Algebraic action on Diminished chords

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Research Article

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Algebraic action on Diminished chords

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Abstract

Diminished chords are always ignored by musicians when they learn music theory. Musicians and music theorists always study the properties and relations of Major and Minor chords or Consonant chords. In Neo-Riemannian theory of TI and PLR groups, he suggest a way of applying Dihedral group, D_{12} to study the relation of Consonant chords. In this paper we will introduce some concepts of group theory to understand the algebraic relationship between the Diminished chords.

Keywords: Dihedral group, neo-Riemannian theory, TI group, Diminished chords.

1 Group theory

1.1 Dihedral group

A dihedral group is the group of symmetries of a regular polygon, which includes rotations through an angle and reflections about some axes. In algebra, D_n refers to the dihedral group.

A regular polygon with n sides has $2n$ different symmetries, i.e. n rotational symmetries and n reflection symmetries. Usually, we take $n \geq 3$ here. The associated rotations and reflections make up the dihedral group D_n .

Figure 1: The above figure shows the axes of symmetry

If n is odd, each axis of symmetry is the lines that pass through each vertex and the midpoint of its opposite edge. If n is even, then the axes of symmetry is the lines that pass through a vertex and its opposite vertex and also the lines that pass through the midpoint of an edge to the midpoint of the opposite edge of the polygon. In either case, there are n axes of symmetry.

Reflecting in one axis of symmetry followed by reflecting in another axis of symmetry produces a rotation through twice the angle between the axes.

Figure 2: The above figure illustrates the symmetries of the rotations of a triangle.

1.2 Group action

Let $(G, \cdot)$ be a group and let S be a non-empty set. We say that G *acts on* S if we are given a function

$$\begin{aligned} G \times S &\rightarrow S \\ (g, s) &\mapsto g \star s \end{aligned}$$

such that;

- (i) $e \star s = s$ for all $s \in S$, where e is the identity element in G
- (ii) $(g_1 \cdot g_2) \star s = g_1 \star (g_2 \star s)$ for all $g_1, g_2 \in G$ and $s \in S$.

Given an action of G on S we can define the following sets.

Let $s \in S$. Define the *orbit* of s as,

$$\text{Orb}(s) = \{g \star s : g \in G\}.$$

Note that $\text{Orb}(s)$ is a subset of S , equal to all the images of the element s under the action of the elements of the group G . We also define the *stabilizer* of s as,

$$\text{Stab}(s) = \{g \in G : g \star s = s\}.$$

Note that $\text{Stab}(s)$ is a subset of G . In fact, it is a subgroup. Note that every element $g \in G$ defines a function $S \rightarrow S$ by $s \mapsto gs$.

2 Music theory

Music is basically the combination of sounds. Sounds are produced by vibrations that propagates in a medium, different frequency of vibration produce differnt sounds. For example,a middle C note in a piano produces a frequency of 262Hz, whereas a middle F note produces a frequency of 349.228Hz. In music theory, an Octave is the interval above a musical note which produces twice the frequency of that lower note. For example, the note A_4 produces a frequency of 440Hz,while the note A_5 produces a frequency of 880Hz and A_3 produces the frequency of 220Hz.

There are 12 notes in an octave. The interval from a note to the next note is called a Semi-tone.

A semi-tone higher to a note is denoted by the symbol $\sharp$, pronounced as Sharp.A semi-tone lower note is denoted by the symbol $\flat$. For example, $C^\sharp$ denote the note which is immediately next to the C note and $D^\flat$ denotes the note which is immediately before D.

Note: $C^\sharp$ and $D^\flat$ denotes the same musical note.

Note: The C in the table represents the 'Middle C' of the piano and the note A is in a standard tuning at 440Hz.

Notes	C	$C^\sharp/D^\flat$	D	$D^\sharp/E^\flat$	E	F	$F^\sharp/G^\flat$	G	$G^\sharp/A^\flat$	A	$A^\sharp/B^\flat$	B
Pitch (Hz)	261.6	277.1	293.6	311.1	329.6	349.2	369.9	392	415	440	466.1	493.8

Table 1: The above table shows the note names and pitches of the notes in a piano.

