{\mathcal{A}}_{M-1}$ is the contribution from $M-1$ quasiholes. It is convenient to demonstrate this relation in second quantization, where only in the end, $\tilde{\mathcal{A}}_{M-1}$ is computed from first quantization methods [24–26]. So let $|\Psi_\eta^{(M-1)\text{qh}}\rangle = \hat{\psi}_\eta^\dagger |0\rangle$, where $\hat{\psi}_\eta^\dagger$ is an element in the algebra generated by $c_j^\dagger$ s, where $c_j^\dagger$ creates a particle in the orbital $\psi_0^j(z)$. Thus,

$$\hat{\psi}_\eta^\dagger = \sum_{j_1, \dots, j_{N-1}} F_{j_1, \dots, j_{N-1}} c_{j_1}^\dagger \dots c_{j_{N-1}}^\dagger \quad (19)$$

with some coefficients $F_\bullet$ dependent on η .

The statement made earlier that $\psi_\eta^0(z)$ is not occupied in $|\Psi_\eta^{(M-1)\text{qh}}\rangle$ is equivalent to saying that

$$\Lambda(\eta)\hat{\psi}_\eta^\dagger = (-1)^{N-1}\hat{\psi}_\eta^\dagger\Lambda(\eta), \quad \Lambda^\dagger(\eta)\hat{\psi}_\eta = (-1)^{N-1}\hat{\psi}_\eta\Lambda^\dagger(\eta).$$

Trivially, also, $\Lambda(\eta)$ has the same relation with $\hat{\psi}_\eta$, and $\Lambda^\dagger(\eta)$ (or $\frac{d}{dt}\Lambda^\dagger(\eta)$) with $\hat{\psi}_\eta^\dagger$. From normalization,

$$\Lambda(\eta)\Lambda^\dagger(\eta)|0\rangle = |0\rangle = \hat{\psi}_\eta\hat{\psi}_\eta^\dagger|0\rangle. \quad (20)$$

Thus,

$$\begin{aligned} \left\langle \Psi_\eta^{\text{qp}} \left| \frac{d}{dt} \Psi_\eta^{\text{qp}} \right. \right\rangle &= \left\langle 0 \left| \hat{\psi}_\eta \Lambda(\eta) \left(\frac{d}{dt} \Lambda^\dagger(\eta) \right) \hat{\psi}_\eta^\dagger \right| 0 \right\rangle \\ &+ \left\langle 0 \left| \hat{\psi}_\eta \Lambda(\eta) \Lambda^\dagger(\eta) \left(\frac{d}{dt} \hat{\psi}_\eta^\dagger \right) \right| 0 \right\rangle \equiv i\mathcal{A}_1 + i\tilde{\mathcal{A}}_{M-1}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} i\mathcal{A}_1 &= \left\langle 0 \left| \hat{\psi}_\eta \Lambda(\eta) \left(\frac{d}{dt} \Lambda^\dagger(\eta) \right) \hat{\psi}_\eta^\dagger \right| 0 \right\rangle \\ &= \left\langle 0 \left| \hat{\psi}_\eta \hat{\psi}_\eta^\dagger \Lambda(\eta) \left(\frac{d}{dt} \Lambda^\dagger(\eta) \right) \right| 0 \right\rangle \\ &= \left\langle 0 \left| \Lambda(\eta) \left(\frac{d}{dt} \Lambda^\dagger(\eta) \right) \right| 0 \right\rangle = \left\langle \psi_\eta^0 \left| \frac{d}{dt} \psi_\eta^0 \right. \right\rangle, \end{aligned} \quad (22)$$

and

$$\begin{aligned} i\tilde{\mathcal{A}}_{M-1} &= \left\langle 0 \left| \hat{\psi}_\eta \Lambda(\eta) \Lambda^\dagger(\eta) \left(\frac{d}{dt} \hat{\psi}_\eta^\dagger \right) \right| 0 \right\rangle \\ &= \left\langle 0 \left| \Lambda(\eta) \Lambda^\dagger(\eta) \hat{\psi}_\eta \left(\frac{d}{dt} \hat{\psi}_\eta^\dagger \right) \right| 0 \right\rangle \\ &= \left\langle 0 \left| \hat{\psi}_\eta \left(\frac{d}{dt} \hat{\psi}_\eta^\dagger \right) \right| 0 \right\rangle = \left\langle \Psi_\eta^{(M-1)\text{qh}} \left| \frac{d}{dt} \Psi_\eta^{(M-1)\text{qh}} \right. \right\rangle. \end{aligned} \quad (23)$$

This finishes the proof.

Therefore, the quasiparticle charge e^* has a contribution from a particle of charge e and $M-1$ quasiholes of charge $-e/M$, i.e., $e^* = e - e(M-1)/M = e/M$, as expected. In simple terms, the channel fusing two quasiholes with one electron leads to a quasielectron of charge $e/3$ in an $\nu = 1/3$ Laughlin fluid. This is a very intuitive (and exact) mechanism that has been overlooked until now. Notice that we proved that the evaluation of the quasiparticle Berry connection is exact for any N , while the quasihole charge $-e/M$ is only exact asymptotically in the limit $N \rightarrow \infty$ (see Section 2.4 of [8]).

B. Charge density

A consequence of this effective fusion mechanism manifests in the calculation of the quasiparticle charge density $\rho_{\text{qp}}(z)$. We appeal once more to the fact that

$$\Lambda(\eta)|\Psi_\eta^{(M-1)\text{qh}}\rangle = 0. \quad (24)$$

This can be expressed as

$$\int d^2r_j \psi_\eta^0(z_j)^* \Psi_\eta^{(M-1)\text{qh}}(Z_{N-1}) = 0, \quad (25)$$

for $j = 1, \dots, N-1$. Here $d^2r_j = \frac{1}{2i} dz_j^* \wedge dz_j$ is the usual two dimensional measure on the complex plane.

We can write the quasielectron wave function as

$$\Psi_\eta^{\text{qp}}(Z_N) = \frac{1}{\sqrt{N}} \sum_{j=1}^N (-1)^j \psi_\eta^0(z_j) \Psi_\eta^{(M-1)\text{qh}}(Z_{N-1}, \hat{z}_j), \quad (26)$$

where $\hat{z}_j$ means that coordinate z_j is absent.

We want to evaluate

$$\begin{aligned} \rho_{\text{qp}}(z) &= \sum_{j=1}^N \int d^2r_1 \dots d^2\hat{r}_j \dots d^2r_N |\Psi_\eta^{\text{qp}}(Z_N)|^2 \\ &= N \int d^2r_1 \dots d^2r_{N-1} |\Psi_\eta^{\text{qp}}(Z_{N-1}, z_N = z)|^2 \\ &= \left[\sum_{j,j'=1}^N (-1)^{j+j'} \int d^2r_1 \dots d^2r_{N-1} \psi_\eta^0(z_j)^* \psi_\eta^0(z_{j'}) \right. \\ &\quad \times \Psi_\eta^{(M-1)\text{qh}}(Z_{N-1}, \hat{z}_j)^* \Psi_\eta^{(M-1)\text{qh}}(Z_{N-1}, \hat{z}_{j'}) \Big]_{z_N=z}. \end{aligned} \quad (27)$$

We now see that terms with $j \neq j'$ do not contribute. This is so since in such a case, at least one of them is not equal to N , say $j \neq N$, and then (25) gives zero. In the $j = j' = N$ term, the integrals give a value of unity for reasons of normalization and we get

$$\psi_\eta^0(z)^* \psi_\eta^0(z) = \langle \psi_\eta^0 | \hat{\rho}(z) | \psi_\eta^0 \rangle, \quad (28)$$

where $\hat{\rho}(z)$ is the density operator. For $j = j' \neq N$, the integral over the j th variable gives 1, and the rest

FIG. 1. (Color online.) Density profiles (in units of $\rho_0 = \frac{\nu}{2\pi}$) for 1 quasielectron located at the position η of an incompressible $\nu = \frac{1}{3}$ Laughlin fluid with $N = 7$ particles (left and middle panels). The right panel depicts 2 quasiholes in an otherwise $\nu = \frac{1}{3}$ Laughlin fluid with $N = 6$ particles (adding the electron charge density $\frac{1}{2\pi}e^{-\frac{1}{4}|z-\eta|^2}$ leads to the exact same middle panel). Lower panels are contour plots of their 3D plots above. Monte Carlo simulations averaged over more than 2×10^{10} equilibrated configurations.

precisely gives $\langle \Psi_\eta^{(M-1)qh} | \hat{\rho}(z) | \Psi_\eta^{(M-1)qh} \rangle$. We have just shown that

$$\rho_{qp}(z) = \langle \psi_\eta^0 | \hat{\rho}(z) | \psi_\eta^0 \rangle + \langle \Psi_\eta^{(M-1)qh} | \hat{\rho}(z) | \Psi_\eta^{(M-1)qh} \rangle. \quad (29)$$

Here, the first term is just $\frac{1}{2\pi}e^{-\frac{1}{2}|z-\eta|^2}$, while the second one is the local particle density at z of $\Psi_\eta^{(M-1)qh}(Z_{N-1})$, which is governed by a plasma analogy.

