Integral equation for a solitary wave

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Research Article

Keywords: Incompressible inviscid fluid, Solitary wave, Nonlinear integral equation, The Galerkin method

Posted Date: February 4th, 2022

DOI: <https://doi.org/10.21203/rs.3.rs-1312275/v1>

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Abstract The problem is formulated as a boundary-value problem for a strip in the complex potential plane and converted to a boundary-value problem for a half-plane by conformal mapping. The solution is obtained using a Cauchy type integral for the density of which a nonlinear integral equation is derived. Its solution is found with the Galerkin method and the Newton–Raphson technique. The calculated results are compared with the experimental data and the calculations by other researchers. The lower limit of the speed of a solitary wave is found.

Keywords Incompressible inviscid fluid · Solitary wave · Nonlinear integral equation · The Galerkin method

Statements and Declarations The work submitted for publication did not receive financial support from anyone. No researchers other the author participated in it.

Software appendices to the paper with instructions on how to use them are available at:

<https://drive.google.com/drive/folders/1gNnRHqMZ1xEeiZltODz7CBr4cCAHDHV?usp=sharing>

These software applications allow you to get all the numerical results given in the paper.

1 Introduction

The discovery of the solitary wave by Russell [1] was the beginning of theoretical and experimental studies of this phenomenon. The theory was developed in two directions. One of them is the expansion of all functions in series in a small parameter. The first such studies were carried out by Boussinesq [2], Rayleigh [3], Korteweg & de Vries [4]. Their results were refined by McCowan [5], Weinstein [6], Packham [7], Hunt [8], Long [9], Laitone [10], Lenau [11], Fenton [12], Longuet-Higgins [13], Longuet-Higgins & Fenton [14], Hunter & Vanden-Broeck [15], Pennell & Su [16], Pennell [17], Wu, Kao & Zhang [18], Clamond & Dutykh [19], Dutykh & Clamond [20], and also by Duan et al. [21]. Since the convergence of the series in the small parameter is not obvious, attempts were made to prove the existence of the solution for the solitary wave problem and thereby substantiate the mentioned solutions as the series. The proof was presented by Lavrentiev [22], Friedrichs & Hyers [23] and in final form by Amick & Toland [24].

Another direction of research is to reduce the problem's solution to solving some nonlinear integral equation. Such an equation for finite-amplitude waves on a liquid-layer's surface was derived by Nekrasov [25]. However, it does not suit for solitary wave modelling. Milne-Thomson succeeded in modifying Nekrasov's approach and obtaining an integral equation for the solitary wave [26, 27]. But this equation is very complicated. A simpler integro-differential equation was found by Byatt-Smith [28] who applied Fourier's integral transformation. It is shown in [29] that the results of solving Byatt-Smith's equation agree with the results obtained

by Longuet-Higgins & Fenton [14]. A system of integral equations is presented by Hunter & Vanden-Broeck [15]. They have considered the stream in the complex potential plane. The integro-differential equation different from Byatt-Smith's equation was found by Evans & Ford [30]. They used Green's theorem to solve the Laplace equation. Thus the equations derived by now are the integral equation derived by Milne-Thomson [26, 27] (very complicated), the system of equations [15], and also the integro-differential equations derived by Byatt-Smith [28] and Evans & Ford [30].

In this paper the solution of the solitary wave problem is reduced to solving a comparatively simple integral equation. Its deriving relies on the conformal mapping of a half-plane on the strip that is the region of the liquid flow in the complex potential plane. The liquid flow velocity is determined using a Cauchy type integral. The boundary conditions for the velocity field lead to the nonlinear integral equation for the ordinate of the flow region's boundary contour. The equation can be linearized, which makes it possible to determine the minimum possible Froude number.

2 Formulation of the problem

The motion of a surface gravity wave in incompressible inviscid liquid is considered. The problem is solved in two-dimensional formulation. The wave speed is assumed to be constant. With respect to the observer at rest, the wave moves from right to left with speed c . Let us introduce a coordinate system moving together with the wave [27]. In this coordinate system, the flow is stationary. The stream region (Fig. 1) is a layer of infinite length and variable depth

$$H(x) = H_0 + h(x); \quad x \in (-\infty, \infty), \quad (1)$$

where H_0 is the undisturbed liquid layer depth, $h(x)$ is the liquid free surface elevation. The value of H_0 is preset and the function $h(x)$ is determined by solving the problem.

Fig. 1 Solitary wave profile: the stream is symmetric relative to the y -axis

Since the liquid is inviscid and incompressible, the following relations are valid [27]:

$$v_x = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}; \quad v_y = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}, \quad (2)$$

where $v_x(x, y)$, $v_y(x, y)$ are the fluid velocities along the abscissa and ordinate axes, $\varphi(x, y)$ is the velocity potential, and $\psi(x, y)$ is the stream function. Both of these functions satisfy the Laplace equation. Therefore, it is advisable to use a complex variable $z = x + iy$, where i is the unit imaginary number. It can be shown [27] that the complex potential $\chi = \varphi + i\psi$ and the complex conjugate velocity $\bar{v} = v_x - iv_y$ are the holomorphic functions of the variable z . In this case, Eq. (2) are recorded as

$$\frac{d\chi}{dz} = \bar{v}. \quad (3)$$

Along with Eq. (3), the Bernoulli equation is valid for the fluid flow [27]:

$$p + \frac{\rho}{2}(v_x^2 + v_y^2) + \rho gy = C, \quad (4)$$

where $p(x, y)$ is the pressure in the liquid, ρ is liquid's density, g is the acceleration of gravity, C is a constant.

Let us consider the problem's boundary conditions. At infinity, the flow is unperturbed:

$$\lim_{x \rightarrow \pm\infty} h(x) = 0; \quad \lim_{x \rightarrow \pm\infty} v_x(x, y) = c; \quad \lim_{x \rightarrow \pm\infty} v_y(x, y) = 0. \quad (5)$$

The boundary condition of non-leakage is set on the line AB (the channel bottom):

$$v_y(x, -H_0) = 0. \quad (6)$$

The pressure in the liquid in the contour CD (the flow free surface) must be equal to the known external pressure p_0 :

$$p(x, h(x)) = p_0 = \text{const}. \quad (7)$$

The constant C in the Bernoulli equation (4) is determined from the conditions at infinity (5). Thus we have:

$$C = p_0 + \frac{\rho c^2}{2} \quad (8)$$

The exposed problem formulation coincides with the conventional formulation of the solitary wave problem [27].