Figure 3: The above figure shows the names of the piano keys

2.1 Chords

When two or more musical notes are played together it's called a Chord. There are a large family of Chords which produce different kind of harmony and emotion. A chord consisting of exactly three notes is called a Triad. Further chords can be classified as consonant chords and dissonant chords. Generally, Minor triads and major triads are Consonant whereas Diminished triads are dissonant. When a Root note, its minor third and perfect fifth are played together it makes a Minor Triad; when a Root note, its major third and perfect fifth are played together it makes a Major Triad; when a Root note, minor third and its diminished fifth. The following table shows the notes of the three types of triads:

Root	Minor chord	Major chord	Diminished chord
C	$C, D^\sharp, G$	C, E, G	$C, D^\sharp, F^\sharp$
$C^\sharp$	$C^\sharp, E, G^\sharp$	$C^\sharp, F, G^\sharp$	$C^\sharp, E, G$
D	D, F, A	$D, F^\sharp, A$	$D, F, G^\sharp$
$D^\sharp$	$D^\sharp, F^\sharp, A^\sharp$	$D^\sharp, G, A^\sharp$	$D^\sharp, F^\sharp, A$
E	E, G, B	$E, G^\sharp, B$	$E, G, A^\sharp$
F	$F, G^\sharp, C$	F, A, C	$F, G^\sharp, B$
$F^\sharp$	$F^\sharp, A, C^\sharp$	$F^\sharp, A^\sharp, C^\sharp$	$F^\sharp, A, C$
G	$G, A^\sharp, D$	G, B, D	$G, A^\sharp, C^\sharp$
$G^\sharp$	$G^\sharp, B, D^\sharp$	$G^\sharp, C, D^\sharp$	$G^\sharp, B, D$
A	A, C, E	$A, C^\sharp, E$	$A, C, D^\sharp$
$A^\sharp$	$A^\sharp, C^\sharp, F$	$A^\sharp, D, F$	$A^\sharp, C^\sharp, E$
B	$B, D, F^\sharp$	$B, D^\sharp, F^\sharp$	B, D, F

3 Algebraic action

Since it is useful to translate pitch class to integer mod12, we will consider 0 to be C and construct the pitch class as in the following table

Note	C	$C^\sharp$	D	$D^\sharp$	E	F	$F^\sharp$	G	$G^\sharp$	A	$A^\sharp$	B
Mod12	0	1	2	3	4	5	6	7	8	9	10	11

For instance, the number 0 refers to the note C , since $0=0\text{mod}12$ and the number 15 refers to the note $D^\sharp$, since $15=3\text{mod}12$. Thus, we can map every note in music to an element in $\mathbb{Z}_{12}$.

Now we can define a Triad as a set of three elements in $\mathbb{Z}_{12}$. For instance we take a C-major triad $\{C, E, G\}$ as $\{0, 4, 7\}$.

In general any Major triad is defined as $\{x, x + 4, x + 7\}$, any minor triad is defined as $\{x, x + 3, x + 7\}$ and any Diminished triad is defined as $\{x, x + 3, x + 6\}$, where x is the root note of the Triad. Note that this is an ordered set.

3.1 TI group

The Transposition is defined mathematically as a function from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$ such that,

$$T_n(x) := (x + n) \bmod 12$$

where, $n \in \mathbb{Z}_{12}$

and The Inversion is defined as a function from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$ such that,

$$I_n(x) := (-x + n) \bmod 12$$

where, $n \in \mathbb{Z}_2$

These transposition and inversion forms a group under the operation Composition defined as,

$$T_m \circ T_n = T_{(m+n) \bmod 12}$$

$$T_m \circ I_n = I_{(m+n) \bmod 12}$$

$$I_m \circ T_n = I_{(m-n) \bmod 12}$$

$$I_m \circ I_n = T_{(m-n) \bmod 12}$$

For example:

$$T_1(\{0, 3, 6\}) = \{T_1(0), T_1(3), T_1(6)\} = \{1, 4, 7\}$$

$$I_0(\{0, 3, 6\}) = \{I_0(0), I_0(3), I_0(6)\} = \{0, 9, 6\} = \{6, 9, 0\}$$

ie. T_1 maps C diminished triad to $C^\sharp$ diminished triad whereas I_0 maps C diminished triad to $F^\sharp$ diminished triad.