This picture is physically appealing. On one hand, there is no (local) plasma analogy for a state such as $\Psi_\eta^{qp}(Z_N)$, but certain properties such as the Berry connection or the quasiparticle charge density simplify because of the fusion mechanism of fractionalization for any finite N . On the other hand, this same mechanism facilitate numerical computations, such as Monte Carlo [27], of certain physical properties. For example, in Fig. 1 we have checked numerically that the fusion mechanism works for the charge density of an $N = 7$ electron system and $\nu = 1/3$. In this way, we can simulate an arbitrary large system of electrons because there is an "effective plasma analogy" and the Monte Carlo updates become quite efficient. Figure 2 shows Monte Carlo simulations of the radial density for a system of $N = 50$ electrons. We can measure the charge of the quasiparticle by using the expression $\delta\rho_{qp} = 2\pi \int_0^{r_{cut-off}} [\rho_{qp}(r) - \rho_L(r)] r dr$ where, in a finite system, the cut-off radius $r_{cut-off}$ must at least enclose completely the quasiparticle and, at the same time, be sufficiently far from the boundary to avoid boundary effects [28]. Using the Monte Carlo data for $N = 400$ particles (see Fig. 4 in Appendix F) and choosing $r_{cut-off} \leq 30l$, we get a saturation of the fractional charge at the value $\delta\rho_{qp} = 0.3330(30)e$. Similarly, for

two quasiholes we get $\delta\rho_{2qh} = -0.6634(30)e$.

FIG. 2. (Color online.) Left panel: Radial charge density $\rho(r)$ (in units of $\rho_0 = \frac{\nu}{2\pi}$) of a $\nu = 1/3$ Laughlin fluid ($\rho_L(r)$ with $N = 50$ electrons), 2 quasiholes ($N = 49$ electrons), 1 electron (of density $\frac{1}{2\pi}e^{-\frac{1}{4}|z|^2}$), and 1 quasielectron ($N = 50$ electrons) localized at $\eta = 0$. The fusion mechanism dictates that the sum of 2 quasiholes and 1 electron is identical to 1 quasielectron. Right panel: Quasiparticle localized at $\eta = 4 + 3i$.

V. CONSTRUCTIVE SUBSPACE BOSONIZATION

A bosonization map is an example of a duality [11]. Typically, dualities are *dictionaries* constructed as isometries of bond algebras acting on the whole Hilbert space [11]. A weaker notion may involve subspaces defined from a prescribed vacuum and, thus, are Hamiltonian-dependent. This is the case of Luttinger's bosonization [29] that describes, in the thermodynamic limit, collective

low energy excitations about a gapless fermion ground state. Our bosonization is performed with respect to a radically different vacuum- that of the gapped Laughlin state. Unlike most treatments, we will not bosonize the one-dimensional FQH edge (by assuming it to be a Luttinger system) but rather bosonize the entire two-dimensional FQH system. Contrary to gapless collective excitations about the one-dimensional Fermi gas ground state associated with the Luttinger bosonization scheme, our bosonization does not describe modes of arbitrarily low finite energy but rather only the zero-energy (topological) excitations [9] that are present in the gapped Laughlin fluid. As illustrated in [4, 9], the zero-mode subspace $\mathcal{Z} = \bigoplus_{N=0}^{\infty} \mathcal{Z}_N$ is generated by the action of the commutative algebra $\mathbf{A}$ on the Laughlin state $|\psi_M^N\rangle$ for different particle numbers N . Yet another notable difference with the conventional Luttinger bosonization (and conjectured extensions to 2+1 dimensions [30]) is, somewhat similar to earlier continuum renditions (as opposed to our discrete case), e.g., [31], that the indices parameterizing our bosonic excitations, $d \geq 0$, are taken from the discrete positive half-line (angular momentum values) instead of the continuous full real line of the Luttinger system (or plane of [30]). Each zero-energy state in our original (fermionic/bosonic) Hilbert space has an image in the mapped bosonized Hilbert space. Consider the following bosonic creation (annihilation) operators $\mathbf{b}_d^\dagger = \mathcal{O}_d/\sqrt{d\nu}$ ($\mathbf{b}_d = \mathcal{O}_d^\dagger/\sqrt{d\nu}$). Then, $d\nu[\mathbf{b}_d, \mathbf{b}_d^\dagger]_- = \sum_{r=0}^{d-1} \bar{a}_r^\dagger \bar{a}_r$ and, in the thermodynamic limit,

$$\langle \psi_M^N | [\mathbf{b}_d, \mathbf{b}_d^\dagger]_- | \psi_M^N \rangle \| \psi_M^N \|^2 \rightarrow 1. \quad (30)$$

The commutator $[\mathbf{b}_d, \mathbf{b}_{d'}^\dagger]_-$ does not preserve total angular momentum when $d \neq d'$. It follows that, in the thermodynamic limit, within the Laughlin state subspace, $[\mathbf{b}_d, \mathbf{b}_{d'}^\dagger]_- = \delta_{d,d'}$. The field operator $\varphi(z) = \sum_{d \geq 0} \phi_d(z) \mathbf{b}_d$ and its adjoint $\varphi^\dagger(z)$ satisfy $[\varphi(z), \varphi(z')]_- = 0$ and $[\varphi(z), \varphi^\dagger(z')]_- = \{z'|z\}$.

We next construct the operators connecting different particle sectors, that is, the Klein factors that commute with the bosonic degrees of freedom $\mathbf{b}_d, \mathbf{b}_d^\dagger$ and are N -independent. Since $|\psi_M^{N+1}\rangle = \frac{1}{N+1} K_{M,N} |\psi_M^N\rangle$ we define $F_{M,N}^\dagger = \frac{1}{N+1} K_{M,N}$ and $\mathcal{F}_M^\dagger = \sum_{N \geq 0} F_{M,N}^\dagger |\psi_M^N\rangle \langle \psi_M^N|$. This illustrates the relation between the Klein factors of bosonization with the (non-local) Read operator. We then get $\langle \psi_M^{N+1} | [\mathcal{O}_d, F_{M,N}^\dagger]_- | \psi_M^N \rangle = 0$ and $\langle \psi_M^{N+1} | [\mathbf{b}_d^\dagger, \mathcal{F}_M^\dagger]_- | \psi_M^N \rangle = 0$. One can prove a similar relation for $\mathcal{F}_M := (\mathcal{F}_M^\dagger)^\dagger$ and, analogously, for $\mathbf{b}_d^\dagger$ replaced by $\mathbf{b}_d$ (see Appendix G). Since the $\hat{U}_N(\eta)$ operators can be expressed in terms of $\mathbf{b}_d^\dagger$'s, the fractionalization equations (both for quasiparticle as well as quasihole) can be thought of as the dictionary, at the field operator level, for our bosonization. We reiterate that this bosonization within the zero-mode subspace reflects its purely topological character. Indeed, the only Hamiltonian that commutes with the generators of $\mathbf{A}$ is the null operator.

VI. UNIVERSAL EDGE BEHAVIOR

An understanding of the bulk-boundary correspondence in interacting topological matter is a long standing challenge. For FQH fluids, Wen's hypothesis [32] for using Luttinger physics for the edge compounded by further effective edge Hamiltonian descriptions [33, 34] constitutes our best guide for the edge physics. We now advance a conjecture enabling direct analytical calculations. We posit that the asymptotic long-distance behavior of the single-particle edge Green's function may be calculated by evaluating it for the root partition (the DNA) of the corresponding FQH state. As we next illustrate, our computed long-distance behavior shows remarkable agreement with Wen's hypothesis. Our (root pattern) angular momentum basis calculations do not include the effects of boundary confining potentials (if any exist). Most notably, we do not, at all, assume that the FQH edge is a Luttinger liquid or another effective one-dimensional system.

