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Critical non-Hermitian skin effect in a single closed non-Hermitian chain

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ABSTRACT

A critical non-Hermitian skin effect (CNHSE) was recently discovered in which the eigenenergy spectrum of a coupled system comprising two 1D non-Hermitian (NH) chains exhibits an abrupt change when the length of the system increases beyond a critical length. Here, we overturn the conventional wisdom that multiple chains are required for the CNHSE by showing that a similar abrupt transition in the eigenenergy spectrum occurs with increasing length in a single finite 1D chain when the two ends of the chain are connected by a weak terminal coupling to form a closed loop. We illustrate this phenomenon using the classic Hatano-Nelson and Su-Schrieffer-Heeger NH systems, and show that it can co-exist with the conventional CNHSE in coupled NH loops.

INTRODUCTION

Non-Hermitian phenomena [1–4] are one of the most exciting topics that have emerged in condensed matter physics. In general, the breaking of Hermiticity by non-reciprocal couplings between lattice sites or onsite gain/loss terms [5–7] can induce a plethora of unusual phenomena that include exceptional points [5, 8, 9], nodal rings [10], and the extensive localization of eigenstates [11–15], also known as the non-Hermitian skin effect (NHSE). The NHSE can be exploited for ultra-sensitive sensors [16, 17], unidirectional transport [18], and the amplification and attenuation of quantum signals [19, 20].

In particular, Lee et al. recently uncovered a critical non-Hermitian skin effect (CNHSE) that occurs when two non-Hermitian chains with different NHSE decay lengths are coupled together in parallel, i.e., every site in one chain is coupled to its corresponding site on the other chain [21]. In the CNHSE, the eigenenergy spectrum of the coupled chain under open boundary conditions (OBC) resembles the union of the spectra of the individual constituent chains for short lengths of the coupled chain. However, when the length of the coupled chain exceeds a critical threshold, the eigenenergy spectrum undergoes an abrupt transformation and begins to approach the thermodynamic-limit OBC spectrum of the coupled chain [11]. The existing studies on the CNHSE have so far focused on systems consisting of two non-Hermitian chains with different NHSE decay lengths coupled together [21, 22].

The presence of more than one chain was thought to be a necessary ingredient for the CNHSE. In this work, we overturn this conventional wisdom by demonstrating that the CNHSE phenomenon can emerge in the most basic example of a non-reciprocal system, i.e., a *single* Hatano-Nelson (HN) chain, by coupling the two ends of the open chain to form a single closed loop. At short lengths of the closed loop, the energy spectrum of the closed loop resembles that of the OBC spectrum of the open HN chain. However, when the length of the closed loop exceeds a critical limit, which depends on the coupling strength between the ends of the originally open chain, the eigenenergy spectrum of the loop undergoes an abrupt transition and approaches that of the periodic boundary condition (PBC). We term this new phenomenon the single-chain critical non-Hermitian skin effect (SC-CNHSE) to distinguish it from the conventional CNHSE arising in multiple coupled chains. We further show that the onset of SC-CNHSE is not restricted to a closed HN loop, but also occurs in a closed loop of more complicated non-Hermitian systems, indicating that the SC-CNHSE constitutes a general feature of non-Hermitian loops.

RESULTS AND DISCUSSION

We consider a closed HN loop (“CHN”) where the end points at $n = 1$ and $n = N$ of the open HN chain are connected together by a reciprocal terminal coupling Γ illustrated in Fig. 1a. The Hamiltonian of a closed HN loop of length N is given by

$$H_{\text{CHN}} = \sum_{j=1}^{N-1} |j\rangle t_R \langle j+1| + |j+1\rangle t_L \langle j| + \Gamma(|N\rangle \langle 1| + |1\rangle \langle N|). \quad (1)$$

The terms in the first line is the Hamiltonian for the open HN chain with the coupling strengths of t_L and t_R between a site and its left and right neighbors, respectively, and that in the second line the terminal coupling of strength Γ between the ends of the chain. $|j\rangle$ ($\langle j|$) is the ket (bra) vector representing a state localized at site j .