3 Conformal mapping

The formulated problem belongs to the class of problems with an unknown boundary. One of the methods for solving such problems [27] consists in going to the complex potential plane, that is, where the stream region boundaries are known. Since the considered problem is stationary, both the line AB and the line CD are streamlines. This means [27] that along these lines the stream function is constant. The stream function is determined up to a constant [27], and therefore it is possible without loss of generality to assume that $\psi = 0$ is fulfilled on the line AB , where the equality $y = -H_0$ holds. The value of ψ on the line CD is found from conditions at infinity (5) and Eq. (2) as

$$x \rightarrow \pm\infty: \quad \frac{\partial\psi}{\partial y} = c \quad \Rightarrow \quad \psi(y) = (y + H_0)c \quad \Rightarrow \quad \psi(0) = cH_0. \quad (9)$$

As proved by Toland [31], the inequality $v_x > 0$ is valid everywhere in the flow domain. Therefore, the potential φ is a monotone increasing function of the variable x . It appears that the stream region in the complex potential plane is a strip (Fig. 2). If we manage to find a function $\bar{v}(\chi)$ that is holomorphic in this strip, then from Eq. (3) we obtain a dependence $z(\chi)$, which is, essentially, the solution of the problem posed. The function $\bar{v}(\chi)$ must meet boundary conditions. On the contour $A'B'$, in accordance with Eq. (6), we have:

$$\psi = 0: \quad \text{Im} \bar{v} = 0. \quad (10)$$

Fig. 2 The stream region in the complex potential plane is the strip

The boundary condition for the contour $C'D'$ is derived by combining Eq. (4), (7), and (8):

$$\psi = cH_0: \quad |\bar{v}|^2 = v_x^2 + v_y^2 = c^2 - 2gh. \quad (11)$$

Let us transform the stream region from the strip to the upper half-plane (Fig. 3). The function performing the respective conformal mapping is known [32] and recorded as:

$$\chi = \frac{cH_0}{\pi} \ln \zeta. \quad (12)$$

Fig. 3 The stream region in the $\zeta = \xi + i\eta$ plane is the upper half-plane

Using Levi-Civita's approach [27], we introduce a new holomorphic function:

$$f(\zeta) = \ln \frac{\bar{v}}{c} = \ln r + i\theta; \quad r = \left| \frac{\bar{v}}{c} \right| = \frac{\sqrt{v_x^2 + v_y^2}}{c}; \quad \begin{cases} \cos \theta = v_x / \sqrt{v_x^2 + v_y^2} \\ \sin \theta = v_y / \sqrt{v_x^2 + v_y^2} \end{cases}. \quad (13)$$

We arrive at the following boundary value problem. It is necessary to find a function $f(\zeta)$ that is holomorphic in the upper half-plane including the point at infinity. Its values along the real axis are given:

$$\begin{aligned} \eta = 0; \quad \xi \in (-\infty, 0): \quad \operatorname{Re} f(\xi) &= \frac{1}{2} \ln(1 - 2gh/c^2); \\ \eta = 0; \quad \xi \in (0, \infty): \quad \operatorname{Im} f(\xi) &= 0 \end{aligned} \quad (14)$$

The function $h(\xi)$ in Eq. (14) is unknown. After solving the formulated boundary value problem, the equation for its definition can be found from Eq. (3).

4 Solution of the boundary value problem

The holomorphic function according to its values on a certain contour L can be found using a Cauchy type integral [27, 33]. In this case, the contour L is real axis $\eta = 0$. The plus sign is used to denote the vicinity of the contour L on the left, where $\eta > 0$, whereas the minus sign is used to denote the vicinity on the right, where $\eta < 0$.

Let us introduce an auxiliary function

$$\Phi(\zeta) = \frac{f(\zeta)}{\zeta^{1/2}}. \quad (15)$$

The point $\zeta = 0$ is the branch point of the function $\zeta^{1/2}$. Therefore, this point is the source of a branch cut of the function $\zeta^{1/2}$, and this cut is extended to infinity. In this case, it is the negative section of the real axis. The limiting values of the function $\zeta^{1/2}$ on the real axis approached from the upper half-plane are recorded as:

$$\eta = 0: \quad \zeta_+^{1/2}(\xi) = \begin{cases} \sqrt{\xi}, & \xi \in (0, \infty) \\ i\sqrt{-\xi}, & \xi \in (-\infty, 0) \end{cases}. \quad (16)$$

Considering Eq. (16), the boundary values of the function $\Phi(\zeta)$ are reduced to the boundary values only for its imaginary part:

$$\begin{aligned} \eta = 0; \quad \xi \in (-\infty, 0): \quad \operatorname{Im} \Phi_+(\xi) &= -\frac{\ln(1 - 2gh/c^2)}{2\sqrt{-\xi}}; \\ \eta = 0; \quad \xi \in (0, \infty): \quad \operatorname{Im} \Phi_+(\xi) &= 0 \end{aligned} \quad (17)$$

Let us represent the function $\Phi(\zeta)$ as a Cauchy type integral:

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_L \frac{u(t) dt}{t - \zeta} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(t) dt}{t - \zeta}, \quad (18)$$

where the Cauchy type integral density $u(\xi)$ is the function that must be found from the boundary conditions. According to Plemelj's formulae [27, 33], we record the function $\Phi(\zeta)$ on the contour L (real axis) as:

$$\eta = 0; \quad \xi \in (-\infty, \infty): \quad \Phi_+(\xi) = \frac{u(\xi)}{2} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(t) dt}{t - \xi}. \quad (19)$$

The denominator of the integral in the right part of Eq. (19) becomes zero at $t = \xi$. The value of the integral used hereinafter is its Cauchy principal value [27, 33].

