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Vibration control of a nonlinear cantilever beam operating in the 3D space

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Abstract:

This paper addresses a control problem of a nonlinear cantilever beam with translating base in the three-dimensional space, wherein the coupled nonlinear dynamics of the transverse, lateral, and longitudinal vibrations of the beam and the base's motions are considered. The control scheme employs two control inputs applied to the beam's base to control the base's position while simultaneously suppressing the beam's transverse, lateral, and longitudinal vibrations. According to the Hamilton principle, a hybrid model describing the nonlinear coupling dynamics of the beam and the base is established: This model consists of three partial differential equations representing the beam's dynamics and two ordinary differential equations representing the base's dynamics. Subsequently, the control laws are designed to move the base to the desired position and attenuate the beam's vibrations in all three directions. The asymptotic stability of the closed-loop system is proven via the Lyapunov method. Finally, the effectiveness of the designed control scheme is illustrated via the simulation results.

Keywords: *Flexible beam; boundary control; nonlinear vibration; gantry manipulator; Lyapunov method; exponential stability.*

1. Introduction

The systems consisting of an elastic cantilever beam fixed on a translating base are found in various practical engineering applications, such as master fuel assemblies in nuclear refueling machines and robotic manipulators [1, 2]. In these systems, the base's translational motion can produce large-amplitude vibrations of the beam in the three-dimensional space. This vibration becomes a

significant negative factor in association with the system's safety and performance. Therefore, it is necessary to analyze and control the three-dimensional vibration of the beam with a moving base to ensure safety and performance.

The flexible cantilever beam is a distributed parameter system with an infinite number of vibration modes. Its dynamics are characterized by partial differential equations (PDEs) [3]. When a flexible beam is fixed on a translating base, the base's dynamics are described by ordinary differential equations (ODEs) to be considered simultaneously with the beam's dynamics. Furthermore, if the amplitudes of the beam's vibrations are large, three-dimensional analysis of the beam's dynamics should be performed; wherein the nonlinear coupling effects between the transverse, lateral, and longitudinal vibrations are considered, see Fig. 1.

The dynamic behaviors of the beam attached to a moving base have been studied in the literature [4-7]. In [8], the equations of motion of the mass-beam-cart system, which is a beam with translating base, were developed based on the Hamilton principle. The system's natural frequencies were also obtained by using modal analysis. Later, the vibration of a flexible beam fixed on a cart and carrying a moving mass was examined via an experimental study, see Park and Youm [9]. However, most researches on the beam attached to a translating base assume that the base moves along one direction, restricting the beam's vibration to two-dimensional space. In these works, only the transverse vibration was considered. For the beam with a translating base in the three-dimensional space, Shah and Hong [10] addressed the vibration problem of the master fuel assembly in nuclear refueling machines. In their work, the nuclear fuel rod and the trolley were treated as the flexible beam and the base moving on a plane, respectively.

The control problem of distributed parameter systems has been investigated in the literature [11-15]. Contrary to the stationary beam, the vibration of the cantilever beam with a moving base can be suppressed via the control input applied at the base (i.e., the clamped end of the beam) or the beam's tip. In this situation, we aim to simultaneously control of the base's position and the beam's vibration. These objectives can be achieved by either open-loop control [16, 17] or closed-loop control [10, 18, 19-23]: The input shaping control is the most feasible and practical open-loop control technique for beams with a moving base. In [16], model parameters of an underwater system consisting of a beam and a translating base were determined using the model analysis method. Accordingly, the input shaping control law was designed to position the base and suppress the beam's vibration. Pham et al. [17] used the model parameters obtained through an experiment to design the input shaping control law for a non-uniform beam with a moving base. For the closed-loop control technique, Liu and Chao [18] presented an experimental study on implementing the

Fig. 1 An example of nonlinear cantilever beam operating in the 3D space: (a) Gantry robot (www.a-m-c.com/servo-drives-for-gantry-systems), (b) the coordinate systems and motions.

neuro-fuzzy approach to control a beam-cart system. In their work, the piezoelectric transducers located at the beam's tip were used to suppress the transverse vibration. Another closed-loop control of an Euler-Bernoulli beam with a translating base was done by Shah and Hong [10], whereas the control of a Timoshenko beam attached to a moving base was presented by Pham et al. [24].

Most of the published studies on the control flexible beams attached to a translating base consider the linear vibration [10, 16-18]. In these studies, under the assumption of the small-amplitude vibration, the dynamic tension, which makes the beam's dynamics nonlinear, is ignored. However, in such cases of large-amplitude vibration, the omission of the dynamic tension can affect the system's performance and lead to an erroneous result. Additionally, the existing studies assumed that the beam's vibration occurs in a plane; therefore, only transverse or lateral vibration is considered. Even though Shah and Hong [10] investigate both transverse and lateral vibrations of an Euler-Bernoulli beam, they ignored the coupled dynamics of the transverse and lateral vibrations; that is, the transverse vibration does not affect the time-evolution of the lateral vibration and vice versa.

Recently, the beam's three-dimensional vibration (in all three directions) has received significant attention [25-34]. Do and Pan [28] and Do [29] used the Euler-Bernoulli beam model with large-amplitude vibration to model a flexible riser. Consequently, the authors obtained a model describing the system's transverse, lateral, and longitudinal vibrations. Later, He et al. [31], Ji and Liu [32, 33], and Liu et al. [34] investigated the coupled three-dimensional cantilever beam with a tip payload described by a set of PDEs and ODEs. Furthermore, the control of three-dimensional beams, wherein the coupled dynamics of nonlinear vibrations are considered, has been investigated in [25-34]. However, these studies have only dealt with beams attached to a stationary base. Additionally, the authors proposed control strategies wherein control forces/torques are applied to the beam's tip. Although the numerical simulation showed the superior capability of suppressing the beam vibrations, these control strategies may not apply to all situations. For instance, in gantry manipulators, surgeon robots, and flexible liquid handling robots, implementing forces/torques in the beam's tip interferes with the system's operation.

In this paper, the beam's longitudinal vibration and axial deformation (which makes the beam's dynamics nonlinear) are further considered. In such cases of large-amplitude vibration, the omission of longitudinal vibration and axial deformation can affect the system's performance and lead to an erroneous result. Henceforth, the control problem of the nonlinear three-dimensional vibrations of a beam affixed on a translating base is addressed. The considered system is represented as a gantry robot consisting of the gantry, trolley, and flexible robotic arm depicted in Fig. 1. In the gantry robot, the gantry moves along the k -axis, whereas the trolley moves along the j -axis. A flexible robotic arm with a constant length is fixed on the trolley. The Hamilton principle is used to develop a novel hybrid model describing the nonlinear coupling dynamics of the robotic arm's transverse, lateral, and longitudinal vibrations, and the rigid body motions of the gantry and trolley. Employing the Lyapunov method, boundary control laws are developed for simultaneous control of the trolley's position, the gantry's position, and the robotic arm's 3D vibrations. The asymptotic stability of the closed-loop system is verified. Finally, the simulation results are provided.

The published papers in the literature on controlling cantilever-beam vibrations were restricted to: i) Cases where one end of the cantilever was fixed on a stationary base, and the other end had 3D motions [25-34]; and ii) the beam was attached to a translating base, but the considered dynamics were linear by ignoring the coupled dynamics between the transversal and lateral vibrations of the beam [10, 16-18]. Thus far, the control problem considering both the base motion and the coupled 3D dynamics of the beam has not been solved yet. The main contributions of this paper are summarized as follows: i) A novel dynamic model of a flexible beam attached to a translating base, wherein the coupled dynamics of the nonlinear transverse, lateral, and longitudinal vibrations and the base's motions are considered, is developed for the first time. ii) A boundary control strategy using the control forces of the base for position and vibration control simultaneously is designed. iii) The asymptotic stability of the closed-loop system is proven by using the Lyapunov method, and simulation results are provided.

This paper is organized as follows: Section 2 derives the dynamic model of the three-dimensional beam with a translating base. In Section 3, the boundary control based on the Lyapunov method is presented. The simulation results are given in Section 4. The conclusions are drawn in Section 5.