Consider the fermionic Green's function

$$-iG(z, z') = \rho(z, z') = \langle \psi_M^N | \Psi^\dagger(z) \Psi(z') | \psi_M^N \rangle \| \psi_M^N \|^2 \times e^{-\frac{1}{4}(|z|^2 + |z'|^2)}, \quad (31)$$

and coordinates $z = Re^{i\theta}$, $z' = Re^{i\theta'}$, where $R = \sqrt{2(r_{\max} + 1)}$ is the radius of the last occupied orbital and it can be identified with the classical radius of the droplet, i.e. it satisfies $\pi R^2 \cdot \alpha = N$ with $\alpha = N/(r_{\max} + 1)$ being the average density of the (homogeneous) droplet. Then,

$$\rho(z, z') = \frac{e^{-\frac{1}{2}R^2}}{2\pi} \sum_{r=0}^{r_{\max}} \left(\frac{R^2}{2} \right)^r \frac{e^{i(\theta' - \theta)r}}{r!} \frac{\langle \psi_M^N | \bar{c}_r^\dagger \bar{c}_r | \psi_M^N \rangle}{\| \psi_M^N \|^2}. \quad (32)$$

Similarly, the edge Green's function associated with the root partition $|\tilde{\psi}_M^N\rangle$ is

$$\tilde{\rho}(z, z') = \frac{e^{-\frac{1}{2}R^2} e^{iNM(\theta' - \theta)}}{2\pi} \sum_{k=1}^N \left(\frac{R^2}{2} \right)^{(N-k)M} \frac{e^{ikM(\theta - \theta')}}{[(N-k)M]!}, \quad (33)$$

where we used $\langle \tilde{\psi}_M^N | \bar{c}_r^\dagger \bar{c}_r | \tilde{\psi}_M^N \rangle \| \tilde{\psi}_M^N \|^2 = 1$ for $r = 0, M, \dots, (N-1)M$, and 0 otherwise. Thus far, our root partition calculation is exact. We next perform asymptotic analysis. For large k , the largest phase oscillations appear when $\cos(Mk(\theta - \theta')) = (-1)^k$, i.e., for $|\theta - \theta'| = \tilde{\theta}$ near $\pi \frac{1+2l}{M}$ with $l = 0, \dots, m$ and $M = 2m + 1$. This implies that the dominant contributions to the sum originate from small k values. We can then apply Stirling's approximation $[(N-k)M]! \cong \sqrt{\pi R} (R^2/2)^{(N-k)M} e^{-R^2/2}$, where we used $R^2\nu \approx 2N$ (valid since $1 - \nu \ll R^2$) leading to

$$|\tilde{\rho}(z, z')| \cong \frac{1}{4\pi^{3/2}R} \left| \sin \frac{M\tilde{\theta}}{2} \right|. \quad (34)$$

Long distances correspond to $\tilde{\theta}$ near π . As

$$\left| \sin \left(\frac{M\tilde{\theta}}{2} \right) \right| = \left| \sin \left(\frac{\tilde{\theta}}{2} \right) \right|^M - \frac{1}{8} M(M-1)(\tilde{\theta} - \pi)^2 + \mathcal{O}((\tilde{\theta} - \pi)^4), \quad (35)$$

the edge Green function

$$|\tilde{\rho}(\tilde{\theta})| = \frac{1}{4\pi^{3/2}R} \left| \sin \left(\frac{\tilde{\theta}}{2} \right) \right|^M \left(1 + \mathcal{O}((\tilde{\theta} - \pi)^2) \right), \quad (36)$$

or, equivalently, $|\tilde{\rho}(\tilde{\theta})| \propto |z - z'|^{-M}$. This is only valid in the vicinity of $\tilde{\theta} = \pi$ (e.g., demanding the corrections to be $\leq 1\%$, for $M = 3$, restricts us to $[0.96\pi, \pi]$), while Eq. (34) spans a broader range – see Fig. 3. The Green's function was computed by using the tables of characters for permutation groups $S_{N(N-1)}$ for $M = 3$ (up to $N = 8$ and then extrapolating the results), adjusting the method in [35]. The difference between $|\rho|$ and $|\tilde{\rho}|$ vanishes at π as $N^{-1/2}$.

FIG. 3. (Color online.) Edge Green's function corresponding to the Laughlin's (blue line) and the root (orange dotted line) state, for 8 particles. Both $(\sin(\frac{\tilde{\theta}}{2}))^{-3}$ (green dashed line) and $\sin(\frac{3\tilde{\theta}}{2})$ (red dashed line) laws are also shown. The latter is a better approximation of $|\tilde{\rho}|$ in a broader range around π . In the vicinity of blue points Stirling's approximation is valid. Insets: $\log |G(\tilde{\theta})|$ as a function of $\log |\sin(\tilde{\theta}/2)|$ in the range (Right): $[0.6, 1]$ for $\sin(\tilde{\theta}/2)$, for both $N = 8$ (red dashed line) and $N = 7$ (blue line); (Left): $[0.967, 1]$ for $N = 8$ (red dashed line) and $[0.991, 1]$ for $N = 7$ (blue line).

Nevertheless, the long-distance ($\tilde{\theta} \sim \pi$) behavior of the Green's function, in the thermodynamic limit, cannot be reliably determined from small N calculations [36]. For instance, by examining the slope μ of $\log |G(\tilde{\theta})|$ when plotted as a function of $\log |\sin(\tilde{\theta}/2)|$ for $N = 8$ (Fig. 3), we get $\mu = -3.88$ when using the range $[0.967, 1]$ for $\sin(\tilde{\theta}/2)$, while the value for $N = 7$ in the range $[0.991, 1]$ is $\mu = -6$. The deduced numerical value is highly dependent on the range used in the fitting procedure, e.g. for $N = 8$ and the range $[0.6, 1]$ we obtain $\mu = -3.23$

(for linear scale of $\tilde{\theta}$). We established that the asymptotic long-distance behavior of the edge Green's function corresponding to the root state coincides with Wen's conjecture [32].

A. Beyond the LLL

The aforementioned behaviour remains true also beyond the LLL that forms the focus of our work. Indeed, repeating the above calculation when using the DNA [2, 37] of the Jain's 2/5 state, we found $\mu = -3$, in agreement with Wen's hypothesis [32]. In this Jain's state example, our computation captures the (EPP) entanglement structure of the root partition [2] not present in Laughlin states.

In this case we need the exact form of the following orbitals:

$$\psi_{0,r}(z) = \frac{z^r}{\mathcal{N}_{0,r}}, \quad \psi_{1,r}(z) = \frac{z^* z^{r+1} - 2(r+1)z^r}{\mathcal{N}_{1,r}}, \quad (37)$$

with $\mathcal{N}_{0,r} = \sqrt{2\pi 2^r r!}$ and $\mathcal{N}_{1,r} = \sqrt{2\pi 2^{r+2}(r+1)!}$. The fermionic field operator is now $\Psi(z) = \sum_{n,r} \psi_{n,r}(z) \bar{c}_{n,r}$

which leads to the Green's function of the following form:

$$\rho(z, z') = \sum_{n,n'} \sum_{r,r'} \psi_{n,r}^*(z) \psi_{n',r'}(z') \frac{\langle \psi | \bar{c}_{n,r}^\dagger \bar{c}_{n',r'} | \psi \rangle}{\|\psi\|^2} \times e^{-\frac{1}{4}(|z|^2 + |z'|^2)}, \quad (38)$$

where $|\psi\rangle$ is the corresponding ground state. By the angular momentum conservation, $r = r'$ under the above summation, so that

$$\rho(z, z') = \sum_{n,n'} \sum_r \psi_{n,r}^*(z) \psi_{n',r}(z') \frac{\langle \psi | \bar{c}_{n,r}^\dagger \bar{c}_{n',r} | \psi \rangle}{\|\psi\|^2} \times e^{-\frac{1}{4}(|z|^2 + |z'|^2)}, \quad (39)$$

For the “DNA” of the ground state $|\psi\rangle$ we get

$$\tilde{\rho}(z, z') = \sum_{n,n'} \sum_r \psi_{n,r}^*(z) \psi_{n',r}(z') \frac{\langle \text{DNA} | \bar{c}_{n,r}^\dagger \bar{c}_{n',r} | \text{DNA} \rangle}{\|\text{DNA}\|^2} \times e^{-\frac{1}{4}(|z|^2 + |z'|^2)}. \quad (40)$$

The “DNA” of the ground state of the Jain's 2/5 state [2, 37] is of the form $|\text{DNA}\rangle = \prod_{k \geq 0} \widehat{\varphi}_{5k+3} |0\rangle$ with

$$\widehat{\varphi}_r = \alpha_{0,0}(r) \left(\bar{c}_{0,r-1}^\dagger \bar{c}_{0,r+1}^\dagger + \frac{\sqrt{r+2}}{2} \bar{c}_{0,r-1}^\dagger \bar{c}_{1,r+1}^\dagger - \frac{\sqrt{r}}{2} \bar{c}_{1,r-1}^\dagger \bar{c}_{0,r+1}^\dagger \right), \quad (41)$$

where $\alpha_{0,0}$ is an r -dependent factor.