The comparison between Eq. (17) and (19) results to the following definition of the function $u(\xi)$:

$$u(\xi) = \begin{cases} -\frac{i \ln(1 - 2gh/c^2)}{\sqrt{-\xi}}, & \xi \in (-\infty, 0) \\ 0, & \xi \in (0, \infty) \end{cases}. \quad (20)$$

That said boundary conditions (17) are met. It follows from Eq. (13) and (15) that

$$\bar{v}(\zeta) = c \exp \left[-\frac{\zeta^{1/2}}{2\pi} \int_{-\infty}^0 \frac{\ln(1 - 2gh/c^2) dt}{\sqrt{-t}(t - \zeta)} \right]. \quad (21)$$

On purpose to link the complex conjugate velocity $\bar{v}$ with the complex coordinate z we exclude the complex potential χ from Eq. (3) with the help of Eq. (12). The result is:

$$\frac{dz}{d\zeta} = \frac{cH_0}{\pi\zeta\bar{v}} = \frac{H_0}{\pi\zeta} \exp \left[\frac{\zeta^{1/2}}{2\pi} \int_{-\infty}^0 \frac{\ln(1 - 2gh/c^2) dt}{\sqrt{-t}(t - \zeta)} \right]. \quad (22)$$

If the wave profile function $h(t)$ is known, Eq. (21), (22) allow, essentially, finding the relation $\bar{v}(z)$ and then the pressure in the each point of the flow with Bernoulli's equation (4).

5 Equation for the wave profile

Let us record Eq. (22) on the line $C''D''$ (Fig. 3) corresponding to the flow free surface CD (Fig. 1). Considering the Plemelj formulae [27, 33], as well as Eq. (16) and (20), we obtain ($\xi < 0$):

$$\begin{aligned} \lim_{\eta \rightarrow +0} \zeta^{1/2} \int_{-\infty}^0 \frac{\ln(1 - 2gh/c^2) dt}{\sqrt{-t}(t - \zeta)} &= i \sqrt{-\xi} \left[\frac{\pi i}{\sqrt{-\xi}} \ln(1 - 2gh/c^2) + \int_{-\infty}^0 \frac{\ln(1 - 2gh/c^2) dt}{\sqrt{-t}(t - \xi)} \right] \Rightarrow \\ \frac{dz}{d\xi} &= \frac{H_0}{\pi\xi\sqrt{1 - 2gh/c^2}} \exp \left[\frac{i\sqrt{-\xi}}{2\pi} \int_{-\infty}^0 \frac{\ln(1 - 2gh/c^2) dt}{\sqrt{-t}(t - \xi)} \right] \end{aligned} \quad (23)$$

The complex coordinate z on the line CD is recorded as

$$z = x + iy = x + ih. \quad (24)$$

Separating the real and imaginary parts in Eq. (23), we obtain a system of equations for determining the coordinates of the points of the wave profile CD :

$$\begin{cases} \frac{dx}{d\xi} = \frac{H_0}{\pi\xi\sqrt{1 - 2gh/c^2}} \cos \omega \\ \frac{dh}{d\xi} = \frac{H_0}{\pi\xi\sqrt{1 - 2gh/c^2}} \sin \omega \end{cases}; \quad \omega(\xi) = \frac{\sqrt{-\xi}}{2\pi} \int_{-\infty}^0 \frac{\ln(1 - 2gh/c^2) dt}{\sqrt{-t}(t - \xi)}; \quad \xi \in (-\infty, 0). \quad (25)$$

There is only one integral equation (more precisely, an integro-differential) in this system. It is the second equation. After solving it, the function $x(\xi)$ for the wave profile CD is found by simple integration. Note also that the integro-differential equation with respect to h is easily

converted to integral equation (see below).

Let's move on to dimensionless quantities:

$$\begin{aligned} x_* &= x/H_0; \quad y_* = y/H_0; \quad z_* = z/H_0; \quad h_* = h/H_0; \\ F &= \frac{c}{\sqrt{gH_0}}; \quad v_{x*} = \frac{v_x}{\sqrt{gH_0}}; \quad v_{y*} = \frac{v_y}{\sqrt{gH_0}}; \quad \bar{v} = \frac{\bar{v}}{\sqrt{gH_0}}; \quad \Delta p_* = \frac{p - p_0}{\rho g H_0} \end{aligned} \quad (26)$$

where F is the Froude number. The Froude number is the only similarity parameter of this problem.

The Bernoulli equation (4) is then transformed to the form:

$$\Delta p_* = 0.5 \left[F^2 - (v_{x*}^2 + v_{y*}^2) \right] - y_* . \quad (27)$$

Before proceeding to the solution of Eq. (25), it is convenient to introduce new independent variables λ and μ :

$$\xi = -\exp(\pi\lambda); \quad t = -\exp(\pi\mu) . \quad (28)$$

The result is:

$$\begin{cases} \frac{dx_*}{d\lambda} = \frac{\cos \omega}{\sqrt{1 - 2h_*/F^2}} \\ \frac{dh_*}{d\lambda} = \frac{\sin \omega}{\sqrt{1 - 2h_*/F^2}} \end{cases}; \quad \omega(\lambda) = \frac{1}{4} \int_{-\infty}^{\infty} \frac{\ln(1 - 2h_*/F^2) d\mu}{\sinh[0.5\pi(\lambda - \mu)]}; \quad \lambda \in (-\infty, \infty) . \quad (29)$$

It can be verified by direct substitution that $\omega(\lambda)$ and $x_*(\lambda)$ are odd functions and $h_*(\lambda)$ is an even function. It is also convenient to introduce a new unknown function

$$\Psi(\lambda) = 1 - (1 - 2h_*/F^2)^{3/2} . \quad (30)$$

In this case, relations (29) are simplified

$$\begin{cases} \frac{dx_*}{d\lambda} = \frac{\cos \omega}{\sqrt[3]{1 - \Psi}} \\ \frac{d\Psi}{d\lambda} = \frac{3}{F^2} \sin \omega \end{cases}; \quad \omega(\lambda) = \frac{1}{6} \int_0^{\infty} \left\{ \frac{1}{\sinh[0.5\pi(\lambda - \mu)]} + \frac{1}{\sinh[0.5\pi(\lambda + \mu)]} \right\} \ln(1 - \Psi) d\mu . \quad (31)$$