2. Problem formulation

In Fig. 1, the flexible robotic arm is modeled as a uniform Euler-Bernoulli beam of length l . The motions of the gantry and trolley are generated by two control forces f_z and f_y , respectively. The trolley's position and gantry's position are denoted by $y(t)$ and $z(t)$. The beam's vibrations in the i -axis, j -axis, and k -axis are defined as the longitudinal vibration $u(x, t)$, the transverse vibration $w(x, t)$, and the lateral vibration $v(x, t)$. In this study, the subscripts like $(\cdot)_x$ and $(\cdot)_t$ are the partial derivatives of the spatiotemporal functions with respect to x and t , respectively, whereas $\dot{y}$ and $\dot{z}$ denotes the total derivative of $y(t)$ and $z(t)$ with respect to t , respectively. The kinetic energy of the entire gantry, trolley, and beam system is given as follows:

$$K = \frac{1}{2} \rho A \int_0^l [(y + w_t)^2 + u_t^2 + (z + v_t)^2] dx + \frac{1}{2} (m_1 + m_2) \dot{y}^2 + \frac{1}{2} m_2 \dot{z}^2 \quad (1)$$

where ρ and A are the beam's mass density and cross-section area; m_1 and m_2 are the gantry's mass and trolley's mass, respectively. The potential energy due to the axial force, the axial deformation, and the bending moment is given as follows:

$$U = \int_0^l P(x) \left(\frac{w_x^2}{2} + \frac{v_x^2}{2} \right) dx + \frac{1}{2} \int_0^l EA \varepsilon(x, t)^2 dx + \frac{1}{2} EI_y \int_0^l w_{xx}^2 dx + \frac{1}{2} EI_z \int_0^l v_{xx}^2 dx \quad (2)$$

where $P(x) = \rho A(l - x)g$ is the axial force generated by the influence of the gravitational acceleration on the beam's elements [35, 36], E denotes Young's modulus, and I_y and I_z indicate the moments of inertia of the beam. The axial strain $\varepsilon(x, t)$ is given by the following approximation [37]:

$$\varepsilon(x, t) = u_x + w_x^2 / 2 + v_x^2 / 2. \quad (3)$$

The virtual work done on the system by the boundary control inputs and the friction is given as follows.

$$\delta W = f_y \delta y + f_z \delta z - c_w \int_0^l w_t \delta w dx - c_u \int_0^l u_t \delta u dx - c_v \int_0^l v_t \delta v dx \quad (4)$$

where c_w , c_u , and c_v are the structural damping coefficients (note that the subscripts w , u , and v stand for transverse, longitudinal, and lateral, respectively). According to the Hamilton principle, the dynamic model of the considered system and the corresponding boundary conditions are obtained as follows.

$$\rho A (\ddot{y} + w_{tt}) + c_w w_t - (P w_x)_x - EA [w_x (u_x + w_x^2 / 2 + v_x^2 / 2)]_x + EI_y w_{xxxx} = 0, \quad (5)$$

$$w(0, t) = w_x(0, t) = w_{xx}(l, t) = w_{xxx}(l, t) = 0, \quad (6)$$

$$\rho A u_{tt} + c_u u_t - EA [u_x + w_x^2 / 2 + v_x^2 / 2]_x = 0, \quad (7)$$

$$u(0, t) = u_x(l, t) + w_x^2(l, t) / 2 + v_x^2(l, t) / 2 = 0, \quad (8)$$

$$\rho A (\ddot{z} + v_{tt}) + c_v v_t - (P v_x)_x - EA [v_x (u_x + w_x^2 / 2 + v_x^2 / 2)]_x + EI_z v_{xxxx} = 0, \quad (9)$$

$$v(0, t) = v_x(0, t) = v_{xx}(l, t) = v_{xxx}(l, t) = 0, \quad (10)$$

$$(m_1 + m_2) \ddot{y} - c_w \int_0^l w_t dx + EI_y w_{xxx}(0, t) = f_y, \quad (11)$$

$$m_2 \ddot{z} - c_v \int_0^l v_t dx + EI_z v_{xxx}(0, t) = f_z. \quad (12)$$

The dynamics of the considered system are represented by the nonlinear PDE-ODE model in (5)-(12): Equations (5)-(10) are PDEs describing the transverse, longitudinal, and lateral vibrations of the robotic arm, respectively, whereas the ODEs in (11) and (12) represent the dynamics of the gantry and the trolley, respectively. Observably, the beam's motion affects the gantry and trolley's motions, and vice versa. Additionally, if the potential energy due to the axial deformation is ignored (i.e., $\varepsilon^2 = (u_x + w_x^2/2 + v_x^2/2)^2 \cong 0$), the nonlinear terms in (5), (7), and (9) vanish, and the coupling dynamics of the beam's vibrations can be decoupled.

3. Controller design

The control objectives include position control and vibration suppression: i) Move the gantry and trolley carrying the flexible beam to a desired position and ii) suppress the beam's transverse, lateral, and longitudinal vibrations. In this paper, two forces f_z and f_y applied to the gantry and trolley are used as the control inputs to achieve the control objectives. The position errors of the trolley and gantry are defined as follows:

$$e_y = y - y_d, \quad (13)$$

$$e_z = z - z_d \quad (14)$$

where y_d and z_d are the desired positions of the trolley and gantry, respectively. Based on the Lyapunov direct method, we design f_z and f_y to guarantee that the convergences of the vibrations, position errors, and velocities of the trolley and gantry to zero are achieved. The following control forces are proposed to stabilize the considered system.

$$f_y = -K_1 \dot{y} - K_2 e_y - K_3 w_{xx}(0, t), \quad (15)$$

$$f_z = -K_4 \dot{z} - K_5 e_z - K_6 v_{xx}(0, t) \quad (16)$$

where K_i ($i = 1, 2, \dots, 6$) are the control parameters. The implementation of these control laws requires the measurement of $w_{xx}(0, t)$ and $v_{xx}(0, t)$. In the practical system, these signals can be obtained by using strain gauge sensors attached at the clamp end of the beam.

The following lemmas and assumptions are used for stability analysis of the closed-loop system with the control law given in (15) and (16).

Lemma 1 [38] *Let $\varphi(x, t) \in \mathbb{R}$ be a function defined on $x \in [0, l]$ and $t \in [0, \infty)$ that satisfies the boundary condition $\varphi(0, t) = 0$, $\forall t \in [0, \infty)$, the following inequalities hold.*

$$\int_0^l \varphi^2(x, t) dx \leq l^2 \int_0^l \varphi_x^2(x, t) dx, \quad \forall x \in [0, l], \quad (17)$$

$$\varphi^2(x, t) \leq l \int_0^l \varphi_x^2(x, t) dx, \quad \forall x \in [0, l]. \quad (18)$$

Furthermore, if $\varphi(x, t)$ satisfies $\varphi_x(0, t) = 0$, $\forall t \in [0, \infty)$, then the following inequalities hold.

$$\int_0^l \varphi_x^2(x, t) dx \leq l^2 \int_0^l \varphi_{xx}^2(x, t) dx, \quad \forall x \in [0, l], \quad (19)$$

$$\varphi^2(x, t) \leq l^3 \int_0^l \varphi_{xx}^2(x, t) dx, \quad \forall x \in [0, l]. \quad (20)$$

Lemma 2 [39] *Let $\varphi_1(x, t), \varphi_2(x, t) \in \mathbb{R}$ be a function defined on $x \in [0, l]$. Then, the following inequality holds.*

$$\varphi_1(x, t) \varphi_2(x, t) \leq \varphi_1^2(x, t) / \delta + \delta \varphi_2^2(x, t), \quad \forall \delta > 0. \quad (21)$$

Lemma 3 [40] *If $\varphi(x, t): [0, l] \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is uniformly bounded, $\{\varphi(x, t)\}_{x \in [0, l]}$ is equicontinuous on t , and $\lim_{t \rightarrow \infty} \int_0^t \|\varphi(x, \tau)\|^2 d\tau$ exists and is finite, then $\lim_{t \rightarrow \infty} \|\varphi(x, t)\| = 0$.*

Assumption 1 [14] *The transverse vibration $w(x, t)$, the lateral vibration $v(x, t)$, and the longitudinal vibration $u(x, t)$ of a flexible beam satisfy the following inequalities: $u_x^2 \leq w_x^2 / 2$ and $u_x^2 \leq v_x^2 / 2$. Furthermore, by using Lemma 1, we obtain*

$$\int_0^l u^2 dx \leq l^2 \int_0^l u_x^2 dx \leq \frac{l^2}{4} \int_0^l w_x^2 dx + \frac{l^2}{4} \int_0^l v_x^2 dx \leq \frac{l^4}{4} \int_0^l w_{xx}^2 dx + \frac{l^4}{4} \int_0^l v_{xx}^2 dx. \quad (22)$$

Assumption 2 [41] *If the potential energy of the system in (2) is bounded for $\forall t \in [0, \infty)$, then $w_{xx}(x, t)$, $w_{xxx}(x, t)$, $v_{xx}(x, t)$, and $v_{xxx}(x, t)$ are bounded for $\forall t \in [0, \infty)$.*