As a result,

$$\begin{aligned} \tilde{\rho}(z, z') &= \frac{e^{-\frac{1}{4}(|z|^2 + |z'|^2)}}{2\pi} \\ &\times \left\{ \sum_{0 \leq 5l+2 \leq r_{\max}} \frac{(z^* z')^{5l+2}}{2^{5l+2} h_l(5l+2)!} [1 + (5l+5)^2 \right. \\ &\quad \left. + \frac{2(5l+3) - |z'|^2}{4} + (2(5l+3) - |z|^2)(5l+3)] \right. \\ &\quad + \sum_{0 \leq 5l+4 \leq r_{\max}} \frac{(z^* z')^{5l+4}}{2^{5l+4} h_l(5l+4)!} [1 + (5l+3)^2 \\ &\quad \left. + \frac{|z'|^2 - 2(5l+5)}{4} + (|z|^2 - 2(5l+5))(5l+5)] \right\}, \end{aligned} \quad (42)$$

where $h_l = 1 + (5l+3)^2 + (5l+5)^2$.

Henceforth, we will focus on points $z = Re^{i\theta}$ and $z' = Re^{i\theta'}$ that lie on the edge. We next discuss the two contributions to $\tilde{\rho}(z, z')$ in Eq. (42).

We start by discussing the contribution to $\tilde{\rho}(z, z')$ of exponent $5l+2$. With the above polar substitution for the boundary points z and z' , this becomes

$$\begin{aligned} &\frac{e^{-\frac{1}{2}R^2}}{2\pi} \sum_{k=1}^{\mathcal{N}} \left(\frac{R^2}{2} \right)^{5(\mathcal{N}-k)+2} \frac{e^{-i(5(\mathcal{N}-k)+2)(\theta-\theta')}}{h_{\mathcal{N}-k}(5(\mathcal{N}-k)+2)!} \\ &\times \left[1 + (5(\mathcal{N}-k) + 5)^2 \right. \\ &\quad \left. + \frac{2(5(\mathcal{N}-k) + 3) - R^2}{4} \cdot (20(\mathcal{N}-k) + 13) \right], \end{aligned} \quad (43)$$

where $\mathcal{N} = \lfloor \frac{1}{5}(\frac{N-1}{\nu} - 2) \rfloor + 1$ with $\nu = \frac{2}{5}$, and we have used the same change of summation index as in the case of the LLL.

Assume now that only small integer k are of relevance in the above summation - we will check validity of this assumption later on. Then using the Stirling approximation, the fact that $N \cong \frac{R^2\nu}{2} \gg 1$ and $\mathcal{N} - k \cong \mathcal{N}$, we get

$$[5(\mathcal{N}-k)]! \cong \sqrt{\pi} R \left(\frac{R^2}{2} \right)^{5(\mathcal{N}-k)+2} e^{-\frac{R^2}{2}}, \quad (44)$$

and, as a result, for the part with the exponent $5l+2$, we end up with

$$\frac{e^{-i5\mathcal{N}(\theta-\theta')}}{2R\pi^{3/2}} \kappa_2(\mathcal{N}, R) \sum_{k=1}^{\mathcal{N}} e^{i(5k-2)(\theta-\theta')}, \quad (45)$$

where $\kappa_2(\mathcal{N}, R)$ is a certain rational function in $\mathcal{N}$.

Similar to the above, for the part having $5l+4$ as an exponent, we get

$$\frac{e^{-i5\hat{\mathcal{N}}(\theta-\theta')}}{2R\pi^{3/2}} \kappa_4(\hat{\mathcal{N}}, R) \sum_{k=1}^{\hat{\mathcal{N}}} e^{i(5k-4)(\theta-\theta')} \quad (46)$$

with a rational, in $\hat{\mathcal{N}} = \lfloor \frac{1}{5}(\frac{N-1}{\nu} - 4) \rfloor + 1$, function $\kappa_4(\hat{\mathcal{N}}, R)$.

Next, we observe that for large radius R we can without the loss of generality assume that $\hat{\mathcal{N}} = \mathcal{N}$, so that $e^{-i5\hat{\mathcal{N}}(\theta-\theta')}$'s lead to an irrelevant global factor since at the very end we will be interested in the absolute value of the Green's function. We now argue that in the thermodynamic limit, $\frac{\kappa_2}{\kappa_4} \rightarrow 1$. Indeed, since $R^2 \sim \frac{2N}{\nu}$ and $\nu = \frac{2}{5}$ we have $R^2 \sim 5N$. Moreover, we know that $\mathcal{N} \sim \frac{N}{5\nu} \sim \frac{N}{2}$. Hence $R^2 \sim 10\mathcal{N}$. This shows that $\frac{\kappa_2}{\kappa_4} \rightarrow 1$. Furthermore, this also shows that, in the limit $\mathcal{N} \rightarrow \infty$, we have for $\tilde{\rho}$:

$$\frac{e^{-i5\mathcal{N}(\theta-\theta')}}{4R\pi^{3/2}} \left(\sum_{k=1}^{\infty} e^{i(5k-2)(\theta-\theta')} + \sum_{k=1}^{\infty} e^{i(5k-4)(\theta-\theta')} \right), \quad (47)$$

since both κ_2 and κ_4 tends to $\frac{1}{2}$ in this limit.

We next explain why the assumption $\mathcal{N} - k \cong \mathcal{N}$ is valid. Towards this end, one needs to verify that the only k values that matter are the small ones, i.e., that $\cos((5k-2)(\theta-\theta')) \cong (-1)^k$. This is indeed true (in particular around $\theta - \theta' = \pi$, which is exactly our point of interest). Analogous considerations work also for the term of exponent $5l+4$. Therefore, the problem reduces to the evaluation of

$$\begin{aligned} &\frac{1}{4R\pi^{3/2}} \left| \sum_{k=1}^{\infty} e^{i(5k-2)(\theta-\theta')} + e^{i(5k-4)(\theta-\theta')} \right| \\ &= \frac{1}{4R\pi^{3/2}} \left| \frac{ie^{i\frac{3}{2}(\theta-\theta')} \cos(\theta-\theta')}{\sin\left(\frac{5(\theta-\theta')}{2}\right)} \right|. \end{aligned} \quad (48)$$

To ascertain the long distance behavior, we examine angular differences $|\theta - \theta'| = \tilde{\theta}$ close to π , where this asymptotically becomes

$$\frac{1}{4R\pi^{3/2}} \left| \frac{1 - 2\sin^2\left(\frac{\tilde{\theta}}{2}\right)}{\sin\left(\frac{5\tilde{\theta}}{2}\right)} \right| \cong \frac{1}{2R\pi^{3/2}} \frac{1}{\left| \sin\left(\frac{\tilde{\theta}}{2}\right) \right|^3}. \quad (49)$$

The above derived result is in agreement with Wen's conjecture [32] for the FQH Jain's $2/5$ liquid.

VII. CONCLUSIONS

Our approach sheds light on the elusive exact mechanism underlying fractionalized quasielectron excitations in FQH fluids (and formalizes the fractionalization of quasihole excitations [38]). By solving an outstanding open problem [22, 23], our construct underscores the importance of a systematic operator based microscopic approach complementing Laughlin's original quasiparticle wave function Ansatz. The algebraic structure of the LLL is deeply tied to the Newton-Girard relations. We have shown that there are numerous pairs of "dual" operators that are linked to each other via these relations

(including operators associated with the Witt algebra). The Newton-Girard relations typically convert a local operator into a non-local “dual” operator. A main message of the present work is that “derivative operations” on FQH vacua do not lead to exact quasiparticle excitations. The precise mechanism leading to charge fractionalization consists of the joint process of flux and (original) particle insertions. In other words, an elementary fusion channel of quasiholes and an electron generates a quasielectron excitation. For instance, to generate one quasielectron excitation in a $\nu = 1/M$ Laughlin fluid one needs to insert $M - 1$ fluxes, in an $(N - 1)$ -electron fluid, and fuse them with an additional electron. A fundamental difference between quasihole and quasiparticle excitations can be traced back to their M -clustering properties [2]. While quasiholes preserve the M -clustering property of the incompressible (ground state) fluid, quasiparticle states breaks it down. This is at the origin of the asymmetry between these two kinds of excitations. Equivalently, while quasihole wave functions sustain a (local) plasma analogy this is not the case for quasielectrons.