Consideration of Fig. 1, Fig. 3, Eq. (5), (28), (30) and parity conditions leads to the boundary conditions for the functions $x_*(\lambda)$, $\Psi(\lambda)$:

$$x_*(0) = 0; \quad \lim_{\lambda \rightarrow \infty} x_*(\lambda) = \infty; \quad \lim_{\lambda \rightarrow \infty} \Psi(\lambda) = 0 . \quad (32)$$

Let us integrate equations (31). Taking into account the parity and boundary conditions at infinity, we obtain:

$$\begin{cases} x_*(\lambda) = \int_0^{\lambda} \frac{\cos \omega d\gamma}{\sqrt[3]{1 - \Psi}} \\ \Psi(\lambda) = -\frac{3}{F^2} \int_{\lambda}^{\infty} \sin \omega d\gamma \end{cases}; \quad \lambda \in [0, \infty) . \quad (33)$$

The second equality (33) is the integral equation to the derivation of which this work is dedicated. It has a trivial solution $\Psi = 0$. At that rate $h_* = 0$, $x_* = \lambda$. Further research is devoted

to the search for nontrivial solutions. These are eigenfunctions and associated with them eigenvalues F . The equation has a continuous set of eigenvalues $F \in (F_{\inf}, F_{\sup})$, and $F_{\inf} \geq 1$. This inequality is followed from Starr's identity [34]:

$$3V_* = (F^2 - 1)M_* , \quad (34)$$

where M_* and V_* are the dimensionless mass and potential energy of the fluid rise:

$$M_* = 2 \int_0^\infty h_* dx_* > 0; \quad V_* = \int_0^\infty h_*^2 dx_* . \quad (35)$$

6 Linear approach

If the rise of the wave is very small $h_* \ll 1$, the given ratios allow linearization. We believe:

$$\Psi = 3h_*/F^2; \quad \cos \omega = 1; \quad \sin \omega = \omega; \quad \sqrt[3]{1 - \Psi} = 1; \quad \ln(1 - \Psi) = -\Psi . \quad (36)$$

The system of equations (33) is transformed to the form:

$$x_* = \lambda; \quad \Psi(\lambda) = \frac{1}{2F^2} \int_\lambda^\infty d\gamma \int_0^\infty \left\{ \frac{1}{\sinh[0.5\pi(\gamma - \mu)]} + \frac{1}{\sinh[0.5\pi(\gamma + \mu)]} \right\} \Psi(\mu) d\mu . \quad (37)$$

This equation can be simplified by changing the order of integration. As a result, we get:

$$\Psi(\lambda) = -\frac{1}{\pi F^2} \int_0^\infty \ln \left| \tanh[0.25\pi(\lambda + \mu)] \tanh[0.25\pi(\lambda - \mu)] \right| \Psi(\mu) d\mu . \quad (38)$$

One can find an analytical solution of Eq. (38) if the kernel of the equation is represented as a series of degenerate kernels. For this, new variables are introduced:

$$\lambda = -2 \ln \tan(0.5\theta); \quad \mu = -2 \ln \tan(0.5\varphi) . \quad (39)$$

Eq. (38) is transformed to the form:

$$\Psi(\theta) = \frac{2}{\pi^2 F^2} \int_0^{0.5\pi} \ln \left| \frac{\tan[0.5(\varphi + \theta)]}{\tan[0.5(\varphi - \theta)]} \right| \frac{\Psi(\varphi)}{\sin \varphi} d\varphi . \quad (40)$$

We use the well-known series expansion [35]:

$$\ln \left| \frac{\sin[0.5(\varphi + \theta)]}{\sin[0.5(\varphi - \theta)]} \right| = 2 \sum_{k=1}^\infty \frac{\sin k\varphi \sin k\theta}{k} . \quad (41)$$

The following series expansions are easily obtained from it:

$$\begin{aligned} \ln \left| \frac{\cos[0.5(\varphi - \theta)]}{\cos[0.5(\varphi + \theta)]} \right| &= \ln \left| \frac{\sin[0.5((\pi - \varphi) + \theta)]}{\sin[0.5((\pi - \varphi) - \theta)]} \right| = 2 \sum_{k=1}^\infty \frac{(-1)^{k-1} \sin k\varphi \sin k\theta}{k}; \\ \ln \left| \frac{\tan[0.5(\varphi + \theta)]}{\tan[0.5(\varphi - \theta)]} \right| &= 4 \sum_{k=1}^\infty \frac{\sin[(2k-1)\varphi] \sin[(2k-1)\theta]}{2k-1} \end{aligned} \quad (42)$$

Thus, the kernel of the integral equation (38) is represented by a series whose members are degenerate kernels. Therefore, the solution of Eq. (38) is sought in the form of a series:

$$\Psi(\theta) = \sum_{k=1}^{\infty} c_k \sin[(2k-1)\theta] . \quad (43)$$

Substituting expression (43) into equation (40) leads to an infinite system of linear algebraic equations with respect to the coefficients c_k :

$$F^2 c_k = \sum_{m=1}^{\infty} a_{km} c_m; \quad a_{km} = \frac{8}{\pi^2 (2k-1)} \int_0^{0.5\pi} \sin[(2k-1)\varphi] \sin[(2m-1)\varphi] \frac{d\varphi}{\sin \varphi} . \quad (44)$$

Thus, the problem reduces to a linear eigenvalue problem [36]. The eigenvalue is F^2 . The found value F will obviously be the lower bound of the set of possible Froude numbers $F_{\inf}$. An approximate solution is obtained if we confine ourselves to the first N terms of the series. The system of equations has N eigenvalues, but only the largest of them makes sense for this problem. It was used to find the Froude number. The series expansions (41), (42) converge slowly, therefore, to obtain a sufficiently accurate solution, many terms of the series are required. Table 1 presents the calculated values F depending on the number N of members of the series.