Based on the system's mechanical energy, the following Lyapunov function candidate is introduced:

$$V = V_0 + V_1 \quad (23)$$

where

$$\begin{aligned} V_0 = & \frac{\rho A}{2} \left[\int_0^l (\dot{y} + w_t)^2 dx + \int_0^l (\dot{z} + v_t)^2 dx + \int_0^l u_t^2 dx \right] + \left(1 + \frac{2\alpha_2}{\rho A} \right) \left[\int_0^l P \left(\frac{w_x^2}{2} + \frac{v_x^2}{2} \right) dx \right. \\ & + \frac{1}{2} EA \int_0^l \left(u_x + \frac{w_x}{2} + \frac{v_x}{2} \right)^2 dx + EI_y \int_0^l w_{xx}^2 dx + EI_z \int_0^l v_{xx}^2 dx \left. \right] + \frac{1}{2} (m_1 + m_2) \dot{y}^2 \\ & + \frac{1}{2} m_2 \dot{z}^2 + \frac{1}{2} \alpha_1 e_y^2 + \alpha_2 \int_0^l (w_t^2 + u_t^2 + v_t^2) dx + \frac{1}{2} \alpha_3 e_z^2, \end{aligned} \quad (24)$$

$$\begin{aligned} V_1 = & \rho A \beta_1 \int_0^l w w_t dx + \frac{1}{2} \beta_1 c_w \int_0^l w^2 dx + \rho A \beta_2 \int_0^l u u_t dx + \frac{1}{2} \beta_2 c_u \int_0^l u^2 dx + (\beta_3 c_w / \rho A) \int_0^l w \dot{y} dx \\ & + \beta_3 \int_0^l \dot{y} (\dot{y} + w_t) dx + \beta_4 \dot{y} e_y + \beta_5 \int_0^l e_y (\dot{y} + w_t) dx + \rho A \beta_6 \int_0^l v v_t dx + \frac{1}{2} \beta_6 c_v \int_0^l v^2 dx \\ & + (\beta_7 c_v / \rho A) \int_0^l v \dot{z} dx + \beta_7 \int_0^l \dot{z} (\dot{z} + v_t) dx + \beta_8 \dot{z} e_z + \beta_9 \int_0^l e_z (\dot{z} + v_t) dx \end{aligned} \quad (25)$$

where α_i ($i = 1, 2, 3$) and β_j ($j = 1, 2, \dots, 9$) are positive coefficients.

Lemma 4. *The Lyapunov function candidate in (23) is upper and lower bounded as follows.*

$$0 \leq \lambda_1 W_1 \leq V \leq \lambda_2 W_2 \quad (26)$$

where λ_1 and λ_2 are positive constants, and

$$W_1 = \dot{y}^2 + \dot{z}^2 + \int_0^l w_t^2 dx + \int_0^l u_t^2 dx + \int_0^l v_t^2 dx + \int_0^l w_{xx}^2 dx + \int_0^l v_{xx}^2 dx + e_y^2 + e_z^2, \quad (27)$$

$$\begin{aligned} W_2 = & \dot{y}^2 + \dot{z}^2 + \int_0^l w_t^2 dx + \int_0^l u_t^2 dx + \int_0^l v_t^2 dx + \int_0^l P(w_x^2 + v_x^2) dx + \int_0^l (u_x + w_x/2 + v_x/2)^2 dx \\ & + \int_0^l w_{xx}^2 dx + \int_0^l v_{xx}^2 dx + e_y^2 + e_z^2. \end{aligned} \quad (28)$$

Proof of Lemma 4: See Appendix A.

Lemma 5. *Under the control laws (15) and (16), the time derivative of the Lyapunov function candidate in (23) is upper bounded as follows.*

$$\dot{V} \leq -\lambda V, \quad (29)$$

where λ is a positive constant.

Proof of Lemma 5: See Appendix B.

Theorem 1. Consider a hybrid system described by (5)-(12) under control laws (15)-(16) and Assumptions 1 and 2. Control parameters K_i ($i = 1, 2, \dots, 6$) are selected for satisfying the conditions in (A.16), (B.9), (B.17)-(B.22), and (B.26). The asymptotic stability of the closed-loop system in the sense that the transverse vibration $w(x, t)$, lateral vibration $v(x, t)$, longitudinal vibration $u(x, t)$, and position errors (13) and (14) converge to zero is guaranteed. Additionally, the control laws are bounded.

Proof of Theorem: Lemma 4 reveals that the Lyapunov function candidate in (23) is a positive definite. According to Lemma 5, we obtain

$$V(t) \leq e^{-\lambda t} V(0) \leq V(0) < \infty. \quad (30)$$

We define the norm of a spatiotemporal function as follows: $\|w(x, t)\| = \left(\int_0^l w^2(x, t) dx \right)^{1/2}$. Using Lemmas 1 and 4 and Assumption 1, the following inequalities are obtained.

$$w^2(x, t) \leq l^3 \int_0^l w_{xx}^2(x, t) dx \leq l^3 W_1 \leq l^3 V / \lambda_1 < \infty, \quad (31)$$

$$v^2(x, t) \leq l^3 \int_0^l v_{xx}^2(x, t) dx \leq l^3 W_1 \leq l^3 V / \lambda_1 < \infty, \quad (32)$$

$$u^2(x, t) \leq l \int_0^l u_x^2(x, t) dx \leq \frac{1}{2} l \int_0^l w_x^2(x, t) dx \leq \frac{1}{2} l^3 \int_0^l w_{xx}^2(x, t) dx \leq \frac{1}{2} l^3 V \lambda_1 < \infty, \quad (33)$$

$$e_y^2(t) \leq W_1 \leq V / \lambda_1 < \infty, \quad (34)$$

$$\dot{e}_y^2(t) \leq W_1 \leq V / \lambda_1 < \infty, \quad (35)$$

$$e_z^2(t) \leq W_1 \leq V / \lambda_1 < \infty, \quad (36)$$

$$\dot{e}_z^2(t) \leq W_1 \leq V / \lambda_1 < \infty. \quad (37)$$

Inequalities (31)-(37) assure $w(x, t)$, $u(x, t)$, $v(x, t)$, e_y , $\dot{e}_y$, e_z , and $\dot{e}_z$ are all uniformly bounded.

Similarly, we also obtain the boundedness of $\|w(x, t)\|^2$, $\|w_t(x, t)\|^2$, $\|u(x, t)\|^2$, $\|u_t(x, t)\|^2$, $\|v(x, t)\|^2$, and $\|v_t(x, t)\|^2$ based on Lemmas 4 and 5.

$$\begin{aligned} -\|w(x, t)\|^2 &\geq -l^4 W_1 \geq -l^4 V / \lambda_1 \geq l^4 \dot{V} / \lambda \lambda_1 \\ &\Rightarrow \lim_{t \rightarrow \infty} \int_0^t \|w(x, \tau)\|^2 d\tau \leq -l^4 \lim_{t \rightarrow \infty} (V(t) - V(0)) / \lambda \lambda_1 < \infty, \end{aligned} \quad (38)$$

$$\begin{aligned} -\|v(x, t)\|^2 &\geq -l^4 W_1 \geq -l^4 V / \lambda_1 \geq l^4 \dot{V} / \lambda \lambda_1 \\ &\Rightarrow \lim_{t \rightarrow \infty} \int_0^t \|v(x, \tau)\|^2 d\tau \leq -l^4 \lim_{t \rightarrow \infty} (V(t) - V(0)) / \lambda \lambda_1 < \infty, \end{aligned} \quad (39)$$

$$\begin{aligned} -\|u(x, t)\|^2 &\geq -l^2 \int_0^l u_x^2(x, t) dx \geq -\frac{1}{2} l^2 \int_0^l w_x^2(x, t) dx \geq -\frac{1}{2} l^4 W_1 \geq -\frac{1}{2} l^4 V \lambda_1 \geq \frac{1}{2} l^4 \dot{V} \lambda \lambda_1 \\ &\Rightarrow \lim_{t \rightarrow \infty} \int_0^t \|u(x, \tau)\|^2 d\tau \leq -\frac{1}{2} l^4 \lim_{t \rightarrow \infty} (V(t) - V(0)) \lambda \lambda_1 < \infty. \end{aligned} \quad (40)$$

Additionally, the following results also imply that $w(x, t)$, $u(x, t)$, and $v(x, t)$ are equicontinuous in t .

$$d\|w(x, t)\|^2 / dt = 2 \int_0^l w(x, t) w_t(x, t) dx \leq \|w(x, t)\|^2 + \|w_t(x, t)\|^2 < \infty, \quad (41)$$

$$d\|v(x, t)\|^2 / dt = 2 \int_0^l v(x, t) v_t(x, t) dx \leq \|v(x, t)\|^2 + \|v_t(x, t)\|^2 < \infty, \quad (42)$$

$$d\|u(x, t)\|^2 / dt = 2 \int_0^l u(x, t) u_t(x, t) dx \leq \|u(x, t)\|^2 + \|u_t(x, t)\|^2 < \infty. \quad (43)$$

Accordingly, we can conclude that $\lim_{t \rightarrow \infty} \|w(x, t)\| = 0$, $\lim_{t \rightarrow \infty} \|v(x, t)\| = 0$, and $\lim_{t \rightarrow \infty} \|u(x, t)\| = 0$ via Lemma 3.