Figure 1. Eigenspectrum and density distribution of closed Hatano-Nelson loops. **a.** Schematic representation of a closed HN loop containing N nodes in which each node from $n = 1$ to $n = N - 1$ are coupled to their left right neighbors by the non-reciprocal couplings t_L and t_R , respectively, and the N th node is coupled to the first node by the reciprocal terminal coupling Γ . **b.** and **c.** show the eigenenergy spectra of the closed HN loops as functions of the chain length N for $t_L = 2$, $t_R = 1$ at **b.** $\Gamma = 10^{-3}$, and **c.** $\Gamma = 10^{-4}$. The open-chain OBC and PBC eigenenergy spectra are also respectively shown in blue at the bottom and orange on top for comparison. **d.** **e** show the eigenenergies and the normalized density distributions across the lattice sites of the eigenstates of **d.** $N = 30$, and **e.** $N = 50$ closed loops with the terminal coupling of $\Gamma = 10^{-3}$. A darker color at each lattice site indicates a larger density at the site.

Figures 1b and 1c show the eigenvalue spectra of the closed loop in Eq. 1 for the exemplary values of $t_L = 2$ and $t_R = 1$ and two different terminal coupling strengths of $\Gamma = 10^{-3}$ and $\Gamma = 10^{-4}$, respectively. The OBC and PBC spectra of the open chains are also plotted in the figures for comparison. Figure 1b shows that for small values of N of less than 22 (henceforth referred to as the short-chain regime), the eigenenergy spectrum takes the form of the open-chain OBC spectrum and is distributed along a line on the real energy axis. When the length of the closed loop exceeds the critical value, its eigenenergy spectrum abruptly deviates from the open-chain OBC spectrum in a manner reminiscent of the CNHSE, and acquires a finite imaginary component. As the length of the closed loop increases, its eigenenergy spectrum approaches that of the open-chain PBC spectrum (henceforth referred to as the long-chain regime), which takes the form of an ellipse on the complex energy plane. Comparing Fig. 1c, which has a smaller value of $\Gamma = 10^{-4}$ and critical length of 27, to Fig. 1b with a larger value of $\Gamma = 10^{-3}$, it is evident that the transition from the short-chain regime to the long-chain regime occurs at longer chain lengths for smaller values of Γ .

This transition from an OBC-like eigenenergy spectrum to a PBC-like spectrum as the length of the system increases can be understood physically as follows: As the length of the chain increases, the relative proportion of the terminal coupling with the value of Γ between sites 1 and N compared to the number of non-terminal couplings with the values of t_L or t_R between the rest of the lattice sites on the chains decreases. The system therefore behaves as if it is homogeneously coupled by t_L / t_R as its length increases, and the deviation of the eigenenergy spectrum and eigenstates of the closed loop from the open-chain PBC spectrum decreases.

At a deeper level, as the length of the closed loop increases, the loop can support modes with increasingly long wavelengths that span over larger numbers of lattice sites. The sensitivity of these long-wavelength modes to the single-site perturbation caused by the deviation of Γ from t_L and t_R decreases with the wavelength of the mode because the length scale of the perturbation is much smaller than its wavelength. Therefore, these long wavelength

modes behave as if the perturbation does not exist, and the corresponding eigenenergy spectrum approaches that of the PBC spectrum. In contrast, in the shorter closed loops, which can only support short-wavelength modes, the relative length scale of the perturbation is proportionally larger for these short-wavelength modes and has a much larger effect on these modes. In particular, because $|\Gamma| \ll |t_{L,R}|$, the terminal coupling acts as a break in the connection between the sites, i.e., an OBC, along the chain for the short-wavelength modes. The eigenenergy spectra of the shorter chains therefore resembles that of the open-chain OBC spectra in Fig. 1b and c. We provide a more mathematical explanation of why the SC-CNHSE transition occurs in terms of the boundary conditions at the terminal nodes and a description of the long-chain regime in the Supplementary Materials.