N	1	10	50	100	500	1000	3000	8000
F	0.9003	0.9647	0.9790	0.9826	0.9881	0.9897	0.9916	0.9930

Table 1 Calculated values F depending on the number N of members of the series

It follows from the calculation results that the Froude number is less than unity, which contradicts the Starr identity (34). This means that the linearized mathematical model does not describe a solitary wave. Thus, a solitary wave is a purely non-linear effect. Such an assumption was made by Stoker [37]. Now it's proven. The value $F_{\inf} = 1$ is a limiting value and is not included in the range of the Froude number. The range of Froude numbers is limited by the inequality $F > 1$.

7 Numerical solution of the nonlinear equation

7.1 Asymptotic analysis

Let's consider the case $\lambda \rightarrow \infty$. Taking into account the boundary conditions (32), we obtain the asymptotic equalities

$$\ln(1 - \Psi) = -\Psi; \quad \sin \omega = \omega . \quad (45)$$

In the integrand in formula (31), the second term can be neglected. We come to the equation:

$$\frac{d\Psi}{d\lambda} = \frac{1}{2F^2} \int_0^{\infty} \frac{\Psi(\mu) d\mu}{\sinh[0.5\pi(\mu - \lambda)]} . \quad (46)$$

We introduce a new integration variable $s = \mu - \lambda$ and get:

$$\frac{d\Psi}{d\lambda} = \frac{1}{2F^2} \int_{-\lambda}^{\infty} \frac{\Psi(s + \lambda) ds}{\sinh(0.5\pi s)} . \quad (47)$$

Since $\lambda \rightarrow \infty$, equation (47) is transformed to the form:

$$\frac{d\Psi}{d\lambda} = \frac{1}{2F^2} \int_{-\infty}^{\infty} \frac{\Psi(s + \lambda) ds}{\sinh(0.5\pi s)} . \quad (48)$$

Equation (48) has an analytical solution that satisfies the boundary condition (32):

$$\Psi(\lambda) = A e^{-\alpha\lambda}; \quad \alpha > 0; \quad \Psi(s + \lambda) = A e^{-\alpha s} e^{-\alpha\lambda}. \quad (49)$$

Substitution of relations (49) into equation (48) leads to a formula relating the quantities α and F :

$$\alpha = \frac{1}{F^2} \int_0^\infty \frac{\sinh(\alpha s) ds}{\sinh(0.5\pi s)} \Rightarrow F^2 = \frac{\tan \alpha}{\alpha}. \quad (50)$$

The value of the integral is taken from the handbook [35]. Formulas (49), (50) were previously obtained by Byatt-Smith [28] in a different way.

7.2 Integral equation transformation

The integrand in integral (31) contains a singularity. As is known [38], in order for the accuracy of numerical integration to be high, it is necessary to transform this integral into a Riemann integral. This is achieved using integration by parts. We get the expression

$$\omega(\lambda) = \frac{1}{3\pi} \int_0^\infty \ln \left| \frac{\tanh[0.25\pi(\lambda + \mu)]}{\tanh[0.25\pi(\lambda - \mu)]} \right| \left| \frac{\Psi'(\mu)}{1 - \Psi(\mu)} \right| d\mu. \quad (51)$$

We add and subtract to the second factor an expression that turns it to zero at $\mu = \lambda$. As a result, we get:

$$\omega(\lambda) = \frac{1}{3\pi} \left\{ \int_0^\infty \ln \left| \frac{\tanh[0.25\pi(\lambda + \mu)]}{\tanh[0.25\pi(\lambda - \mu)]} \right| \left[\frac{\Psi'(\mu)}{1 - \Psi(\mu)} - \frac{\Psi'(\lambda)}{1 - \Psi(\lambda)} \right] d\mu + \frac{2\Psi'(\lambda)L(e^{-0.5\pi\lambda})}{1 - \Psi(\lambda)} \right\}, \quad (52)$$

where the function $L(x)$ is defined by the expression

$$L(x) = \frac{2}{\pi} \left[\int_x^1 (1-t) \ln \left(\frac{1+t}{1-t} \right) \frac{dt}{t} + 2 \ln 2 - (1+x) \ln(1+x) - (1-x) \ln(1-x) \right]; \quad (53)$$

$$x \in [0, 1]; \quad L(0) = \frac{\pi}{2}; \quad L(1) = 0$$

To find a numerical solution, it is necessary to convert an infinite integration interval into a finite interval. In addition, it should be possible not only to satisfy the boundary condition (32), but also to ensure the asymptotics (49). Equivalence relation

$$\lambda \rightarrow \infty: \quad e^{-\alpha\lambda} \sim 0.5 [1 - \tanh(0.5\alpha\lambda)] \quad (54)$$

is the basis for introducing new integration variables

$$s = \tanh(0.5\alpha\lambda); \quad \tau = \tanh(0.5\alpha\mu). \quad (55)$$

Let us introduce a simplifying notation κ_x for the function $\kappa(x)$:

$$\kappa_x = \kappa(x) = \left[(1-x)/(1+x) \right]^{\frac{\pi}{2\alpha}}; \quad x \in [0, 1]. \quad (56)$$

Relations (31), (32) for functions $\omega(\lambda)$, $\Psi(\lambda)$, $x_*(\lambda)$ are transformed to the form:

$$\omega(q) = \frac{1}{3\pi} \left\{ \int_0^1 \ln \left| \frac{(1-\kappa_q \kappa_\tau)(\kappa_\tau + \kappa_q)}{(1+\kappa_q \kappa_\tau)(\kappa_\tau - \kappa_q)} \right| \left[\frac{(1-\tau^2)\Psi'(\tau)}{1-\Psi(\tau)} - \frac{(1-q^2)\Psi'(q)}{1-\Psi(q)} \right] \frac{d\tau}{(1-\tau^2)} + \right. \\ \left. \frac{\alpha(1-q^2)\Psi'(q)L(\kappa_q)}{1-\Psi(q)} \right\}; \quad \Psi(s) = -\frac{6}{F^2\alpha} \int_s^1 \frac{\sin \omega}{1-q^2} dq; \quad x_*(s) = \frac{2}{\alpha} \int_0^s \frac{\cos \omega}{\sqrt[3]{1-\Psi(q)}(1-q^2)} dq \quad (57)$$

7.3 Application of the Galerkin method

The Galerkin method [38] is used to solve the problem. We represent the desired function as a partial sum of a series:

$$\Psi(s) = \sum_{k=1}^N c_k \mathcal{G}_k(s); \quad \mathcal{G}_k(s) = (1-s)T_k(2s-1); \quad s \in [0,1], \quad (58)$$

where c_k are the unknown coefficients determined in the process of solving the problem, $\mathcal{G}_k(s)$ are the coordinate functions, which are the product of the multiplier $(1-s)$ and the Chebyshev polynomials [32]. This multiplier ensures the correct asymptotics of the solution for $\lambda \rightarrow \infty$. Chebyshev polynomials

$$T_1(x) = 1; \quad T_2(x) = x; \quad T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x); \quad k = 3, 4, \dots; \quad x \in [-1,1] \quad (59)$$

form a complete system of functions [38], which is a sufficient condition for the convergence of the solution with increasing number N .