3.2 Musical clock

The idea of converting Musical notes to elements in $\mathbb{Z}_{12}$ help us to construct a Polygon with 12 vertices, where each vertex corresponds to a musical note. Thus we can now use Dihedral group to study the Algebraic structure of Diminished chords. The following figure shows us how the musical notes and elements in $\mathbb{Z}_{12}$ are related with the 12-gon.

Figure 4: The Musical clock

Using this Musical clock we can give an algebraic structure to all the Chords by connecting the vertices corresponding to the notes in the chord with a straight line. The following figures show the structure of some chords.

Figure 5: C-major triad

Figure 6: C-minor triad

Figure 7: C-diminished triad

3.3 The action

We define the set of diminished triads as $\mathbb{S} = \{(x, x+3, x+6) \mid x \in \mathbb{Z}_{12}\}$. Let the dihedral group $(D_{12}, \star)$ act on the set of diminished triads. The action is defined as a map from

$$D_{12} \times \mathbb{S} \rightarrow \mathbb{S}$$

$$(r^i f^j, s) \mapsto r^i f^j \cdot s = r^i f^j(s)$$

where, $i \in \mathbb{Z}_{12}$, $j \in \mathbb{Z}_2$, r is the rotation and f is the reflection. Then,

1. $\exists e \in D_{12}$ such that $(e, s) = (r^0 f^0, s) = r^0 f^0 \cdot s = s, \forall s \in \mathbb{S}$
2. $g_1 \cdot (g_2 \cdot s) = g_1 \cdot (g_2(s)) = g_1(g_2(s)) = g_1 \star g_2(s), \forall g_1, g_2 \in D_{12}, s \in \mathbb{S}$
3. $r^{12} = f^2 = e$ and $f r f = r^{-1}$

Theorem 3.1. *The TI group $(TI, \circ)$ is isomorphic to a Dihedral group $(D_{12}, \star)$*

Proof. $TI = \langle T_n, I_m \rangle$, where $T_{12} = I_2 = e$ and $I_1 T_1 I_1 = T_{-1}$

$D_{12} = \langle r^n, f^n \rangle$, where $r^{12} = f^2 = e$ and $f r f = r^{-1}$

Let $\phi : TI \rightarrow D_{12}$ such that $\phi(T_n) = r^n$ and $\phi(I_n) = f^n$

Since ϕ maps the generator of TI group to generator of D_{12} and they have same orders, the mapping ϕ is an isomorphism. $\square$

3.4 Orbits and Stabilizers

The orbit of $s \in \mathbb{S}$ is $\{g \cdot s \mid g \in TI\}$.

Since the action of T_n can map any triad to another triad, the orbit of any $s \in \mathbb{S}$ is $\mathbb{S}$.

The stabilizer of s is $\{g \in TI \mid g \cdot s = s\}$.

Clearly, T_0 stabilizes every element of $\mathbb{S}$ since $T_0(x, x+3, x+6) = (x, x+3, x+6)$ and there is an other stabilizer for every $s \in \mathbb{S}$ which is a reflection about the axis passing through the vertices $(x+3) \bmod 12$ and $(x+9) \bmod 12$.

Corollary 3.1. *The action of TI group on the set of Diminished triads is Transitive.*

4 Geometric structure

To study the geometric structure, we divide the set of Diminished triads into three families, where any two Triads in a family share two notes.

For example: C diminished triad and $F^\sharp$ diminished triad belong to the same family.

The following table shows the Families of diminished triads.

Family	Triads			
1	C	$D^\sharp$	$F^\sharp$	A
2	$C^\sharp$	E	G	$A^\sharp$
3	D	F	$G^\sharp$	B

By connecting the common notes in the triads we can clearly see that each family of diminished triads is Tetrahedral ie. Triangular Pyramid shaped. The following figures show the Geometric structure of Diminished triads. In the following figures each face corresponds to a unique Diminished triad.

Figure 8: Family 1

Figure 9: Family 2

Figure 10: Family 3

4.1 Relation and Jump

Now we define a function from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$ as

$$R_n(x) = (x + 3n) \bmod 12$$

This new function is called as *Relation* since it takes a diminished chord to another diminished chord in its family.