Figure 1d and 1e shows the density distribution at each lattice site (i.e., the square of the absolute value of the wavefunction) for each eigenstate for the chain lengths of $N = 30$ and $N = 50$, respectively, corresponding to the long-chain regime of Fig. 1b. For both values of N , the eigenstates are localized near the right edge of the chain, i.e., at site $n = N$ because of the NHSE. Moreover, comparing Fig. 1d and Fig. 1e, it can be seen that the decay rates of the eigenstates from the right edge towards the interior of the loops are largely similar relative to the respective total lengths of the loops. This is manifestation of the scale-free nature of the SC-CNHSE, which we prove mathematically for HN loops in the Supplementary Materials.

We now demonstrate the generality of the single-chain CNHSE in more complex closed loop systems beyond the closed HN loop. The first example is the non-Hermitian Su-Schrieffer-Heeger (SSH) chain. In contrast to the HN chain in which each unit cell contains only a single node, the unit cell of the SSH chain contains two nodes with the characteristic feature that the inter-cell and intra-cell couplings differ from each other. The Hamiltonian of a closed SSH chain containing N unit cells with the intra-unit cell couplings of $t_{L,R,1}$ in the left and right directions, inter-unit cell coupling of $t_{L,R,2}$, and terminal coupling of Γ between the first and last sites can be written as

$$\begin{aligned} H_{\text{SSH}} = & \left(\sum_j^N |2j-1\rangle t_{R1} \langle 2j| + |2j\rangle t_{L1} \langle 2j-1| \right) \\ & + \left(\sum_j^{N-1} |2j\rangle t_{R2} \langle 2j+1| + |2j+1\rangle t_{L2} \langle 2j| \right) \\ & + \Gamma(|1\rangle \langle 2N| + |2N\rangle \langle 1|) \end{aligned} \quad (2)$$

where the first line represents the inter-unit cell couplings, the second line the intra-unit cell couplings, and the last line the terminal coupling between the first and last lattice sites. The system represented by the Hamiltonian in Eq. (2) is schematically illustrated in Fig. 2a. Fig. 2b shows the occurrence of the SC-CNHSE for an exemplary parameter set in a SSH chain. A clear signature of the SC-CNHSE can be seen at around $N = 20$. Here, a transition occurs in the energy spectrum from the open-chain OBC form comprising two separated lines on the real energy axis, which characterizes the short-chain regime, to the open-chain PBC form comprising two ellipses on the complex energy plane symmetrically distributed about the imaginary axis, which characterizes the long-chain regime. Fig. 2c and 2d show the density distributions for two chains with the same parameters as in Fig. 2b at the lengths of 40 and 80 unit cells, respectively, which fall within the long-chain regime. Note that despite the loop in Fig. 2d being twice as long as that in Fig. 2c, the spatial density distributions of the eigenmodes for both chains relative to the total length are broadly similar. This again demonstrates the scale-free characteristic of the SC-CNHSE effect.

By applying the standard condition that the loci of the eigenenergies in the thermodynamic-limit OBC open-chain energy spectrum is given by energy values at which the middle values of β have the same absolute values [23], the inverse decay skin length of the open-chain SSH system under OBC, $(\text{Ln } |\beta|)$ can be derived to be

$$\text{Ln } |\beta| = \frac{1}{2} \text{Ln } \left| \frac{t_{L1} t_{L2}}{t_{R1} t_{R2}} \right|. \quad (3)$$

Eq. (3) implies that there is a critical set of parameters at which the inverse skin length is 0, i.e., there is no localization of the eigenstates near the edges even when the system is non-Hermitian. This occurs when the critical condition

$$|t_{L1} t_{L2}| = |t_{R1} t_{R2}| \quad (4)$$

is satisfied [12]. Figure 2e shows the eigenenergy spectra for a closed SSH chain with an exemplary set of parameters satisfying Eq. (4) along with the PBC and OBC spectra of the corresponding open SSH chains. Both the OBC and PBC spectra of the open chains have the same form, i.e., both consist of two discontinuous lines on the real axis. The correspondence between the two implies that there is no skin effect in the system, as expected. The absence of