In accordance with the Galerkin method, the residual of equation (57) must be orthogonal to the coordinate functions:

$$\Omega_k(c_m) = \int_0^1 \left[\Psi(s) + \frac{6}{F^2\alpha} \int_s^1 \frac{\sin \omega}{1-q^2} dq \right] \mathcal{G}_k(s) ds = \sum_{m=1}^N a_{km} c_m + \frac{6}{F^2\alpha} \int_0^1 \mathcal{G}_k(s) ds \int_s^1 \frac{\sin \omega}{1-q^2} dq = 0; \quad (60)$$

$$a_{km} = \int_0^1 \mathcal{G}_k(s) \mathcal{G}_m(s) ds; \quad k, m = 1, \dots, N$$

Equalities (60) for a given Froude number, and, consequently, a given value α (see Eq. (50)) represent a system of N nonlinear equations with respect to the coefficients c_m . This system of equations is solved by the Newton–Raphson method. Let n be the iteration number, $\Delta c_m^n = c_m^n - c_m^{n-1}$. The increments Δc_m^n are found from a system of linear equations, where the quantities related to the $n-1$ iteration are known:

$$\sum_{m=1}^N \left(\frac{\partial \Omega_k}{\partial c_m} \right)^{n-1} \Delta c_m^n = -\Omega_k^{n-1}. \quad (61)$$

The elements of the matrix of coefficients of the system of equations (61) are determined by the formulas:

$$\frac{\partial \Omega_k}{\partial c_m} = a_{km} + \frac{6}{F^2\alpha} \int_0^1 \mathcal{G}_k(s) ds \int_s^1 \frac{\cos \omega}{1-q^2} \frac{\partial \omega}{\partial c_m} dq, \quad (62)$$

where

$$\begin{aligned}
\frac{\partial \omega}{\partial c_m} = \frac{1}{3\pi} & \left\{ \int_0^1 \ln \left| \frac{(1 - \kappa_q \kappa_\tau)(\kappa_\tau + \kappa_q)}{(1 + \kappa_q \kappa_\tau)(\kappa_\tau - \kappa_q)} \right| \left[(1 - \tau^2) \frac{\partial}{\partial c_m} \left(\frac{\Psi'(\tau)}{1 - \Psi(\tau)} \right) - \right. \right. \\
& \left. \left. (1 - q^2) \frac{\partial}{\partial c_m} \left(\frac{\Psi'(q)}{1 - \Psi(q)} \right) \right] \frac{d\tau}{(1 - \tau^2)} + \alpha (1 - q^2) L(\kappa_q) \frac{\partial}{\partial c_m} \left(\frac{\Psi'(q)}{1 - \Psi(q)} \right) \right\}; \\
\frac{\partial}{\partial c_m} \left(\frac{\Psi'(\tau)}{1 - \Psi(\tau)} \right) &= \frac{1}{(1 - \Psi(\tau))} \left[\mathcal{G}'_m(\tau) + \frac{\Psi'(\tau) \mathcal{G}_m(\tau)}{1 - \Psi(\tau)} \right]
\end{aligned} \quad (63)$$

After solving system (60), the function $x_*(s)$ is found using formula (57), and the function $h_*(s)$ is found from relation (30). Formulas for calculating the mass and potential energy of the liquid elevation are obtained in the form:

$$M_* = \frac{4}{\alpha} \int_0^1 \frac{h_* \cos \omega}{\sqrt[3]{1 - \Psi}(1 - s^2)} ds; \quad V_* = \frac{2}{\alpha} \int_0^1 \frac{h_*^2 \cos \omega}{\sqrt[3]{1 - \Psi}(1 - s^2)} ds. \quad (64)$$

Of interest are also the dependences of velocity and pressure on spatial coordinates. Let us first find the velocity on the free surface of the liquid (the line CD in Fig. 1). From equalities (22), (28)-(30) it follows:

$$\frac{dz_*}{d\xi} = \frac{F}{\pi \xi \bar{v}_*}; \quad \frac{dz_*}{d\lambda} = \frac{e^{i\omega}}{\sqrt[3]{1 - \Psi}} = \frac{F}{\bar{v}_*} \Rightarrow v_{x*} = F \sqrt[3]{1 - \Psi} \cos \omega; \quad v_{y*} = F \sqrt[3]{1 - \Psi} \sin \omega. \quad (65)$$

The velocity at the internal points of the region is determined by formula (21). The integral included in it is transformed by changing the integration variables (28) and (55), as well as taking into account the evenness of the function $\Psi(\mu)$. We get:

$$\bar{v}_*(\zeta) = F \exp \left[\frac{2}{3\alpha} (1 + \zeta) \zeta^{1/2} \int_0^1 \frac{\kappa_\tau (1 + \kappa_\tau^2) \ln(1 - \Psi)}{(1 - \tau^2) \left[(1 + \zeta^2) \kappa_\tau^2 + \zeta (1 + \kappa_\tau^4) \right]} d\tau \right]. \quad (66)$$

The increment of a complex variable ζ for a given increment Δz is found from expression (22):

$$\Delta \zeta = \frac{\pi}{F} \zeta \bar{v}_*(\zeta) \Delta z. \quad (67)$$

The initial conditions for the problem of determining the dependence $\bar{v}_*(z)$ are the quantities $z, \zeta, \bar{v}$ on the line CD . They are known as a result of solving the system of equations (60). Further, formulas (67) and (66) are successively used. After the velocity field $\bar{v}_*(z)$ is found, the dependence of pressure on the coordinates is determined by formula (27).