Furthermore, Lemmas 4 and 5 also imply

$$-e_y^2(t) \geq -W_1(t) \geq -V(t)/\lambda_1 \geq \dot{V}(t)/\lambda_1 \lambda \Rightarrow \lim_{t \rightarrow \infty} \int_0^t e_y^2(\tau) d\tau \leq -\lim_{t \rightarrow \infty} (V(t) - V(0))/\lambda_1 \lambda < \infty, \quad (44)$$

$$-e_z^2(t) \geq -W_1(t) \geq -V(t)/\lambda_1 \geq \dot{V}(t)/\lambda_1 \lambda \Rightarrow \lim_{t \rightarrow \infty} \int_0^t e_z^2(\tau) d\tau \leq -\lim_{t \rightarrow \infty} (V(t) - V(0))/\lambda_1 \lambda < \infty. \quad (45)$$

Based on Barbalat's Lemma, we can conclude that $\lim_{t \rightarrow \infty} |e_y| = 0$ and $\lim_{t \rightarrow \infty} |e_z| = 0$.

Inequality (30) implies the boundedness of $V(t)$. It follows that the potential energy function is also a bounded function. Under Assumption 2, $w_{xx}(x, t)$, $w_{xxx}(x, t)$, $v_{xx}(x, t)$, and $v_{xxx}(x, t)$ are bounded. Inequalities (34)-(37) reveal that e_y , $\dot{e}_y$, e_z , and $\dot{e}_z$ are also bounded. Finally, we can conclude that the control laws in (15) and (16) are bounded. Theorem 1 is proven.

4. Simulation results

In this section, numerical simulations are performed to illustrate the effectiveness of the proposed control laws. The system parameters used in the numerical simulation are shown in Table 1. According to these system parameters, the control gains in (15) and (16) are selected as $K_1 = 750$, $K_2 = 950$, $K_3 = 1.12 \times 10^4$, $K_4 = 550$, $K_5 = 750$, and $K_6 = 2.34 \times 10^4$; parameters α_i , β_j , and δ_k have been also chosen such that the conditions in (A.16), (B.9), (B.17)-(B.22), and (B.26) are satisfied. The simulation is performed by using MATLAB, wherein the finite difference method is utilized to determine the approximate solutions for the equations of motion. The approximate solutions' accuracy and simulation speed depend on the sizes of the time and space steps (i.e., Δt and Δx , respectively). By using a large time step size, the approximate solutions of PDEs are determined quickly. However, a time step size that is too large reduces the accuracy of the solution and further leads to instabilities. Contrarily, the quality of the solutions can be improved by selecting the smaller step size. In this case, the simulation duration increases significantly. Therefore, selecting appropriate step sizes is necessary to guarantee a balance between accuracy and simulation speed. In this paper, the time and space step sizes are selected as follows: $\Delta t = 10^{-5}$ and $\Delta x = 0.075$.

The dynamic behaviors of the system are investigated in three cases: i) Using the traditional PD control law, ii) using the zero-vibration (ZV) input shaping control, and iii) using the proposed control law (15) and (16). For the input shaping control, the ZV input shapers are designed based on

Table 1. System parameters.

Parameter	Definition	Value
l	Beam length	1.5 m
h	Beam height	0.01 m
b	Beam width	0.006 m
A	Beam's cross-section area	$0.6 \times 10^{-4} \text{ m}^2$
I_y	Beam's initial moment	$1.8 \times 10^{-10} \text{ m}^4$
I_z	Beam's initial moment	$5 \times 10^{-10} \text{ m}^4$
ρ	Beam's mass density	2700 kg/m^2
E	Young's modulus	$69 \times 10^9 \text{ Pa}$
m_1	Gantry's mass	30 kg
m_2	Trolley's mass	20 kg
c_w	Transverse damping coefficient	0.05 Ns/m
c_v	Lateral damping coefficient	0.05 Ns/m
c_u	Longitudinal damping coefficient	0.05 Ns/m
y_d	Trolley's desired position	6 m
z_d	Gantry's desired position	4 m

the cantilever beam's natural frequencies and damping ratios. The natural frequencies are determined via the solution of the frequency equation (i.e., see [16]), whereas the damping ratios are calculated by using the logarithmic decrement algorithm (i.e., see [17]).

Figures 2 and 3 illustrate the system's responses under different controllers. Figure 2 shows the trolley's position and gantry's position, whereas Fig. 3 reveal the vibrations of the beam's tip. It shows that the PD controller, input shaping controller, and the proposed controller can move the trolley and gantry to the desired position (i.e., Fig. 2). However, the traditional PD controller cannot deal with the beam's vibrations (i.e., Fig. 3). In this case, vibration suppression was done only based on structural damping; therefore, it requires a significant amount of time. Contrary to the PD control, the system's vibrations under the input shaping control and proposed control law are quickly suppressed (i.e., Fig. 3). Most of the beam's tip oscillations are eliminated when the trolley and gantry reach the desired position (i.e., at $t \approx 4s$). Furthermore, the proposed control law shows the outstanding capability of suppressing the vibrations compared with the input shaping control (i.e., see the magnified graphs in Fig. 3). The control forces under the proposed control law and vibration suppression of all elements of the beam and are illustrated in Fig. 4 and Fig. 5.

Figures 6 and 7 reveal the robustness of the proposed control law. In Fig. 6, we consider the system under the influence of disturbances. Two boundary disturbances, $d_y(t) = 10\sin(20\pi t)$ and $d_z(t) = 8\sin(20\pi t)$, are applied to the trolley and gantry, respectively. As shown in Fig. 6, the proposed control law can still eliminate most of the vibrations of the beam system under boundary disturbance. The sensitivity of the proposed control law to the measurement noises of the sensors is considered in Fig. 7. In this case, 20% noises are added in the feedback signals $w_{xxx}(0, t)$ and $v_{xxx}(0, t)$. Observably, the measurement noises have no significant effects on the responses of the closed-loop system under the proposed control law. The simulation results show that the proposed control law is not too sensitive to disturbances and measurement noises.

Fig. 2. (a) Trolley's position and (b) gantry's position.

Fig. 3 Vibrations of the beam's tip: (a) Transverse vibration $w(l, t)$, (b) lateral vibration $v(l, t)$, and (c) longitudinal vibration $u(l, t)$.

Fig. 4 Control forces under the proposed control law

Fig. 5 Vibrations of the three-dimensional flexible beam under the proposed control law: (a) Transverse vibration $w(x, t)$, (b) lateral vibration $v(x, t)$, and (c) longitudinal vibration $u(x, t)$.

5. Conclusions

This paper investigated the vibration suppression problem of a three-dimensional cantilever beam fixed on a translating base. The equations of motions describing the nonlinear coupling dynamics of the beam's transverse, lateral, longitudinal vibrations, the gantry, and the trolley are developed based on the Hamilton principle. Accordingly, the control laws were designed. The asymptotic stability of the closed-loop system in the sense that the beam's transverse vibration, lateral vibration, longitudinal vibration, and gantry's position error and trolley's position error converge to zero was proven via the Lyapunov method. Simulation results showed the effectiveness of the proposed control laws.

In practical gantry systems, the length of the robotic arm changes over time, and the system is subjected to disturbances. Our future work will address extending the current control strategy to a varying-length beam. Furthermore, an experimental study will also be conducted.

Fig. 6 Vibrations of the beam's tip under boundary disturbances: (a) Transverse vibration $w(l, t)$, (b) lateral vibration $v(l, t)$, and (c) longitudinal vibration $u(l, t)$.

Appendix A

The proof of Lemma 4 is shown in here. By using Lemmas 1-2, we obtain

$$\left| \int_0^l w w_t dx \right| \leq (l^4 / \delta_0) \int_0^l w_{xx}^2 dx + \delta_0 \int_0^l w_t^2 dx, \quad (\text{A.1})$$

$$\left| \int_0^l u u_t dx \right| \leq \frac{1}{4} l^4 \delta_1 \left(\int_0^l w_{xx}^2 dx + \int_0^l v_{xx}^2 dx \right) + \delta_1 \int_0^l u_t^2 dx, \quad (\text{A.2})$$

$$\left| \int_0^l w \dot{y} dx \right| \leq l^4 \int_0^l w_{xx}^2 dx + l \dot{y}^2, \quad (\text{A.3})$$

$$\left| \int_0^l \dot{y} w_t dx \right| \leq (l / \delta_2) \dot{y}^2 + \delta_2 \int_0^l w_t^2 dx, \quad (\text{A.4})$$

$$|\dot{y} e_y| \leq \dot{y}^2 + e_y^2, \quad (\text{A.5})$$

$$\left| \int_0^l e_y (\dot{y} + w_t) dx \right| \leq 2l e_y^2 + l \dot{y}^2 + \int_0^l w_t^2 dx, \quad (\text{A.6})$$

Fig. 7 Vibrations of the beam's tip under measurement noises of feedback signals: (a) Transverse vibration $w(l, t)$, (b) lateral vibration $v(l, t)$, and (c) longitudinal vibration $u(l, t)$.