Figure 2. Eigenspectrum and density distribution of closed Su-Schrieffer-Heeger (SSH) loops. **a.** A schematic representation of a closed SSH chain with N unit cells and the intra-unit cell couplings of $t_{L,R,1}$, inter-unit cell couplings of $t_{L,R,2}$, and terminal coupling between the first and last node of Γ . Each unit cell is denoted by a dashed box. **b.** The energy spectrum of a closed SSH loop with $t_{L1} = 2.9$, $t_{L1} = 1.1$, $t_{L2} = 0.775$, $t_{R2} = 1.225$, and $\Gamma = 10^{-3}$ as a function of N . The open-chain OBC and PBC spectra are also plotted in the figure for comparison. **c.** and **d.** show the eigenenergy spectra and density distributions across the lattice sites for the eigenmodes of the closed SSH loop in **b.** at $N = 40$ and $N = 80$ respectively. Darker colors indicate higher densities. (Note that the number of sites is twice the number of unit cells because each unit cell contains two sites.) **e.** The energy spectrum of a closed SSH loop at $\Gamma = 10^{-3}$, and the critical parameter values of $t_{L1} = 2.9$, $t_{L1} = 1.1$, $t_{L2} = 0.55$ and $t_{R2} = 1.45$. **f.** The eigenenergy spectrum and density distribution across the lattice sites for the eigenmodes of the critical closed SSH chain in **e.** for $N = 80$.

the NHSE is further demonstrated in the plot of the density distributions of the closed-chain eigenstates at $N = 80$ in Fig. 2f, which clearly shows that the absence of exponential localization of the eigenmodes near either of the edges. As explained in the Supplementary Materials, the transition from the short- to long-chain regime in the SC-CNHSE depends on the wavefunction at one end of the chain being exponentially larger than that at the other end in the NHSE. Thus, the absence of any exponential localization in Fig. 2e implies the absence of SC-CNHSE in this

particular closed SSH chain when the coupling parameters satisfy Eq. (4).

We next demonstrate the co-existence of the conventional and single-chain CNHSE effects in a system comprising two closed HN loops denoted as chains A and B with the respective coupling constants of $t_{(L,R),A}$ and $t_{(L,R),B}$ coupled in parallel. Each site in chain A is coupled to its corresponding site in chain B by a reciprocal coupling t_C , and the first and last nodes of each chain are coupled by a reciprocal terminal coupling Γ to form a closed loop, as shown schematically in Fig. 3a. The Hamiltonian for this system with N sites in each chain can be written as

$$\begin{aligned}
 H_{\text{PHN}} = & \sum_j^{N-1} \left(|j, A\rangle t_{R,A} \langle j+1, A| + |j, B\rangle t_{R,B} \langle j+1, B| \right. \\
 & \left. |j+1, A\rangle t_{L,A} \langle j, A| + |j+1, B\rangle t_{L,B} \langle j, B| \right) \\
 & + \sum_j^N (|j, A\rangle t_c \langle j, B| + |j, B\rangle t_c \langle j, A|) \\
 & + \Gamma(|1, A\rangle \langle N, A| + |1, B\rangle \langle N, B|).
 \end{aligned} \tag{5}$$

Fig. 3b shows the eigenenergy spectrum of the system in which Γ was set to 0 while t_C has a finite value. This