7.4 Branching of the solution

The system of nonlinear equations (60) determines the expansion coefficients (58) as implicit functions of the Froude number. The solution of system (60) is not unique. So, for example, there is a trivial solution $c_m = 0$. Within one branch of solutions, an infinitesimal increment dF corresponds to uniquely defined infinitesimal increments dc_k . The branching point of solutions is determined by the condition that the Jacobian vanishes [39]

$$J = \det \left(\frac{\partial \Omega_k}{\partial c_m} \right) = 0. \quad (68)$$

Therefore, when carrying out calculations, it is necessary to control the sign of the Jacobian.

7.5 Solution convergence

The trapezoidal method was used to calculate the integrals. The accuracy of calculations depends on the number of partitions of the integration interval M and the number of coordinate functions N . These values were successively increased until the calculation error became less than the specified value.

8 Calculation results

8.1 Convergence analysis of the solution

The solution of the system of equations (60) is sought for a given Froude number when equality (50) is satisfied. It is also necessary to set the initial approximation of the unknown quantities Ψ_k^0 . For the iterative process to converge, it must be sufficiently close to the result of the solution. It is possible, as was done in this study, to choose a Froude number slightly larger than unity and use the Boussinesq–Rayleigh formula [40]:

$$h_* = (F^2 - 1) \cosh^{-2} \left[\frac{x_* \sqrt{3(F^2 - 1)}}{2F} \right]. \quad (69)$$

In this case, the variables x_* and λ are identified. When changing the Froude number, the previous numerical solution is used as a zero approximation.

The measure of the error of the iterative process was determined by the formula:

$$\varepsilon = \sqrt{\sum_{m=1}^N (\Delta c_m^n)^2 / \sum_{m=1}^N (c_m^n)^2}. \quad (70)$$

All calculation results presented below were obtained under the condition $\varepsilon < 10^{-13}$. To obtain such an accuracy of the solution, 7–12 iterations are usually sufficient.

Let us denote the maximum elevation of the wave $\beta = h_{*max}$. Calculations show that the dependence graphs $\beta(F)$ obtained for $N=10$ and for $N=20$ are visually indistinguishable. Some typical numerical results are presented in Table 2.

F	1.001	1.08	1.12	1.16	1.20	1.24	1.28	1.29
$N=10$	0.002098	0.1682	0.2588	0.6320	0.3547	0.5695	0.7050	0.7567
$N=20$	0.002048	0.1682	0.2588	0.6322	0.3547	0.5696	0.7047	0.7526

Table 2 Calculated values β depending on the Froude number for different number of coordinate functions

Obviously, the number of coordinate functions $N=20$ provides an accuracy sufficient for solving hydrodynamic problems. Therefore, below are the results of calculations obtained for $N=20$.

8.2 Solution features

The graph of the calculated dependence of the maximum wave height on the Froude number $\beta(F)$ is shown in Fig. 4. The boundary of the region of admissible values of this function is also shown here. This region is limited by the inequality (see Eq. (29)):

$$h_* < 0.5F^2. \quad (71)$$

Of greatest interest is the region of large values F (Fig. 5). To understand the behavior of the

function $\beta(F)$ in this region, one should refer to the dependence $J(F)$ (see Eq. (68)), the graph of which is shown in Fig. 6.

Thus, the Jacobian changes sign, and, consequently, there is a branching of the solution. And this means that there is an upper bound F_{sup} . Its approximate value obtained in this study turned out to be $F_{\text{sup}} = F_{\text{max}} = 1.2941$, while $\beta = 0.7908$. Longuet-Higgins & Fenton [14] obtained $F_{\text{max}} = 1.294$ for $\beta = 0.790$, Tanaka [41] obtained $F_{\text{max}} = 1.294$ for $\beta = 0.8122$.

The upper branch of the solution turns out to be short, because it rests against the boundary curve. This means that there is a maximum value of β_{max} . The problem of determining this quantity has been the subject of a number of studies. Some of their results are presented in Table 3.

Fig. 4 Dependence of the maximum wave height β on the Froude number F . Ratio for the boundary line is $\beta = 0.5F^2$

Fig. 5 Dependence of the maximum wave height β on the Froude number F (fragment)

Fig. 6 Dependence of the Jacobian J on the Froude number F in the vicinity of the branching point of the solution

Researcher	$\beta_{\max}$
Yamada [42]	0.827
Lenau [11]	0.8281
Longuet-Higgins & Fenton [14]	0.827
Williams [43]	0.8332
Hunter & Vanden-Broeck [15]	0.8332
Tanaka [41]	0.8324
Pennell [17]	0.8261
Evans & Ford [30]	0.8332
Wu at al. [18]	0.8331

Table 3 The values of the maximum wave elevation found by various researchers

In the present study, the value $\beta_{\max} = 0.8331$ at $F = 1.2909$ was obtained, which is consistent with the data given in Table 3.

According to Stokes [40], the wave crest with the highest possible elevation has a sharp point with an angle 120° . It is interesting to check whether the calculated wave profile at the maximum elevation satisfies the Stokes condition. This is done. Figure 7 shows graphs of the calculated wave profile at $\beta_{\max}$ and a straight line

$$h_* = \beta_{\max} - x_*/\sqrt{3} \quad , \quad (72)$$

inclined to the y-axis at an angle 60° . As follows from Fig. 7, the calculation results are consistent with the Stokes condition.

Fig. 7 Wave profile with maximum elevation

8.3 Comparison of calculation results with experimental data

Experiments to determine the dependence $\beta(F)$ were carried out by Daily & Stephan [44], as well as by Bukreev [45]. Their results, compared with the results of calculations, are shown in Fig. 8.

Fig. 8 Comparison of the results of calculating the dependence $\beta(F)$ with experimental data

In the experiments by Daily & Stephan [44], Bukreev [45], as well as Duan et al. [21], the wave profile was determined for various values of the maximum elevation β . To compare the calculation results with experimental data, the value F was determined by interpolation of the calculated dependence $\beta(F)$ (see Fig. 4), and then equation (33) was solved.