Example 4.1. $R_1(\{0, 3, 6\}) = \{R_1(0), R_1(3), R_1(6)\} = \{3, 6, 9\}$

$R_2(\{2, 5, 8\}) = \{R_2(2), R_2(5), R_2(8)\} = \{8, 11, 2\}$

In other words, R_1 takes C -diminished triad to $D^\sharp$ -diminished triad and R_2 takes D -diminished triad to $G^\sharp$ -diminished triad

Next we define another function from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{12}$ as

$$J(x) = \begin{cases} x + 1 & \text{if } x \bmod 3 = 0, 1 \\ x - 2 & \text{if } x \bmod 3 = 2 \end{cases}$$

This new function is called as *Jump* since it takes a diminished chord to a diminished chord in another family.

Example 4.2. $J(\{0, 3, 6\}) = \{J(0), J(3), J(6)\} = \{1, 4, 7\}$

$J(\{8, 11, 2\}) = \{J(8), J(11), J(2)\} = \{6, 9, 0\}$

In other words, J takes C -diminished triad to $C^\sharp$ -diminished triad and $G^\sharp$ -diminished triad to $F^\sharp$ -diminished triad

Proposition 4.1. The set of functions JR forms an abelian group under the operation ‘composition’.

Proof. The set JR has the following elements, $JR = \{R_0, R_1, R_2, R_3, J^1, J^2, R_1J^1, R_1J^2, R_2J^1, R_2J^2, R_3J^1, R_3J^2\}$

Here any element is in the form R_nJ^m , where $n \in \mathbb{Z}_4$ and $m \in \mathbb{Z}_3$. The function J^0 is composing J zero times and the function J^3 is same as the function R_0

Let R_nJ^m, R_pJ^q be any two elements in the set JR . Now $R_nJ^m \circ R_pJ^q = R_{n+p}J^{m+q}$ Since $n, p \in \mathbb{Z}_4, n + p \in \mathbb{Z}_4$. So, $R_{n+p} \in JR$. Also $J^{m+q} \in \{J^1, J^2, R_0\}$ since $m, p \in \{1, 2, 3\}$ Thus the set JR is closed under composition.

Let R_nJ^m, R_pJ^q, R_aJ^b be any three elements from the set JR .

$$R_nJ^m \circ (R_pJ^q \circ R_aJ^b) = R_nJ^m \circ (R_{p+a}J^{q+b}) = R_{n+p+a}J^{m+q+b}$$

$$(R_nJ^m \circ R_pJ^q) \circ R_aJ^b = (R_{n+p}J^{m+q}) \circ R_aJ^b = R_{n+p+a}J^{m+q+b}$$

So associativity holds.

The identity element $e = R_0$

Claim: $R_0 \circ R_nJ^m = R_nJ^m$, where $n \in \mathbb{Z}_4$ and $m \in \mathbb{Z}_3$

Let $R_nJ^m(x) = y$. Now $R_0(y) = (y + 3(0)) \bmod 12 = y \Rightarrow R_0 \circ R_nJ^m = R_nJ^m$

Similarly, $R_nJ^m \circ R_0 \Rightarrow R_nJ^m(R_0(x)) = R_nJ^m(x)$ Thus identity exists in the set JR .

Let R_nJ^m be an arbitrary element in the set JR . Then the element $R_{4-n}J^{3-m}$ is the inverse of R_nJ^m , since $R_nJ^m \circ R_{4-n}J^{3-m} = R_0$. Thus inverse exists for every element in the set JR under the operation composition.