Figure 3. Co-existence of conventional and single-chain critical non-Hermitian skin effect **a.** A schematic representation of two Hatano-Nelson chains each containing N nodes and the respective coupling strengths of $t_{L,R,A}$ and $t_{L,R,B}$. Each node in one chain is coupled to its corresponding node in the other chain by the inter-chain coupling t_C . **b.** The energy spectrum of the coupled open chain (“coupled OC”) as a function of the chain length. The OBC spectrum in the thermodynamic limit of the open chain is shown as well. **c.** The energy spectrum of the coupled closed loop as a function of the chain length. **d.** The energy spectrum of the same loop in c. as a function of the chain length for longer lengths of the chain. The different regimes in each of these systems are indicated by the colors of the dots. Common parameters used : $t_{L,A} = 2$, $t_{L,B} = 3$, and $t_{R,A} = t_{R,B} = 1$

corresponds effectively to two open-chain NH systems that are coupled to each other in parallel. Such a system exhibits the conventional CNHSE where the eigenenergy spectrum switches from the form corresponding to the short-chain regime, which consists of a line on the real axis formed from the union of the OBC spectra of the two individual chains,

to the form corresponding to the long-chain regime. The transition occurs at a critical chain length N of around 20. Fig. 3c shows the eigenenergy spectrum of the coupled closed chain system where the terminal coupling Γ is now set to a small but finite value of 10^{-11} . The same transition of the eigenenergy spectrum from the straight line of the open-chain OBC spectra to the elliptical form of the coupled open chain occurs at around the same critical value of N . Interestingly, the presence of a finite terminal Γ coupling that transforms the NH chains into a closed loop also induces a second transition at around $N = 52$. At this critical point, the eigenenergy spectrum switches from the single ellipse to the intermediate form comprising a straight line enclosed within a larger ellipse. The ellipse corresponds to the PBC spectrum of chain A, which has transitioned to the long-chain regime because of its shorter SC-CNHSE critical length, whereas the straight line corresponds to the OBC spectrum of chain B, which has a longer SC-CNHSE critical length and thus is still within the SC-CNHSE short-chain regime. Fig. 3d shows the eigenenergy spectrum of the same system as Fig. 3c for a larger range of N . The eigenenergy spectrum undergoes another transition when the energy spectra of the states that are more associated with chain B also make the SC-CNHSE transition at around $N = 76$. The spectrum of these states transform from the straight line OBC form to the elliptical PBC form, so that the spectrum of the coupled system takes the form of two concentric ellipses. These results clearly demonstrate the SC and conventional CNHSEs can both occur in the same system, and that the different length scales at which they are dominant translate into multiple transitions of the eigenenergy spectra.

In summary, we reported the single-chain CNHSE (SC-CNHSE), which differs from the conventional CNHSE in two key aspects: (i) The conventional CNHSE arises when two non-Hermitian chains with different non-Hermitian skin lengths are coupled together in parallel. In contrast, the SC-CNHSE arises simply by coupling the two ends of a single non-Hermitian chain. (ii) In the conventional CNHSE, the eigenenergy spectrum of the coupled system undergoes a transition from the union of the OBC spectra of the individual constituent chains to the OBC spectrum of the coupled system when the system size exceeds a certain critical length. In the SC-CNHSE case, the eigenenergy spectrum transitions from the open-chain OBC spectrum to the PBC spectrum. Despite these differences, the single-chain CNHSE shares the common feature of scale-free localization with the conventional CNHSE. The SC-CNHSE phenomenon is general. We demonstrated its occurrence in the SSH loop as well as in a system where two HN chains are coupled together in parallel. In the former, the SC-CNHSE vanishes when the coupling constants are tuned to values where the NHSE is suppressed in open SSH chains. In the latter, we demonstrate the co-existence of the conventional and SC-CNHSE and the resulting multiple transitions of the eigenenergy spectrum as the system length is varied. The ubiquity of the SC-CNHSE effect in various non-Hermitian systems coupled with the sharp and robust changes of their eigenenergy spectra may be utilized for potential applications such as electrical switching, sensing, and multiple-state data storage.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon request.

CODE AVAILABILITY

All codes used to produce the findings of this study are available from the corresponding author upon request.

AUTHOR CONTRIBUTIONS

Z. B. S. and S. M. R. conceived the idea of the project, performed the calculations, and wrote the initial draft manuscript. M. B. A. J. supervised the project. All authors participated in the discussion of the results and the finalization of the manuscript.

COMPETING INTERESTS

The authors declare no competing interests.

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