$$\left| \int_0^l v v_t dx \right| \leq (l^4 / \delta_3) \int_0^l v_{xx}^2 dx + \delta_3 \int_0^l v_t^2 dx, \quad (\text{A.7})$$

$$\left| \int_0^l v \dot{z} dx \right| \leq l^4 \int_0^l v_{xx}^2 dx + l \dot{z}^2, \quad (\text{A.8})$$

$$\left| \int_0^l \dot{z} v_t dx \right| \leq (l / \delta_4) \dot{z}^2 + \delta_4 \int_0^l v_t^2 dx, \quad (\text{A.9})$$

$$|\dot{z} e_z| \leq \dot{z}^2 + e_z^2, \quad (\text{A.10})$$

$$\left| \int_0^l e_z (\dot{z} + v_t) dx \right| \leq 2l e_z^2 + l \dot{z}^2 + \int_0^l v_t^2 dx. \quad (\text{A.11})$$

According to (A.1)-(A.11), the lower bound of V_0 and V_1 are given by

$$\begin{aligned} V_0 \geq & \frac{1}{2} \rho A \int_0^l u_t^2 dx + \left(\frac{1}{2} + \alpha_2 / \rho A \right) \left[EI_y \int_0^l w_{xx}^2 dx + EI_z \int_0^l v_{xx}^2 dx \right] + \frac{1}{2} (m_1 + m_2) \dot{y}^2 \\ & + \frac{1}{2} m_2 \dot{z}^2 + \frac{1}{2} \alpha_1 e_y^2 + \alpha_2 \int_0^l w_t^2 dx + \alpha_2 \int_0^l u_t^2 dx + \alpha_2 \int_0^l v_t^2 dx + \frac{1}{2} \alpha_3 e_z^2, \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned}
V_1 \geq & [\beta_3 l(1 - c_w / \rho A - 1 / \delta_2) - \beta_4 - \beta_5 l] \dot{y}^2 + [\beta_7 l(1 - c_v / \rho A - 1 / \delta_4) - \beta_8 - \beta_9 l] \dot{z}^2 \\
& - (\rho A \beta_1 \delta_1 + \beta_3 \delta_2 + \beta_5) \int_0^l w_t^2 dx - \rho A \delta_1 \beta_2 \int_0^l u_t^2 dx - (\rho A \beta_6 \delta_3 + \beta_7 \delta_4 + \beta_9) \int_0^l v_t^2 dx \\
& - l^4 (\rho A \beta_1 / \delta_1 + \rho A \beta_2 / 4 \delta_1 + \beta_3 c_w / \rho A) \int_0^l w_{xx}^2 dx \\
& - l^4 (\rho A \beta_6 / \delta_3 + \beta_7 c_v / \rho A + \rho A \beta_2 / 4 \delta_1) \int_0^l v_{xx}^2 dx - (\beta_4 + 2 \beta_5 l) e_y^2 - (\beta_8 + 2 \beta_9 l) e_z^2. \quad (A.13)
\end{aligned}$$

Based on (A.12) and (A.13), the lower bound of the Lyapunov function candidate is obtained as follows.

$$\begin{aligned}
V \geq & [(m_1 + m_2) / 2 + \beta_3 l(1 - c_w / \rho A - 1 / \delta_2) - \beta_4 - \beta_5 l] \dot{y}^2 \\
& + [m_2 / 2 + \beta_7 l(1 - c_v / \rho A - 1 / \delta_4) - \beta_8 - \beta_9 l] \dot{z}^2 \\
& + [\alpha_2 - (\rho A \beta_1 \delta_0 + \beta_3 \delta_2 + \beta_5)] \int_0^l w_t^2 dx + [\alpha_2 + \rho A / 2 - \delta_1 \rho A \beta_2] \int_0^l u_t^2 dx \\
& + [\alpha_2 - (\rho A \beta_6 \delta_3 + \beta_7 \delta_4 + \beta_9)] \int_0^l v_t^2 dx \\
& + [EI_y (1 / 2 + \alpha_2 / \rho A) - l^4 (\rho A \beta_1 / \delta_0 + \rho A \beta_2 / 4 \delta_1 + \beta_3 c_w / \rho A)] \int_0^l w_{xx}^2 dx \\
& + [EI_z (1 / 2 + \alpha_2 / \rho A) - l^4 (\rho A \beta_6 / \delta_3 + \rho A \beta_2 / 4 \delta_1 + \beta_7 c_v / \rho A)] \int_0^l v_{xx}^2 dx \\
& + [\alpha_1 / 2 - (\beta_4 + 2 \beta_5 l)] e_y^2 + [\alpha_5 / 2 - (\beta_8 + 2 \beta_9 l)] e_z^2 \\
\geq & \lambda_1 W_1 \quad (A.14)
\end{aligned}$$

where

$$\lambda_1 = \min \left\{ \begin{array}{c} (m_1 + m_2) / 2 + \beta_3 l(1 - c_w / \rho A - 1 / \delta_2) - \beta_4 - \beta_5 l \\ m_2 / 2 + \beta_7 l(1 - c_v / \rho A - 1 / \delta_4) - \beta_8 - \beta_9 l \\ \alpha_2 - (\rho A \beta_1 \delta_0 + \beta_3 \delta_2 + \beta_5) \\ \alpha_2 + \rho A / 2 - \delta_1 \rho A \beta_2 \\ \alpha_2 - (\rho A \beta_6 \delta_3 + \beta_7 \delta_4 + \beta_9) \\ EI_y (1 / 2 + \alpha_2 / \rho A) - l^4 (\rho A \beta_1 / \delta_0 + \rho A \beta_2 / 4 \delta_1 + \beta_3 c_w / \rho A) \\ EI_z (1 / 2 + \alpha_2 / \rho A) - l^4 (\rho A \beta_6 / \delta_3 + \rho A \beta_2 / 4 \delta_1 + \beta_7 c_v / \rho A) \\ \alpha_1 / 2 - (\beta_4 + 2 \beta_5 l) \\ \alpha_5 / 2 - (\beta_8 + 2 \beta_9 l) \end{array} \right\}. \quad (A.15)$$

and α_i , β_j , and δ_k (wherein $i = 1, 2, \dots, 6$, $j = 1, 2, \dots, 9$, and $k = 1, 2, \dots, 4$) satisfy

$$\lambda_1 > 0. \quad (A.16)$$

Similarly, the upper bound of the Lyapunov function candidate can be obtained by using (A.1)-(A.11), that is

$$\begin{aligned}
V \leq & [\rho A l + (m_1 + m_2) / 2 + \beta_4 + \beta_5 l + \beta_3 l(1 + c_w / \rho A + 1 / \delta_2)] \dot{y}^2 \\
& + [\rho A l + m_2 / 2 + \beta_8 + \beta_9 l + \beta_7 l(1 + c_v / \rho A + 1 / \delta_4)] \dot{z}^2 \\
& + (\rho A + \rho A \beta_1 \delta_0 + \beta_3 \delta_2 + \beta_5 + \alpha_2) \int_0^l w_t^2 dx + (\rho A + \rho A \beta_6 \delta_3 + \beta_7 \delta_4 + \beta_9 + \alpha_2) \int_0^l v_t^2 dx \\
& + (\rho A / 2 + \delta_1 \rho A \beta_2 + \alpha_2) \int_0^l u_t^2 dx \\
& + (1 / 2 + \alpha_2 / \rho A) \int_0^l P(w_x^2 + v_x^2) dx + (1 / 2 + \alpha_2 / \rho A) EA \int_0^l (u_x + w_x^2 / 2 + v_x^2 / 2)^2 dx \\
& + [EI_y (1 / 2 + \alpha_2 / \rho A) + l^4 (\rho A \beta_1 / \delta_0 + \beta_1 c_w / 2 + \rho A \beta_2 / 4 \delta_1 + \beta_2 c_w / 8 + \beta_3 c_w / \rho A)] \int_0^l w_{xx}^2 dx \\
& + [EI_z (1 / 2 + \alpha_2 / \rho A) + l^4 (\rho A \beta_6 / \delta_3 + \beta_6 c_v / 2 + \rho A \beta_2 / 4 \delta_1 + \beta_2 c_v / 8 + \beta_7 c_v / \rho A)] \int_0^l v_{xx}^2 dx \\
& + (\alpha_1 / 2 + \beta_4 + 2 \beta_5 l) e_y^2 + (\alpha_3 / 2 + \beta_8 + 2 \beta_9 l) e_z^2 \\
\leq & \lambda_2 W_2 \quad (A.17)
\end{aligned}$$

where

$$\lambda_2 = \max \left\{ \begin{array}{l} \rho A l + (m_1 + m_2) / 2 + \beta_4 + \beta_3 l + \beta_3 l (1 + c_w / \rho A + 1 / \delta_2) \\ \rho A l + m_2 / 2 + \beta_8 + \beta_9 l + \beta_7 l (1 + c_v / \rho A + 1 / \delta_4) \\ \rho A + \rho A \beta_1 \delta_0 + \beta_3 \delta_2 + \beta_5 + \alpha_2 \\ \rho A + \rho A \beta_6 \delta_3 + \beta_7 \delta_4 + \beta_9 + \alpha_2 \\ \rho A / 2 + \delta_1 \rho A \beta_2 + \alpha_2 \\ (1 / 2 + \alpha_2 / \rho A) \\ (1 / 2 + \alpha_2 / \rho A) E A \\ E I_y (1 / 2 + \alpha_2 / \rho A) + l^4 (\rho A \beta_1 / \delta_0 + \beta_1 c_w / 2 + \rho A \beta_2 / 4 \delta_1 + \beta_2 c_w / 8 + \beta_3 c_w / \rho A) \\ E I_z (1 / 2 + \alpha_2 / \rho A) + l^4 (\rho A \beta_6 / \delta_3 + \beta_6 c_v / 2 + \rho A \beta_2 / 4 \delta_1 + \beta_2 c_v / 8 + \beta_7 c_v / \rho A) \\ (\alpha_1 / 2 + \beta_4 + 2 \beta_5 l) \\ (\alpha_3 / 2 + \beta_8 + 2 \beta_9 l) \end{array} \right\}. \quad (\text{A.18})$$

Based on (A.14)-(A.15) and (A.17)-(A.18), Lemma 4 is proven.