We explicitly *constructed the quasiparticle (quasielectron) wave function*. Our found fusion mechanism of quasiparticle generation is not only the mathematically exact (for an arbitrary number of particles) field-theoretic operator procedure but it is also behind the exact analytic computation of other quasiparticle properties, such as the Berry connection or charge density. This is a truly unprecedented remarkable result that we have numerically confirmed via detailed Monte Carlo simulations.

Intriguingly, within our field-theoretic framework, we

find that the Laughlin state is a condensate of a non-local Read type operator. Our approach allows for a constructive (zero-energy) subspace bosonization of the full two-dimensional system that further evinces the non-local topological character of the problem and, once again, cements links to Read’s operator. The constructed Klein operator associated with this angular momentum (and flux counting) root state based bosonization scheme is none other than Read’s non-local operator. We suspect that this angular momentum (flux counting) based mapping might relate to real-space flux attachment (and attendant Chern-Simons) type bosonization schemes [30, 39]. Lastly, we illustrated how the long-distance behavior of edge excitations associated with the root partition component (DNA) of the *bulk* FQH ground state may be readily calculated. Strikingly, the asymptotic long-distance edge physics derived in this manner matches Wen’s earlier hypothesis in the cases that we tested. This agreement hints at a possibly general powerful *computational recipe for predicting edge physics*.

VIII. ACKNOWLEDGEMENTS

We thank J. Jain, H. Hansson, and S. Simon for useful comments. G.O. acknowledges support from the US Department of Energy grant DE-SC0020343. A.B. acknowledges the Polish-US Fulbright Commission for the possibility of visiting Indiana University, Bloomington, during the Fulbright Junior Research Award scholarship. Work by A.S. has been supported by the National Science Foundation Grant No. DMR-2029401.

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Appendix A: Newton-Girard dual pairs

Let $d \geq 0$, and consider operators $\mathcal{O}_d$ and e_d , defined for both fermions and bosons. Since they are operator analogues of the symmetric polynomials $p_d(z) = \sum_{i=1}^N z_i^d$ and $s_d(z) = \sum_{1 \leq i_1 \leq \dots \leq i_d \leq N} z_{i_1} \dots z_{i_d}$, respectively, one may expect that the operator version of the Newton-Girard relation, known from the theory of symmetric polynomials, holds also in this case:

$$de_d + \sum_{k=1}^d (-1)^k \mathcal{O}_k e_{d-k} = 0. \quad (\text{A1})$$

Indeed, this was already proven in [9]. Then, $\mathcal{O}_d$ operators can be expressed recursively in terms of e_d 's:

$$\mathcal{O}_d = (-1)^{d-1} \left(de_d + \sum_{k=1}^{d-1} (-1)^k \mathcal{O}_k e_{d-k} \right), \quad (\text{A2})$$

so that any operator $\mathcal{O}_d$ can be represented in terms of e_d 's only, and vice versa. Moreover, this can be achieved by a (many-variable) polynomial relation. In order to write an explicit expression for $\mathcal{O}_d$ in terms of e_d 's operators we start with the Newton-Girard relation, (A1), and for any positive d we get

$$\begin{cases} e_1 = \mathcal{O}_1, \\ 2e_2 = e_1 \mathcal{O}_1 - \mathcal{O}_2, \\ \vdots \\ de_d = e_{d-1} \mathcal{O}_1 - e_{d-2} \mathcal{O}_2 + \dots + (-1)^d e_1 \mathcal{O}_{d-1} \\ \quad \quad \quad + (-1)^{d-1} \mathcal{O}_d. \end{cases} \quad (\text{A3})$$

Since, again by the Newton-Girard relation, every coefficient which is a multiple of e_k is also in the algebra $\mathcal{A}$ of $\mathcal{O}_d$ operators, this leads to a system of d linear equations in the commutative algebra $\mathcal{A}$. We

would like to solve this linear system for unknowns $(\mathcal{O}_1, -\mathcal{O}_2, \mathcal{O}_3, \dots, (-1)^d \mathcal{O}_{d-1}, \mathcal{O}_d)$. By Cramer's theorem applied to the ring $\mathbf{A}$, the necessary and sufficient condition for the existence of the unique solution is the invertibility in $\mathbf{A}$ of the determinant of the matrix encoding this system. In our case this is a lower triangular matrix with diagonal $(1, \dots, 1, (-1)^{d-1})$, so its determi-

nant, equal to $(-1)^{d-1}$, is clearly an invertible element in $\mathbf{A}$. Therefore, again by Cramer's theorem, we get (see also [42, p. 28]):

$$\begin{aligned} \mathcal{O}_d &= (-1)^{d-1} \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & e_1 \\ e_1 & 1 & 0 & \dots & 0 & 2e_2 \\ e_2 & e_1 & 1 & \dots & 0 & 3e_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{d-2} & e_{d-3} & e_{d-4} & \dots & 1 & (d-1)e_{d-1} \\ e_{d-1} & e_{d-2} & e_{d-3} & \dots & e_1 & de_d \end{pmatrix} \\ &= \det \begin{pmatrix} e_1 & 1 & 0 & 0 & \dots & 0 \\ 2e_2 & e_1 & 1 & 0 & \dots & 0 \\ 3e_3 & e_2 & e_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ de_d & e_{d-1} & e_{d-2} & e_{d-3} & \dots & e_1 \end{pmatrix}. \end{aligned}$$

Expanding the above determinant one can finally find

$$\begin{aligned} \mathcal{O}_d &= (-1)^d d \sum_{\substack{r_1+2r_2+\dots+dr_d=d \\ r_1, \dots, r_d \geq 0}} \frac{(r_1 + \dots + r_d - 1)!}{r_1! \dots r_d!} \\ &\quad \times (-1)^{r_1 + \dots + r_d} e_1^{r_1} \dots e_d^{r_d} \\ &= (-1)^d d \sum_{k=1}^d \frac{1}{k} \hat{B}_{d,k}(-e_1, \dots, -e_{d-k-1}), \end{aligned} \quad (\text{A4})$$

where $\hat{B}_{n,k}(x_1, \dots, x_{n-k+1}) = \frac{k!}{n!} B_{n,k}(1!x_1, 2!x_2, \dots, (n-k+1)!x_{n-k+1})$ and $B_{n,k}$ is the (exponential) Bell polynomial [43].

Let now f_d be the operator representing, in the second quantization picture, the elementary symmetric polynomial with the z_i variables substituted by partial derivatives, $s_d(\partial_z)$, and let $\mathcal{Q}_d$ be the second-quantized version of the operator $p_d(\partial_z)$. Since the Newton-Girard relation holds for $s_d(\partial_z)$ and $p_d(\partial_z)$, an analogous one is expected for the pair of operators f_d and $\mathcal{Q}_d$.

First, we claim that the second-quantized version f_d of the differential operator $s_d(\partial_z)$ is

$$f_d = \frac{1}{d!} \sum_{r_1, \dots, r_d \geq 0} r_1 \dots r_d \bar{a}_{r_1-1}^\dagger \dots \bar{a}_{r_d-1}^\dagger \bar{a}_{r_d} \dots \bar{a}_{r_1}. \quad (\text{A5})$$

We start with the fermionic case. For $\beta_1 > \dots > \beta_N \geq 0$ consider the monic monomial $r_\beta(Z_N) = z_1^{\beta_1} \dots z_N^{\beta_N}$, where $\beta = (\beta_1, \dots, \beta_N)$ and $Z_N = (z_1, \dots, z_N)$. r_β is a homogeneous polynomial of degree $|\beta| = \beta_1 + \dots + \beta_N$. Let $R_\beta(Z_N) = N! \hat{\mathcal{A}} r_\beta(Z_N)$ be defined by the total antisymmetrization $\hat{\mathcal{A}}$ of r_β in variables forming Z_N . This

polynomial corresponds, in second quantization, to the operator $\bar{c}_\beta^\dagger = \bar{c}_{\beta_1}^\dagger \dots \bar{c}_{\beta_N}^\dagger$, the Slater determinant.

Notice that for $d > N$, $f_d \bar{c}_\beta = 0$, and similarly $s_d(\partial_z) R_d(Z_N) = 0$. Therefore, below we implicitly assume that d is at most N . We notice that

$$\begin{aligned} s_d(\partial_z) R_\beta(Z_N) &= N! \hat{\mathcal{A}} s_d(\partial_z) r_\beta(Z_N) \\ &= N! \hat{\mathcal{A}} \mathcal{S}_{(i_1, \dots, i_d)} \left(\partial_{z_{i_1}} \dots \partial_{z_{i_d}} z_1^{\beta_1} \dots z_N^{\beta_N} \right) \\ &= N! \hat{\mathcal{A}} \mathcal{S}_{(i_1, \dots, i_d)} \left(\beta_{i_1} \dots \beta_{i_d} z_1^{\beta_1-1} \dots z_{i_1-1}^{\beta_{i_1}-1} \dots z_{i_d}^{\beta_{i_d}-1} \dots z_N^{\beta_N} \right) \\ &= \mathcal{S}_{(i_1, \dots, i_d)} \left(\beta_{i_1} \dots \beta_{i_d} R_{(\beta_1, \dots, \beta_{i_1}-1, \dots, \beta_{i_d}-1, \dots, \beta_N)}(z) \right). \end{aligned} \quad (\text{A6})$$

Here $\mathcal{S}_{(i_1, \dots, i_d)}$ denotes the total symmetrization in $i_1, \dots, i_d$ with $i_j \neq i_k$ for every pair (j, k) of indices $j, k \in \{1, \dots, d\}$ such that $j \neq k$.