Let R_nJ^m and R_pJ^q be any two elements in the set JR . Then,

$$R_nJ^m \circ R_pJ^q = R_{n+p}J^{m+q}.$$

$$R_pJ^q \circ R_nJ^m = R_{p+n}J^{q+m} = R_{n+p}J^{m+q}$$

Thus the set JR forms an abelian group under the operation composition. □

4.2 The orders of elements in JR group

The orders of each element in JR group is,

$$\begin{aligned}
\langle (R_0) \rangle &= \{(R_0)\} \\
\langle (R_1) \rangle &= \{(R_1), (R_2), (R_3), (R_0)\} \\
\langle (R_2) \rangle &= \{(R_2), (R_0)\} \\
\langle (R_3) \rangle &= \{(R_3), (R_2), (R_1), (R_0)\} \\
\langle (J^1) \rangle &= \{(J^1), (J^2), (R_0)\} \\
\langle (J^2) \rangle &= \{(J^2), (J^1), (R_0)\} \\
\langle (R_1 J^1) \rangle &= \{(R_1 J^1), (R_2 J^2), (R_3), (J^1), (R_1 J^2), (R_2), (R_3 J^1), (J^2), (R_1), (R_2 J^1), (R_3 J^2), (R_0)\} \\
\langle (R_1 J^2) \rangle &= \{(R_1 J^2), (R_2 J^1), (R_3), (J^2), (R_1 J^1), (R_2), (R_3 J^2), (J^1), (R_1), (R_2 J^2), (R_3 J^1), (R_0)\} \\
\langle (R_2 J^1) \rangle &= \{(R_2 J^1), (J^2), (R_2), (J^1), (R_2 J^2), (R_0)\} \\
\langle (R_2 J^2) \rangle &= \{(R_2 J^2), (J^1), (R_2), (J^2), (R_2 J^1), (R_0)\} \\
\langle (R_3 J^1) \rangle &= \{(R_3 J^1), (R_2 J^2), (R_1), (J^1), (R_3 J^2), (R_2), (R_1 J^1), (J^2), (R_3), (R_2 J^1), (R_1 J^2), (R_0)\} \\
\langle (R_3 J^2) \rangle &= \{(R_3 J^2), (R_2 J^1), (R_1), (J^2), (R_3 J^1), (R_2), (R_1 J^2), (J^1), (R_3), (R_2 J^2), (R_1 J^1), (R_0)\}
\end{aligned}$$

Element	Order
R_0	1
R_1	4
R_2	2
R_3	4
J^1	3
J^2	3
$R_1 J^1$	12
$R_1 J^2$	12
$R_2 J^1$	6
$R_2 J^2$	6
$R_3 J^1$	12
$R_3 J^2$	12

Since JR group is generated by a single element it is a cyclic group.

4.3 Subgroups of JR group

From the fundamental theorem of cyclic groups, for a finite cyclic group of order n, every subgroup's order is a divisor of n, and there is exactly one subgroup for each divisor.

Order	Subgroup
1	$\{R_0\}$
2	$\{R_0, R_2\}$
3	$\{R_0, J^1, J^2\}$
4	$\{R_0, R_1, R_2, R_3\}$
6	$\{R_0, R_2, J^1, J^2, R_2 J^1, R_2 J^2\}$
12	$\{R_0, R_1, R_2, R_3, J^1, J^2, R_1 J^1, R_1 J^2, R_2 J^1, R_2 J^2, R_3 J^1, R_3 J^2\}$

4.4 The action of JR on the set of Diminished chords

The set of diminished triads, $\mathbb{S} = \{(x, x+3, x+6), x \in \mathbb{Z}_{12}\}$. The action is defined as a map from

$$JR \times \mathbb{S} \rightarrow \mathbb{S}$$

$$(R_n J^m, s) \rightarrow (R_n J^m(s)) = (R_n J^m(x), R_n J^m(x+3), R_n J^m(x+6))$$

where, $n \in \mathbb{Z}_4$, $m \in \mathbb{Z}_3$, $s \in \mathbb{S}$ such that;

(i) $R_0(s) = s$, for all $s \in \mathbb{S}$

(ii) $(R_n J^m \circ R_p J^q)(s) = R_n J^m(R_p J^q(s))$, for all $R_n J^m, R_p J^q \in JR$ and $s \in \mathbb{S}$

4.4.1 Orbits

For any $s \in \mathbb{S}$,

$$Orb(s) = \{R_n J^m(s) : R_n J^m \in JR\}$$

$Orb(s) = \mathbb{S}$, for all $s \in \mathbb{S}$

Thus, the JR group acts transitively on the set of diminished triads.

4.4.2 Stabilizer

$$Stab(s) = \{R_n J^m \in JR : R_n J^m(s) = s\}$$

Clearly, R_0 is the stabilizer of this action.

4.5 The span of diminished triads by the action of R

Figure 11: The function R moves the chords clockwise in the above circle

4.6 The span of diminished triads by the action of J

(a) Family 1

(b) Family 2

(c) Family 3

(d) Family 3

Figure 12: The function J moves the chords clockwise in the above circle

4.7 Structure

Figure 13: The complete structure

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