Appendix B

The proof of Lemma 5 is shown here. The time derivative of V_0 is derived as follows.

$$\begin{aligned} \dot{V}_0 = & \rho A \int_0^l (\dot{y} + w_t)(\ddot{y} + w_{tt}) dx + \rho A \int_0^l (\dot{z} + v_t)(\ddot{z} + v_{tt}) dx + \rho A \int_0^l u_t u_{tt} dx \\ & + (1 + 2\alpha_2 / \rho A) \left[\int_0^l P(w_x w_{xt} + v_x v_{xt}) dx + E A \int_0^l (u_x + w_x^2 / 2 + v_x^2 / 2) (u_{xt} + w_x w_{xt} + v_x v_{xt}) dx \right. \\ & + E I_y \int_0^l w_{xx} w_{xxt} dx + E I_z \int_0^l v_{xx} v_{xxt} dx \left. \right] + (m_1 + m_2) \dot{y} \ddot{y} + m_2 \dot{z} \ddot{z} + \alpha_1 e_y \dot{y} + \alpha_3 e_z \dot{z} \\ & + 2\alpha_2 \left(\int_0^l w_t w_{tt} dx + \int_0^l u_t u_{tt} dx + \int_0^l v_t v_{tt} dx \right). \end{aligned} \quad (\text{B.1})$$

For notational convenience, ε is used instead of $(u_x + w_x^2 / 2 + v_x^2 / 2)$ (i.e., ε is the axial strain given in (3)). Substituting the dynamic model in (5)-(12) into (B.1) yields

$$\begin{aligned} \dot{V}_0 = & -c_w \int_0^l w_t^2 dx - c_u \int_0^l u_t^2 dx - c_v \int_0^l v_t^2 dx - c_w \int_0^l \dot{y} w_t dx - c_v \int_0^l \dot{z} v_t dx \\ & + [P w_x \dot{y}]_0^l + [P v_x \dot{z}]_0^l + [E A \varepsilon w_x \dot{y}]_0^l + [E A \varepsilon v_x \dot{z}]_0^l - [E I_y \dot{y} w_{xxx}]_0^l - [E I_z \dot{z} v_{xxx}]_0^l \\ & + (1 + 2\alpha_2 / \rho A) \left\{ [P w_x w_t]_0^l + [P v_x v_t]_0^l + [E A \varepsilon u_t]_0^l + [E A \varepsilon w_x w_t]_0^l + [E A \varepsilon v_x v_t]_0^l \right. \\ & + E I_y [w_{xx} w_{xt} - w_t w_{xxx}]_0^l + E I_z [v_{xx} v_{xt} - v_t v_{xxx}]_0^l \left. \right\} + (m_1 + m_2) \dot{y} \ddot{y} + m_2 \dot{z} \ddot{z} \\ & - (2c_w \alpha_2 / \rho A) \int_0^l w_t^2 dx - (2c_u \alpha_2 / \rho A) \int_0^l u_t^2 dx - (2c_v \alpha_2 / \rho A) \int_0^l v_t^2 dx \\ & - 2\alpha_2 \dot{y} \int_0^l w_t dx - 2\alpha_2 \dot{z} \int_0^l v_t dx + \alpha_1 e_y \dot{y} + \alpha_3 e_z \dot{z}. \end{aligned} \quad (\text{B.2})$$

By using the boundary conditions and $P(l) = 0$, (B.2) can be rewritten as

$$\begin{aligned} \dot{V}_0 = & -c_w \int_0^l w_t^2 dx - c_u \int_0^l u_t^2 dx - c_v \int_0^l v_t^2 dx - \dot{y} c_w \int_0^l w_t dx - \dot{z} c_v \int_0^l v_t dx + E I_y \dot{y} w_{xxx}(0, t) + E I_z \dot{z} v_{xxx}(0, t) \\ & + (m_1 + m_2) \dot{y} \ddot{y} + m_2 \dot{z} \ddot{z} - (2c_w \alpha_2 / \rho A) \int_0^l w_t^2 dx - (2c_u \alpha_2 / \rho A) \int_0^l u_t^2 dx - (2c_v \alpha_2 / \rho A) \int_0^l v_t^2 dx \\ & - 2\alpha_2 \dot{y} \int_0^l w_t dx - 2\alpha_2 \dot{z} \int_0^l v_t dx + \alpha_1 e_y \dot{y} + \alpha_3 e_z \dot{z} \end{aligned}$$

$$\begin{aligned}
&= -c_w (1 + 2\alpha_2 / \rho A) \int_0^l w_t^2 dx - c_u (1 + 2\alpha_2 / \rho A) \int_0^l u_t^2 dx - c_v (1 + 2\alpha_2 / \rho A) \int_0^l v_t^2 dx + \dot{y}f_y + \dot{z}f_z \\
&\quad - 2\alpha_2 \ddot{y} \int_0^l w_t dx - 2\alpha_2 \ddot{z} \int_0^l v_t dx + \alpha_1 e_y \dot{y} + \alpha_3 e_z \dot{z}.
\end{aligned} \tag{B.3}$$

The time derivative of V_1 is derived as follows.

$$\begin{aligned}
\dot{V}_1 &= \rho A \beta_1 \int_0^l w_t^2 dx + \rho A \beta_1 \int_0^l w w_{tt} dx + \beta_1 c_w \int_0^l w w_t dx + \rho A \beta_2 \int_0^l u_t^2 dx + \rho A \beta_2 \int_0^l u u_{tt} dx + \beta_2 c_u \int_0^l u u_t dx \\
&\quad + (\beta_3 c_w / \rho A) \int_0^l w_t \dot{y} dx + (\beta_3 c_w / \rho A) \int_0^l w \ddot{y} dx + \beta_3 \int_0^l \ddot{y} (\dot{y} + w_t) dx + \beta_3 \int_0^l \dot{y} (\ddot{y} + w_{tt}) dx + \beta_4 \ddot{y} e_y \\
&\quad + \beta_4 \dot{y}^2 + \beta_5 \int_0^l \dot{y} (\dot{y} + w_t) dx + \beta_5 \int_0^l e_y (\ddot{y} + w_{tt}) dx + \rho A \beta_6 \int_0^l v_t^2 dx + \rho A \beta_6 \int_0^l v v_{tt} dx + \beta_6 c_v \int_0^l v v_t dx \\
&\quad + (\beta_7 c_v / \rho A) \int_0^l v_t \dot{z} dx + (\beta_7 c_v / \rho A) \int_0^l v \ddot{z} dx + \beta_7 \int_0^l \ddot{z} (\dot{z} + v_t) dx + \beta_7 \int_0^l \dot{z} (\ddot{z} + v_{tt}) dx + \beta_8 \ddot{z} e_z \\
&\quad + \beta_8 \dot{z}^2 + \beta_9 \int_0^l \dot{z} (\dot{z} + v_t) dx + \beta_9 \int_0^l e_z (\ddot{z} + v_{tt}) dx.
\end{aligned} \tag{B.4}$$