Observe that when acting by the operator f_d on the state $\bar{c}_\beta^\dagger |0\rangle$ the symmetrization $\mathcal{S}_{(i_1, \dots, i_d)}$ is explicitly involved. Moreover, each term under the symmetrization produces

$$\beta_{i_1} \dots \beta_{i_d} \bar{c}_{\beta_{i_1}-1}^\dagger \dots \bar{c}_{\beta_{i_d}-1}^\dagger \bar{c}_{\beta_{i_d}} \dots \bar{c}_{\beta_{i_1}} \bar{c}_{\beta_1}^\dagger \dots \bar{c}_{\beta_N}^\dagger |0\rangle. \quad (\text{A7})$$

To finish the proof it is enough to show that when placing the fermionic operators in the canonical order no sign is generated out of the reordering permutation. By counting signs appearing in the reordering, one can easily check that this is indeed the case.

For bosons, in the definition of $R_\beta(Z_N)$ we replace the total antisymmetrization by a symmetrization, which corresponds to having a permanent instead of a Slater determinant. The reasoning in (A6) follows with only minor adjustments. Now $\mathcal{S}_{(i_1, \dots, i_d)}$ denotes the total symmetrization in $i_1, \dots, i_d$ without the additional assumption that the indices are pairwise distinct, contrary to the fermionic case. Obviously, no sign counting in the reordering is needed in the bosonic case.

In order to justify that $\mathcal{Q}_d = \sum_{r>0} r(r-1) \dots (r-d) \bar{a}_{r-d}^\dagger \bar{a}_r$ is the second-quantized version of the $p_d(\partial_z)$ polynomial, we will show that this operator satisfies the Newton-Girard equation. First notice that mimicking the computation from [9] one can easily prove that

$$[f_d, \bar{a}_r^\dagger]_- = r \bar{a}_{r-1}^\dagger f_{d-1}, \quad [f_d, \bar{a}_r]_- = -(r+1) f_{d-1} \bar{a}_{r-1}, \quad (\text{A8})$$

for any $d > 0$ and $r \geq 0$. Moreover, we have

$$f_d = \frac{1}{d} \sum_{r \geq 0} r \bar{a}_{r-1}^\dagger f_{d-1} \bar{a}_r. \quad (\text{A9})$$

Next, iterating this equation sufficiently many times we

find ($f_1 = \mathcal{Q}_1$)

$$\begin{aligned}
df_d &= \sum_{r \geq 0} r \left(f_{d-1} \bar{a}_{r-1}^\dagger - (r-1) \bar{a}_{r-2}^\dagger f_{d-2} \right) \bar{a}_r \\
&= f_{d-1} \mathcal{Q}_1 - \sum_{r \geq 0} r(r-1) \bar{a}_{r-2}^\dagger f_{d-2} \bar{a}_r \\
&= f_{d-1} \mathcal{Q}_1 - \sum_{r \geq 0} r(r-1) \left(f_{d-2} \bar{a}_{r-2}^\dagger - (r-2) \bar{a}_{r-3}^\dagger f_{d-3} \right) \bar{a}_r \\
&= f_{d-1} \mathcal{Q}_1 + (-1) f_{d-2} \mathcal{Q}_2 \\
&\quad + (-1)^2 \sum_{r \geq 0} r(r-1)(r-2) \bar{a}_{r-3}^\dagger f_{d-3} \bar{a}_r \\
&= \dots = - \sum_{k=1}^d (-1)^k f_{d-k} \mathcal{Q}_k.
\end{aligned} \tag{A10}$$

By induction, each f_d commutes with every $\mathcal{Q}_k$, so that we end up with the Newton-Girard relation for the operators f_d and $\mathcal{Q}_k$:

$$df_d + \sum_{k=1}^d (-1)^k \mathcal{Q}_k f_{d-k} = 0. \tag{A11}$$

1. The Witt algebra

For $d \geq 0$ we consider also the differential operators $\ell_d = - \sum_{i=1}^N z_i^{d+1} \partial_{z_i}$. They satisfy $[\ell_d, \ell_{d'}]_- = (d-d') \ell_{d+d'}$, i.e. form the positive Witt algebra $\mathfrak{W}_+$. Another set of operators satisfying the same algebra is given by $\hat{\ell}_d = - \sum_{r \geq 0} r \bar{a}_{r+d}^\dagger \bar{a}_r$. We claim that $\hat{\ell}_d$ is the second quantized version of ℓ_d . Indeed, we will show it explicitly for fermions and left the adjustments in the bosonic case to the reader as they are exactly the same as we discussed above in the case of the f_d operator. We have $\ell_d r_\beta(Z_N) = - \sum_{i=1}^N \beta_i z_1^{\beta_1} \dots z_i^{\beta_i+d} \dots z_N^{\beta_N}$. Since ℓ_d is symmetric under transpositions (i, j) for any two indices i, j under the summation, from the very definition of the totally antisymmetrized polynomial $R_\beta(Z_N)$ we have also that $\ell_d R_\beta(Z_N) = N! \hat{\ell}_d r_\beta(Z_N)$, and, as a result, we end up with $\ell_d R_\beta(Z_N) = - \sum_{j=1}^N \beta_j R_{(\beta_1, \dots, \beta_{j-1}, \beta_j+d, \beta_{j+1}, \dots, \beta_N)}(Z_N)$. On the other hand, for the Slater determinant corresponding to the polynomial $R_\beta(Z_N)$ we have $\hat{\ell}_d \bar{c}_\beta^\dagger |0\rangle = \sum_{j=1}^N \beta_j \bar{c}_{\beta_1}^\dagger \dots \bar{c}_{\beta_{j-1}}^\dagger \bar{c}_{\beta_j+d}^\dagger \bar{c}_{\beta_{j+1}}^\dagger \dots \bar{c}_{\beta_N}^\dagger |0\rangle$, where we have used the fact that the total number of transpositions required to properly order the creation operators is even. It remains to notice that the Slater determinant $\bar{c}_{\beta_1}^\dagger \dots \bar{c}_{\beta_{j-1}}^\dagger \bar{c}_{\beta_j+d}^\dagger \bar{c}_{\beta_{j+1}}^\dagger \dots \bar{c}_{\beta_N}^\dagger$ corresponds to the polynomial $R_{(\beta_1, \dots, \beta_{j-1}, \beta_j+d, \beta_{j+1}, \dots, \beta_N)}(Z_N)$.

Appendix B: Properties of the quasihole operator

The quasihole operator $\hat{U}_N(\eta) = \sum_{d=0}^N (-\eta)^{N-d} e_d$ satisfies the following important relations:

$$\hat{U}_N(\eta) \bar{a}_r^\dagger = -\eta \bar{a}_r^\dagger \hat{U}_N(\eta) + \bar{a}_{r+1}^\dagger \hat{U}_N(\eta), \tag{B1}$$

$$\bar{a}_r \hat{U}_N(\eta) = -\eta \hat{U}_N(\eta) \bar{a}_r + \hat{U}_N(\eta) \bar{a}_{r-1}. \tag{B2}$$

Since the first of those relations was already demonstrated in [9], we present here, for completeness, the proof of the second one. This follows as a result of a straightforward computation:

$$\begin{aligned}
\hat{U}_N(\eta) \bar{a}_r &= (-\eta)^{\hat{N}} \sum_{d \geq 0} (-\eta)^{-d} e_d \bar{a}_r \\
&= (-\eta)^{\hat{N}} \sum_{d \geq 0} (-e_{d-1} \bar{a}_{r-1} + \bar{a}_r e_d) \\
&= -(-\eta)^{\hat{N}} (-\eta)^{-1} \sum_{d \geq 0} (-\eta)^{-d-1} e_{d-1} \bar{a}_{r-1} \\
&\quad + \bar{a}_r (-\eta)^{-1} \sum_{d \geq 0} (-\eta)^{-d} e_d \\
&= \frac{1}{\eta} \hat{U}_N(\eta) \bar{a}_{r-1} - \frac{1}{\eta} \bar{a}_r \hat{U}_N(\eta).
\end{aligned} \tag{B3}$$

Appendix C: Laughlin sequence states

We start with the following

Lemma 1. The M th power of the quasihole operator is given by

$$\hat{U}_N(z)^M = (-1)^{MN} \sum_{r \geq 0} z^r \hat{S}_{MN-r}^\#, \tag{C1}$$

where $\hat{S}_l^\# = (-1)^l \sum_{n_1 + \dots + n_M = l} e_{n_1} \dots e_{n_M}$ for $l > 0$, and 0 otherwise, as defined in [18, 44].