Fig. 9 shows a comparison of the calculation results with the experimental data by Daily & Stephan [44].

Fig. 9 Comparison of the calculated wave profiles with those obtained in the experiments by Daily & Stephan [44]

As can be seen from the graphs, the experimental and theoretical results differ slightly, which indicates the adequacy of the mathematical model.

Bukreev [45] determined the wave profile at $\beta = 0.55$. As it turned out, the experimental results are very well approximated by the formula [45]:

$$h_*/\beta = \cosh^{-2}(0.8813x_*/l_*) , \quad (73)$$

where l_* is the value x_* for which $h_*/\beta = 0.5$. Figure 10 shows the experimental wave profile constructed according to equation (73) and the theoretical profile constructed based on the results of calculations by the developed method. The discrepancy between theoretical and experimental results is insignificant.

Fig. 10 Comparison of the calculated wave profile with that obtained in Bukreev's experiments [45]

Fig. 11 Comparison of the calculated wave profiles with those obtained in the experiments by Duan et al. [21]

Let us now compare the theoretical results with the experiments of Duan et al. [21] (Fig. 11). Note that the results in [21] are presented in a laboratory (fixed) frame of reference. Therefore, on the graphs given in [21], the abscissa shows the value $t\sqrt{g/H_0}$, where t is the time. When passing to a reference frame moving together with the wave (at the initial moment, the coordinates of both frames coincide), we obtain:

$$x = ct \Rightarrow x_* = \frac{ct}{H_0} = \frac{ct\sqrt{g}}{\sqrt{gH_0}\sqrt{H_0}} = Ft\sqrt{\frac{g}{H_0}}. \quad (74)$$

The discrepancy between theoretical and experimental results is insignificant.

In the experiments by Constantin et al. [46], the change in pressure along the ordinate axis under the wave top, that is, at $x_* = 0$, was studied. The experimental results compared with the calculation data are shown in Fig. 12. Calculations are based on Eq. (66), (67) and (27).

(a) $\beta = 0.0972$

(b) $\beta = 0.131$

(c) $\beta = 0.166$

(d) $\beta = 0.200$

(e) $\beta = 0.235$

(f) $\beta = 0.0700$

Fig. 12 Comparison of the calculated pressure with that obtained in the experiments by Constantin et al. [46]

Here the discrepancy between theory and experiment is greater than in the previous cases. However, it is still such that we can talk about their coordination.

8.4 Comparison of calculation results with theoretical results by other researchers

The most important numerical result is the dependence $\beta(F)$. Fig. 13 shows a graph of this dependence obtained by the developed method. It is compared with the graph of the Boussinesq–Rayleigh dependence [40] (Eq. (69)), as well as with the results of numerical calculations by Longuet-Higgins & Fenton [14] and Dutykh & Clamond [20]. The Boussinesq–Rayleigh approximation turns out to be very accurate in almost the entire range of the Froude number.

Along with the dependence $\beta(F)$, the dependences of the mass and potential energy of the wave on its elevation $M_*(\beta)$ and $V_*(\beta)$ can be of interest. The graphs of these dependencies, calculated by the described method, are shown in Fig. 14, 15. They are compared with the results of calculations by Dutykh & Clamond [20].

Fig. 13 Comparison of the results of calculating the dependence $\beta(F)$ with the theoretical results by other researchers

Fig. 14 Comparison of the results of calculating the dependence $M_*(\beta)$ with the theoretical results by Dutykh & Clamond [20]

Fig. 15 Comparison of the results of calculating the dependence $V_*(\beta)$ with the theoretical results by Dutykh & Clamond [20]

Let us now compare the calculated wave profiles with those calculated by other researchers. Fig. 16 shows the wave profiles obtained by the developed method, calculated by the Boussinesq-Rayleigh formula (69) and found by Byatt-Smith [28]. Figure 17 presents similar results for two Froude numbers obtained in the present study, as well as by Dutykh & Clamond [20].

In addition to the wave profile, the dependence of the wave contour velocity $v_{x*}(x_*, h_*)$, $v_{y*}(x_*, h_*)$ on the coordinate x_* may be of interest. Fig. 18 shows the results of calculating these dependencies with the described method at $F = 1.066$. These results are compared with those obtained by Dutykh & Clamond [20].

It can be concluded that the calculation results agree satisfactorily with the results obtained by Byatt-Smith [28], Longuet-Higgins & Fenton [14] and Dutykh & Clamond [20].

Fig. 16 Designed wave profile as compared with the results of the calculations by Eq. (69) and the results obtained by Byatt-Smith [28]

Fig. 17 Designed wave profile as compared with the results obtained by Dutykh & Clamond [20]

Fig. 18 Designed wave surface speed as compared with the results obtained by Dutykh & Clamond [20]

9. Conclusion

A radical simplification of any boundary value problem is its reduction to the solution of an integral equation. Such an equation is obtained in this article. Although it is non-linear, it is quite simple. Its derivation is based on the conformal mapping of a half-plane onto a strip and the representation of the complex conjugate velocity as a Cauchy-type integral whose density is found from the boundary conditions using Plemelj's formulas. The numerical solution of the integral equation is found by the Galerkin method using the Newton–Raphson iterative method. The convergence of the solution with an increase in the number of coordinate functions is shown.

It was possible to find a linear approximation of the problem, which, as expected, describes a solitary wave with an infinitely small amplitude. In this case, a linear homogeneous integral equation is obtained. Its own value allows you to determine the minimum possible Froude number. An analytical solution of the problem is found by expanding the kernel of the integral equation into a series of degenerate kernels. But it turned out that the calculated Froude number is less than one, which contradicts the Starr identity (34). This means that the resulting linear integral equation cannot serve as a mathematical model of a solitary wave. When describing a solitary wave, nonlinear effects cannot be neglected, as suggested earlier by Stoker [37]. And,

therefore, the lower bound of the Froude numbers is equal to one.

Comparison of calculation results with experimental data and theoretical results of other researchers leads to conclusions about the adequacy of the mathematical model and the acceptable accuracy of the numerical solution of the integral equation. Due to its simplicity, the resulting integral equation can be used in applied research, for example, in tsunami modeling.

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