Using the dynamic model and boundary conditions in (5)-(12) yields

$$\begin{aligned}
\dot{V}_1 &= (\beta_4 + \beta_5 l) \dot{y}^2 + (\beta_8 + \beta_9 l) \dot{z}^2 + \rho A \beta_1 \int_0^l w_t^2 dx + \rho A \beta_2 \int_0^l u_t^2 dx + \rho A \beta_6 \int_0^l v_t^2 dx \\
&\quad + \beta_1 \int_0^l w (P w_x)_x dx + \beta_1 \int_0^l w (E A \varepsilon w_x)_x dx + \beta_2 \int_0^l E A u \varepsilon_x dx + \beta_6 \int_0^l v (P v_x)_x dx \\
&\quad + \beta_6 \int_0^l v (E A \varepsilon v_x)_x dx - E I_y \beta_1 \int_0^l w w_{xxx} dx - E I_z \beta_6 \int_0^l v v_{xxx} dx + (c_w \beta_3 l / (m_1 + m_2)) \\
&\quad + \beta_5 \int_0^l \dot{y} w_t dx + (c_w \beta_4 l / (m_1 + m_2) - c_w \beta_5 / \rho A) \int_0^l e_y w_t dx \\
&\quad + (c_v \beta_7 l / m_2 + \beta_9) \int_0^l \dot{z} v_t dx + (c_v \beta_8 / m_2 - c_v \beta_9 / \rho A) \int_0^l e_z v_t dx \\
&\quad + (\beta_3 c_w / \rho A - \rho A \beta_1) \int_0^l w \ddot{y} dx + (\beta_7 c_v / \rho A - \rho A \beta_6) \int_0^l v \ddot{z} dx \\
&\quad + \beta_3 \int_0^l \ddot{y} w_t dx + \beta_7 \int_0^l \ddot{z} v_t dx + (\beta_3 l / (m_1 + m_2)) \dot{y} f_y \\
&\quad + (\beta_4 l / (m_1 + m_2)) e_y f_y + (\beta_7 l / m_2) \dot{z} f_z + (\beta_8 / m_2) e_z f_z \\
&\quad + E I_y (\beta_3 / \rho A - \beta_3 l / (m_1 + m_2)) \dot{y} w_{xxx}(0, t) + E I_y (\beta_5 / \rho A - \beta_4 l / (m_1 + m_2)) e_y w_{xxx}(0, t) \\
&\quad + E I_z (\beta_7 / \rho A - l \beta_7 / m_2) \dot{z} v_{xxx}(0, t) + E I_z (\beta_9 / \rho A - \beta_8 l / m_2) e_z v_{xxx}(0, t).
\end{aligned} \tag{B.5}$$

According to (B.3) and (B.5), the time derivative of V is derived as follows.

$$\begin{aligned}
\dot{V} &= \dot{y} \left[(\beta_3 l / (m_1 + m_2) + 1) f_y + (\beta_4 + \beta_5 l) \dot{y} + E I_y (\beta_3 / \rho A - \beta_3 l / (m_1 + m_2)) w_{xxx}(0, t) + \alpha_1 e_y \right] \\
&\quad + \dot{z} \left[(\beta_7 l / m_2 + 1) f_z + (\beta_8 + \beta_9 l) \dot{z} + E I_z (\beta_7 / \rho A - l \beta_7 / m_2) v_{xxx}(0, t) + \alpha_3 e_z \right] \\
&\quad - \left[c_w (1 + 2\alpha_2 / \rho A) - \rho A \beta_1 \right] \int_0^l w_t^2 dx - \left[c_u (1 + 2\alpha_2 / \rho A) - \rho A \beta_2 \right] \int_0^l u_t^2 dx \\
&\quad - \left[c_v (1 + 2\alpha_2 / \rho A) - \rho A \beta_6 \right] \int_0^l v_t^2 dx \\
&\quad + \beta_1 \int_0^l w (P w_x)_x dx + \beta_1 \int_0^l w (E A \varepsilon w_x)_x dx - E I_y \beta_1 \int_0^l w w_{xxx} dx + \beta_2 \int_0^l E A u \varepsilon_x dx \\
&\quad + \beta_6 \int_0^l v (P v_x)_x dx + \beta_6 \int_0^l v (E A \varepsilon v_x)_x dx - E I_z \beta_6 \int_0^l v v_{xxx} dx \\
&\quad + (\beta_3 c_w / \rho A - \rho A \beta_1) \int_0^l w \ddot{y} dx + (c_w \beta_3 l / (m_1 + m_2) + \beta_5) \int_0^l \dot{y} w_t dx \\
&\quad + (\beta_7 c_v / \rho A - \rho A \beta_6) \int_0^l v \ddot{z} dx + (\beta_7 l c_v / m_2 + \beta_9) \int_0^l \dot{z} v_t dx \\
&\quad - c_w (\beta_5 / \rho A - \beta_4 l / (m_1 + m_2)) \int_0^l e_y w_t dx - c_v (\beta_9 / \rho A - \beta_8 l / m_2) \int_0^l e_z v_t dx \\
&\quad + (\beta_3 - 2\alpha_2) \ddot{y} \int_0^l w_t dx + E I_y (\beta_5 / \rho A - \beta_4 l / (m_1 + m_2)) e_y w_{xxx}(0, t) + (\beta_4 l / (m_1 + m_2)) e_y f_y \\
&\quad + (\beta_7 - 2\alpha_2) \ddot{z} \int_0^l v_t dx + E I_z (\beta_9 / \rho A - \beta_8 l / m_2) e_z v_{xxx}(0, t) + (\beta_8 l / m_2) e_z f_z.
\end{aligned} \tag{B.6}$$

By using integration by parts and Lemma 1, the following inequality and equation are obtained.

$$\begin{aligned}
& \beta_1 \int_0^l w(Pw_x)_x dx + \beta_1 \int_0^l w(EA\varepsilon w_x)_x dx + \beta_2 \int_0^l EAu\varepsilon_x dx + \beta_6 \int_0^l v(Pv_x)_x dx + \beta_6 \int_0^l v(EA\varepsilon v_x)_x dx \\
& \leq -\beta_1 \int_0^l Pw_x^2 dx - \beta_6 \int_0^l Pv_x^2 dx + \delta_5 EA l^2 \left(\int_0^l w_{xx}^2 dx + \int_0^l v_{xx}^2 dx \right) \\
& - \left(\beta_2 - \left((2\beta_1 - \beta_2)^2 + (2\beta_6 - \beta_2)^2 \right) / 4\delta_5 \right) EA \int_0^l \varepsilon^2 dx,
\end{aligned} \tag{B.7}$$

$$-\beta_1 EI_y \int_0^l ww_{xxx} dx = -\beta_1 EI_y \int_0^l w_{xx}^2 dx; \quad -EI_z \beta_6 \int_0^l vv_{xxx} dx = -\beta_6 EI_z \int_0^l v_{xx}^2 dx \tag{B.8}$$

where δ_5 is a positive constant. It is noted that the inequalities $w_x^4 \ll w_x^2$ and $v_x^4 \ll v_x^2$ are used in (B.7). By letting β_i satisfy the following conditions

$$\begin{aligned}
& \beta_3 = \beta_7 = 2\alpha_2, \quad \beta_3 c_w / \rho A = \rho A \beta_1, \quad \beta_7 c_v / \rho A = \rho A \beta_6, \\
& \beta_5 / \rho A - \beta_4 / (m_1 + m_2) \geq 0, \quad \beta_9 / \rho A - \beta_8 / m_2 \geq 0,
\end{aligned} \tag{B.9}$$

and using Lemmas 1 and 2 for the terms $\int_0^l \dot{w}_t dx$, $\int_0^l e_y w_t dx$, $\int_0^l \dot{v}_t dx$, and $\int_0^l e_z v_t dx$ of (B.6), we obtain

$$\begin{aligned}
\dot{V} \leq & - \left[c_w (1 + 2\alpha_2 / \rho A) - \rho A \beta_1 - c_w \delta_7 (\beta_5 / \rho A - \beta_4 / (m_1 + m_2)) - \delta_6 (c_w \beta_3 l / (m_1 + m_2) + \beta_5) \right] \int_0^l w_t^2 dx \\
& - \left[c_u (1 + 2\alpha_2 / \rho A) - \rho A \beta_2 \right] \int_0^l u_t^2 dx \\
& - \left[c_v (1 + 2\alpha_2 / \rho A) - \rho A \beta_6 - \delta_9 c_v (\beta_9 / \rho A - \beta_8 / m_2) - \delta_8 (\beta_7 l c_v / m_2 + \beta_9) \right] \int_0^l v_t^2 dx \\
& - (\beta_1 EI_y - \delta_5 EA l^2) \int_0^l w_{xx}^2 dx - (\beta_6 EI_z - \delta_5 EA l^2) \int_0^l v_{xx}^2 dx - \max(\beta_1, \beta_6) \int_0^l P(w_x^2 + v_x^2) dx \\
& - \left(\beta_2 - \left((2\beta_1 - \beta_2)^2 + (2\beta_6 - \beta_2)^2 \right) / 4\delta_5 \right) \int_0^l EA \varepsilon^2 dx + D_y + D_z
\end{aligned} \tag{B.10}$$

where

$$\begin{aligned}
D_y = & \left[(\beta_3 l / (m_1 + m_2) + 1) f_y + \left((c_w \beta_3 l / (m_1 + m_2) + \beta_5) l / \delta_6 + \beta_4 + \beta_3 l \right) \dot{y} \right. \\
& + EI_y (\beta_3 / \rho A - \beta_3 l / (m_1 + m_2)) w_{xxx}(0, t) + \alpha_1 e_y \left. \right] \dot{y} \\
& + (c_w l (\beta_5 / \rho A - \beta_4 / (m_1 + m_2)) / \delta_7) e_y^2 + EI_y (\beta_5 / \rho A - \beta_4 / (m_1 + m_2)) e_y w_{xxx}(0, t) \\
& + (\beta_4 / (m_1 + m_2)) e_y f_y,
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
D_z = & \left[(\beta_7 l / m_2 + 1) f_z + (\beta_8 + \beta_9 l + (\beta_7 l c_v / m_2 + \beta_9) l / \delta_8) \dot{z} \right. \\
& + EI_z (\beta_7 / \rho A - \beta_7 l / m_2) v_{xxx}(0, t) + \alpha_5 e_z \left. \right] \dot{z} + (c_v l (\beta_9 / \rho A - \beta_8 / m_2) / \delta_9) e_z^2 \\
& + EI_z (\beta_9 / \rho A - \beta_8 / m_2) e_z v_{xxx}(0, t) + (\beta_8 / m_2) e_z f_z.
\end{aligned} \tag{B.12}$$