Before we proceed with the proof we remark that our M corresponds to $M+1$ in [18, 44], and we are working with the disk geometry as opposed to the infinitely thick cylinder geometry used therein. Moreover, we use the phase $(-1)^{MN}$ which is appropriate both for fermions and bosons.

Proof. First, we notice that

$$\begin{aligned}
\hat{U}_N(z)^M &= \left(\sum_{d=0}^N (-z)^{N-d} e_d \right)^M \\
&= \sum_{k_0 + \dots + k_N = M} (-z)^{Nk_0 + (N-1)k_1 + \dots + 1 \cdot k_{N-1} + 0 \cdot k_N} \\
&\quad \times \binom{M}{k_0, \dots, k_N} e_0^{k_0} \dots e_N^{k_N}.
\end{aligned} \tag{C2}$$

Next, we observe that

$$\begin{aligned}
& Nk_0 + (N-1)k_1 + \dots + 1 \cdot k_{N-1} + 0 \cdot k_N \\
&= N(k_0 + \dots + k_N) - (0 \cdot k_0 + 1 \cdot k_1 + \dots + N \cdot k_N) \\
&= MN - (0 \cdot k_0 + 1 \cdot k_1 + \dots + N \cdot k_N),
\end{aligned} \tag{C3}$$

where we use the constraint $k_0 + \dots + k_N = M$ imposed under the summation. Therefore, this leads to

$$\begin{aligned}
\hat{U}_N(z)^M &= \sum_{k_0 + \dots + k_N = M} (-z)^{NM - (0 \cdot k_0 + \dots + N \cdot k_N)} \\
&\quad \times \binom{M}{k_0, \dots, k_N} e_0^{k_0} \dots e_N^{k_N}
\end{aligned} \tag{C4}$$

Continuing along these lines we get

$$\begin{aligned}
\hat{U}_N(z)^M &= \sum_{r \geq 0} \sum_{k_0 + \dots + k_N = M} \delta_{r, MN - (0 \cdot k_0 + \dots + N \cdot k_N)} (-z)^r \\
&\quad \times \binom{M}{k_0, \dots, k_N} e_0^{k_0} \dots e_N^{k_N} \\
&= \sum_{r \geq 0} z^r (-1)^r \sum_{k_0 + \dots + k_N = M} \delta_{MN - r, 0 \cdot k_0 + \dots + N \cdot k_N} \\
&\quad \times \binom{M}{k_0, \dots, k_N} e_0^{k_0} \dots e_N^{k_N} \\
&= \sum_{r \geq 0} z^r (-1)^r \sum_{n_1 + \dots + n_M = MN - r} e_{n_1} \dots e_{n_M} \\
&= (-1)^{MN} \sum_{r \geq 0} z^r \hat{S}_{MN-r}^\#
\end{aligned} \tag{C5}$$

□

From the above Lemma,

$$K_{M,N} = (-1)^{MN} \sum_{r \geq 0} \bar{a}_r^\dagger \hat{S}_{MN-r}^\# \tag{C6}$$

after integration over z , and together with the recurrence relation [18, 44],

$$|\psi_M^N\rangle = \frac{(-1)^{M(N-1)}}{N} \sum_{r \geq 0} \bar{a}_r^\dagger \hat{S}_{M(N-1)-r}^\# |\psi_M^{N-1}\rangle, \tag{C7}$$

results in

$$|\psi_M^N\rangle = \frac{1}{N} K_{M,N-1} |\psi_M^{N-1}\rangle. \tag{C8}$$

We now prove by induction that $|\psi_M^N\rangle = \frac{1}{N!} \mathcal{K}_M^N |0\rangle$.

Indeed, we have

$$\begin{aligned}
\frac{1}{N!} \mathcal{K}_M^N |0\rangle &= \left(\frac{1}{N} \mathcal{K}_M \right) \left(\frac{1}{(N-1)!} \mathcal{K}_M^{N-1} |0\rangle \right) \\
&= \frac{1}{N} \mathcal{K}_M |\psi_M^{N-1}\rangle \\
&= \sum_{N' \geq 0} \frac{1}{N} \int D[z] \Lambda^\dagger(z) \hat{U}_{N'}(z)^M |\psi_M^{N'}\rangle \langle \psi_M^{N'} | \psi_M^{N-1} \rangle \\
&= \frac{1}{N} K_{M,N-1} |\psi_M^{N-1}\rangle = |\psi_M^N\rangle.
\end{aligned} \tag{C9}$$

Appendix D: Quasihole operator fractionalization

One of the consequence of the Lemma 1 is the operator relation satisfied by the quasihole,

$$\Lambda(\eta) |\psi_M^{N+1}\rangle = \hat{U}_N(\eta)^M |\psi_M^N\rangle. \tag{D1}$$

This can be proven with the help of Lemma 1 and the following identity

$$\Lambda(z) |\psi_M^{N+1}\rangle = (-1)^{MN} \sum_{r \geq 0} z^r \hat{S}_{MN-r}^\# |\psi_M^N\rangle. \tag{D2}$$

Appendix E: The quasiparticle (quasielectron) problem

Here we elaborate on the quasiparticle definition within the framework of second quantization. First, we stress that the operator f_d differs from $e_d^\dagger$ exactly by the factors $r_1 \dots r_d$ in each term under the summation. This is one of the reasons behind the differences between the various quasiparticle (quasielectron) wave function proposals. Let us concentrate for convenience on the quasielectron case. Interestingly, one can check by an explicit computation that both these operations produce exactly the same (up to a global factor) polynomials out of the ones representing Laughlin 3- and 4-particle states. Moreover, they also agree (again, up to a global factor) with the action of $(\bar{c}_0^\dagger e_2^{-1} \bar{c}_0)^3$ and $(\bar{c}_0^\dagger e_3^{-1} \bar{c}_0)^3$, respectively [45]. The first difference can be observed for $N = 5$, where in $\Psi^\dagger(0) |\psi_3^4\rangle$ the highest occupied orbital is equal to nine, while the action by f_4^3 on $|\psi_3^5\rangle$ produces terms with the twelfth orbital occupied. This issue is caused by the lack of presence of the $\bar{c}_0$ and $\bar{c}_0^\dagger$ operators. Moreover, the numerical coefficients also differ as a result of the presence of additional $r_1 \dots r_d$ factors in the second-quantized version of the $s_d(\partial_z)$ operator.

1. Comparison with other proposals

Our approach is field-theoretic and, therefore, allows an algebraic proof of operator fractionalization. There

are, however, other first-quantization approaches proposing corrections to Laughlin's quasielectron state. It is fair, then, to compare all these proposals. To this end, in the following we'll consider the case of a quasielectron localized at the center of a disk ($\eta = 0$) in a $\nu = \frac{1}{3}$ QH fluid.

a. Laughlin's quasielectron

The (holomorphic part of the) wave function for Laughlin's quasielectron is

$$\Psi_0^L(Z_N) = \prod_{i=1}^N (2\partial_{z_i}) \prod_{j < k}^N (z_j - z_k)^3. \quad (\text{E1})$$

In particular, for $N = 3$ this leads to the second-quantized form

$$|\Psi_0^L(Z_3)\rangle = (-288\bar{c}_5^\dagger\bar{c}_1^\dagger\bar{c}_0^\dagger + 720\bar{c}_4^\dagger\bar{c}_2^\dagger\bar{c}_0^\dagger - 2880\bar{c}_3^\dagger\bar{c}_2^\dagger\bar{c}_1^\dagger)|0\rangle. \quad (\text{E2})$$

The angular momenta counting leads to the conclusion that Laughlin's quasielectron operator cannot describe fractionalization. Indeed, we can observe this effect directly here by applying the operator $\prod_{i=1}^3 (2\partial_{z_i})$ three times on the 3-particle Laughlin state. In this case we get immediately zero, which is obviously not the required result, $\Psi^\dagger(0)|\psi_3^2\rangle$. A modification of Laughlin's original proposal could, in principle (due to the angular momenta match), work, but as already mentioned, the difference can be easily found already in case of $N = 5$.

b. Approach based on conformal block

The (holomorphic part) of the quasielectron wave function resulting from the CFT approach [20] is given by

$$\Psi_0^{\text{CFT}}(Z_N) = \sum_{i=1}^N (-1)^i \prod_{\substack{j < k \\ j, k \neq i}}^N (z_j - z_k)^3 \partial_{z_i} \prod_{l \neq i}^N (z_l - z_i)^2, \quad (\text{E3})$$

since a single CFT quasielectron localized at zero agrees with Jain's approach [21] based on composite fermions [20, p. 37 – discussion below Eqn. (70)].

In this case, again for $N = 3$, we get the second-quantized form

$$|\Psi_0^{\text{CFT}}(Z_3)\rangle = (5\bar{c}_5^\dagger\bar{c}_1^\dagger\bar{c}_0^\dagger - 10\bar{c}_4^\dagger\bar{c}_2^\dagger\bar{c}_0^\dagger + 40\bar{c}_3^\dagger\bar{c}_2^\dagger\bar{c}_1^\dagger)|0\rangle. \quad (\text{E4})$$

One can also try to compare with the wave function proposed in [22, Eq. (11)] for a quasielectron in a state with angular momentum one, in order to match the angular momenta counting. In such a case we end up with

$$|\Psi_0^{\text{CFT}; (1)}(Z_3)\rangle = (2\bar{c}_5^\dagger\bar{c}_2^\dagger\bar{c}_0^\dagger - 10\bar{c}_4^\dagger\bar{c}_3^\dagger\bar{c}_0^\dagger + 10\bar{c}_4^\dagger\bar{c}_2^\dagger\bar{c}_1^\dagger)|0\rangle. \quad (\text{E5})$$

c. Our quasielectron

In first quantization (the holomorphic part of) our quasielectron wave function is

$$\Psi_0^{\text{qe}}(Z_N) = (-1)^N \sum_{i=1}^N (-1)^i \prod_{\substack{j < k \\ j, k \neq i}}^N (z_j - z_k)^3 \prod_{l \neq i}^N z_l^2. \quad (\text{E6})$$

Indeed, from our second-quantized prescription for the quasielectron we deduce that

$$\Psi_0^{\text{qe}}(Z_N) = \sum_{j=1}^N \frac{\psi_3^N(z_1, \dots, z_{j-1}, 0, z_{j+1}, \dots, z_N)}{z_1 \dots \widehat{z_j} \dots z_N}, \quad (\text{E7})$$

where $\widehat{z_j}$ means that the variable z_j is not present in the product. This expression immediately leads to (E6).

The action of our quasielectron operator, $\bar{c}_0^\dagger e_2^\dagger \bar{c}_0$, on the Laughlin's 3-particle state $|\psi_3^3\rangle$ produces the second-quantized state

$$|\Psi_0^{\text{qe}}(Z_3)\rangle = (\bar{c}_5^\dagger\bar{c}_2^\dagger\bar{c}_0^\dagger - 3\bar{c}_4^\dagger\bar{c}_3^\dagger\bar{c}_0^\dagger)|0\rangle, \quad (\text{E8})$$

which differs from all the proposals above. Moreover, we notice that applying the operator $\bar{c}_0^\dagger e_2^\dagger \bar{c}_0$ three times on the state $|\psi_3^3\rangle$ give us $-3\bar{c}_2^\dagger\bar{c}_1^\dagger\bar{c}_0^\dagger|0\rangle$, which is exactly the action of $\bar{c}_0^\dagger = \Psi^\dagger(0)$ on Laughlin state $|\psi_3^2\rangle$, as required by fractionalization.

Moreover, at general quasielectron location η , we can present the quasielectron wavefunction in the following mixed first/second quantized form:

$$\Psi_\eta^{\text{qe}}(Z_N) = \Lambda(\eta)^\dagger \frac{1}{\prod_{i=1}^{N-1} (z_i - \eta)} \Lambda(\eta) \psi_M^N(Z_N). \quad (\text{E9})$$

In particular, the squeezing arguments of the main text imply that despite the rational factors, the resulting wave function is still analytic up to Gaussian factors, i.e., is in the LLL.

Appendix F: Numerical computation of fractional charge

Using the method presented in [28] we can calculate the fractional charge of the quasielectron:

$$\delta\rho_{\text{qp}} = 2\pi \int_0^{r_{\text{cut-off}}} [\rho_{\text{qp}}(r) - \rho_L(r)] r dr, \quad (\text{F1})$$

with the cut-off radius $r_{\text{cut-off}}$ chosen as discussed in the main text. The results for $N = 400$ particles (obtained using Monte Carlo data, with 400 grid points) are shown in Fig. 4. Numerical integration is performed with the help of the trapezoidal method using the raw Monte Carlo data.

FIG. 4. (Color online.) The charge $\delta\rho_{\text{qp}}$ of the quasielectron (located at $\eta = 0$) as a function of the cut-off radius $r_{\text{cut-off}}$ (blue points), with the range of $r_{\text{cut-off}}$ chosen so that boundary effects do not affect the result. Saturation of $\delta\rho_{\text{qp}}$ is clearly visible for $r_{\text{cut-off}} \geq 12\ell$. The mean value for the charge in the range $r_{\text{cut-off}} \in [12\ell, 30\ell]$ is then computed: $\delta\rho_{\text{qp}} = 0.3330(30)e$ (solid line). Monte Carlo simulations averaged over more than 2×10^{10} equilibrated configurations.

Appendix G: Klein factors for the subspace bosonization

We have defined $F_{M,N}^\dagger = \frac{1}{N+1}K_{M,N}$ and $\mathcal{F}_M^\dagger = \sum_{N \geq 0} F_{M,N}^\dagger = \sum_{N \geq 0} F_{M,N}^\dagger |\psi_M^N\rangle \langle \psi_M^N|$. Since $[\mathcal{O}_d, \bar{a}_r^\dagger]_- =$

$\bar{a}_{r+d}^\dagger$ we immediately get

$$[\mathcal{O}_d, F_{M,N}^\dagger]_- = \frac{(-1)^{MN}}{N+1} \sum_{r \geq 0} \bar{a}_{r+d}^\dagger \hat{S}_{MN-r}^\# \quad (\text{G1})$$

The operator $\bar{a}_{r+d}^\dagger \hat{S}_{MN-r}^\#$, for any $d > 0$, annihilates given state or changes its angular momentum by $MN + d$, so as a result, $\langle \psi_M^{N+1} | [\mathcal{O}_d, F_{M,N}^\dagger]_- | \psi_M^N \rangle = 0$, since the N -particle $\frac{1}{M}$ -Laughlin state has an angular momentum equal to $J = \frac{MN(N-1)}{2}$. Next, we also have

$$\begin{aligned} & \langle \psi_M^{N+1} | [\mathcal{O}_d, \mathcal{F}_M^\dagger]_- | \psi_M^N \rangle \\ &= \sum_{N' \geq 0} \langle \psi_M^{N+1} | F_{M,N}^\dagger [\mathcal{O}_d, |\psi_M^{N'}\rangle \langle \psi_M^{N'}|]_- | \psi_M^N \rangle \\ &= \sum_{N' \geq 0} \langle \psi_M^{N+1} | F_{M,N}^\dagger \mathcal{O}_d |\psi_M^{N'}\rangle \langle \psi_M^{N'} | \psi_M^N \rangle \\ &\quad - \sum_{N' \geq 0} \langle \psi_M^{N+1} | F_{M,N}^\dagger |\psi_M^{N'}\rangle \langle \psi_M^{N'} | \mathcal{O}_d | \psi_M^N \rangle \\ &= \langle \psi_M^{N+1} | \mathcal{O}_d F_{M,N}^\dagger | \psi_M^N \rangle \| \psi_{M,N} \|^2 \\ &\quad - \sum_{N' \geq 0} \langle \psi_M^{N+1} | \psi_M^{N'+1} \rangle \langle \psi_M^{N'} | \mathcal{O}_d | \psi_M^N \rangle \\ &= \langle \psi_M^{N+1} | \mathcal{O}_d | \psi_M^{N+1} \rangle \| \psi_M^N \|^2 - \langle \psi_M^N | \mathcal{O}_d | \psi_M^N \rangle \| \psi_M^{N+1} \|^2 \\ &= 0, \end{aligned} \quad (\text{G2})$$

for any $N \geq 0$ and any $d > 0$. Similar computations can be performed also for the commutators with $\mathcal{O}_d^\dagger$ as well as for the operator $\mathcal{F}_M = (\mathcal{F}_M^\dagger)^\dagger$.