In (B.10)-(B.12), δ_i ($i = 6, 7, 8, 9$) are positive constants. Substituting the control laws in (15) and (16) into (B.11) and (B.12), respectively, yields

$$\begin{aligned}
D_y = & - \left[k_1 - \beta_4 - \beta_3 l - (c_w \beta_3 l / (m_1 + m_2) + \beta_5) l / \delta_6 \right] \dot{y}^2 \\
& - \left[k_2 \beta_4 / (\beta_3 l + m_1 + m_2) - c_w l (\beta_5 / \rho A - \beta_4 / (m_1 + m_2)) / \delta_7 \right] e_y^2 \\
& + (\beta_3 EI_y / \rho A - \beta_3 l EI_y / (m_1 + m_2) - k_3) \dot{y} w_{xxx}(0, t) + (\alpha_1 - k_2 - k_1 \beta_4 / (\beta_3 l + m_1 + m_2)) e_y \dot{y} \\
& + (\beta_3 EI_y / \rho A - \beta_4 EI_y / (m_1 + m_2) - k_3 \beta_4 / (\beta_3 l + m_1 + m_2)) e_y w_{xxx}(0, t),
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
D_z = & - \left[k_4 - \beta_8 - \beta_9 l - (\beta_7 l c_v / m_2 + \beta_9) l / \delta_8 \right] \dot{z}^2 \\
& - \left[k_5 \beta_8 / (\beta_7 l + m_2) - c_v l (\beta_9 / \rho A - \beta_8 / m_2) / \delta_9 \right] e_z^2 \\
& + (\beta_7 EI_z / \rho A - \beta_7 l EI_z / m_2 - k_6) \dot{z} v_{xxx}(0, t) + (\alpha_5 - k_5 - k_4 \beta_8 / (\beta_7 l + m_2)) e_z \dot{z} \\
& + (\beta_9 EI_z / \rho A - \beta_8 EI_z / m_2 - k_6 \beta_8 / (\beta_7 l + m_2)) e_z v_{xxx}(0, t)
\end{aligned} \tag{B.14}$$

where

$$k_i = (\beta_3 l / (m_1 + m_2) + 1) K_i, \quad i = 1, 2, 3, \quad (\text{B.15})$$

$$k_j = (\beta_7 l / m_2 + 1) K_j, \quad j = 4, 5, 6. \quad (\text{B.16})$$

If the following conditions hold

$$EI_y (\beta_3 / \rho A - \beta_3 l / (m_1 + m_2)) - k_3 = 0, \quad (\text{B.17})$$

$$\alpha_1 - k_2 - k_1 \beta_4 / (\beta_3 l + m_1 + m_2) = 0, \quad (\text{B.18})$$

$$EI_z (\beta_9 / \rho A - \beta_8 / m_2) - k_6 \beta_8 / (\beta_7 l + m_2) = 0, \quad (\text{B.19})$$

$$EI_z (\beta_7 / \rho A - \beta_7 l / m_2) - k_6 = 0, \quad (\text{B.20})$$

$$\alpha_5 - k_5 - k_4 \beta_8 / (\beta_7 l + m_2) = 0, \quad (\text{B.21})$$

$$EI_y (\beta_5 / \rho A - \beta_4 / (m_1 + m_2)) - k_3 \beta_4 / (\beta_3 l + m_1 + m_2) = 0, \quad (\text{B.22})$$

then (B.10) can be rewritten as follows.

$$\begin{aligned} \dot{V} \leq & - \left[k_1 - \beta_4 - \beta_5 l - (c_w \beta_3 l / (m_1 + m_2) + \beta_5) l / \delta_6 \right] \dot{y}^2 \\ & - \left[k_4 - \beta_8 - \beta_9 l - (\beta_7 l c_v / m_2 + \beta_9) l / \delta_8 \right] \dot{z}^2 \\ & - \left[k_2 \beta_4 / (\beta_3 l + m_1 + m_2) - c_w l (\beta_5 / \rho A - \beta_4 / (m_1 + m_2)) / \delta_7 \right] e_y^2 \\ & - \left[k_5 \beta_8 / (\beta_7 l + m_2) - c_v l (\beta_9 / \rho A - \beta_8 / m_2) / \delta_9 \right] e_z^2 \\ & - \left[c_w (1 + 2\alpha_2 / \rho A) - \rho A \beta_1 - \delta_6 (c_w \beta_3 l / (m_1 + m_2) + \beta_5) - c_w \delta_7 (\beta_5 / \rho A - \beta_4 / (m_1 + m_2)) \right] \int_0^l w_t^2 dx \\ & - \left[c_v (1 + 2\alpha_2 / \rho A) - \rho A \beta_6 - \delta_8 (c_v \beta_7 l / m_2 + \beta_9) - \delta_9 c_v (\beta_9 / \rho A - \beta_8 / m_2) \right] \int_0^l v_t^2 dx \\ & - \left[c_u (1 + 2\alpha_2 / \rho A) - \rho A \beta_2 \right] \int_0^l u_t^2 dx - E (\beta_1 I_y - \delta_5 A l^2) \int_0^l w_{xx}^2 dx - E (\beta_6 I_z - \delta_5 A l^2) \int_0^l v_{xx}^2 dx \\ & - \max(\beta_1, \beta_6) \int_0^l P(w_x^2 + v_x^2) dx - \left(\beta_2 - ((2\beta_1 - \beta_2)^2 + (2\beta_6 - \beta_2)^2 / 4\delta_5) \right) \int_0^l EA \varepsilon^2 dx. \end{aligned} \quad (\text{B.23})$$

Inequality (B.23) leads to the following result

$$\dot{V} \leq -\lambda_3 W_2 \quad (\text{B.24})$$

where

$$\lambda_3 = \min \left\{ \begin{array}{c} k_1 - \beta_4 - \beta_5 l - (c_w \beta_3 l / (m_1 + m_2) + \beta_5) l / \delta_6 \\ k_2 \beta_4 / (\beta_3 l + m_1 + m_2) - c_w (\beta_5 / \rho A - \beta_4 / (m_1 + m_2)) l / \delta_7 \\ \max(\beta_1, \beta_6) \\ k_4 - \beta_8 - \beta_9 l - (c_v \beta_7 l / m_2 + \beta_9) l / \delta_8 \\ k_5 \beta_8 / (\beta_7 l + m_2) - c_v l (\beta_9 / \rho A - \beta_8 / m_2) / \delta_9 \\ c_u (1 + 2\alpha_2 / \rho A) - \rho A \beta_2 \\ c_w (1 + 2\alpha_2 / \rho A) - \rho A \beta_1 - \delta_6 (c_w \beta_3 l / (m_1 + m_2) + \beta_5) - c_w \delta_7 (\beta_5 / \rho A - \beta_4 / (m_1 + m_2)) \\ \left[c_v (1 + (1 + 2\alpha_2 / \rho A)) - \rho A \beta_6 - \delta_8 (c_v \beta_7 l / m_2 + \beta_9) - \delta_9 c_v (\beta_9 / \rho A - \beta_8 / m_2) \right] \\ E (\beta_1 I_y - \delta_5 A l^2) \\ E (\beta_6 I_z - \delta_5 A l^2) \\ \beta_2 - ((2\beta_1 - \beta_2)^2 + (2\beta_6 - \beta_2)^2) / 4\delta_5 \end{array} \right\}. \quad (\text{B.25})$$

The coefficients α_i , β_j , and δ_k ($i = 1, 2, \dots, 6$, $j = 1, 2, \dots, 9$, and $k = 1, 2, \dots, 8$) satisfy the condition

$$\lambda_3 \geq 0. \quad (\text{B.26})$$

Based on Lemma 4, the following inequality is derived

$$\dot{V} \leq -\lambda_3 W_2 \leq -(\lambda_3 / \lambda_2) V \Rightarrow \dot{V} \leq -\lambda V \quad (\text{B.27})$$

where $\lambda = \lambda_3 / \lambda_2$. Accordingly, Lemma 5 is proven.

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Statements and Declarations

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Competing Interests

The authors declare that they have no conflict of interest.

Author Contributions

All authors contributed to the study conception and design. Modeling, control design, and simulations were performed by Phuong-Tung Pham. The first draft of the manuscript was written by Phuong-Tung Pham. Keum-Shik Hong and Quoc Chi Nguyen contributed to validation and manuscript revisions. All authors read and approved the final manuscript.

Data Availability

All data generated or analysed during this study are included in this published article.